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Uniformization of projective klt varieties by bounded symmetric domains

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Abstract

Using results from non-abelian Hodge theory for klt spaces developed by Greb, Kebekus, Peternell and Taji, we deduce necessary and sufficient conditions for projective varieties with klt singularities to be uniformized by bounded symmetric domains. This generalizes a well known result of Simpson to the singular setting. We apply this to obtain explicit Miyaoka-Yau-type equalities to characterize singular quotients of the four classical irreducible bounded symmetric domains, and the polydisk.

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1 Introduction

The problem of finding necessary and sufficient conditions for a complex projective variety to have a Hermitian symmetric space of non-compact type as its universal cover is a classical one, and has produced important contributions.

For example, Yau showed in [20] that surfaces of general type which satisfy equality in the Miyaoka-Yau inequality are uniformized by the unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$. This result has since been generalized to quasi-projective varieties (see [4]), and projective varieties with Kawamata log terminal (klt) singularities (see [9, 11]).

In [9], the authors formulated a singular version of the Miyaoka-Yau inequality in terms of so-called $\mathbb{Q}$-Chern classes, known as the $\mathbb{Q}$-Miyaoka-Yau inequality, which is satisfied by varieties of general type with klt singularities. They showed that if a complex projective variety of general type with klt singularities and nef canonical divisor satisfies equality in the $\mathbb{Q}$-Miyaoka-Yau inequality, then the associated canonical model is the quotient of the unit ball by the action of a discrete group.

The present work is inspired by the article [18], in which Simpson formulated necessary and sufficient conditions for a smooth complex projective variety to have an arbitrary Hermitian symmetric space of non-compact type as its universal cover. These

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conditions require the existence of a uniformizing variation of Hodge structure on the variety. A sufficient condition for uniformization by the polydisk, and the classical result of uniformization by the unit ball are some consequences of Simpson's work. In this article our goal is to generalize the main results of [18] to the klt setting, and as a consequence derive necessary and sufficient conditions for a complex projective klt variety with ample canonical divisor to be uniformized by a Hermitian symmetric space of non compact type.

The main result of this article is the following analog of Simpson's characterization of smooth compact quotients of bounded symmetric domains.

Theorem 1.1 *Let X be a n-dimensional complex projective klt variety with ample canonical divisor K_X. Let $\mathcal{D} = G_0/K_0$ be a Hermitian symmetric space of non-compact type. Then $X \cong \mathcal{D}/\Gamma$, where Γ is a discrete cocompact subgroup of* $\mathrm{Aut}(\mathcal{D})$*, whose action on $\mathcal{D}$ is fixed point free in codimension one, if and only if:*

1. *The smooth locus X_{reg} admits a uniformizing system of Hodge bundles (P, θ) for* $\mathrm{Aut}(\mathcal{D})$.
2. *X satisfies the $\mathbb{Q}$-Chern class equality $\widehat{ch}_2(\mathcal{E}') \cdot [K_X]^{n-2} = 0$, where $\mathcal{E}' := j_*(P \times_K \mathfrak{g})$, and $j : X_{reg} \hookrightarrow X$ is the inclusion.*

Typically, a bounded symmetric domain $\mathcal{D}$ is written as a quotient $\mathcal{D} = G_0/K_0$, where G_0 is a semisimple real algebraic Lie group, and $K_0 \subset G_0$ is a maximal compact subgroup which is unique up to conjugation. For a fixed $\mathcal{D}$, G_0 may not be unique; however, the canonical choice is the automorphism group $G_0 = \mathrm{Aut}(\mathcal{D})$. In condition 1 of Theorem 1.1, a uniformizing system of Hodge bundles on X_{reg} is, roughly speaking, the data of a reduction of structure group of the tangent bundle $\mathcal{T}_{X_{reg}}$ (see Definition 2.12 for a precise definition). In 2, the term $\widehat{ch}_2(\mathcal{E}')$ denotes the second $\mathbb{Q}$-Chern character of $\mathcal{E}'$ (Remark 2.18) and the equality $\widehat{ch}_2(\mathcal{E}') \cdot [K_X]^{n-2} = 0$ is a Miyaoka-Yau-type equality, which is numerical condition on the Chern classes of X. The group K is the complexification of K_0 and $\mathfrak{g}$ denotes the complexified Lie algebra of G_0 i.e., $\mathfrak{g} := \mathrm{Lie}(G_0) \otimes_{\mathbb{R}} \mathbb{C}$.

The proof of Theorem 1.1 consists mainly of two parts, one for each implication of the statement. The easier of the two is to show that if the universal cover of X is $\mathcal{D}$, then X satisfies 1 and 2. We apply Selberg's lemma to see that the quotient map $\mathcal{D} \to X$ factors through a smooth projective variety Y, such that the map $Y \to X$ is Galois and quasi-étale. By Simpson's uniformization theorem in the smooth case, Y satisfies conditions 1 and 2. We then use a simple argument to show that these descend to X.

The other direction is more technical. We use the non-abelian Hodge correspondance for klt spaces developed by Greb, Kebekus, Peternell and Taji in [11] to show that conditions 1 and 2 are equivalent to the existence of a so-called uniformizing variation of Hodge structure (P, θ) on the smooth locus X_{reg}. Roughly speaking, we show that there is a system of Hodge bundles $\mathcal{E} := P \times_K \mathfrak{g}$ on X_{reg} such that the reflexive sheaf $\mathcal{E}' := j_*\mathcal{E}$ carries a flat metric. Then, we pass to a Galois quasi-étale cover $\gamma : Y \to X$ such that the induced map of étale fundamental groups $\widehat{\pi}_1(Y_{reg}) \to \widehat{\pi}_1(Y)$ is an isomorphism. On Y, the reflexive sheaf $\mathcal{F} := \gamma^{[*]}\mathcal{E}'$ is actually locally free. It turns out that $\mathcal{F}$ contains the tangent sheaf $\mathcal{T}_Y$ as a direct summand, which means that $\mathcal{T}_Y$ is locally free and thus Y is smooth. Again, by Simpson's uniformization theorem in the smooth setting, it follows that Y is uniformized by a bounded symmetric domain, and we conclude by an additional simple argument that the same is true for X.

We apply Theorem 1.1 to obtain explicit characterizations of quotients of the polydisk, and each of the four classical irreducible bounded symmetric domains. When $\mathcal{D} = \mathcal{H}_n$ is the Siegel upper half-space for example, we have the following.

Theorem 1.2 *Let X be a projective klt variety of dimension $n(n+1)/2$ such that the canonical divisor K_X is ample. Then $X \cong \mathcal{H}_n/\Gamma$, where Γ is a discrete cocompact subgroup of $\mathrm{Aut}(\mathcal{H}_n) = PSp(2n, \mathbb{R})$, whose action on $\mathcal{D}$ is fixed point free in codimension one, if and only if X satisfies*

- $\mathcal{T}_{X_{reg}} \cong \mathrm{Sym}^2(\mathcal{E})$
- $[2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(\mathcal{E}')^2] \cdot [K_X]^{\dim X - 2} = 0,$

where $\mathcal{E}$ is a vector bundle of rank n on X_{reg}, and $\mathcal{E}' := j_*\mathcal{E}$.

2 Preliminaries

In this section, we introduce objects and notation that will be used to formulate our main results. We follow the conventions of [9, Section 2], [11, Sections 2 and 3] and of [18].

A closed subset Z of a normal quasi-projective variety X is called *small* if the codimension of Z in X is at least two. An open subset U of X is called *big* if the complement $X \setminus U$ is small. We call a variety X *of general type* if the canonical divisor K_X is big.

2.1 Higgs sheaves and stability

Definition 2.1 *(Higgs sheaves)* Let X be a smooth, quasi-projective variety. A *Higgs sheaf* on X is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a coherent sheaf of $\mathcal{O}_X$-modules, and $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ is an Ω^1_X-valued operator called the *Higgs field*, such that the composed morphism

$$\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \Omega^1_X \xrightarrow{\theta \otimes \mathrm{Id}} \mathcal{E} \otimes \Omega^1_X \otimes \Omega^1_X \xrightarrow{\mathrm{Id} \otimes \wedge} \mathcal{E} \otimes \Omega^2_X$$

is zero. This is an integrability condition and is usually denoted $\theta \wedge \theta = 0$.

Remark 2.2 For applications to uniformization (Sections 4-6), we work exclusively with Higgs sheaves over the smooth locus X_{reg} of a projective klt variety X (see Section 2.4). If $\mathcal{F} \subset \mathcal{E}$ is any subsheaf, then X_{reg} being quasi-projective and $\Omega^1_{X_{reg}}$ being locally free implies that the functor $- \otimes \Omega^1_{X_{reg}}$ is exact. In particular, $\mathcal{F} \otimes \Omega^1_{X_{reg}} \subset \mathcal{E} \otimes \Omega^1_{X_{reg}}$ holds.

Following the notation of [9, Section 2], we denote by $N^1(X)_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space of numerical Cartier divisor classes on a normal quasi-projective variety X. For any sheaf $\mathcal{F}$ on X whose determinant is $\mathbb{Q}$-Cartier, we denote the corresponding element of $N^1(X)_{\mathbb{Q}}$ by $[\mathcal{F}] := [\det\mathcal{F}]$. Using [9, Construction 2.17], we can compute the intersection products of Weil and Cartier divisors. We denote the intersection product of a coherent sheaf $\mathcal{F}$ with numerical classes $\alpha_1, ..., \alpha_{n-1} \in N^1(X)_{\mathbb{Q}}$ by $[\mathcal{F}] \cdot \alpha_1 \cdots \alpha_{n-1} \in \mathbb{Q}$. The *degree* of $\mathcal{F}$ with respect to a nef $\mathbb{Q}$-Cartier divisor H is defined as $\deg_H(\mathcal{F}) := [\mathcal{F}] \cdot [H] \cdots [H] = [\mathcal{F}] \cdot [H]^{n-1} \in \mathbb{Q}$, where $[H]$ denotes the class of H in $N^1(X)_{\mathbb{Q}}$.

Definition 2.3 *(Slope with respect to a nef divisor)* Let X be a normal projective variety of dimension n, and let H be a nef, $\mathbb{Q}$-Cartier divisor on X. If $\mathcal{F}$ is a torsion free Higgs sheaf on X, the *slope* of $\mathcal{F}$ with respect to H is given by

$$\mu_H(\mathcal{F}) := \frac{\deg_H(\mathcal{F})}{\operatorname{rank}(\mathcal{F})} = \frac{[\mathcal{F}] \cdot [H]^{n-1}}{\operatorname{rank}(\mathcal{F})}.$$

We call $\mathcal{F}$ *semistable* with respect to H if for any Higgs subsheaf $\mathcal{E} \subset \mathcal{F}$ with $0 < \operatorname{rank}(\mathcal{E}) < \operatorname{rank}(\mathcal{F})$, we have $\mu_H(\mathcal{E}) \leq \mu_H(\mathcal{F})$. We call $\mathcal{F}$ *stable* with respect to H if the strict inequality holds for all $\mathcal{E}$.

We would like to use notions of slope and stability which work for Higgs sheaves defined over the smooth locus of a normal projective variety, as developed in [10, Section 2]. For completeness, we state the relevant definitions which appear therein.

Note that all the sheaves with which we work are assumed to be algebraic. For coherent analytic sheaves $\mathcal{F}$ on the analytic space associated to X_{reg}, the pushforward $j_*\mathcal{F}$ is in general not analytic, as remarked in [10, Remark 2.24]. We will view X_{reg} as a quasi-projective variety over $\mathbb{C}$, hence coherent sheaves on X_{reg} will always be algebraic.

Definition 2.4 *(Slope for sheaves on the smooth locus)* Let X be a normal projective variety of dimension n, and let $\mathcal{E}$ be a torsion free Higgs sheaf of rank r on the smooth locus X_{reg}. If H is a nef, $\mathbb{Q}$-Cartier divisor on X, define the slope of $\mathcal{E}$ with respect to H as

$$\mu_H(\mathcal{E}) := \frac{\deg_H(j_*\mathcal{E})}{r} = \frac{[j_*\mathcal{E}] \cdot [H]^{n-1}}{r}$$

where $j : X_{reg} \to X$ is the inclusion.

As earlier, we call $\mathcal{E}$ *semistable* with respect to H if for any Higgs subsheaf $\mathcal{F} \subset \mathcal{E}$ on X_{reg} with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$, we have $\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})$. We call $\mathcal{E}$ *stable* with respect to H if the strict inequality holds.

2.2 Uniformizing systems of Hodge bundles

We borrow definitions from [18]. The variety X is assumed to be smooth and quasi-projective through Definitions 2.5-2.12.

Definition 2.5 *(System of Hodge sheaves)* A *system of Hodge sheaves* on X is a Higgs sheaf (E, θ) as in Definition 2.1, together with a splitting $E = \bigoplus_{p,q} E^{p,q}$ such that restricting the Higgs field θ to each $E^{p,q}$, we get $\mathcal{O}_X$-linear maps $\theta|_{E^{p,q}} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_X$. If E is locally free, we say it is a *system of Hodge bundles*.

Let E be a system of Hodge sheaves. A *subsystem of Hodge sheaves* F of E is a sub-Higgs sheaf of E which has the same decomposition $F = \bigoplus_{p,q} F^{p,q}$ as E, with Higgs field given by $\theta|_F : F^{p,q} \to F^{p-1,q+1} \otimes \Omega^1_X$. A system of Hodge sheaves E is *semistable* (resp. *stable*) with respect to a nef, $\mathbb{Q}$-Cartier divisor if for any proper subsystem of Hodge sheaves $F \subset E$, we have $\mu_H(F) \leq \mu_H(E)$ (resp. $\mu_H(F) < \mu_H(E)$). We call E *polystable* if it is a direct sum of stable subsystems of Hodge sheaves of the same slope.

We are interested in automorphism groups of bounded symmetric domains, which are real Lie groups with some additional properties.

Definition 2.6 *(Hodge group)* A *Hodge group* is a semisimple real algebraic Lie group G_0, together with a Hodge decomposition of the complexified Lie algebra

$$\mathfrak{g} = \bigoplus_p \mathfrak{g}^{p,-p}$$

such that $[\mathfrak{g}^{p,-p}, \mathfrak{g}^{r,-r}] \subset \mathfrak{g}^{p+r,-p-r}$ and such that $(-1)^{p+1}\,\mathrm{Tr}(\mathrm{ad}(U) \circ \mathrm{ad}(\bar{V})) > 0$ for $U, V \in \mathfrak{g}^{p,-p} \setminus \{0\}$. Here, $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ denotes the adjoint representation of the Lie algebra $\mathfrak{g}$.

Let G_0 be a Hodge group as in the above definition, and let $K_0 \subset G_0$ be the subgroup corresponding to the Lie algebra $\mathfrak{k} = \mathfrak{g}_0^{0,0}$. It is the subgroup of elements k such that $ad(k)$ preserves the Hodge decomposition of $\mathfrak{g}$. In particular $\mathrm{ad}(k)$ preserves the positive definite form $(-1)^{p+1}\mathrm{Tr}(\mathrm{ad}(U) \circ \mathrm{ad}(\bar{V}))$ so K_0 is compact.

Remark 2.7 Henceforth, we will fix the Hodge group G_0 to be linear, because this is the case for applications to uniformization (see Sections 4-6 and [18, Section 9]). We denote by G and K the complexifications of G_0 and K_0, respectively.

Definition 2.8 *(Principal system of Hodge bundles)* A *principal system of Hodge bundles* on X for a Hodge group G_0 is a principal K-bundle P on X together with a morphism of vector bundles

$$\theta : \mathcal{T}_X \to P \times_K \mathfrak{g}^{-1,1}$$

such that $[\theta(u), \theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_X$.

The morphism θ maps local sections of $\mathcal{T}_X$ to elements of $\mathfrak{g}^{-1,1}$, thus the Lie bracket appearing in the above definition is the usual Lie bracket of $\mathfrak{g}$ extended to the bundle $P \times_K \mathfrak{g}^{-1,1}$.

Remark 2.9 Let (P, θ) be a principal system of Hodge bundles on X as in Definition 2.8. We endow the bundle $P \times_K \mathfrak{g} = \bigoplus_{i \in \{-1,0,1\}} P \times_K \mathfrak{g}^{i,-i}$ with the structure of a system of Hodge bundles as follows. Consider the $\mathcal{O}_X$-linear morphism

$$\begin{aligned} P \times_K \mathfrak{g} &\to \mathrm{Hom}(\mathcal{T}_X, P \times_K \mathfrak{g}) \cong P \times_K \mathfrak{g} \otimes \Omega^1_X \\ a &\mapsto (v \mapsto [\theta(v), a]), \end{aligned}$$

where $[,]$ denotes the Lie bracket of $\mathfrak{g}$. By slight abuse of notation, we also denote this morphism by θ. Since $[\mathfrak{g}^{p,-p}, \mathfrak{g}^{r,-r}] \subset \mathfrak{g}^{p+r,-p-r}$, it follows that the image of $P \times_K \mathfrak{g}^{i,-i}$ via θ lands in $P \times_K \mathfrak{g}^{i-1,-i+1} \otimes \Omega^1_X$. It is straightforward to check that $\theta \wedge \theta = 0$, i.e., θ is a Higgs field and $P \times_K \mathfrak{g}$ is a system of Hodge bundles. This justifies the name in Definition 2.8.

Definition 2.10 *(Hodge group of Hermitian type)* A Hodge group G_0 of *Hermitian type* is a Hodge group such that the Hodge decomposition of $\mathfrak{g}$ has only types $(1, -1)$, $(0, 0)$, and $(-1, 1)$, and such that G_0 has no compact factors.

In the case of a Hodge group of Hermitian type, $K_0 \subset G_0$ is a maximal compact subgroup, and the quotient $\mathcal{D} = G_0/K_0$ is a *Hermitian symmetric space of non-compact type*, and it can also be realized as a *bounded symmetric domain*. Moreover, all bounded symmetric domains arise in this way.

We use the following result of Simpson on Lie algebras of Hodge groups of Hermitian type multiple times later, so we state it here.

Lemma 2.11 ([18, Corollary 9.3]) *Let $\mathfrak{g}$ be the complexified Lie algebra of a Hodge group of Hermitian type. Let $W \subset \mathfrak{g}$ be a sub-Hodge structure such that $[\mathfrak{g}^{-1,1}, W] \subset W$, where $[\cdot, \cdot]$ denotes the Lie bracket of $\mathfrak{g}$. Then*

$$\dim(W^{-1,1}) \geq \dim(W^{1,-1})$$

and if equality holds then W is a direct sum of ideals of $\mathfrak{g}$.

Definition 2.12 *(Uniformizing system of Hodge bundles)* Let G_0 be a Hodge group of Hermitian type. A *uniformizing system of Hodge bundles* on X for G_0 is a principal system of Hodge bundles (P, θ) on X such that the map $\theta : \mathcal{T}_X \to P \times_K \mathfrak{g}^{-1,1}$ is an isomorphism of sheaves.

Remark 2.13 A uniformizing system of Hodge bundles on X corresponds to a reduction of structure group of $\mathcal{T}_X$ from $GL(n, \mathbb{C})$ to K, where the map $K \to GL(n, \mathbb{C})$ is given by the adjoint representation.

Remark 2.14 Let $W \subset P \times_K \mathfrak{g}$ be a subsystem of Hodge sheaves of a uniformizing system of Hodge bundles $(P \times_K \mathfrak{g}, \theta)$. Then W is locally a sub-Hodge structure of $\mathfrak{g}$. From Remark 2.9, we know that the Higgs field θ is locally given by the Lie bracket of $\mathfrak{g}$. More precisely, the Higgs field restricted to W sends a local section $w \in W$ to $(v \mapsto [\theta(v), w]) \in P \times_K \mathfrak{g} \otimes \Omega^1_X$, where $v \in \mathfrak{g}^{-1,1}$. Since $\theta(W) \subset W \otimes \Omega^1_X$ by definition, it follows that $[\theta(v), w] \in W$ for all $v \in \mathfrak{g}^{-1,1}$ and $w \in W$, i.e., we have $[\mathfrak{g}^{-1,1}, W] \subset W$. In particular, Lemma 2.11 holds locally for W, and we conclude that $\mathrm{rank}(W^{-1,1}) \geq \mathrm{rank}(W^{1,-1})$.

Definition 2.15 *(Uniformizing variation of Hodge structure)* A *uniformizing variation of Hodge structure* on X is a uniformizing system of Hodge bundles (P, θ) together with a flat metric on the associated bundle $P \times_K \mathfrak{g}$.

The table below shows bounded symmetric domains $\mathcal{D} = G_0/K_0$, and the associated groups G_0, K_0, and their respective complexifications G, K, which we use to prove our uniformization statements (Sections 5-6).

Bounded symmetric domain	G_0	K_0	G	K
Polydisk ($\mathbb{H}^n$)	$SL(2, \mathbb{R})^n$	$U(1)^n$	$SL(2, \mathbb{C})^n$	$(\mathbb{C}^*)^n$
Type CI ($\mathcal{H}_n$)	$Sp(2n, \mathbb{R})$	$U(n)$	$Sp(2n, \mathbb{C})$	$GL(n, \mathbb{C})$
Type DIII ($\mathcal{D}_n$)	$SO^*(2n)$	$U(n)$	$SO^*(2n, \mathbb{C})$	$GL(n, \mathbb{C})$
Type BDI ($\mathcal{B}_n$)	$SO(2, n)$	$SO(2) \times SO(n)$	$SO(2+n, \mathbb{C})$	$SO(2, \mathbb{C}) \times SO(n, \mathbb{C})$
Type AIII ($\mathcal{A}_{pq}$)	$SU(p, q)$	$S(U(p) \times U(q))$	$SL(p+q, \mathbb{C})$	$S(GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$

2.3 Klt spaces and $\mathbb{Q}$-Chern classes

Let X be a klt space. After excluding a suitable subset $Z \subset X$ of codimension ≥ 3, only quotient singularities remain, and $X \setminus Z$ can be equipped with the structure of a $\mathbb{Q}$-variety that admits a global, Cohen-Macaulay cover (see [9, Section 3.4]). Thus following Mumford's work [17], Chern classes can be defined on $X \setminus Z$. Since $\operatorname{codim}_X(Z) \geq 3$, one can construct on any klt space of dimension $n \geq 3$ intersection products with first and second $\mathbb{Q}$-Chern classes of any reflexive sheaf $\mathcal{E}$ on X. The associated symmetric $\mathbb{Q}$-multilinear forms can be written as follows.

$$\begin{aligned}
\widehat{c}_1(\mathcal{E}) : N^1(X)_{\mathbb{Q}}^{n-1} &\to \mathbb{Q}, \quad (\alpha_1, \ldots, \alpha_{n-1}) \mapsto \widehat{c}_1(\mathcal{E}) \cdot \alpha_1 \cdot \cdots \cdot \alpha_{n-1} \\
\widehat{c}_1(\mathcal{E})^2 : N^1(X)_{\mathbb{Q}}^{n-2} &\to \mathbb{Q}, \quad (\alpha_1, \ldots, \alpha_{n-2}) \mapsto \widehat{c}_1(\mathcal{E})^2 \cdot \alpha_1 \cdot \cdots \cdot \alpha_{n-2} \\
\widehat{c}_2(\mathcal{E}) : N^1(X)_{\mathbb{Q}}^{n-1} &\to \mathbb{Q}, \quad (\alpha_1, \ldots, \alpha_{n-2}) \mapsto \widehat{c}_2(\mathcal{E}) \cdot \alpha_1 \cdot \cdots \cdot \alpha_{n-2}.
\end{aligned}$$

The subject of $\mathbb{Q}$-Chern classes has been covered in greater detail in [9, Section 3]. We will use the following important observation about the behaviour of $\mathbb{Q}$-Chern classes under quasi-étale covers.

Lemma 2.16 ([9, Lemma 3.16]) *Let $\gamma : Y \to X$ be a quasi-étale morphism between projective klt spaces. Then the following equalities hold for all reflexive sheaves $\mathcal{E}$ on X, and all numerical classes $\alpha_1, \ldots, \alpha_{n-1} \in N^1(X)_{\mathbb{Q}}$*

$$\widehat{c}_1(\gamma^{[*]}\mathcal{E}) \cdot \gamma^*\alpha_1 \cdot \cdots \cdot \gamma^*\alpha_{n-1} = (\deg \gamma)\widehat{c}_1(\mathcal{E}) \cdot \alpha_1 \cdot \cdots \cdot \alpha_{n-1},$$

where $\gamma^{[*]}\mathcal{E} = (\gamma^*\mathcal{E})^{\vee\vee}$. *Analogous statements hold for* $\widehat{c}_1(\gamma^{[*]}\mathcal{E})^2$ *and* $\widehat{c}_2(\gamma^{[*]}\mathcal{E})$.

Remark 2.17 We will often use the notation $\widehat{c}_1(\mathcal{F}) \cdot [H]^{n-1}$ in place of $[\mathcal{F}] \cdot [H]^{n-1}$ to denote the degree of a sheaf $\mathcal{F}$ with respect to a nef $\mathbb{Q}$-Cartier divisor H.

Remark 2.18 The first and second $\mathbb{Q}$-Chern characters of a reflexive sheaf $\mathcal{E}$ are defined in terms Chern classes as follows.

$$\widehat{ch}_1(\mathcal{E}) := \widehat{c}_1(\mathcal{E}) \quad \text{and} \quad \widehat{ch}_2(\mathcal{E}) := \frac{1}{2}(\widehat{c}_1(\mathcal{E})^2 - 2\widehat{c}_2(\mathcal{E}))$$

2.4 Harmonic bundles

In order to prove the sufficiency part of Theorem 1.1, we make use of the existence of the so-called maximally quasi-étale covers.

Theorem 2.19 ([8, Theorem 1.14]) *Let X be a projective klt variety. Then X admits a quasi-étale cover $\gamma : Y \to X$ such that the natural map of étale fundamental groups $\widehat{\pi}_1(Y_{reg}) \to \widehat{\pi}_1(Y)$ is an isomorphism. This is known as a* maximally quasi-étale cover.

We also crucially use the fact that any reflexive polystable Higgs sheaf on the smooth locus of a projective klt variety with vanishing first and second $\mathbb{Q}$-Chern classes is locally free and extends to a Higgs bundle on any maximally-quasi-étale cover.

Theorem 2.20 ([11, Theorem 5.1 and Proposition 3.17]) *Let X be a projective klt space of dimension $n \geq 2$. Let H be an ample divisor on X and use H to equip X_{reg} with a Kähler metric. Let $(\mathcal{E}_{X_{reg}}, \theta_{X_{reg}})$ be a reflexive Higgs sheaf on X_{reg} and let $\mathcal{E}$ denote the reflexive extension of $\mathcal{E}_{X_{reg}}$ to X.*

Suppose that the Higgs sheaf $(\mathcal{E}_{X_{reg}}, \theta_{X_{reg}})$ is polystable with respect to H, and the $\mathbb{Q}$-Chern characters of $\mathcal{E}$ satisfy

$$\widehat{ch}_1(\mathcal{E}) \cdot [H]^{n-1} = 0 \quad \text{and} \quad \widehat{ch}_2(\mathcal{E}) \cdot [H]^{n-2} = 0.$$

Then $\mathcal{E}_{X_{reg}}$ is locally free on X_{reg} and for any maximally quasi-étale cover $\gamma : Y \to X$, the sheaf $\mathcal{F} := \gamma^{[*]}\mathcal{E}$ is locally free on Y and all the Chern classes of $\mathcal{F}$ vanish.

3 Simpson's uniformization theorem

Our strategy to prove Theorem 1.1 for projective klt varieties is to use the results in Section 2.4 and reduce to the smooth case, at which point one can apply the results of [18].

In the case where X is a smooth compact complex manifold, Simpson derives the following necessary and sufficient conditions for X to be uniformized by a Hermitian symmetric space $\mathcal{D}$ of noncompact type.

Theorem 3.1 ([18, Proposition 9.1 and Theorem 2]) *Let X be a smooth compact complex manifold and let $\widetilde{X}$ be the universal cover of X. Then $\widetilde{X}$ is isomorphic to the bounded symmetric domain $\mathcal{D}$ if and only if X admits a uniformizing variation of Hodge structure for some Hodge group G_0 with $\mathcal{D} = G_0/K_0$.*

In other words, $\widetilde{X} \cong \mathcal{D}$ if and only if

1. *There is a uniformizing system of Hodge bundles (P, θ) for a Hodge group G_0 of Hermitian type corresponding to $\mathcal{D}$, such that $P \times_K \mathfrak{g}$ is polystable with respect to K_X as a Higgs bundle, and*
2. *The Chern class equality $c_2(P \times_K \mathfrak{g}) \cdot [K_X]^{n-2} = 0$ holds.*

4 Extending Simpson's result to the klt case

We are now ready to prove Theorem 1.1. In order to do so, we first prove that any uniformizing system of Hodge bundles on a projective klt variety of general type is polystable.

4.1 Auxiliary remarks

The following observation on the semistability of the tangent sheaf of a variety of general type is a slight generalization of a result of H. Guenancia ([12, Theorem A]). It allows us to relax the assumption that the tangent sheaf has negative degree, and thus strengthen Simpson's uniformization result ([18, Corollary 9.7]).

Proposition 4.1 ([9], Theorem 7.1) *Let X be a projective klt variety of general type whose canonical divisor K_X is nef. Then the sheaves $\mathcal{T}_X$ and $\Omega_X^{[1]}$ are semistable with respect to K_X.*

As a special case of [10, Lemma 2.26], we see that the (semi)stability of a sheaf on a normal projective variety X in the sense of Definition 2.4 is equivalent to the (semi)stability of the sheaf restricted to the smooth locus X_{reg}.

Lemma 4.2 *Let $\mathcal{E}$ be a torsion-free coherent sheaf on a normal projective variety X and let H be a nef $\mathbb{Q}$-Cartier divisor on X. Then $\mathcal{E}$ is semistable (resp. stable) with respect to H if and only if $\mathcal{E}|_{X_{reg}}$ is semistable (resp. stable) with respect to H.*

Proof Let $j : X_{reg} \to X$ denote the inclusion. If for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$ we have $\mu_H(\mathcal{F}) > \mu_H(\mathcal{E})$, then it follows that $\mu_H(\mathcal{F}|_{X_{reg}}) > \mu_H(\mathcal{E}|_{X_{reg}})$. Indeed, by Definition 2.4 the slope of $\mathcal{F}|_{X_{reg}}$ with respect to H on X_{reg} is defined as the slope of $j_*\mathcal{F}$ with respect to H on X, i.e., we have

$$\mu_H(\mathcal{F}|_{X_{reg}}) := \mu_H(j_*\mathcal{F}|_{X_{reg}}) = \frac{[j_*\mathcal{F}_{X_{reg}}] \cdot [H]^{n-1}}{\mathrm{rank}(\mathcal{F})}.$$

Using the equality of Weil divisors

$$[j_*\mathcal{F}|_{X_{reg}}] = [\mathcal{F}^{\vee\vee}] = [\mathcal{F}]$$

on X, it follows that $\mu_H(j_*\mathcal{F}|_{X_{reg}}) = \mu_H(\mathcal{F}^{\vee\vee}) = \mu_H(\mathcal{F})$. Similarly, we have $\mu_H(\mathcal{E}|_{X_{reg}}) = \mu_H(\mathcal{E})$.

Conversely, the same argument shows that if $\mu_H(\mathcal{F}|_{X_{reg}}) > \mu_H(\mathcal{E}|_{X_{reg}})$ then we have $\mu_H(\mathcal{F}) > \mu_H(\mathcal{E})$. □

Remark 4.3 In particular, if X is a projective klt variety of general type with K_X nef, the vector bundles $\mathcal{T}_{X_{reg}}$ and $\Omega^1_{X_{reg}}$ on X_{reg} are semistable with respect to K_X.

An important observation due to A. Langer is that the stability of a system of Hodge sheaves is equivalent to the stability of the underlying Higgs sheaf (see [15, Proposition 8.1])

We give the following independent proof of this result for systems of Hodge sheaves on the smooth locus of a projective variety X with klt singularities.

Lemma 4.4 *Let X be a projective variety with klt singularities and let $(E = \bigoplus_{p=0}^n E^p, \theta)$ be a torsion free system of Hodge sheaves on the smooth locus X_{reg}. Then E is (semi)stable as a system of Hodge sheaves if and only if E is (semi)stable as a Higgs sheaf, in the sense of Definition 2.4.*

Proof By definition, we have $\theta : E^p \to E^{p-1} \otimes \Omega^1_{X_{reg}}$ for all $0 \le p \le n$. It is clear that if E is (semi)stable as a Higgs sheaf, then it is (semi)stable as a system of Hodge sheaves. It remains to show the converse, i.e., that the (semi)stability of E as a system of Hodge sheaves implies the (semi)stability of E as a Higgs sheaf.

To this end, let $F \subset E$ be a Higgs subsheaf. The idea is to construct a subsystem of Hodge sheaves $F' = \bigoplus_{p=0}^n F'^p$ of E such that F and F' have the same rank and first Chern class. For each i, define $G_i = F \cap \bigoplus_{p<i} E^p$, where $0 \le i \le n+1$. So, for example, $G_0 = 0$, $G_1 = F \cap E_0$, and $G_{n+1} = F \cap \bigoplus_{p<n+1} E^p = F$. Thus, the G_i's give a filtration of F

$$0 = G_0 \subset G_1 \subset \cdots \subset G_n \subset G_{n+1} = F.$$

Since $\theta(F) \subset F \otimes \Omega^1_{X_{reg}}$, and $\theta(\bigoplus_{p<i} E^p) \subset (\bigoplus_{p<i-1} E^p) \otimes \Omega^1_{X_{reg}}$, it follows that $\theta(G_i) \subset G_{i-1} \otimes \Omega^1_{X_{reg}}$ for all $0 \le i \le n+1$. Let $\{F'^i\}_{i=0}^n$ be the quotients of this filtration, i.e., $F'^i = G_{i+1}/G_i$ for all i. Consider the sequence of maps

$$G_{i+1} \hookrightarrow \bigoplus_{p<i+1} E^p \to E^i$$

where the first map is the inclusion and the second is the projection. The kernel of the composed map is G_i, so the image of G_{i+1} in E^i is isomorphic to the quotient $F'^i = G_{i+1}/G_i$. Thus, we have $F'^i \subset E^i$ for all $0 \le i \le n$. Moreover, since $\theta(G_i) \subset G_{i-1} \otimes \Omega^1_{X_{reg}}$, and tensoring with $\Omega^1_{X_{reg}}$ is exact, it follows that $\theta(F'^i) \subset F'^{i-1} \otimes \Omega^1_{X_{reg}}$ for all $0 \le i \le n$. Hence $F' = \bigoplus_{p=0}^n F'^p \subset \bigoplus_{p=0}^n E^p$ is a subsystem of Hodge sheaves.

To see that F and F' have the same numerical invariants, we look at the series of short exact sequences

$$
\begin{aligned}
&0 \to G_0 = 0 \to G_1 \to F'^0 \to 0 \\
&0 \to G_1 \to G_2 \to F'^1 \to 0 \\
&\vdots \\
&0 \to G_{n-1} \to G_n \to F'^{n-1} \to 0 \\
&0 \to G_n \to G_{n+1} = F \to F'^n \to 0
\end{aligned}
$$

From the first exact sequence, it follows that $\mathrm{rank}(G_1) = \mathrm{rank}(F'^0)$. This implies that $\mathrm{rank}(G_2) = \mathrm{rank}(F'^0) + \mathrm{rank}(F'^1)$, and repeating this gives $\mathrm{rank}(G_i) = \mathrm{rank}(F'^0) + \cdots + \mathrm{rank}(F'^{i-1})$. From the last exact sequence, we get $\mathrm{rank}(F) = \mathrm{rank}(F'^0) + \cdots + \mathrm{rank}(F'^n) = \mathrm{rank}(F')$. The same computation holds for the first Chern classes, so we have $c_1(F) = c_1(F')$. □

Recall that any bounded symmetric domain $\mathcal{D}$ can be expressed as a quotient $\mathcal{D} = G_0/K_0$, where G_0 is a Hodge group of Hermitian type with maximal compact subgroup K_0. Their complexifications are denoted G and K respectively. The Lie algebra $\mathfrak{g}$ of G decomposes as $\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$. We know from the discussion in Section 3.1 that any smooth, projective quotient X of $\mathcal{D}$ satisfies $\mathcal{T}_X \cong P \times_K \mathfrak{g}^{-1,1}$, where P is a principal K-bundle on X. Moreover, X admits a system of Hodge bundles $P \times_K \mathfrak{g}$ which has a Hodge decomposition induced from that of $\mathfrak{g}$.

We will now consider the system of Hodge bundles $P \times_K \mathfrak{g}$ over the smooth locus X_{reg} of a projective klt variety X. The semistability of the bundle $P \times_K \mathfrak{g}^{0,0}$ is necessary to show that the system of Hodge bundles $P \times_K \mathfrak{g}$ is polystable as a Higgs bundle in the sense of Definition 2.4. This will play a central role in the proofs of statements that appear later. We observe the following.

Lemma 4.5 *Let X be a projective klt variety of general type with K_X nef, and let (P, θ) define a uniformizing system of Hodge bundles for a Hodge group G_0 of Hermitian type on X_{reg}. Then the vector bundle $P \times_K \mathfrak{g}^{0,0}$ is semistable with respect to K_X of degree* 0.

Proof Let $\mathcal{T}_{X_{reg}} = \bigoplus V_i$ be a decomposition corresponding to irreducible representations of $K \to \mathrm{Aut}(\mathfrak{g}^{-1,1})$. First, suppose that the group G is connected. Let $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ be the decomposition into simple ideals. Then $\mathfrak{g}^{-1,1} = \bigoplus \mathfrak{g}_i^{-1,1}$ is the decomposition into irreducible representations of K. Since $\mathcal{T}_{X_{reg}}$ is semistable by Proposition 4.1, it follows that $V_i \cong P \times_K \mathfrak{g}_i^{-1,1}$ and their duals $P \times_K \mathfrak{g}_i^{1,-1}$ are also semistable. Since the Lie bracket $\mathfrak{g}_i^{-1,1} \otimes \mathfrak{g}_i^{1,-1} \to \mathfrak{g}_i^{0,0}$ is surjective (see [18, proof of Proposition 9.6]), we get a surjective map of vector bundles

$$(P \times_K \mathfrak{g}_i^{-1,1}) \otimes (P \times_K \mathfrak{g}_i^{1,-1}) \to P \times_K \mathfrak{g}_i^{0,0},$$

where the left hand side is a semistable vector bundle of degree zero and the right hand side is a vector bundle of degree zero. It follows that $P \times_K \mathfrak{g}_i^{0,0}$ is also semistable. Indeed, if it is not, then neither is the dual bundle $(P \times_K \mathfrak{g}_i^{0,0})^\vee$. Note that $(P \times_K \mathfrak{g}_i^{0,0})^\vee$ is also of degree zero and is a subbundle of $(P \times_K \mathfrak{g}_i^{-1,1}) \otimes (P \times_K \mathfrak{g}_i^{1,-1})$. Thus, if some subsheaf of $(P \times_K \mathfrak{g}_i^{0,0})^\vee$ destabilizes it, then it also destabilizes $(P \times_K \mathfrak{g}_i^{-1,1}) \otimes (P \times_K \mathfrak{g}_i^{1,-1})$, which is a contradiction. Since $P \times_K \mathfrak{g}^{0,0} = \bigoplus_i P \times_K \mathfrak{g}_i^{0,0}$, it follows that $P \times_K \mathfrak{g}^{0,0}$ is also a semistable vector bundle of degree zero.

Now suppose that G is not connected. Let G' be the connected component of G, let $K' = K \cap G'$, and let $f' : Y' \to X_{reg}$ be a finite étale cover on which the structure group of P can be reduced to K'. Note that Y' can be completed to a projective klt variety Y such that K_Y is nef, and there is a finite quasi-étale map $f : Y \to X$, which restricts to f' on Y'. Since the V_i's are semistable with respect to K_X, the pullbacks $f'^* V_i$ are semistable with respect to $f^* K_X = K_Y$. If we decompose $\mathcal{T}_Y = \bigoplus_k V'_k$ corresponding to irreducible components of the representation $K' \to \mathrm{Aut}(\mathfrak{g}^{-1,1})$, the V'_k's are direct summands of $f'^* V_i$, so they are also semistable with respect to K_Y. Since $V'_k \cong f'^*(P \times_K \mathfrak{g}_k^{-1,1}) = f'^* P \times_{K'} \mathfrak{g}_k^{-1,1}$, and G' is connected, it follows that $f'^*(P \times_K \mathfrak{g}^{0,0})$ is semistable with respect to K_Y of degree zero. Since f' is finite and étale, it follows that $P \times_K \mathfrak{g}^{0,0}$ is also semistable with respect to K_X of degree zero. Indeed, if not, then the pullback along f' of any destabilizing subsheaf of $P \times_K \mathfrak{g}^{0,0}$ will destabilize $f'^*(P \times_K \mathfrak{g}^{0,0})$. □

The following observation is at the heart of the proof of Theorem 1.1, and we again split the proof into two cases, namely G connected and disconnected.

Proposition 4.6 *Let X be a projective klt variety with ample canonical divisor K_X and let (P, θ) define a uniformizing system of Hodge bundles for a Hodge group G_0 on X_{reg}. Then the system of Hodge bundles $P \times_K \mathfrak{g}$ is K_X-polystable as a Higgs bundle on X_{reg}. If $\mathfrak{g}$ is a simple Lie algebra, then it is K_X-stable.*

Remark 4.7 The system of Hodge bundles $P \times_K \mathfrak{g}$ on X_{reg} being K_X-polystable is equivalent to the restriction $(P \times_K \mathfrak{g})|_{X'}$ being K_X-polystable on X', for any big open subset $X' \subset X_{reg}$. Thus, we may replace X_{reg} by any big open subset X' in Proposition 4.6.

Proof of Proposition 4.6 First, suppose that the complexification G of G_0 is connected. By assumption, there is an isomorphism of vector bundles $\theta : \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$ on X_{reg}. Write $\mathcal{T}_{X_{reg}} = \bigoplus_i V_i$, where each V_i corresponds to an irreducible component of the representation $K \hookrightarrow \mathrm{Aut}(\mathfrak{g}^{-1,1})$. We know from Remark 4.3 that $\mathcal{T}_{X_{reg}}$ is semistable with respect to K_X, and since K_X is ample by assumption, $\mathcal{T}_{X_{reg}}$ has negative slope with respect to K_X. It follows that all direct summands V_i are semistable with respect to K_X and have the same negative slope as $\mathcal{T}_{X_{reg}}$.

Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be a decomposition of $\mathfrak{g}$ as a direct sum of simple ideals. Then each $P \times_K \mathfrak{g}_i$ is a subsystem of Hodge bundles of $P \times_K \mathfrak{g}$ (see [18, p.903]), and we have a direct sum decomposition $P \times_K \mathfrak{g} = \bigoplus_i P \times_K \mathfrak{g}_i$. Note that if G is not connected, we do not in general have a global decomposition of $P \times_K \mathfrak{g}$ into the bundles $P \times_K \mathfrak{g}_i$ associated to the simple ideals $\mathfrak{g}_i$ of $\mathfrak{g}$. Let $\mathfrak{g}^{-1,1} = \bigoplus_i \mathfrak{g}_i^{-1,1}$ be the decomposition into irreducible representations of $K \subset \mathrm{Aut}(\mathfrak{g}^{-1,1})$. Then $V_i \cong P \times_K \mathfrak{g}_i^{-1,1}$ for all i, and the bundles $P \times_K \mathfrak{g}_i^{-1,1}$ and their duals $P \times_K \mathfrak{g}_i^{1,-1}$ are semistable with respect to K_X. From the proof of Lemma 4.5, we know that the bundle $P \times_K \mathfrak{g}_i^{0,0}$ is semistable with respect to K_X of degree zero. Thus, each $P \times_K \mathfrak{g}_i$ is a system of Hodge bundles of degree zero.

By Lemma 4.4, to show that $P \times_K \mathfrak{g}$ is polystable as a Higgs bundle, it is sufficient to show that it is polystable as a system of Hodge bundles on X_{reg}. Let $W \subset P \times_K \mathfrak{g}_i$ be a subsystem of Hodge sheaves. Then W is locally a sub-Hodge structure of $\mathfrak{g}_i$. Since $W^{-1,1} \subset P \times_K \mathfrak{g}_i^{-1,1}$ we have $\mu_{K_X}(W^{-1,1}) \leq \mu_{K_X}(P \times_K \mathfrak{g}_i^{-1,1}) < 0$. The last inequality holds because K_X is assumed to be ample. Similarly, we have $\mu_{K_X}(W^{1,-1}) \leq \mu_{K_X}(P \times_K \mathfrak{g}_i^{1,-1})$ and $\mu_{K_X}(W^{0,0}) \leq \mu_{K_X}(P \times_K \mathfrak{g}_i^{0,0}) = 0$ by Lemma 4.5. These inequalities can be rewritten as follows.

$$\begin{aligned}
\widehat{c}_1(W^{-1,1}) \cdot [K_X]^{n-1} &\leq \mathrm{rank}(W^{-1,1})\mu_{K_X}(P \times_K \mathfrak{g}_i^{-1,1}),\\
\widehat{c}_1(W^{0,0}) \cdot [K_X]^{n-1} &\leq \mathrm{rank}(W^{0,0})\mu_{K_X}(P \times_K \mathfrak{g}_i^{0,0}) = 0, \quad \text{and}\\
\widehat{c}_1(W^{1,-1}) \cdot [K_X]^{n-1} &\leq \mathrm{rank}(W^{1,-1})\mu_{K_X}(P \times_K \mathfrak{g}_i^{1,-1}).
\end{aligned}$$

Since $(P \times_K \mathfrak{g}_i^{-1,1})^\vee \cong P \times_K \mathfrak{g}_i^{1,-1}$, we have $\mu_{K_X}(P \times_K \mathfrak{g}_i^{-1,1}) = -\mu_{K_X}(P \times_K \mathfrak{g}^{1,-1})$ and we compute

$$\begin{aligned}
\mu_{K_X}(W) &= \frac{\widehat{c}_1(W^{-1,1}) \cdot [K_X]^{n-1} + \widehat{c}_1(W^{0,0}) \cdot [K_X]^{n-1} + \widehat{c}_1(W^{1,-1}) \cdot [K_X]^{n-1}}{\mathrm{rank}(W)}\\
&\leq \frac{(\mathrm{rank}(W^{-1,1}) - \mathrm{rank}(W)^{1,-1})\mu_{K_X}(P \times_K \mathfrak{g}_i^{-1,1})}{\mathrm{rank}(P \times_K \mathfrak{g}_i)}. \qquad (1)
\end{aligned}$$

We know from Lemma 2.11 and Remark 2.14 that $\mathrm{rank}(W^{-1,1}) \geq \mathrm{rank}(W^{1,-1})$. Then the inequality (1) implies that $\mu_{K_X}(W) \leq 0$, and if equality holds, then $W = P \times_K \mathfrak{g}_i$ because $\mathfrak{g}_i$ is simple. Thus, each $P \times_K \mathfrak{g}_i$ is K_X-stable as a Higgs bundle of degree zero, and since $P \times_K \mathfrak{g} = \bigoplus_i P \times_K \mathfrak{g}_i$, it follows that $P \times_K \mathfrak{g}$ is K_X-polystable as a Higgs bundle on X_{reg}, of degree zero. Note that if $\mathfrak{g}$ is a simple Lie algebra, then the only non-trivial ideal of $\mathfrak{g}$ is itself. So in this case, $P \times_K \mathfrak{g}$ is even K_X-stable as a Higgs bundle on X_{reg}.

Now suppose G is not connected. Let G' denote the connected component of G and let $K' = K \cap G'$. Let $f : Y' \to X_{reg}$ be a finite étale cover over which the structure group of P can be reduced from K to K', and let Y be a normal projective completion of Y' such that f can be extended to a quasi-étale map $\gamma : Y \to X$. Note that Y is again klt, $Y' \subset Y_{reg}$ is a big open subset, and $K_Y = \gamma^* K_X$ is still ample. Thus $\mathcal{T}_{Y'}$ is semistable with respect to K_Y of negative degree.

We can write $\mathcal{T}_{Y'} = f^*\mathcal{T}_{X_{reg}} = f^*(P \times_K \mathfrak{g}^{-1,1}) = f^*P \times_{K'} \mathfrak{g}^{-1,1}$. Then $\mathcal{T}_{Y_{reg}} = P' \times_{K'} \mathfrak{g}^{-1,1}$, where P' is a principal K'-bundle on Y_{reg} such that $P'|_{Y'} = f^*P$. Since G' is connected, we know that the system of Hodge bundles $P' \times_{K'} \mathfrak{g}$ is K_Y-polystable as a Higgs bundle on Y_{reg}. Therefore by Remark 4.7, $f^*(P \times_K \mathfrak{g})$ is K_Y-polystable as a Higgs bundle on Y'.

If $W \subset P \times_K \mathfrak{g}$ is any saturated subsystem of Hodge sheaves with $\deg(W) \geq 0$, then f^*W is a subsystem of Hodge sheaves of $f^*(P \times_K \mathfrak{g})$ of degree zero, and is therefore a direct summand of $f^*(P \times_K \mathfrak{g})$ by [18, Proposition 3.3]. This implies that locally on Y', f^*W is a sub-Hodge structure of $\mathfrak{g}$. Then by Lemma 2.11, f^*W is locally a direct sum of simple ideals of $\mathfrak{g}$, i.e., f^*W is locally a direct sum of the $P \times_{K'} \mathfrak{g}_i$. Hence the same holds for W on X_{reg}. From the discussion in [18, p.903], there is a unique finest global decomposition

$$P \times_K \mathfrak{g} = \bigoplus_j \mathcal{E}_j \tag{2}$$

on X_{reg} such that each $\mathcal{E}_j$ is a subsystem of Hodge bundles of $P \times_K \mathfrak{g}$, and is locally a direct sum of simple ideals of $\mathfrak{g}$. We want to show that each $\mathcal{E}_j$ in the decomposition 2 is K_X-stable as a system of Hodge bundles of degree zero on X_{reg}.

There is similarly a finest global decomposition $f^*P \times_{K'} \mathfrak{g} = \bigoplus_k \mathcal{E}'_k$ on Y', and since $f^*P \times_{K'} \mathfrak{g}$ is K_Y-polystable as a system of Hodge bundles of degree zero, each $\mathcal{E}'_k$ is K_Y-stable as a system of Hodge bundles of degree zero. The pullbacks $f^*\mathcal{E}_j$ are direct sums of the $\mathcal{E}'_k$, thus each $\mathcal{E}_j$ also has degree zero.

Now we argue as in the proof of [18, Corollary 9.4]. If $W \subset \mathcal{E}_j$ is any saturated subsystem of Hodge sheaves of degree ≥ 0, then W must have degree zero, and we know that W is a locally a direct sum of simple ideals of $\mathfrak{g}$. Therefore we must have $W = \mathcal{E}_j$ by minimality of the decomposition (2). Thus $\mathcal{E}_j$ is stable with respect to K_X as a system of Hodge bundles of degree zero. It follows that $P \times_K \mathfrak{g}$ is K_X-polystable as a system of Hodge bundles, which is equivalent to $P \times_K \mathfrak{g}$ being K_X-polystable as a Higgs bundle on X_{reg}, by Lemma 4.4. □

We are now ready to prove Theorem 1.1.

4.2 Proof of Theorem 1.1

We split the proof into two steps to make it more readable. The first step is to show that if $X \cong \mathcal{D}/\Gamma$, then X satisfies the two conditions of Theorem 1.1. The second step is to prove the converse.

Proof **Step I**. Suppose we can write $X \cong \mathcal{D}/\Gamma$, for a Hermitian symmetric space $\mathcal{D}$ of noncompact type. Let $G_0 = \mathrm{Aut}(\mathcal{D})$ be the full automorphism group of $\mathcal{D}$, then $\Gamma \subset G_0$. In this case G_0 is a Hodge group of Hermitian type, and K_0 is a maximal compact subgroup of G_0. Let G and K denote complexifications of G_0 and K_0 respectively. Since the smooth locus X_{reg} is (the analytic space associated to) a quasi-projective variety, its fundamental group $\pi_1(X_{reg})$ is finitely generated, and isomorphic to Γ. Then from Selberg's lemma (see [1]), it follows that Γ admits a normal torsion free subgroup $\widehat{\Gamma}$ of finite index. Thus the quotient map $\mathcal{D} \to \mathcal{D}/\Gamma = X$ factors as

$$\mathcal{D} \xrightarrow{\pi} \mathcal{D}/\widehat{\Gamma} \xrightarrow{\gamma} \mathcal{D}/\Gamma = X,$$

where $\mathcal{D}/\widehat{\Gamma} = Y$ is smooth and projective because $\widehat{\Gamma}$ acts freely and cocompactly on $\mathcal{D}$. The map $\gamma : Y \to X$ is quasi-étale and Galois with group $H = \Gamma/\widehat{\Gamma}$. Note that the holomorphic tangent bundle of $\mathcal{D}$ satisfies $\mathcal{T}_{\mathcal{D}} \cong \tilde{P} \times_K \mathfrak{g}^{-1,1}$, where $\tilde{P}$ is a principal K-bundle on $\mathcal{D}$. Since Y is uniformized by $\mathcal{D}$, it follows from the classical Theorem 3.1 that Y admits a uniformizing system of Hodge bundles (P', θ) for the Hodge group G_0, such that $c_1(P' \times_K \mathfrak{g}) \cdot [K_Y]^{n-1} = c_2(P' \times_K \mathfrak{g}) \cdot [K_Y]^{n-2} = 0$. There is hence an isomorphism of vector bundles $\mathcal{T}_Y \cong P' \times_K \mathfrak{g}^{-1,1}$ on Y, where $P' = \tilde{P}/\widehat{\Gamma}$.

By the purity of branch locus, $\gamma : Y \to X$ branches only over the singular locus of X. Let $Y^o = \gamma^{-1}(X_{reg})$. Then $\gamma|_{Y^o} : Y^o \to X_{reg}$ is étale, and $Y^o \subset Y$ is a big open subset. We have $\mathcal{T}_{Y^o} = \mathcal{T}_Y|_{Y^o} = P'' \times_K \mathfrak{g}^{-1,1}$, where $P'' = P'|_{Y^o}$. The action of H on Y restricts to a free action on Y^o and lifts to a left action on $\mathcal{T}_{Y^o}$. On X_{reg}, we have $\mathcal{T}_{X_{reg}} \cong \mathcal{T}_{Y^o}/H$, and the isomorphism $\mathcal{T}_{Y^o} \cong P'' \times_K \mathfrak{g}^{-1,1}$ is H-equivariant because the isomorphism $\mathcal{T}_{\mathcal{D}} \cong \tilde{P} \times_K \mathfrak{g}^{-1,1}$ is Γ-equivariant. Thus there is an isomorphism $\theta : \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$, where $P \cong P''/H$ is a principal K-bundle on X_{reg}. It follows that (P, θ) is a uniformizing system of Hodge bundles on X_{reg}.

Let $\mathcal{E}$ be the system of Hodge bundles $P \times_K \mathfrak{g}$ on X_{reg}, and let $\mathcal{E}'$ denote the unique extension of $\mathcal{E}$ to X as a reflexive sheaf. Then, $\gamma^{[*]}\mathcal{E}' \cong P' \times_K \mathfrak{g}$ on Y, because $\gamma^{[*]}\mathcal{E}$ and $P' \times_K \mathfrak{g}$ are both reflexive and agree on the big open subset Y^o of Y. Moreover, we have $K_Y = \gamma^* K_X$. From the behaviour of $\mathbb{Q}$-Chern classes under quasi-étale covers (Lemma 2.16), it follows that

$$
\begin{aligned}
(a)\ \widehat{c}_1(\mathcal{E}') \cdot [K_X]^{n-1} &= (\deg(\gamma))^{-1}\widehat{c}_1(\gamma^{[*]}\mathcal{E}') \cdot [\gamma^* K_X]^{n-1} \\
&= (\deg(\gamma))^{-1} c_1(P' \times_K \mathfrak{g}) \cdot [K_Y]^{n-1} \\
(b)\ \widehat{c}_2(\mathcal{E}') \cdot [K_X]^{n-2} &= (\deg(\gamma))^{-1}\widehat{c}_2(\gamma^{[*]}\mathcal{E}') \cdot [\gamma^* K_X]^{n-2} \\
&= (\deg(\gamma))^{-1} c_2(P' \times_K \mathfrak{g}) \cdot [K_Y]^{n-2}.
\end{aligned}
$$

The right hand sides of equalities (a) and (b) are zero, again by Theorem 3.1. Thus it follows that $\widehat{ch}_2(\mathcal{E}') \cdot [K_X]^{n-2} = 0$. This proves one implication of Theorem 1.1.

Step II. Conversely, suppose that X is a projective klt variety which satisfies the two conditions of Theorem 1.1. Let G_0, K_0, G, and K be as in Step I. By assumption, we have an isomorphism of vector bundles $\theta : \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$ on X_{reg}.

Let $\gamma : Y \to X$ be a Galois, maximally quasi-étale cover, which we know exists by Theorem 2.19 and let H denote the Galois group. Note that Y is again klt ([14, Proposition 5.20]) and $K_Y = \gamma^* K_X$ is ample. Since γ branches only over the singular locus of X, the restricted map $\gamma|_{Y^o} : Y^o \to X_{reg}$ is étale. Here $Y^o := \gamma^{-1}(X_{reg}) \subset Y_{reg}$, and note that Y^o is a big open subset of Y. Then $\mathcal{T}_{Y^o} = (\gamma|_{Y^o})^* \mathcal{T}_{X_{reg}} \cong (\gamma|_{Y^o})^*(P \times_K \mathfrak{g}^{-1,1}) = P' \times_K \mathfrak{g}^{-1,1}$, where $P' = (\gamma|_{Y^o})^* P$ is a principal K-bundle on Y^o. Thus there is an isomorphism of vector bundles $\theta' : \mathcal{T}_{Y_{reg}} \cong P'' \times_K \mathfrak{g}^{-1,1}$ on Y_{reg}, where P'' is a principal K-bundle on Y_{reg} such that $P''|_{Y^o} = P'$. Note that $\mathcal{T}_{Y_{reg}}$ has negative degree, and is semistable with respect to K_Y, again by Remark 4.3.

Let $\mathcal{F} = P'' \times_K \mathfrak{g}$, then $\mathcal{F}$ is a system of Hodge bundles on Y_{reg} with Higgs field coming from θ'. By Proposition 4.6, we know that $\mathcal{F}$ is K_Y-polystable as a Higgs bundle on Y_{reg}, and of degree zero. Let $\mathcal{F}'$ and $\mathcal{E}'$ be the unique extensions of $\mathcal{F}$

to Y and of $P \times_K \mathfrak{g}$ to X respectively, as reflexive sheaves. Since $\gamma^{[*]}\mathcal{E}'$ and $\mathcal{F}'$ agree on the big open subset Y^o, we have $\mathcal{F}' \cong \gamma^{[*]}\mathcal{E}'$. By assumption, the equality $\widehat{ch}_2(\mathcal{E}') \cdot [K_X]^{n-2} = 0$ holds, and by the behaviour of $\mathbb{Q}$-Chern classes under quasi-étale covers (Lemma 2.16), we have $\widehat{ch}_2(\mathcal{F}') \cdot [K_Y]^{n-2} = 0$. Note that $\mathcal{F}'$ has degree zero, so in particular, we also have $\widehat{ch}_1(\mathcal{F}') \cdot [K_Y]^{n-1} = 0$. Then it follows from Theorem 2.20 that the sheaf $\mathcal{F}'$ is in fact locally free. Since $\mathcal{T}_Y$ is a direct summand of $\mathcal{F}'$, we conclude that $\mathcal{T}_Y$ is locally free. The solution to the Lipman-Zariski conjecture for klt spaces ([7, Theorem 16.1]) then asserts that Y is smooth. We may thus apply Simpson's classical Theorem 3.1 to conclude that Y is uniformized by $G_0/K_0 = \mathcal{D}$.

This means that $Y \cong \mathcal{D}/\Gamma'$, where Γ' is a discrete, cocompact, torsion free group of automorphisms of $\mathcal{D}$, and $X \cong Y/H$, where H acts fixed point freely in codimension one on Y by automorphisms. By arguments analogous to those in the proof of [9, Theorem 1.3], (specifically, step (1.3.3) $\implies$ (1.3.1)), there is an exact sequence of groups $1 \to \Gamma' \to \Gamma \to H \to 1$, where Γ is a discrete, cocompact subgroup of $\mathrm{Aut}(\mathcal{D})$ acting properly discontinuously, and fixed point freely in codimension one, such that $X \cong \mathcal{D}/\Gamma$. This concludes the proof of Theorem 1.1. $\square$

Corollary 4.8 *Let $\mathcal{D}$ be a Hermitian symmetric space of non-compact type, and let X be a projective klt quotient of $\mathcal{D}$ by a Γ as in Theorem 1.1, with K_X ample. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the conjugate variety X^σ is also a quotient of $\mathcal{D}$.*

Proof The variety X^σ again has klt singularities and ample canonical divisor K_{X^σ}. From the proof of Theorem 1.1 we know that X admits a smooth, quasi-étale cover $\gamma : Y \to X$ such that Y is uniformized by $\mathcal{D}$. The conjugate variety $Y^\sigma = X^\sigma \times_X Y$ is again smooth, and by the result for smooth projective varieties ([18, Corollary 9.5]), it follows that Y^σ is also uniformized by $\mathcal{D}$. Thus Y^σ admits a uniformizing system of Hodge bundles $\mathcal{E}$ which is polystable with respect to K_Y, and has vanishing first and second Chern classes.

The map $\gamma^\sigma : Y^\sigma \to X^\sigma$ is also quasi-étale and finite, since these properties are preserved under base change. Thus there is a uniformizing system of Hodge bundles $\mathcal{F}$ on X^σ_{reg} which is polystable with respect to K_{X^σ}, such that the reflexive extension $\mathcal{F}'$ of $\mathcal{F}$ to X^σ pulls back to $\mathcal{E}$, i.e., $(\gamma^\sigma)^{[*]}(\mathcal{F}') \cong \mathcal{E}$. By the behaviour of $\mathbb{Q}$-Chern classes under quasi-etale covers (Lemma 2.16), the first and second $\mathbb{Q}$-Chern classes of $\mathcal{F}'$ vanish. Then we conclude by Theorem 1.1 that X^σ is uniformized by $\mathcal{D}$. $\square$

5 Characterizing quotients of a polydisk

In this section, we apply Theorem 1.1 to extend [18, Corollary 9.7] to the klt setting, with some improvements.

5.1 Sufficient conditions

Theorem 5.1 *Let X be an n-dimensional projective klt variety of general type whose canonical divisor K_X is nef. Suppose that the tangent sheaf $\mathcal{T}_X$ of X splits as a direct sum*

$$\mathcal{T}_X = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \tag{3}$$

where each direct summand $\mathcal{L}_i$ is a reflexive sheaf of rank 1 on X. Then the canonical model X_{can} of X admits a Galois quasi-étale cover $\gamma : Y \to X_{can}$, where Y is a smooth projective variety whose universal cover is the polydisk $\mathbb{H}^n$.

Note that Theorem 5.1 does not require a Chern class equality. This is because we can show that the splitting of the tangent sheaf as in (3) implies the Chern class vanishing condition from Theorem 1.1. This was shown by Beauville in [2, Lemma 3.1] for smooth complex manifolds, using which he obtained an improved version of Simpson's characterization of smooth compact polydisk quotients ([2, Theorem B]). See also [5, Proposition 3.2] for a variant of [2, Lemma 3.1] using different methods.

We make use of the following result from [6], which can be viewed as a generalization of [2, Lemma 3.1] to the setting of orbifolds.

Proposition 5.2 ([6, Proposition 5.1]) *Let (X, Δ) be an n-dimensional klt pair, where X is a complex projective variety, and Δ has standard coefficients, i.e., $\Delta = \sum_i (1 - \frac{1}{m_i})\Delta_i$ with integers $m_i \geq 2$ and the Δ_i irreducible and pairwise distinct. Suppose the tangent bundle $\mathcal{T}_{(X^\circ, \Delta^\circ)}$ of the orbifold locus (X°, Δ°) contains an orbifold line bundle $\mathcal{L}$ as a direct summand. Then for any $\mathbb{Q}$-Cartier divisors $D_1, D_2, \ldots, D_{n-2}$ on X, we have*

$$\widetilde{c}_1(\mathcal{L})^2 \cdot [D_1] \cdots [D_{n-2}] = 0.$$

The orbifold locus (X°, Δ°) of a klt pair (X, Δ) is the locus where (X, Δ) has quotient singularities. The term $\widetilde{c}_1(\mathcal{L})$ denotes the *first orbifold Chern class* of $\mathcal{L}$. We refer the reader to [6, Section 2] for definitions and further details.

To prove Theorem 5.1, we apply Proposition 5.2 to the special case where $\Delta = 0$ and obtain the following.

Lemma 5.3 *Let X be an n-dimensional projective klt variety with K_X ample and suppose that the tangent sheaf of X splits as a direct sum $\mathcal{T}_X = \bigoplus_{i=1}^n \mathcal{L}_i$ of reflexive sheaves of rank 1. Then we have*

$$\widehat{c}_1(\mathcal{L}_i)^2 \cdot [K_X]^{n-2} = 0$$

for each $1 \leq i \leq n$.

Proof The orbifold locus of X is the subset $X' := X \setminus Z$, where $Z \subset X$ is the subset of X containing points that are not quotient singularities. Each reflexive sheaf $\mathcal{L}_i|_{X'}$ is then an orbifold line bundle on X' and the orbifold Chern class $\widetilde{c}_1(\mathcal{L}_i|_{X'})$ is precisely the $\mathbb{Q}$-Chern class $\widehat{c}_1(\mathcal{L}_i|_{X'})$.

Since K_X is ample, we can choose a sufficiently increasing and divisible sequence of numbers $0 << m_1 << \cdots << m_{n-2}$ and a general tuple of elements $(H_1, ..., H_{n-2}) \in \prod_{i=1}^{n-2} |m_i K_X|$ and form the complete intersection surface $S := H_1 \cap \cdots \cap H_{n-2} \subset X$. Note that S is irreducible, normal and has klt singularities. Since klt singularities are quotient singularities in codimension 2 by [7, Proposition 9.3], S is contained entirely in X'. Thus, we have

$$\widehat{c}_1(\mathcal{L}_i)^2 \cdot [K_X]^{n-2} = \widehat{c}_1(\mathcal{L}_i|_S)^2 = \widehat{c}_1(\mathcal{L}_i|_{X'})^2 \cdot [K_X]^{n-2}.$$

From Proposition 5.2, it follows that $\widehat{c}_1(\mathcal{L}_i|_{X'})^2[K_X]^{n-2} = 0$ and we are done. □

The proof of Theorem 5.1 also requires the following key observations.

Proposition 5.4 *Let X be an n-dimensional projective klt variety of general type whose canonical divisor K_X is nef. Suppose that the tangent sheaf $\mathcal{T}_X$ of X splits as in (3). Then, the tangent sheaf of the canonical model X_{can} of X also splits as a direct sum of rank 1 reflexive sheaves of negative degree.*

Proof Let $\pi : X \to X_{can}$ be the birational crepant morphism to the canonical model X_{can}. We know from [11] that X_{can} is also projective and klt and its canonical divisor $K_{X_{can}}$ is ample.

Let $U \subset X$ be the largest open subset of X restricted to which π is an isomorphism. Let $V \subset X_{can}$ be the open subset defined as the intersection $V := \pi(U) \cap X_{can,reg}$, where $X_{can,reg}$ denotes the smooth locus of X_{can}. Since $X_{can} \setminus \pi(U)$ and $X_{can} \setminus X_{can,reg}$ are both codimension ≥ 2 subsets of X_{can}, it follows that $X_{can} \setminus V$ is also a codimension ≥ 2 subset of X_{can}.

The restriction $\mathcal{T}_{X_{reg}}|_{\pi^{-1}(V)} = \mathcal{T}_{\pi^{-1}(V)}$ splits as a direct sum of line bundles because we have, by assumption, that $\mathcal{T}_{X_{reg}} \cong \mathcal{L}_1|_{X_{reg}} \oplus \cdots \oplus \mathcal{L}_n|_{X_{reg}}$, where each $\mathcal{L}_i|_{X_{reg}}$ is a line bundle on X_{reg}. Since we have $\pi^{-1}(V) \cong V$, it follows that the tangent bundle $\mathcal{T}_V$ of V also splits as a direct sum of line bundles, say $\mathcal{T}_V = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$. Let $\mathcal{M}'_i$ denote the unique extension of $\mathcal{M}_i$ as a reflexive sheaf of rank 1 on X_{can}, for all $1 \leq i \leq n$. Then, by the uniqueness of the extensions, we have $\mathcal{M}'_1 \oplus \cdots \oplus \mathcal{M}'_n \cong \mathcal{T}_{X_{can}}$ i.e., $\mathcal{T}_{X_{can}}$ splits as a direct sum of rank 1 reflexive sheaves.

We know from Proposition 4.1 that the tangent sheaf $\mathcal{T}_{X_{can}}$ is semistable with respect to $K_{X_{can}}$. Since $K_{X_{can}}$ is ample, we have $[K_{X_{can}}]^n > 0$ and it follows that

$$\mu_{K_{X_{can}}}(\mathcal{T}_{X_{can}}) = \frac{[\det(\mathcal{T}_{X_{can}})] \cdot [K_{X_{can}}]^{n-1}}{\operatorname{rank}(\mathcal{T}_{X_{can}})} = -\frac{[K_{X_{can}}]^n}{n} < 0.$$

For each direct summand $\mathcal{M}'_i$ of $\mathcal{T}_{X_{can}}$, we have $\mu_{K_{X_{can}}}(\mathcal{M}'_i) \leq \mu_{K_{X_{can}}}(\mathcal{T}_{X_{can}}) < 0$ by the semistability of $\mathcal{T}_{X_{can}}$. This implies that $\widehat{c}_1(\mathcal{M}'_i) \cdot [K_{X_{can}}]^{n-1} < 0$ for all $1 \leq i \leq n$. In other words, all direct summands $\mathcal{M}'_i$ of $\mathcal{T}_{X_{can}}$ have (the same) negative degree. □

Proposition 5.5 *Let X be an n-dimensional projective klt variety such that the tangent sheaf $\mathcal{T}_X$ of X splits as a direct sum of rank 1 reflexive sheaves as in (3). Let $\gamma : Y \to X$ be a Galois quasi-étale cover. Then, the tangent sheaf $\mathcal{T}_Y$ of Y also splits as a direct sum of rank 1 reflexive sheaves.*

Proof Note that since $\gamma : Y \to X$ is quasi-étale, Y is again projective and klt of dimension n. By purity of branch locus, we know that γ branches only over the singular locus of X. It follows that $\gamma^{-1}(X_{reg}) = Y^o$, where Y^o denotes the open subset of Y restricted to which the map γ is étale. Note that $Y^o \subset Y_{reg}$, because any singular point of Y necessarily belongs to the branching locus of γ. Since γ is finite, the codimension of the branching locus of γ in Y is equal to the codimension of the singular locus of X in X. This implies that Y^o is a big open subset of Y.

Since the restricted map $\gamma : Y^o \to X_{reg}$ is finite and étale, we have $K_{Y^o} \cong \gamma^* K_{X_{reg}}$ and $\mathcal{T}_{Y^o} \cong \gamma^* \mathcal{T}_{X_{reg}} \cong \gamma^*(\mathcal{L}_1|_{X_{reg}}) \oplus \cdots \oplus \gamma^*(\mathcal{L}_n|_{X_{reg}})$, where the $\mathcal{L}_i$'s are the rank 1 reflexive sheaves appearing in the direct sum decomposition of $\mathcal{T}_X$. Let $\mathcal{M}_i$ denote the unique extension of $\gamma^*(\mathcal{L}_i|_{X_{reg}})$ as a reflexive sheaf over Y, for all $1 \le i \le n$. Then, by the uniqueness of the reflexive extension, we have $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n \cong \mathcal{T}_Y$ as claimed. □

Proposition 5.6 *Let X be a projective klt variety of dimension n, and let the canonical divisor K_X be ample. Suppose that the tangent sheaf $\mathcal{T}_X$ of X splits as a direct sum of rank 1 reflexive sheaves as in (3), and suppose that X is maximally quasi-étale, i.e., there is an isomorphism of étale fundamental groups $\widehat{\pi}_1(X_{reg}) \cong \widehat{\pi}_1(X)$. Then X is smooth.*

Proof Since $\mathcal{T}_{X_{reg}} \cong \mathcal{L}_1|_{X_{reg}} \oplus \cdots \oplus \mathcal{L}_n|_{X_{reg}}$, and we have assumed K_X to be ample, the semistability of $\mathcal{T}_{X_{reg}}$ with respect to K_X (Remark 4.3) implies that each line bundle $\mathcal{L}_i|_{X_{reg}}$ appearing in the direct sum decomposition of $\mathcal{T}_{X_{reg}}$ has negative degree i.e., $c_1(\mathcal{L}_i|_{X_{reg}}) \cdot [K_X]^{n-1} < 0$ for all $1 \le i \le n$.

Let $G_0 = SL(2, \mathbb{R})^n$, then G_0 is a connected Hodge group corresponding to the polydisk $\mathbb{H}^n$, which is a Hermitian symmetric space of noncompact type. Let $G = SL(2, \mathbb{C})^n$, a complexification of G_0, and let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})^n$, the Lie algebra of G. Since G_0 is a Hodge group of *Hermitian type*, $\mathfrak{g}$ has the following Hodge decomposition

$$\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1},$$

with $[\mathfrak{g}^{p,-p}, \mathfrak{g}^{q,-q}] \subset \mathfrak{g}^{p+q,-p-q}$ for $p, q \in \{-1, 0, 1\}$, where $[\cdot, \cdot]$ denotes the Lie bracket of $\mathfrak{g}$. Note that $\mathbb{H}^n \cong G_0/K_0$, where $K_0 = U(1)^n$ is a maximal compact subgroup of G_0. Let K denote the complexification of K_0, then we have $K \cong (\mathbb{C}^*)^n$, and K sits inside $SL(2, \mathbb{C})^n$ as the set of all n-tuples of diagonal matrices.

Since $\mathcal{T}_{X_{reg}}$ splits as a direct sum of line bundles, it admits a reduction in structure group from $GL(n, \mathbb{C})$ to $K = (\mathbb{C}^*)^n$. Let P denote the principal $K = (\mathbb{C}^*)^n$-bundle over X_{reg} associated to $\mathcal{T}_{X_{reg}}$. Note that K acts on $\mathfrak{g}^{-1,1}$ via the adjoint representation, which in the present case is simply by scaling. Thus $\mathbb{C}^{\oplus n}$ and $\mathfrak{g}^{-1,1}$ are isomorphic as K-representations, and it follows from the associated bundle construction that there is an isomorphism of vector bundles

$$\theta : \mathcal{T}_{X_{reg}} \longrightarrow P \times_K \mathfrak{g}^{-1,1} \tag{4}$$

such that $[\theta(u), \theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_{X_{reg}}$. Thus, (P, θ) is a uniformizing system of Hodge bundles on X_{reg}.

We know from Proposition 4.6 that $E := P \times_K \mathfrak{g}$ is K_X-polystable as a Higgs bundle on X_{reg}.

Note that $E \cong \mathcal{T}_{X_{reg}} \oplus \Omega^1_{X_{reg}} \oplus \mathcal{O}^{\oplus n}_{X_{reg}}$, and let $\mathcal{F}_X$ denote the unique extension of E as a reflexive sheaf on X. Then we have $\mathcal{F}_X \cong \mathcal{T}_X \oplus \Omega^{[1]}_X \oplus \mathcal{O}^{\oplus n}_X$, and by construction it follows that $\widehat{c}_1(\mathcal{F}_X) = 0$. Recall that $\mathcal{T}_X = \bigoplus_i \mathcal{L}_i$ by assumption. It follows from Lemma 5.3 that $\widehat{c}_1(\mathcal{L}_i)^2 \cdot [K_X]^{n-2} = 0$ for all $1 \le i \le n$. Using the formula $\widehat{ch}_2(\mathcal{F}_X) = \frac{1}{2}(\widehat{c}_1(\mathcal{F}_X)^2 - 2\widehat{c}_2(\mathcal{F}_X))$, we compute

$$
\begin{aligned}
&-\widehat{c}_2(\mathcal{F}_X)\cdot[K_X]^{n-2} = \widehat{ch}_2(\mathcal{F}_X)\cdot[K_X]^{n-2} = 2\widehat{ch}_2(\mathcal{T}_X)\cdot[K_X]^{n-2}\\
&= \sum_i 2\widehat{ch}_2(\mathcal{L}_i)\cdot[K_X]^{n-2} = \sum_i \widehat{c}_1(\mathcal{L}_i)^2\cdot[K_X]^{n-2} = 0.
\end{aligned}
$$

Since X is maximally quasi-étale by assumption, Theorem 2.20 implies that $\mathcal{F}_X$ is locally free. Moreover, $\mathcal{T}_X$ is a direct summand of $\mathcal{F}_X = \mathcal{T}_X \oplus \Omega_X^{[1]} \oplus \mathcal{O}_X^{\oplus n}$, so we conclude that $\mathcal{T}_X$ is locally free. Then the solution of the Lipman-Zariski conjecture for klt spaces ([7, Theorem 16.1]) asserts that X is smooth, and we are done. □

Proof of Theorem 5.1 Suppose X is an n-dimensional projective klt variety that satisfies the assumptions of Theorem 5.1. Recall from Proposition 5.4 that the canonical model X_{can} of X is projective and klt, and its tangent sheaf splits as a direct sum of rank 1 reflexive sheaves of negative degree. Moreover, the canonical divisor $K_{X_{can}}$ of X_{can} is ample. Let $\gamma : Y \to X_{can}$ be a Galois maximally quasi-étale cover (see Lemma 2.19). Then Y is again klt and the canonical divisor $K_Y = \gamma^* K_{X_{can}}$ is ample. Moreover, by Proposition 5.5 and the semistability of $\mathcal{T}_Y$ with respect to K_Y (Proposition 4.1), we see that $\mathcal{T}_Y$ splits as a direct sum of rank 1 reflexive sheaves of negative degree. Thus, Y satisfies the assumptions of Proposition 5.6, and we conclude that Y is smooth. It follows that Y also satisfies the assumptions of [18, Corollary 9.7], therefore Y is uniformized by the polydisk $\mathbb{H}^n$. □

5.2 Necessary conditions

Proposition 5.7 *Let X be a n-dimensional projective klt variety with K_X ample, and such that $X \cong \mathbb{H}^n/\widehat{\Gamma}$, where $\widehat{\Gamma} \subset PSL(2,\mathbb{R})^n \rtimes \sigma_n = \mathrm{Aut}(\mathbb{H}^n)$ is a discrete cocompact subgroup that acts fixed point freely in codimension one. Then X admits a smooth quasi-étale cover $\gamma : Y \to X$ such that K_Y is ample, and the tangent bundle $\mathcal{T}_Y$ splits as a direct sum of line bundles.*

Proof Due to the structure of the automorphism group $Aut(\mathbb{H}^n) = PSL(2,\mathbb{R})^n \rtimes \sigma_n$, we know that $\widehat{\Gamma}$ can be expressed as an extension

$$1 \to \Gamma' \to \widehat{\Gamma} \to H \to 1,$$

where $\Gamma' \subset PSL(2,\mathbb{R})^n$ is normal of finite index in $\widehat{\Gamma}$, and $H \subset \sigma_n$ is finite. The quotient map $\pi : \mathbb{H}^n \to X$ thus factors as

$$\mathbb{H}^n \to \mathbb{H}^n/\,\Gamma' \to \mathbb{H}^n/\widehat{\Gamma} = X,$$

where $Y' = \mathbb{H}^n/\,\Gamma'$ is also klt and projective, and the map $Y' \to X$ is Galois and quasi-étale. Using Selberg's lemma we know that Γ' has a torsion free normal subgroup Γ of finite index. The quotient map $\pi' : \mathbb{H}^n \to Y'$ thus factors as

$$\mathbb{H}^n \to \mathbb{H}^n/\,\Gamma \to \mathbb{H}^n/\,\Gamma' = Y'.$$

Since Γ acts freely and cocompactly on $\mathbb{H}^n$, the quotient $Y = \mathbb{H}^n/\Gamma$ is a smooth projective variety. The map $Y \to Y'$ is Galois and quasi-étale, and after composing with the map $Y' \to X$, we get a quasi-étale map $Y \to X$. The tangent bundle of $\mathbb{H}^n$ can be expressed as a direct sum $\mathcal{T}_{\mathbb{H}^n} \cong P \times_K \mathfrak{g}^{-1,1}$, where $K = (\mathbb{C}^*)^n$ is maximal compact inside $SL(2, \mathbb{C})^n$ and $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})^n$. The action of Γ on $\mathbb{H}^n$ lifts to a left action of Γ on $\mathcal{T}_{\mathbb{H}^n}$ via pushforward of tangent spaces, so it follows that $\mathcal{T}_Y \cong P' \times_K \mathfrak{g}^{-1,1}$, where $P' \cong P/\Gamma$ as principal K-bundles on Y. Since $\mathcal{T}_Y$ also has structure group $K = (\mathbb{C}^*)^n$, it splits as a direct sum of line bundles. Since the map $\gamma : Y \to X$ is finite and K_X is ample by assumption, it follows that $K_Y = \gamma^* K_X$ is also ample. The semistability of $\mathcal{T}_Y$ with respect to K_Y then implies that the line bundles appearing in the direct sum decomposition of $\mathcal{T}_Y$ all have negative degree. $\square$

Remark 5.8 If X is a smooth projective quotient of the polydisk, then we *do not* have a representation $\pi_1(X) \to PSL(2, \mathbb{R})^n$ in general. Therefore, the action of $\pi_1(X)$ on $\mathbb{H}^n$ is in general not compatible with the reduction in structure group of $\mathcal{T}_{\mathbb{H}^n}$ from $GL(n, \mathbb{C})$ to $(\mathbb{C}^*)^n = K$. Hence, the splitting of the tangent bundle into a direct sum of line bundles does not in general descend to the quotient variety X.

6 Characterizing quotients of classical irreducible bounded symmetric domains

In this section, we characterize quotients of each of the four classical irreducible bounded symmetric domains.

We treat in detail the case of the domain of type $AIII$, as this is the most interesting. Indeed, the automorphism group of this domain could be disconnected (as in the polydisk case), in which case the necessary and sufficient criteria for uniformization are formulated separately. A special case of this domain is the unit ball, and we recover classical results characterizing ball quotients in this case.

The characterizations of quotients of the other three types of bounded symmetric domains are proved almost identically to the $AIII$ case, thus we omit their proofs and only give statements.

6.1 Domains of type *AIII*

The Hermitian symmetric space of type $AIII$, denoted $\mathcal{A}_{pq}$, can be expressed as the quotient G_0/K_0, where we may take $G_0 = SU(p, q)$ and its maximal compact subgroup $K_0 = S(U(p) \times U(q))$, for $p, q \in \mathbb{Z}_{\geq 1}$ such that $pq > 1$. The domain $\mathcal{A}_{pq}$ can also be realized as follows (see [16]).

$$\mathcal{A}_{pq} = \{Z \in M(p, q, \mathbb{C}) \cong \mathbb{C}^{pq} : I_q - \bar{Z}^T Z > 0\},$$

where "> 0" again means positive definiteness. The complex dimension of $\mathcal{A}_{pq}$ is pq. The complexified Lie algebra of $SU(p, q)$ is $\mathfrak{g} = \mathfrak{sl}(p + q, \mathbb{C})$. The Lie algebra $\mathfrak{g}$ has complex dimension $(p + q)^2 - 1$, and again using the list in [13, p.341], can be expressed as the set of complex $(p+q)\times(p+q)$ matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A is a $p\times p$ matrix, B is a $p\times q$ matrix, C is a $q\times p$ matrix, and D is a $q\times q$ matrix, and $Tr(A)+Tr(D)=0$. Note that $G_0 = SU(p,q)$ is a Hodge group of Hermitan type and $\mathfrak{g}$ has a Hodge decomposition $\mathfrak{g} = \mathfrak{g}^{-1,1}\oplus\mathfrak{g}^{0,0}\oplus\mathfrak{g}^{1,-1}$ as in Definition 2.10, where $\mathfrak{g}^{-1,1}$ consists of matrices of the form $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, $\mathfrak{g}^{0,0}$ consists of matrices of the form $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, and $\mathfrak{g}^{1,-1}$ consists of matrices of the form $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$. The Lie algebras $\mathfrak{g}^{-1,1}$, $\mathfrak{g}^{0,0}$, and $\mathfrak{g}^{1,-1}$ have complex dimensions pq, p^2+q^2-1, and pq, respectively.

Let P be a principal $K = S(GL(p,\mathbb{C})\times GL(q,\mathbb{C}))$-bundle on $\mathcal{A}_{pq}$ such that there is an isomorphism $\theta : \mathcal{T}_{\mathcal{A}_{pq}} \cong P\times_K \mathfrak{g}^{-1,1}$. Then there are vector bundles $\mathcal{V}$ and $\mathcal{W}$ of ranks p and q on $\mathcal{A}_{pq}$, such that the system of Hodge bundles $P\times_K\mathfrak{g}$ is isomorphic to $\mathcal{E}nd_0(\mathcal{V}\oplus\mathcal{W})$, the bundle of trace zero endomorphisms of $\mathcal{V}\oplus\mathcal{W}$. From the Hodge decomposition of $\mathfrak{g}$, we get $P\times_K\mathfrak{g}^{-1,1}\cong\mathcal{H}om(\mathcal{V},\mathcal{W})\cong\mathcal{T}_{\mathcal{A}_{pq}}$, $P\times_K\mathfrak{g}^{0,0}\cong(\mathcal{E}nd(\mathcal{V})\oplus\mathcal{E}nd(\mathcal{W}))_0$, and $P\times_K\mathfrak{g}^{1,-1}\cong\mathcal{H}om(\mathcal{W},\mathcal{V})\cong\Omega^1_{\mathcal{A}_{pq}}$. The domain $\mathcal{A}_q$ corresponding to $p=1$ is the unit ball $\mathbb{B}^q\subset\mathbb{C}^q$. In this case, we have $\mathcal{V}\cong\Omega^1_{\mathbb{B}^q}$, $\mathcal{W}=\mathcal{O}_{\mathbb{B}^q}$, and the system of Hodge bundles is given by $P\times_K\mathfrak{g}\cong\mathcal{E}nd_0(\Omega^1_{\mathbb{B}^q}\oplus\mathcal{O}_{\mathbb{B}^q})$.

Proposition 6.1 *Let X be a projective klt variety of dimension pq for some $p,q\in\mathbb{Z}_{\geq 1}$, $pq>1$, with K_X ample, such that the tangent bundle of the regular locus X_{reg} of X satisfies $\mathcal{T}_{X_{reg}}\cong\mathcal{H}om(\mathcal{V},\mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are vector bundles of ranks p and q on X_{reg}. Let $\mathcal{V}'$ and $\mathcal{W}'$ denote the reflexive extensions of $\mathcal{V}$ and $\mathcal{W}$ to X, and suppose that the Chern class equality*

$$[2(p+q)(\widehat{c}_2(\mathcal{V}')+\widehat{c}_2(\mathcal{W}'))-(p+q-1)(\widehat{c}_1(\mathcal{V}')^2+\widehat{c}_1(\mathcal{W}')^2) \\ +2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')]\cdot[K_X]^{pq-2}=0 \tag{5}$$

holds on X. Then $X\cong\mathcal{A}_{pq}/\Gamma$, where $\Gamma\subset\mathrm{Aut}(\mathcal{A}_{pq})$ is a discrete cocompact subgroup, and acts fixed point freely in codimension one on the domain $\mathcal{A}_{pq}$ of type $AIII$.

Proof Let $j : X_{reg}\to X$ denote the natural inclusion. Then we have $\mathcal{T}_X\cong j_*\mathcal{T}_{X_{reg}}\cong j_*\mathcal{H}om(\mathcal{V},\mathcal{W})\cong\mathcal{H}om(\mathcal{V}',\mathcal{W}')$ by the uniqueness of the reflexive extension. Let $G_0=SU(p,q)$ and $K_0=S(U(p)\times U(q))$ be the maximal compact subgroup of G_0, and let G and K denote complexifications of G_0 and K_0, respectively. We choose $G=SL(p+q,\mathbb{C})$ and $K=S(GL(p,\mathbb{C})\times GL(q,\mathbb{C}))$.

Let $\mathcal{E}nd_0(\mathcal{V}\oplus\mathcal{W})$ denote the bundle of trace zero endomorphisms of $\mathcal{V}\oplus\mathcal{W}$. We deduce again from [3, Table 2], that the Lie algebra $\mathfrak{g}=\mathfrak{sl}(p+q,\mathbb{C})$ and the typical fiber $\mathrm{End}_0(\mathbb{C}^p,\mathbb{C}^q)$ of $\mathcal{E}nd_0(\mathcal{V}\oplus\mathcal{W})$ are isomorphic as K-representations. Moreover, the vector bundle $\mathcal{E}nd_0(\mathcal{V}\oplus\mathcal{W})$ admits a reduction in structure group from $GL((p+q)^2-1,\mathbb{C})$ to $K=S(GL(p,\mathbb{C})\times GL(q,\mathbb{C}))$. Thus there is a principal K-bundle P on X_{reg} associated to $\mathcal{E}nd_0(\mathcal{V}\oplus\mathcal{W})$ i.e., we have $P\times_K\mathfrak{g}\cong\mathcal{E}nd_0(\mathcal{V}\oplus\mathcal{W})$. The Hodge decomposition of $\mathfrak{g}$ together with identifying isomorphic K-representations using [3, Table 2] gives $P\times_K\mathfrak{g}^{-1,1}\cong\mathcal{H}om(\mathcal{V},\mathcal{W})$, $P\times_K\mathfrak{g}^{0,0}\cong(\mathcal{E}nd(\mathcal{V})\oplus\mathcal{E}nd(\mathcal{W}))_0$, and

$P \times_K \mathfrak{g}^{1,-1} \cong \mathcal{H}om(\mathcal{W}, \mathcal{V})$. Since $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{V}, \mathcal{W})$ by assumption, we have the following isomorphism

$$\theta : \mathcal{T}_{X_{reg}} \to P \times_K \mathfrak{g}^{-1,1}$$

such that $[\theta(u), \theta(v)] = 0$ for all local sections u, v of $\mathcal{T}_{X_{reg}}$. It follows that (P, θ) gives a uniformizing system of Hodge bundles on X_{reg}.

The system of Hodge bundles $E = P \times_K \mathfrak{g} \cong \mathcal{T}_{X_{reg}} \oplus (\mathcal{E}nd(\mathcal{V}) \oplus \mathcal{E}nd(\mathcal{W}))_0 \oplus \Omega^1_{X_{reg}}$ is in particular a Higgs bundle with Higgs field $\widehat{\theta} : E \to E \otimes \Omega^1_{X_{reg}}$ given by sending a local section u of E to the local section $v \mapsto [\theta(v), u]$ of $\mathcal{H}om(\mathcal{T}_{X_{reg}}, E) \cong E \otimes \Omega^1_{X_{reg}}$. Note that E fits into the following short exact sequence

$$0 \to E = \mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W}) \to \mathcal{E}nd(\mathcal{V} \oplus \mathcal{W}) \to \mathcal{L} \to 0 \tag{6}$$

where $\mathcal{L}$ is a line bundle of degree zero. Thus we have $c_1(E) = c_1(\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})) = 0$. From Proposition 4.6, it follows that E is K_X-polystable as a Higgs bundle on X_{reg}, and in fact E is K_X-stable as a Higgs bundle on X_{reg} because $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{C})$ is a simple Lie algebra.

Let $\mathcal{F}_X$ denote the reflexive extension of E to X. Then $\mathcal{F}_X$ also fits into an exact sequence

$$0 \to \mathcal{F}_X \to \mathcal{E}nd(\mathcal{V}' \oplus \mathcal{W}') \to \mathcal{L}' \to 0$$

where $\mathcal{L}'$ is a coherent sheaf of degree 0. Thus $\widehat{c}_1(\mathcal{F}_X) \cdot [K_X]^{pq-1} = \widehat{c}_1(\mathcal{E}nd(\mathcal{V}' \oplus \mathcal{W}')) \cdot [K_X]^{pq-1} = 0$, and $\widehat{c}_2(\mathcal{F}_X) \cdot [K_X]^{pq-2} = \widehat{c}_2(\mathcal{E}nd(\mathcal{V}' \oplus \mathcal{W}')) \cdot [K_X]^{pq-2}$. Moreover, we compute using standard formulae

$$\begin{aligned} \widehat{c}_2(\mathcal{F}_X) \cdot [K_X]^{n-2} = {} & [2(p+q)(\widehat{c}_2(\mathcal{V}') + \widehat{c}_2(\mathcal{W}')) - (p+q-1)(\widehat{c}_1(\mathcal{V}')^2 + \widehat{c}_1(\mathcal{W}')^2) \\ & + 2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')] \cdot [K_X]^{pq-2}, \end{aligned}$$

so by the assumption of the Proposition we have $\widehat{c}_2(\mathcal{F}_X) \cdot [K_X]^{pq-2} = \widehat{ch}_2(\mathcal{F}_X) \cdot [K_X]^{pq-2} = 0$.

Thus the conditions of Theorem 1.1 are satisfied, and it follows that $X \cong \mathcal{A}_{pq}/\Gamma$, where $\Gamma \subset \mathrm{Aut}(\mathcal{A}_{pq})$ is a discrete, cocompact subgroup acting fixed point freely in codimension one on the domain $\mathcal{A}_{pq}$. This concludes the proof. □

In order to determine the necessary conditions for uniformization by $\mathcal{A}_{pq}$, we consider two separate cases, namely when (i) $p \neq q$, and (ii) $p = q$. This is because the group of automorphisms $\mathrm{Aut}(\mathcal{A}_{pq})$ of $\mathcal{A}_{pq}$ is connected when $p \neq q$, and is a quotient of $SU(p, q)$ by a discrete central subgroup. However when $p = q$, $\mathrm{Aut}(\mathcal{A}_{pp})$ has two connected components, and is a quotient of $SU(p, p) \rtimes \mathbb{Z}_2$ by a discrete central subgroup (see [19]).

Necessary conditions when $p \neq q$

From [19, p. 114], we know that the automorphism group of $\mathcal{A}_{pq}$ when $p \neq q$ is $\mathrm{Aut}(\mathcal{A}_{pq}) = PSU(p, q)$. This is a connected Lie group and is a quotient of

$SU(p,q)$ by a discrete central subgroup. Thus we can choose the associated Hodge group to be $G_0 = SU(p,q)$, and its complexification as $G = SL(p+q,\mathbb{C})$, which is also connected. Thus the sufficient conditions of Proposition 6.1 are also necessary conditions for a projective klt variety with ample canonical divisor to be uniformized by $\mathcal{A}_{pq}$.

Proposition 6.2 *Let X be a projective klt variety of dimension pq, $p \neq q$ with K_X ample, such that $X = \mathcal{A}_{pq}/\widehat{\Gamma}$, where $\widehat{\Gamma} \subset Aut(\mathcal{A}_{pq})$ is a discrete, cocompact subgroup acting fixed point freely in codimension one on $\mathcal{A}_{pq}$. Then the tangent bundle of the regular locus of X satisfies $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{E},\mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of ranks p and q on X_{reg}. Moreover, X satisfies the Chern class equality 5.*

Proof By Theorem 1.1, the smooth locus X_{reg} of X admits a uniformizing system of Hodge bundles (P,θ) corresponding to the Hodge group $G_0 = SU(p,q)$, such that the system of Hodge bundles $E = P \times_K \mathfrak{g}$ is K_X-polystable as a Higgs bundle on X_{reg}. Moreover, X satisfies the $\mathbb{Q}$-Chern class equality $\widehat{c}_2(E') \cdot [K_X]^{pq-2} = 0$, where E' denotes the unique reflexive extension of E to X. Hence there is an isomorphism $\theta : \mathcal{T}_{X_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$. Recall from the proof of Proposition 6.1 that $\mathfrak{g}^{-1,1}$ is isomorphic as a K-representation to $\mathrm{Hom}(V,W)$, where V and W are complex p and q-dimensional vector spaces. So we can write $\mathcal{T}_{X_{reg}} \cong P \times_K \mathrm{Hom}(V,W) = \mathcal{H}om(\mathcal{V},\mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are vector bundles of ranks p and q on X_{reg}. We also have $\Omega^1_{X_{reg}} \cong \mathcal{H}om(\mathcal{W},\mathcal{V}) \cong P \times_K \mathfrak{g}^{1,-1}$, and $(\mathcal{E}nd(\mathcal{V}) \oplus \mathcal{E}nd(\mathcal{W}))_0 \cong P \times_K \mathfrak{g}^{0,0}$, from which it follows that the system of Hodge bundles $P \times_K \mathfrak{g}$ is isomorphic to $\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})$. Let $\mathcal{V}'$ and $\mathcal{W}'$ denote the reflexive extensions of $\mathcal{V}$ and $\mathcal{W}$ to X. Then the Chern class equality $\widehat{c}_2(E') \cdot [K_X]^{pq-2} = 0$ is equivalent to

$$\widehat{c}_2(\mathcal{E}nd_0(\mathcal{V}' \oplus \mathcal{W}')) \cdot [K_X]^{pq-2} = 0.$$

Using the short exact sequence (6), and the expression for the second $\mathbb{Q}$-Chern class of a direct sum of reflexive sheaves, the above equality can be rephrased as

$$\begin{aligned}&[2(p+q)(\widehat{c}_2(\mathcal{V}') + \widehat{c}_2(\mathcal{W}')) - (p+q-1)(\widehat{c}_1(\mathcal{V}')^2 + \widehat{c}_1(\mathcal{W}')^2)\\ &\quad + 2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')] \cdot [K_X]^{pq-2} = 0.\end{aligned}$$

Thus, it follows that X satisfies the Chern class equality (5), which completes the proof. □

Putting together Propositions 6.1 and 6.2, we arrive at the following necessary and sufficient condition for a projective klt variety X with ample canonical divisor to be uniformized by the Hermitian symmetric space $\mathcal{A}_{pq}$ $(p \neq q)$ of type $AIII$.

Theorem 6.3 *Let X be a projective klt variety of dimension $pq \geq 2$, $p,q \in \mathbb{Z}_{\geq 1}$, $p \neq q$, such that the canonical divisor K_X is ample. Then $X \cong \mathcal{A}_{pq}/\Gamma$, where $\Gamma \subset \mathrm{Aut}(\mathcal{A}_{pq})$ is a discrete, cocompact subgroup, and acts fixed point freely in codimension one on $\mathcal{A}_{pq}$, if and only if X satisfies*

- $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{V},\mathcal{W})$

- $[2(p+q)(\widehat{c}_2(\mathcal{V}')+\widehat{c}_2(\mathcal{W}'))-(p+q-1)(\widehat{c}_1(\mathcal{V}')^2+\widehat{c}_1(\mathcal{W}')^2)+2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')]\cdot [K_X]^{pq-2}=0$,

where $\mathcal{V}$ and $\mathcal{W}$ are vector bundles of ranks p and q on X_{reg}, and $\mathcal{V}'$ and $\mathcal{W}'$ denote the reflexive extensions of $\mathcal{V}$ and $\mathcal{W}$ to X.

Note that when $p = 1$ and $q = n$, the associated domain is the unit ball $\mathbb{B}^n$. In this case we have $\mathcal{V} = \Omega^1_{X_{reg}}$, $\mathcal{W} = \mathcal{O}_{X_{reg}}$, and the system of Hodge bundles is given by $P \times_K \mathfrak{g} = \mathcal{E}nd_0(\Omega^1_{X_{reg}} \oplus \mathcal{O}_{X_{reg}})$. The condition on the structure of the tangent bundle is tautological, and the Chern class condition is the well known Bogomolov-Miyaoka-Yau equality satisfied by ball quotients.

Necessary conditions when $p = q$

In this case the group of holomorphic automorphisms has two connected components. Again from [19] we know that $Z \mapsto Z^T$ is an automorphism of $\mathcal{A}_{pp}$ of order 2 that is not contained in the connected component of $\mathrm{Aut}(\mathcal{A}_{pp})$. This defines a group homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{A}_{pp})$, i.e., there is a splitting $\mathrm{Aut}(\mathcal{A}_{pp}) = \mathrm{Aut}^0(\mathcal{A}_{pp}) \rtimes \mathbb{Z}_2$, where $\mathrm{Aut}^0(\mathcal{A}_{pp}) = PSU(p,p)$. It follows that $\mathrm{Aut}(\mathcal{A}_{pp})$ is a quotient of $SU(p,p) \rtimes \mathbb{Z}_2$ by a discrete central subgroup. Let G denote the chosen complexification of $SU(p,p) \rtimes \mathbb{Z}_2$, and K the complexification of $K_0 = S(U(p) \times U(p)) \rtimes \mathbb{Z}_2$. Then the tangent bundle of the regular locus of a projective klt quotient of $\mathcal{A}_{pp}$ admits a reduction in structure group to K. However, as in the polydisc case, such a reduction in structure group does not give a meaningful condition on the tangent sheaf. Therefore, we state the result up to a 2:1 quasi-étale cover.

Proposition 6.4 *Let X be a projective klt variety of dimension p^2 with K_X ample, such that X is a quotient of $\mathcal{A}_{pp}$ by a discrete, cocompact subgroup Γ of $\mathrm{Aut}(\mathcal{A}_{pp})$ acting fixed point freely in codimension one. Then X admits a 2:1 quasi-étale cover X' such that the tangent bundle of the regular locus $X'_{reg} \subset X'$ satisfies $\mathcal{T}_{X'_{reg}} \cong \mathcal{H}om(\mathcal{E},\mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of rank p on X_{reg}. Moreover, X' satisfies the Chern class equality 5 for $p = q$.*

Proof Since the automorphism group $\mathrm{Aut}(\mathcal{A}_{pp})$ has two connected components, the group Γ can be expressed as an extension of a normal subgroup Γ' by a subgroup of $\mathbb{Z}_2$. Thus, the quotient map $\mathcal{A}_{pp} \to X$ factors as

$$\mathcal{A}_{pp} \to \mathcal{A}_{pp}/\Gamma' \to \mathcal{A}_{pp}/\Gamma = X,$$

where $X' = \mathcal{A}_{pp}/\Gamma'$, and the map $X' \to X$ is a Galois, 2:1 quasi-étale cover. Note that X' is again projective and klt, and $K_{X'}$ is ample. Again, from Theorem 1.1, it follows that the smooth locus X'_{reg} of X' admits a uniformizing system of Hodge bundles (P,θ) corresponding to the Hodge group $G_0 = SU(p,p)$, such that the system of Hodge bundles $E = P \times_K \mathfrak{g}$ is $K_{X'}$-polystable as a Higgs bundle on X'_{reg}. Moreover, X' satisfies the $\mathbb{Q}$-Chern class equality $\widehat{c}_2(E') \cdot [K_{X'}]^{p^2-2} = 0$, where E' is the reflexive extension of E to X. Therefore, there is an isomorphism $\theta : \mathcal{T}_{X'_{reg}} \cong P \times_K \mathfrak{g}^{-1,1}$ of vector bundles. Recall from the proof of Proposition 6.1 that $\mathfrak{g}^{-1,1}$ is isomorphic as a K-representation to $\mathrm{Hom}(V,W)$, where V and W

are p-dimensional complex vector spaces. So we can write the tangent bundle as $\mathcal{T}_{X'_{reg}} \cong P \times_K \mathrm{Hom}(V, W) = \mathcal{H}om(\mathcal{V}, \mathcal{W})$, where $\mathcal{V}$ and $\mathcal{W}$ are vector bundles of rank p on X'_{reg}. Moreover, we have $\Omega^1_{X'_{reg}} \cong P \times_K \mathfrak{g}^{1,-1} \cong \mathcal{H}om(\mathcal{W}, \mathcal{V})$, and $P \times_K \mathfrak{g}^{0,0} \cong (\mathcal{E}nd(\mathcal{V}) \oplus \mathcal{E}nd(\mathcal{W}))_0$. Thus the system of Hodge bundles $P \times_K \mathfrak{g}$ is isomorphic to $\mathcal{E}nd_0(\mathcal{V} \oplus \mathcal{W})$.

Let $\mathcal{V}'$ and $\mathcal{W}'$ denote the reflexive extensions of $\mathcal{V}$ and $\mathcal{W}$ to X'. Then $E' \cong \mathcal{E}nd_0(\mathcal{V}' \oplus \mathcal{W}')$, and the $\mathbb{Q}$-Chern class equality $\widehat{c}_2(E') \cdot [K_{X'}]^{p^2-2} = 0$ is equivalent to

$$[\widehat{c}_2(\mathcal{E}nd_0(\mathcal{V}', \mathcal{W}'))] \cdot [K_{X'}]^{p^2-2} = 0.$$

Expanding this using standard formulae for Chern classes, we arrive at the following expression

$$[4p(\widehat{c}_2(\mathcal{V})' + \widehat{c}_2(\mathcal{W}')) - (2p-1)(\widehat{c}_1(\mathcal{V}')^2 + \widehat{c}_1(\mathcal{W}')^2) + 2\widehat{c}_1(\mathcal{V}')\widehat{c}_1(\mathcal{W}')] \cdot [K_{X'}]^{p^2-2} = 0.$$

Thus X' satisfies the Chern class equality as claimed, and this completes the proof. □

6.2 Domains of type *CI*

Theorem 6.5 *Let X be a projective klt variety of dimension $n(n+1)/2$ such that the canonical divisor K_X is ample. Then $X \cong \mathcal{H}_n/\Gamma$, where Γ is a discrete cocompact subgroup of $\mathrm{Aut}(\mathcal{H}_n) = PSp(2n, \mathbb{R})$, whose action on $\mathcal{D}$ is fixed point free in codimension one, if and only if X satisfies*

- $\mathcal{T}_{X_{reg}} \cong \mathrm{Sym}^2(\mathcal{E})$
- $[2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(\mathcal{E}')^2] \cdot [K_X]^{\dim X - 2} = 0,$

where $\mathcal{E}$ is a vector bundle of rank n on X_{reg}, and $\mathcal{E}'$ denotes the reflexive extension of $\mathcal{E}$ to X.

6.3 Domains of type *DIII*

Theorem 6.6 *Let X be a projective klt variety of dimension $n(n-1)/2$ such that the canonical divisor K_X is ample. Then $X \cong \mathcal{D}_n/\Gamma$, where $\Gamma \subset PSO^*(2n, \mathbb{R})$ is a discrete, cocompact subgroup, and acts fixed point freely in codimension one on $\mathcal{D}_n$, if and only if X satisfies*

- $\mathcal{T}_{X_{reg}} \cong \bigwedge^2(\mathcal{E})$
- $[2\widehat{c}_2(X) - \widehat{c}_1(X)^2 + 2n\widehat{c}_2(\mathcal{E}') - (n-1)\widehat{c}_1(\mathcal{E}')^2] \cdot [K_X]^{\dim X - 2} = 0,$

where $\mathcal{E}$ is a vector bundle of rank n on X_{reg}, and $\mathcal{E}'$ denotes the reflexive extension of $\mathcal{E}$ to X.

6.4 Domains of type *BDI*

Theorem 6.7 *Let X be a projective klt variety of dimension $n \geq 3$, such that the canonical divisor K_X is ample. Then $X \cong \mathcal{B}_n/\Gamma$, where $\Gamma \subset \mathrm{Aut}(\mathcal{B}_n)$ is a discrete,* *cocompact subgroup acting fixed point freely in codimension one on $\mathcal{B}_n$, if and only if X satisfies*

- $\mathcal{T}_{X_{reg}} \cong \mathcal{H}om(\mathcal{W}, \mathcal{L})$
- $[\widehat{c}_2(\mathcal{W}' \wedge \mathcal{W}') + 2\widehat{c}_2(\mathcal{W}') - n\widehat{c}_1(\mathcal{L}')^2] \cdot [K_X]^{n-2} = 0,$

where $\mathcal{W}$ is an orthogonal vector bundle of rank n, $\mathcal{L}$ is a line bundle on X_{reg}, and $\mathcal{W}'$ and $\mathcal{L}'$ denote the reflexive extensions of $\mathcal{W}$ and $\mathcal{L}$ to X.

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