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Modular forms for GL$(r, \mathbb{F}_q[T])$: t-expansions of the basic forms

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Abstract
We give closed formulas for the first few expansion coefficients of the basic modular forms for $\mathrm{GL}(r, \mathbb{F}_q[T])$. Here the rank r is larger or equal to 3, and the forms in question include the coefficient forms $g_1, \ldots, g_r$ and the Eisenstein series E_{q^i-1} ($i \in \mathbb{N}$).

Keywords Drinfeld modular forms · Eisenstein series · Coefficient forms · t-Expansions

Mathematics Subject Classification Primary 11F52 ; Secondary 11G09

1 Introduction

In recent years, Drinfeld modular forms for groups like $\Gamma := \mathrm{GL}(r, \mathbb{F}_q[T])$ have been introduced and investigated in more detail in [4] and in the author's series of papers starting with [12–14], see also [17]. As in the case of rank $r = 2$, where a large amount of literature exists, it is desirable to dispose of examples and numerical data allowing a deeper arithmetical study. For that case, see e.g. the works of Bosser [5], Choi [6, 7], Lopez [18], Vincent [21], Armana [1] and, more recently, of Bandini–Valentino [2] and Dalal–Kumar [8]. With the present paper, the author aims to provide some information about the t-expansions of the most basic modular forms for Γ, namely the Eisenstein series E_{q^k-1} and the coefficient forms g_k, both of weight $q^k - 1$ and type 0.

Each modular form for Γ has a t-expansion

$$f(\boldsymbol{\omega}) = \sum_{n \geq 0} a_n(\boldsymbol{\omega}')t^n(\boldsymbol{\omega}),$$

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where $a_n(\boldsymbol{\omega}')$ is a weak modular form of rank $r' := r-1$, and $\boldsymbol{\omega}' = (\omega_2, \ldots, \omega_r)$ is the tail of $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$, a point of the Drinfeld space Ω^r. We are able to give closed formulas for the a_n if

(a) $f = E_{q^k-1}$ and $n < 3(q-1)q^{k+r-2}$, see Theorem 3.8;
(b) $f = g_k$ and $n < 3(q-1)q^{r-1}$ (Theorem 4.1);

and, in a less satisfactory manner, if

(c) $f = h$, the $(q-1)$-th root of the discriminant form $\Delta = g_r$, and $n \leq (q-1)(q^{2r-1} - q^r)$ (Theorem 5.5).

For (a) and (b) we have to assume $q > 2$, otherwise (as $\sum_{c \in \mathbb{F}_q} c \neq 0$ for $q = 2$, in contrast with $q > 2$) some annoying extra terms appear that lead to weaker error estimates.

In case (a), there are precisely $k+1$ non-vanishing coefficients a_n below the given bound with $n = q^k - q^i$ $(0 \leq i \leq k)$; in case (b), there are precisely $2k$ (if $k < r$) or $2r-1$ (if $k = r$) non-vanishing a_n below the bound, all with $n \leq q^k - 1$; in case (c), we have no similar description.

For (a), we start with the known A-expansion involving

$$\sum_{a \in A \text{ monic}} G_{q^k-1}(t_a)$$

of Theorem 2.11. Due to the structure of the quantities t_a and of the Goss polynomials G_{q^k-1}, there is heavy cancellation in the partial sums

$$\sum_{a \text{ monic of degree } d} G_{q^k-1}(t_a),$$

which leads to strong estimates and thereby to Theorem 3.8. For (b) we could try to use (a) and the well-known relationship (2.4.6) between the E_{q^i-1} and the g_j. This, however, turns out hopelessly complicated. Instead, we use the presentation of the generic Drinfeld module $\phi = \phi^{\boldsymbol{\omega}}$ of rank r (where $\boldsymbol{\omega}$ varies through the Drinfeld symmetric space Ω^r) as a Tate object over $\phi' = \phi^{\boldsymbol{\omega}'}$, its counterpart of rank $r' = r-1$. This leads to the more efficient recursion formula (4.3.6) for the g_k, which is better adapted to our problem. Together with Proposition 4.4 (a technical estimate for the Tate object), it allows to show the formula (4.1.1) for the non-vanishing coefficients, that is, Theorem 4.1.

The situation is different for (c), i.e., for the modular form h of weight $(q^r-1)/(q-1)$ and type 1. In contrast with the Eisenstein series E_{q^k-1} and the coefficient forms g_k (whose t-expansions are sparse), but in accordance with the case of rank 2 (see [11]), h has "many" non-vanishing coefficients. We restrict to estimating the contributions of monic $a \in A$ of degree ≥ 2 in the product formula

$$h = (h')^q t \prod_{a \in A \text{ monic}} S_a(t)^{q^r-1}.$$

This leads to the error estimate in Theorem 5.5, where we replace the above product by

$$(h')^q t \prod_{a \text{ monic of degree } 1} S_a(t)^{-1}.$$

So our results are twofold: determining the first few terms of an expansion, and estimating the error, that is the length of the following string of vanishing coefficients. While for (a) and (c) the first task is easy, as we already dispose of the expansions (2.11.1) and (2.12.2) and we are reduced to work out the error estimates, this is different for (b). Here, the closed form of the starting string given in Theorem 4.1 is far from obvious, and could be guessed only after calculating a number of examples. Therefore Theorem 4.1 should be regarded as the principal result of the paper.

We assume throughout that the rank r is larger or equal to 3, which is not a serious restriction. In the rank-2 case there is a plentitude of literature with partially much sharper results than those that could be obtained from the methods of the present paper. Again in the case of rank 2, usually an arithmetic normalization of the uniformizer t is made, where the period $\overline{\pi}$ of the Carlitz module (the counterpart to the classical $2\pi\imath$) enters. Such a normalization is also possible in the present higher rank case (see [20]), but applying it would lead us too far away, and would obscure the overall picture. This is why we decided to use (as in [4] or [14]) the natural geometric normalization (2.5.1) of t. But see Remark 4.5.

2 Modular forms and their expansions

(See, e.g. [10, 12, 14].)

2.1 Throughout, $\mathbb{F} = \mathbb{F}_q$ is the finite field with q elements, of characteristic p, and $r \geq 3$ a natural number, the **rank**. Both q and r are fixed and usually omitted from notation.

$A = \mathbb{F}[T]$ (resp. $K = \mathbb{F}(T)$) is the polynomial ring (resp. the field of rational functions) in an indeterminate T. Further, $K_\infty = \mathbb{F}((T^{-1}))$ is the completion of K at its infinite place, with completed algebraic closure C_∞. The absolute value $|.|$ on C_∞ is normalized such that $|T| = q$.

$$\Omega = \Omega^r = \mathbb{P}^{r-1}(C_\infty) \smallsetminus \bigcup H(C_\infty)$$

(where H runs through the hyperplanes defined over K_∞) with its structure as a rigid-analytic space denotes the Drinfeld symmetric space. Homogeneous coordinates $(\omega_1 : \omega_2 : \ldots : \omega_r)$ are normalized such that $\omega_r = 1$, hence

$$\Omega = \{\boldsymbol{\omega} \in C_\infty^r \mid \omega_1, \ldots, \omega_r \ K_\infty\text{-linearly independent and } \omega_r = 1\}.$$

2.2 The group $\Gamma := \mathrm{GL}(r, A)$ acts on Ω through fractional linear transformations

$$\gamma\boldsymbol{\omega} = \mathrm{aut}(\gamma, \boldsymbol{\omega})^{-1}\gamma \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_r \end{pmatrix}, \tag{2.2.1}$$

where the right hand side is the matrix product and

$$\mathrm{aut}(\gamma, \boldsymbol{\omega}) := \sum_{1 \le i \le r} \gamma_{r,i}\omega_i. \tag{2.2.2}$$

A holomorphic function f on Ω is a **weak modular form** (or **weakly modular**) of **weight** $k \in \mathbb{Z}$ and **type** $m \in \mathbb{Z}/(q-1)$ if

$$f(\gamma\boldsymbol{\omega}) = \mathrm{aut}(\gamma, \boldsymbol{\omega})^k (\det \gamma)^{-m} f(\boldsymbol{\omega}) \tag{2.2.3}$$

holds for $\gamma \in \Gamma$ and $\boldsymbol{\omega} \in \Omega$.

2.3 An A-lattice in C_∞ is a finitely generated (thus free of some rank ρ) A-submodule Λ of C_∞ which is discrete in the sense that Λ intersects with each ball in at most finitely many points. Hence $\Lambda = \bigoplus_{1 \le i \le \rho} A\omega_i$ with ρ K_∞-linearly independent elements ω_i. With each lattice Λ, we associate

- its **exponential function**

$$e^\Lambda(z) = z \prod_{\lambda \in \Lambda}{}' (1 - z/\lambda)$$

  (the primed product $\prod'$ (resp. sum $\sum'$) is over the non-zero elements of the index set). It is everywhere convergent and has a power series expansion

$$e^\Lambda(z) = \sum_{i \ge 0} \alpha_i(\Lambda) z^{q^i}; \tag{2.3.1}$$

- the **Eisenstein series**

$$E_k(\Lambda) = \sum_{\lambda \in \Lambda}{}' \lambda^{-k}$$

  ($k > 0$ and $k \equiv 0 \pmod{q-1}$); otherwise it vanishes);
- the **Drinfeld module** $\phi^\Lambda = C_\infty/\Lambda$ of rank ρ.

The latter is an exotic A-module structure given by the a-th operator polynomial ϕ_a^Λ for each $0 \ne a \in A$, which makes the following diagram with exact rows commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & C_\infty & \xrightarrow{e^\Lambda} & C_\infty & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \phi_a^\Lambda} & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & C_\infty & \xrightarrow{e^\Lambda} & C_\infty & \longrightarrow & 0 \end{array} \tag{2.3.2}$$

ϕ^Λ is uniquely determined already by ϕ_T^Λ, which has shape

$$\phi_T^\Lambda(X) = TX + g_1(\Lambda)X^q + \cdots + g_\rho(\Lambda)X^{q^\rho}. \tag{2.3.3}$$

The only restriction on ϕ_T^Λ is that the **discriminant**

$$\Delta(\phi^\Lambda) := \Delta(\Lambda) := g_\rho(\Lambda)$$

be non-zero.

2.4 Let $C_\infty\{\{\tau\}\}$ be the ring of formal power series in a non-commutative variable τ with commutation rule

$$\tau c = c^q \tau \quad \text{for } c \in C_\infty, \tag{2.4.1}$$

with sub-ring $C_\infty\{\tau\}$ of polynomials. Then $C_\infty\{\{\tau\}\}$ may be identified via $\tau^n \rightsquigarrow z^{q^n}$ with the ring of $\mathbb{F}$-linear power series of shape $\sum c_n z^{q^n}$, where the product of the latter is given by insertion. The map

$$\phi^\Lambda : a \longmapsto \phi_a^\Lambda \tag{2.4.2}$$

is a homomorphism of $\mathbb{F}$-algebras from A to $C_\infty\{\tau\}$, and we can write

$$e^\Lambda(z) = \sum \alpha_i z^{q^i} = \sum \alpha_i \tau^i, \quad \text{where } \alpha_i = \alpha_i(\Lambda). \tag{2.4.3}$$

Define the series $(\beta_i)_{i\in\mathbb{N}_0}$ by $\beta_i = -E_{q^i-1}(\Lambda)$ (with $E_0(\Lambda) = -1$, so $\beta_0 = 1$) and $\log^\Lambda \in C_\infty\{\{\tau\}\}$ by

$$\log^\Lambda = \sum \beta_i \tau^i. \tag{2.4.4}$$

Then

$$e^\Lambda \cdot \log^\Lambda = \log^\Lambda \cdot e^\Lambda = 1 \quad \text{in} \quad C_\infty\{\{\tau\}\}, \tag{2.4.5}$$

i.e., the two series are mutual inverses. The equation

$$\phi_T^\Lambda \cdot e^\Lambda = e^\Lambda \cdot T$$

from (2.3.2) leads to

$$\log^\Lambda \cdot \phi_T^\Lambda = T \cdot \log^\Lambda. \tag{2.4.6}$$

Comparing coefficients in (2.4.5) gives

$$\sum_{i+j=k} \alpha_i \beta_j^{q^i} = \sum_{i+j=k} \beta_i \alpha_j^{q^i} = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k > 0. \end{cases} \tag{2.4.7}$$

Similarly, from (2.4.6),

$$\sum_{i+j=k} \beta_i g_j^{q^i} = T\beta_k \quad f(k \geq 0), \tag{2.4.8}$$

where $g_0 = T$ and all the α_i, β_j, g_k depend on Λ. These formulas allow to recursively calculate any two of the series (α_i), (β_j), (g_k) from the third. For example,

$$g_k = -\sum_{1\leq j<k} \beta_{k-j} g_j^{q^{k-j}} - [k]\beta_k = \sum_{1\leq j<k} E_{q^{k-j}-1} g_j^{q^{k-j}} + [k]E_{q^k-1}, \tag{2.4.9}$$

where $[k]$ is short for $T^{q^k} - T \in A$ and $E_\ell = E_\ell(\Lambda)$.

2.5 For $\boldsymbol{\omega} \in \Omega$ let $\Lambda_{\boldsymbol{\omega}}$ be the lattice $\sum_{1\leq i\leq r} A\omega_i$. We further let $\boldsymbol{\omega}' = (\omega_2, \ldots, \omega_r) \in \Omega' = \Omega^{r-1}$ and $\Lambda' = \Lambda_{\boldsymbol{\omega}'} = \sum_{2\leq i\leq r} A\omega_i$. Consider the function on Ω:

$$t(\boldsymbol{\omega}) = (e^{\Lambda'}(\omega_1))^{-1}. \tag{2.5.1}$$

It is well-defined, holomorphic and nowhere vanishing on Ω, and satisfies

$$t(\widetilde{\boldsymbol{\omega}}) = t(\boldsymbol{\omega}) \quad \text{if} \quad \widetilde{\boldsymbol{\omega}} = (\omega_1 + \lambda', \omega_2, \ldots, \omega_r) \quad \text{for} \quad \lambda' \in \Lambda'. \tag{2.5.2}$$

If f is a weak modular form then (2.2.3) implies that f satisfies the same rule $f(\widetilde{\boldsymbol{\omega}}) = f(\boldsymbol{\omega})$ and, moreover, that f has a Laurent expansion

$$f(\boldsymbol{\omega}) = \sum_{n\in\mathbb{Z}} a_n(\boldsymbol{\omega}')t^n(\boldsymbol{\omega}) \tag{2.5.3}$$

for sufficiently small values of t, where the coefficients a_n depend only on $\boldsymbol{\omega}'$.

2.5.4 Now f is called **modular** if the coefficients $a_n(\boldsymbol{\omega}')$ vanish identically for $n < 0$, and is a **cusp form** if even $a_n(\boldsymbol{\omega}') \equiv 0$ for $n \leq 0$.

Remark 2.6 (i) Suppose that f has weight k and type m. Then the coefficient functions a_n are in fact weakly modular of weight $k - n$ and type m, see e.g. [4].

(ii) If $a_n \neq 0$ then $n \equiv k + m \pmod{q-1}$.

(iii) The boundary condition "$a_n \equiv 0$ if $n < 0$" for a weak modular form f to be in fact modular is equivalent with "f is bounded on the canonical fundamental domain $\mathbf{F}$ of Γ", see [14], but we will not use this fact.

2.7 Associating the lattice $\Lambda_{\boldsymbol{\omega}}$ with $\boldsymbol{\omega} \in \Omega$, we will regard the $\alpha_i = \alpha_i(\Lambda_{\boldsymbol{\omega}}) = \alpha_i(\boldsymbol{\omega})$, and ditto β_i, g_i, E_k, as functions on Ω. Then α_i, β_i, g_i are actually modular forms of weight $q^i - 1$, and E_k is modular of weight k, all of type zero.

Define $M_{k,m}$ as the C_∞-vector space of modular forms of weight k and type m, and

$$\mathbf{M} := \bigoplus_{\substack{k\in\mathbb{N}_0 \\ m\in\mathbb{Z}/(q-1)}} M_{k,m} \tag{2.7.1}$$

as the bigraded C_∞-algebra of modular forms, with sub-algebra

$$\mathbf{M}_0 := \bigoplus_{k \in \mathbb{N}_0} M_{k,0} \tag{2.7.2}$$

of forms of type 0. Then:

Theorem 2.8 ([4, 12])

(i) $\mathbf{M}_0$ *is the polynomial ring* $C_\infty[g_1, \dots, g_r]$ *in the algebraically independent forms* $g_1, \dots, g_r$*;*
(ii) $\mathbf{M} = \mathbf{M}_0[h]$*, where* $h \in M_{(q^r-1)/(q-1),1}$ *is a* $(q-1)$*-th root of the discriminant* $\Delta = g_r$*.*

Remark 2.9 (i) The various $M_{k,m}$ are linearly independent as functions on Ω; hence their sum is in fact direct.

(ii) We choose the normalization of h such that $h^{q-1} = (-1)^{r-1}\Delta$. This allows the formula (2.12.2) for h.

(iii) As results from (2.4.8) and (2.4.9), we can also write

$$\mathbf{M}_0 = C_\infty[\alpha_1, \dots, \alpha_r] = C_\infty[\beta_1, \dots, \beta_r] = C_\infty[E_{q-1}, \dots, E_{q^r-1}].$$

2.10 Given $\boldsymbol{\omega} \in \Omega$, we let $\phi^{\boldsymbol{\omega}}$ be the Drinfeld A-module of rank r associated with the lattice $\Lambda_{\boldsymbol{\omega}}$, i.e., $\phi^{\boldsymbol{\omega}} = \phi^{\Lambda_{\boldsymbol{\omega}}}$. Similarly, writing $\boldsymbol{\omega} = (\omega_1, \boldsymbol{\omega}')$ as in 2.5, $\phi^{\boldsymbol{\omega}'} = \phi^{\Lambda_{\boldsymbol{\omega}'}}$, of rank $r' = r-1$. Quite generally, we write objects related to $\phi^{\boldsymbol{\omega}'}$, for example functions of type $\alpha_i, \beta_i, g_i, \dots$with a prime $\alpha_i', \beta_i', g_i', \dots$to distinguish them from the "same" objects related to $\phi^{\boldsymbol{\omega}}$.

For $0 \neq a \in A$, of degree d and leading coefficient $\mathrm{sgn}(a) \in \mathbb{F}^*$, the operator polynomial $\phi_a^{\boldsymbol{\omega}'}(X)$ (see (2.3.3)) of $\phi^{\boldsymbol{\omega}'}$ has shape

$$\phi_a^{\boldsymbol{\omega}'}(X) = aX + {}_a\ell_1'(\boldsymbol{\omega}')X^q + {}_a\ell_2'(\boldsymbol{\omega}')X^{q^2} + \dots + {}_a\ell_{(r-1)d}'(\boldsymbol{\omega}')X^{q^{(r-1)d}}, \tag{2.10.1}$$

with last coefficient

$${}_a\ell_{(r-1)d}'(\boldsymbol{\omega}') = \mathrm{sgn}(a)\Delta'(\boldsymbol{\omega}')^e =: \Delta_a'(\boldsymbol{\omega}') \neq 0, \tag{2.10.2}$$

where the exponent e is $(q^{(r-1)d} - 1)/(q^{r-1} - 1)$. Define

$$\begin{aligned} S_a(X) &:= \Delta_a'(\boldsymbol{\omega}')^{-1} X^{q^{(r-1)d}} \phi_a^{\boldsymbol{\omega}'}(X^{-1}) \\ &= 1 + \frac{{}_a\ell_{(r-1)d-1}'}{\Delta_a'} X^{q^{(r-1)d} - q^{(r-1)d-1}} + \dots + \frac{a}{\Delta_a'} X^{q^{(r-1)d}-1}. \end{aligned} \tag{2.10.3}$$

(We omit reference to $\boldsymbol{\omega}'$ of $S_a(X)$ and of its coefficients $\frac{{}_a\ell_i'}{\Delta_a'}$.) Let $\mathcal{O}' = \mathcal{O}(\Omega')$ be the ring of holomorphic functions on $\Omega' = \Omega^{r-1}$. As $\phi' = \phi^{\boldsymbol{\omega}'}$ varies with $\boldsymbol{\omega}'$ over Ω',

$S_a(X) \in \mathcal{O}'[X]$ is a polynomial with coefficients in $\mathcal{O}'$. If $a \in \mathbb{F}^*$ then $S_a(X) = 1$; if $a = T + c$ with $c \in \mathbb{F}$ then

$$S_a(X) = 1 + \frac{g'_{r-1}}{\Delta'} X^{q^{r-1}-q^{r-2}} + \cdots + \frac{g'_1}{\Delta'} X^{q^{r-1}-q} + \frac{T+c}{\Delta'} X^{q^{r-1}-1}. \tag{2.10.4}$$

With notation as before, let t_a be the function

$$t_a(\boldsymbol{\omega}) = t(a\omega_1, \boldsymbol{\omega}') = (\Delta'_a)^{-1} t^{q^{(r-1)d}} / S_a(t), \tag{2.10.5}$$

where the right hand side is regarded as a power series in t with coefficients in $\mathcal{O}'$.

By means of S_a and t_a, we may describe the t-expansions of some modular forms, as follows.

Theorem 2.11 ([10, 16]) *The special Eisenstein series E_{q^k-1} has the t-expansion*

$$E_{q^k-1}(\boldsymbol{\omega}) = E'_{q^k-1}(\boldsymbol{\omega}') - \sum_{a \in A \text{ monic}} G_{q^k-1}(t_a), \tag{2.11.1}$$

where $G_{q^k-1}(X)$ is the polynomial

$$G_{q^k-1}(X) = \sum_{0 \le i < k} \beta'_i X^{q^k - q^i} \tag{2.11.2}$$

with $\beta'_i = \beta'_i(\boldsymbol{\omega}') = -E'_{q^i-1}(\boldsymbol{\omega}')$. It converges for $|t|$ small enough, locally uniformly in $\boldsymbol{\omega}'$.

Theorem 2.12 ([3, 9, 16]) *The discriminant form $\Delta = g_r$ has a product expansion*

$$\Delta(\boldsymbol{\omega}) = -\Delta'(\boldsymbol{\omega}')^q t^{q-1} \prod_{a \in A \text{ monic}} S_a(t)^{(q^r-1)(q-1)}. \tag{2.12.1}$$

Accordingly, the function h (with the condition $h^{q-1} = (-1)^{r-1}\Delta$) may be chosen such that

$$h(\boldsymbol{\omega}) = h'(\boldsymbol{\omega}')^q t \prod_{a \text{ monic}} S_a(t)^{q^r-1}. \tag{2.12.2}$$

We will use these to determine the first few coefficients a_n of

$$f(\boldsymbol{\omega}) = \sum_{n \ge 0} a_n(\boldsymbol{\omega}') t^n(\boldsymbol{\omega})$$

for $f \in \{g_1, \ldots, g_r, h, E_{q-1}, E_{q^2-1}, \ldots\}$.

3 The special Eisenstein series E_{q^k-1}

3.1 The **support** supp(P) of a polynomial or power series $P(X)$ is the set of indices $n \in \mathbb{N}_0$ such that the n-th coefficient of P is non-zero. The vanishing order or briefly **order** of P is $o(P) := \min \operatorname{supp}(P)$. We write $P(X) = o(X^n)$ if $o(P) \geq n$. For example,

$$o(G_{q^k-1}) = q^k - q^{k-1} \tag{3.1.1}$$

(as long as $\beta'_{k-1} \neq 0$, see (2.11.2)).

An **affine-linear** map of $\mathbb{F}$-vector spaces is the sum of a linear and a constant map. The crucial property for our purposes is the easy observation: Let $f : V \to W$ be an affine-linear map of $\mathbb{F}$-vector spaces with $0 < \dim V < \infty$. Then

$$\sum_{v \in V} f(v) = 0 \tag{3.1.2}$$

except possibly in the case where $q = 2$ and $\dim V = 1$.

3.2 As in 2.10, we let $a \in A$ be a monic (i.e., sgn(a) = 1) element of degree $d \geq 1$. Then

$$o(S_a(X) - 1) = q^{(r-1)d} - q^{(r-1)d-1} \tag{3.2.1}$$

and

$$o(t_a) = q^{(r-1)d}, \tag{3.2.2}$$

t_a being considered as a power series in t. (We have used the fact that neither of the coefficient functions ${}_a\ell'_i(\boldsymbol{\omega}')$ vanishes identically, see e.g. [13].)

Write

$$a = \sum_{0 \leq i \leq d} c_i T^i \quad \text{with} \quad c_i \in \mathbb{F}, c_d = 1. \tag{3.2.3}$$

As

$$\phi_a^{\boldsymbol{\omega}'} = \sum c_i \phi_{T^i}^{\boldsymbol{\omega}'} \quad \text{and} \quad \phi_{T^i}^{\boldsymbol{\omega}'} = \phi_T^{\boldsymbol{\omega}'} \circ \phi_T^{\boldsymbol{\omega}'} \circ \cdots \circ \phi_T^{\boldsymbol{\omega}'}$$

(i factors, successively inserted, $\phi_1^{\boldsymbol{\omega}'}(X) = X$), the coefficients of $\phi_a^{\boldsymbol{\omega}'}(X)$ are affine-linear functions of $(c_0, \ldots, c_{d-1})$, and the same holds for the coefficients of the reciprocal polynomial $S_a(X)$ (see (2.10.3)). Here we use that by (2.10.2),

$$\Delta'_{(d)} := \Delta'_a = \Delta'^{(q^{(r-1)d}-1)/(q^{r-1}-1)} \tag{3.2.4}$$

depends only on the degree d of a.

3.3 To get control of the Eisenstein series E_{q^k-1} via Theorem 2.11, we must evaluate the expression

$$\sum_{a \in A \text{ monic}} t_a^{q^k-q^i} = \Big(\sum_a t_a^{q^{k-i}-1}\Big)^{q^i}.$$

Therefore we define

$$\Theta(j,d) := \sum_{\substack{a \in A \text{ monic} \\ \deg a = d}} t_a^{q^j - 1} \quad (j, d \geq 1), \tag{3.3.1}$$

a power series in t with coefficients in $\mathcal{O}'$. Our aim is to get good lower estimates for the order $o(\Theta(j,d))$. By the definition (2.10.5) of t_a and (3.2.4),

$$o(\Theta(j,d)) = q^{(r-1)d}(q^j - 1) + o(\Sigma(j,d)), \tag{3.3.2}$$

where

$$\Sigma(j,d) := \sum_{\substack{a \text{ monic} \\ \deg a = d}} S_a^{1-q^j} \in \mathcal{O}'[[t]]. \tag{3.3.3}$$

3.4 We will evaluate $\Sigma(j,d)$ via the scheme

$$\Sigma(j,d) = \sum_a S_a^{1-q^j} = \sum_a \frac{S_a}{S_a^{q^j}} = \sum_a S_a(1-(S_a^{q^j}-1)+(S_a^{q^j}-1)^2-\cdots). \tag{3.4.1}$$

As for each $j \geq 0$ the support $\mathrm{supp}(S_a^{q^j} - 1)$ is contained in the interval $[q^{(r-1)d+j} - q^{(r-1)d+j-1}, q^{(r-1)d+j} - q^j]$ (see (2.10.3)), the supports of $S_a(t)$, $S_a^{q^j}(t) - 1$, and $(S_a^{q^j} - 1)^2$ are mutually disjoint. Therefore

$$\mathrm{supp}(S_a^{1-q^j}) = \mathrm{supp}(S_a) \,\dot\cup\, \mathrm{supp}(S_a^{q^j} - 1) \,\dot\cup\, R \tag{3.4.2}$$

with some subset $R \subset \mathbb{N}$ and $\min(R) = 2(q^{(r-1)d+j} - q^{(r-1)d+j-1})$. Again, this description depends only on the degree of a, but not on a itself. For $n \in \mathbb{N}_0$ let $\kappa_{n,j}(a)$ be the n-coefficient of $S_a^{1-q^j}$. It follows from (3.2.3) and (3.4.2) that the map

$$(c_0, \ldots, c_{d-1}) \longmapsto \kappa_{n,j}\Big(\sum_{0 \leq i < d} c_i T^i + T^d\Big) \tag{3.4.3}$$

is in fact affine-linear if $n \in \mathrm{supp}(S_a) \cup \mathrm{supp}(S_a^{q^j} - 1)$. Looking at (3.1.2), we get our first result.

Proposition 3.5 *Suppose that $q > 2$ or $d > 1$. Then $\sum_{a \text{ monic}, \deg a = d} \kappa_{n,j}(a)$ vanishes for $n < 2(q^{(r-1)d+j} - q^{(r-1)d+j-1})$. Hence $o(\Sigma(j,d)) \geq 2(q^{(r-1)d+j} - q^{(r-1)d+j-1})$. If $q = 2$ and $d = 1$,*

$$\Sigma(j,1) = S_T^{1-q^j} + S_{T+1}^{1-q^j} = \Delta'^{-1} t^{2^{r-1}-1} + \Delta'^{-2^j} t^{2^{r+j-1}-2^j} + o(t^{2^{r+j-1}}),$$

where $\Delta' \in \mathcal{O}'$ is the discriminant form of rank $r-1$.

Proof The first part follows from (3.4.3), (3.4.2) and (3.1.2), the second from inspection. □

Note that the exponent 2^{r+j-1} in the $(q, d) = (2, 1)$ case is the specialization of the general error exponent $2(q^{(r-1)d+j} - q^{(r-1)d+j-1})$. Hence the non-vanishing terms $\Delta'^{-1}t^{2^{r-1}-1}$ and $\Delta'^{-2^j}t^{2^{r+j-1}-2^j}$ for $(q, d) = (2, 1)$ force $o(\Sigma(j, 1))$ to be smaller than expected from the general case. As a consequence, we get weaker lower estimates for some quantities, e.g. in Corollary 3.7, Theorem 3.8, Proposition 4.4, and Theorem 4.1.

Remark 3.6 It is easy to check that the various $\mathrm{supp}((S_a^{q^j} - 1)^\ell)$ are mutually disjoint as long as $\ell \leq q$, and so

$$\mathrm{supp}(S_a^{1-q^j}) = \mathrm{supp}(S_a) \bigcup_{1\leq\ell\leq q}^{\cdot} \mathrm{supp}((S_a^{q^j-1})^\ell) \,\dot\cup\, R'$$

with some remainder $R' \subset \mathbb{N}$. For $n \in \mathrm{supp}((S_a^{q^j}-1)^\ell)$ the function $(c_0, \ldots, c_{d-1}) \mapsto \kappa_{n,j}(\sum c_i T^i + T^d)$ of (3.4.3) is m-polynomial for some m with $0 \leq m \leq \ell$. Here, f m-polynomial means $f(v) = \sum_{0\leq i\leq m} f_i(v)$, where $f_i(cv) = c^i f(v)$ for $c \in \mathbb{F}$.

The vanishing of $\sum_{v\in V} f(v)$ can be shown under assumptions on the size of q. E.g., $\sum_{v\in V} f_2(v) = 0$ for a 2-homogeneous map is assured if there exists $c \in \mathbb{F}^*$ such that $c^2 \neq 1$, i.e., $q > 3$. Hence, making assumptions on the size of q, the bound in Proposition 3.5 may be sharpened.

Corollary 3.7 *The quantity* $\Theta(j, d)$ *of* (3.3.1) *satisifes* $\Theta(j, d) = o(t^N)$ *with* $N = 3q^{(r-1)d+j} - 2q^{(r-1)d+j-1} - q^{(r-1)d}$ *if* $(q, d) \neq (2, 1)$ *and* $N = 2^{r+j-1} - 1$ *if* $(q, d) = (2, 1)$.

Proof 3.5 + (3.3.2). □

Now we can describe the t-expansion of the special Eisenstein series as follows.

Theorem 3.8 *We have*

$$E_{q^k-1}(\boldsymbol{\omega}) = E'_{q^k-1}(\boldsymbol{\omega}') - G_{q^k-1}(t) + o(t^N) = \sum_{0\leq i\leq k} E'_{q^i-1}(\boldsymbol{\omega}')t^{q^k-q^i} + o(t^N) \tag{3.8.1}$$

with $N = 3(q-1)q^{k+r-2}$ *for* $q > 2$ *and* $N = 2^{k-1}(2^r - 1)$ *for* $q = 2$.

Proof Omitting in (2.11.1) the terms $G_{q^k-1}(t_a)$ with $\deg a > 0$ leads to an error

$$o\Big(\sum_{\substack{a \text{ monic} \\ \deg a=1}} t_a^{q^k-q^{k-1}}\Big) = o(t^{q^{k-1}\cdot o(\Theta(1,1))}).$$

By Corollary 3.7, the exponent N results as stated. □

3.9 What can be said about the expansions of non-special Eisenstein series E_k, where $0 < k \equiv 0 \pmod{q-1}$? Again, there is an expansion like (2.11.1)

$$E_k(\boldsymbol{\omega}) = E'_k(\boldsymbol{\omega}') - G_k(t) - \sum_{\substack{a \text{ monic} \\ \deg a \geq 1}} G_k(t_a) \tag{3.9.1}$$

with a monic polynomial $G_k(X) \in \mathcal{O}'[X]$ of degree k, see [15]. But for k not of the special form $k = q^j - 1$, G_k behaves erratic, and its order $\nu(k) := o(G_k(X))$ doesn't grow with k, as is the case with G_{q^k-1}. At least,

$$\nu = \nu(k) \geq \max(2, q-1),$$

so we have the crude estimate

$$o\Big(\sum_{\substack{a \text{ monic} \\ \deg a = d}} t_a^\nu\Big) = \nu q^{(r-1)d} + o\Big(\sum_a S_a^{-\nu}\Big)$$

with

$$\begin{aligned} o\Big(\sum_a S_a^{-\nu}\Big) &\geq o(S_a^{-\nu}) \quad \text{(one fixed monic } a \text{ of degree } d) \\ &\geq o(S_a) \\ &= q^{(r-1)d} - q^{(r-1)d-1}. \end{aligned}$$

Thus

$$\begin{aligned} o\Big(\sum_{\substack{a \text{ monic} \\ \deg a = d}} t_a^\nu\Big) &= (\nu+1) q^{(r-1)d} - q^{(r-1)d-1} \\ &= q^{(r-1)d+1} - q^{(r-1)d-1} \end{aligned}$$

and finally

$$E_k(\boldsymbol{\omega}) = E'_k(\boldsymbol{\omega}') - G_k(t) + o(t^N) \tag{3.9.2}$$

with $N = q^r - q^{r-2}$. This formula is meaningful for such k less than N for which $G_k(X)$ is accessible.

4 The coefficient forms g_k

Our aim is to show the following formula for the starting terms of the t-expansions of the forms g_k $(1 \leq k \leq r)$. As before, $g'_i = g'_i(\boldsymbol{\omega}')$ is the corresponding coefficient form in rank $r' = r - 1$.

Theorem 4.1 *As a power series in t, we have*

$$g_k = g'_k - g'^{\,q}_{k-1} t^{q-1} + g'_{k-1} t^{q^k - q^{k-1}} - g'^{\,q}_{k-2} t^{q^k - q^{k-1} + q - 1} + g'_{k-2} t^{q^k - q^{k-2}} - \cdots - g'^{\,q}_1 t^{q^k - q^2 + q - 1} + g'_1 t^{q^k - q} - [1] t^{q^k - 1} + o(t^N)$$

with $N = 3(q-1)q^{r-1}$ if $q > 2$, $N = 3 \cdot 2^{r-1} - 1$ if $q = 2$, and $[1] = T^q - T$. In closed form,

$$g_k = g'_k + \sum_{1 \le i \le k-1} \left(g'_i t^{q^k - q^i} - g'^{\,q}_i t^{q^k - q^{i+1} + q - 1} \right) - [1] t^{q^k - 1} + o(t^N). \tag{4.1.1}$$

Note that (4.1.1) gives the $2k-2$ intermediate terms in reverse t-order.

Corollary 4.2 *The coefficient of t^n in g_k (a priori a weak modular form of weight $q^k - 1 - n$ and type 0 in $\boldsymbol{\omega}'$) is in fact a modular form for $n < N$ (and vanishes for $q^k - 1 < n < N$).*

A possible approach to Theorem 4.1 would be to use (2.4.8), which recursively determines g_k from the g_i with $i < k$ and the $E_{q^j - 1}$, and our knowledge of the expansion of $E_{q^i - 1}$. However, the strategy developed below is both less complicated and yields stronger results.

4.3 Recall that $\mathcal{O}'$ is the ring of holomorphic functions on

$$\Omega' = \Omega^{r-1} = \{\boldsymbol{\omega}' = (\omega_2, \ldots, \omega_r) \in C_\infty^{r-1} \mid \omega_i\ K_\infty\text{-linearly independent}, \omega_r = 1\}.$$

As usual, we write $\boldsymbol{\omega} = (\omega_1, \boldsymbol{\omega}')$ for $\boldsymbol{\omega} \in \Omega$. We regard the generic Drinfeld module $\phi = \phi^{\boldsymbol{\omega}}$ ($\boldsymbol{\omega}$ varying through Ω) as a Tate object (see, e.g. [19], and [10] for the case of rank two) over the ring

$$\mathcal{R} := \mathcal{O}'((t)) \tag{4.3.1}$$

of formal Laurent series in t over $\mathcal{O}'$, endowed with the t-adic topology. That is, the coefficients of ϕ are replaced with their respective t-expansions. Let $\phi' = \phi^{\boldsymbol{\omega}'}$ be the generic Drinfeld module of rank $r-1$ base extended to $\mathcal{R}$, and let

$$L := \langle t^{-1} \rangle \tag{4.3.2}$$

be the A-lattice generated by t^{-1} via ϕ', i.e.,

$$L = \{\phi'_a(t^{-1}) \mid a \in A\}.$$

Let $a \in A$ have degree d. For fixed $\boldsymbol{\omega}$ (and thus also for $\boldsymbol{\omega}$ varying),

$$\begin{aligned} \phi'_a(t^{-1}) &= \phi'_a(e^{\Lambda'_{\boldsymbol{\omega}'}}(\omega_1)) \\ &= e^{\Lambda'_{\boldsymbol{\omega}'}}(a\omega_1) = t_a^{-1}(\boldsymbol{\omega}) = \Delta'_{(d)}(\boldsymbol{\omega}') t(\boldsymbol{\omega})^{-q^{(r-1)d}}(\boldsymbol{\omega}) S_a(t(\boldsymbol{\omega})). \end{aligned}$$

Hence $\phi'_a(t^{-1})$ is "large" in $\mathcal{R}$ and grows with d, and L is a lattice in the sense of 2.3. As in 2.4, we define the function

$$\begin{aligned} e^L : \mathcal{R} &\longrightarrow \mathcal{R}, \\ z &\longmapsto z \prod_{\lambda \in L}{}' (1 - z/\lambda) \end{aligned} \tag{4.3.3}$$

where the product converges everywhere and defines an $\mathbb{F}$-linear function. We have properties similar to those in 2.3 and 2.4; in particular,

- $e^L(z) = \sum_{i \geq 0} \alpha_i(L) z^{q^i} = \sum_{i \geq 0} \alpha_i(L) \tau^i \in \mathcal{R}\{\{\tau\}\}$;
- there is a formal inverse $\log^L = \sum_{i \geq 0} \beta_i(L) \tau^i$ of e^L in $\mathcal{R}\{\{\tau\}\}$;
- $\beta_i(L) = -E_{q^i-1}(L)$ with $E_{q^i-1}(L) = \sum'_{\lambda \in L} {\lambda'}^{1-q^i}$ if $i > 0$ and $E_0(L) = -1$,

and thus the relations (2.4.7) for the present data. Instead of diagram (2.3.2), we dispose for each $0 \neq a \in A$ of the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & \mathcal{R} & \xrightarrow{e^L} & \mathcal{R} \\ & & \downarrow \phi'_a & & \downarrow \phi'_a & & \downarrow \phi_a \\ 0 & \longrightarrow & L & \longrightarrow & \mathcal{R} & \xrightarrow{e^L} & \mathcal{R}. \end{array} \tag{4.3.4}$$

That is, e^L is an "analytic homomorphism" of the Drinfeld module ϕ' to ϕ.

From $\phi_T \circ e^L = e^L \circ \phi'_T$, we find

$$\log^L \circ \phi_T = \phi'_T \circ \log^L, \tag{4.3.5}$$

a substitute for (2.4.6). Comparing coefficients and isolating the term g_k yields the recursion formula (analogous with (2.4.9)):

$$g_k = \sum_{1 \leq i \leq k} E_{q^i-1}(L) g_{k-i}^{q^i} - \sum_{0 \leq i \leq k} g'_i E_{q^{k-i}-1}^{q^i}(L) \qquad (k \geq 1) \tag{4.3.6}$$

for the coefficients g_k of ϕ_T (and, of course, the g'_i are those of ϕ'_T). Our proof of Theorem 4.1 will be based on (4.3.6) and the next result.

Proposition 4.4 *In the ring $\mathcal{R}$ we have*

$$E_{q^i-1}(L) = -t^{q^i-1} + o(t^N) \tag{4.4.1}$$

with

$$N = \begin{cases} 3q^{r+i-1} - (2q^{i-1}+1)q^{r-1}, & \text{if } q > 2, \\ 2^{r+i} - 2^{r-1} - 1, & \text{if } q = 2. \end{cases} \tag{4.4.2}$$

Proof

$$
\begin{aligned}
E_{q^i-1}(L) &= \sum_{\lambda\in\Lambda}{}' \lambda^{1-q^i} \\
&= \sum_{a}{}' \phi'_a(t^{-1})^{1-q^i} \\
&= -\sum_{d\in\mathbb{N}_0} \sum_{\substack{a \text{ monic} \\ \deg a = d}} \phi'_a(t^{-1})^{1-q^i} \\
&= -\sum_{d\in\mathbb{N}_0} t^{-q^{(r-1)d}(1-q^i)} {\Delta'_{(d)}}^{1-q^i} \sum_{\substack{a \text{ monic} \\ \deg a = d}} S_a(t)^{1-q^i} \\
&= -t^{q^i-1} - \sum_{d\in\mathbb{N}} {\Delta'_{(d)}}^{1-q^i} t^{q^{(r-1)d}(q^i-1)} \Sigma(i,d) \\
&= -t^{q^i-1} + o(t^N)
\end{aligned}
$$

by Corollary 3.7. □

Proof of Theorem 4.1 (4.3.6) and (4.4.1) yield

$$
\begin{aligned}
g_1 &= E_{q-1}(L)T^q - TE_{q-1}(L) - g'_1 E_0^q(L) \\
&= [1]E_{q-1}(L) + g'_1 = g'_1 - [1]t^{q-1} + o(t^N)
\end{aligned}
$$

in accordance with the assertion. (By the way, this agrees up to a better error term for $q = 2$ with the result Theorem 3.8 for E_{q-1}, taking $g_1 = [1]E_{q-1}$ into account.) Note that all the error terms $o(t^{N'})$ that occur in Proposition 4.4 are such that $N' \geq N = 3(q-1)q^{r-1}$ resp. $3\cdot 2^{r-1} - 1$. Therefore, we let "$\equiv$" denote congruence modulo t^N in $\mathcal{R}$. Assume that (4.1.1) holds for k' with $1 \leq k' < k$. By Proposition 4.4 and (4.3.6),

$$
g_k \equiv -\sum_{1\leq i\leq k} t^{q^i-1} g_{k-i}^{q^i} + \sum_{0\leq i\leq k} g'_i t^{q^k-q^i}.
$$

From the induction hypothesis, the latter is congruent to

$$
\begin{aligned}
a &:= \sum_{0\leq i\leq k} g'_i t^{q^k-q^i} - t^{q^k-1}T^{q^k} \\
&\quad - \sum_{1\leq i<k} t^{q^i-1}\Big\{g'_{k-i} + \sum_{1\leq j\leq k-i-1} \big(g'_j t^{q^{k-i}-q^j} - g'^q_j t^{q^{k-i}-q^{j+1}+q-1}\big) - [1]t^{q^{k-i}-1}\Big\}^{q^i}.
\end{aligned}
$$

To complete the induction, we must show that $a = b$, where

$$
b := g'_k + \sum_{1\leq i\leq k-1} \big(g'_i t^{q^k-q^i} - g'^q_i t^{q^k-q^{i+1}+q-1}\big) - [1]t^{q^k-1}
$$

is the principal part of the right hand side of (4.1.1).

Cancelling some obvious terms and multiplying the remaining by -1, we have to verify that

$$b^* := \sum_{1\le i\le k-1} g_i'^q t^{q^k-q^{i+1}+q-1} - (T^{q^k}-T^q)t^{q^k-1}$$

equals

$$\begin{aligned} a^* &:= \sum_{1\le i\le k-1} t^{q^i-1}\Big\{g'_{k-i} + \sideset{}{'}\sum_{1\le j\le k-i-1} g'_j t^{q^{k-i}-q^j} - g_j'^q t^{q^{k-i}-q^{j+1}+q-1} - [1]t^{q^{k-i}-1}\Big\}^{q^i} \\ &= a_1+a_2+a_3 \end{aligned}$$

with

$$\begin{aligned} a_1 &= \sum_{1\le i\le k-1} g_{k-i}'^{q^i} t^{q^i-1} \\ a_2 &= \sum_{1\le i\le k-1}\sum_{1\le j\le k-i-1} \big(g_j'^{q^i} t^{q^k-q^{i+j}+q^i-1} - g_j'^{q^{i+1}} t^{q^k-q^{i+j+1}+q^{i+1}-1}\big) \\ a_3 &= -\sum_{1\le i\le k-1} [1]^{q^i} t^{q^k-1}. \end{aligned}$$

Now the double sum a_2 is

$$\begin{aligned} \sum_{1\le i\le k-1}\sum_{1\le j\le k-i-1} \cdots &= \sum_{1\le j\le k-1}\sum_{1\le i\le k-j-1} \cdots \\ &= \sum_{1\le j\le k-1} \big(g_j'^q t^{q^k-q^{j+1}+q-1} - g_j'^{q^{k-j}} t^{q^{k-j}-1}\big), \end{aligned}$$

as it is telescoping in i; therefore

$$a_2 = \sum_{1\le j\le k-1} g_j'^q t^{q^k-q^{j+1}+q-1} - a_1.$$

Finally,

$$\sum_{1\le i\le k-1} [1]^{q^i} = (T^q-T)^q + (T^q-T)^{q^2} + \cdots + (T^q-T)^{q^{k-1}} = T^{q^k}-T^q,$$

and so

$$a_3 = -(T^{q^k}-T^q)t^{q^k-1}.$$

Together we find $a^* = b^*$, and the proof is complete. □

Remark 4.5 In principle, our results so far also apply to the case of rank $\boxed{r=2}$. Here the quantities g'_k are "modular forms of rank 1"; more precisely, $g'_k = 0$ for $k > 1$ and g'_1 equals the T-coefficient of ϕ'_T, where ϕ is the Drinfeld module associated with the rank-1-lattice A. As is well-known, $g_1' = \overline{\pi}^{q-1}$ with $\overline{\pi}$ a period (well-defined up to $(q-1)$-th roots of unity) of the Carlitz module. In the rank-2 case, the forms g_1 and g_2 are usually denoted by g and Δ and normalized differently from our present convention. Viz,

$$g = \overline{\pi}^{1-q} g_1, \qquad \Delta = \overline{\pi}^{1-q^2} g_2, \tag{4.5.1}$$

so g, Δ are the forms g_{new}, Δ_{new} of [10] Sect. 6. Furthermore

$$\tilde{t} := \text{the uniformizer labelled } t \text{ in [10]} = \overline{\pi}^{-1} t (\text{of the present paper}). \tag{4.5.2}$$

The expansions w.r.t $\tilde{t}$ derived from Theorem 4.1 are (for $q > 2$)

$$g = 1 - [1]\tilde{t}^{q-1} + o(\tilde{t}^{3(q-1)q}) \tag{4.5.3}$$

$$\Delta = -\tilde{t}^{q-1} + \tilde{t}^{q^2-q} - [1]\tilde{t}^{q^2-1} + o(\tilde{t}^{3(q-1)q}) \tag{4.5.4}$$

in accordance with (but weaker than) those of [10] (10.11) and (10.3), respectively.

5 The *h*-function

5.1 We derive similar results for the h-function

$$h = h'^q t \prod_{a \in A \text{ monic}} S_a(t)^{q^r-1}. \tag{5.1.1}$$

For $d \in \mathbb{N}$ put

$$\Pi(d)(X) := \prod_{\substack{a \text{ monic} \\ \deg a = d}} S_a(X). \tag{5.1.2}$$

For typographical reasons, we work with $s := r - 1 = \text{rank of } \phi'$ instead of r. For $a = T + c$ ($c \in \mathbb{F}$) of degree 1,

$$\phi_a'(X) = aX + g_1' X^q + \cdots + g_{s-1}' X^{q^{s-1}} + \Delta' X^{q^s}, \tag{5.1.3}$$

where only the coefficient a of X depends on a. Thus

$$S_a(X) = 1 + \frac{g_{s-1}}{\Delta'} X^{q^s - q^{s-1}} + \cdots + \frac{g_1'}{\Delta} X^{q^s - q} + \frac{a}{\Delta'} X^{q^s - 1} = S_T(X) + cW \tag{5.1.4}$$

with $W = \Delta'^{-1} X^{q^s - 1}$. From the rule

$$\prod_{c \in \mathbb{F}} (x + cy) = x^q - xy^{q-1} \tag{5.1.5}$$

we get

$$\Pi(1) = S_T^q - S_T W^{q-1}, \tag{5.1.6}$$

which starts

$$\Pi(1) = 1 - \Delta'^{1-q} X^{(q-1)(q^s-1)} + \text{higher terms.} \qquad (5.1.7)$$

5.2 Next, we treat $\Pi(2)$. For $a = T^2 + c_1 T + c_0$,

$$S_a(X) = 1 + m_1 X^{q^{2s}-q^{2s-1}} + \cdots + m_{2s-1} X^{q^{2s}-q} + \frac{a}{\Delta_2'} X^{q^{2s}-1} \qquad (5.2.1)$$

with the coefficients $m_n = \frac{{}_a\ell'_{2s-n}}{\Delta'_{(2)}}$ of $X^{q^{2s}-q^n}$, where

- $\Delta'_{(2)} = (\Delta')^{q^{2s}+1}$ is independent of a;
- $m_0 = 1, m_1, \ldots, m_{s-1}$ are independent of a;
- $m_s, \ldots, m_{2s-1}$ depend affine-linearly on c_1, but not on c_0.

Writing $S_a(X) = Z_0(c_1) + c_0 W_0$ with $Z_0(c_1) = S_{T^2+c_1T}$, $W_0 = (\Delta'_{(2)})^{-1} X^{q^{2s}-1}$, we find for fixed c_1:

$$S(c_1) := \prod_{c_0 \in \mathbb{F}} S_{T^2+c_1T+c_0}(X) = Z_0(c_1)^q - Z_0(c_1) W_0^{q-1}. \qquad (5.2.2)$$

We repeat the argument and write $S(c_1) = Z_1 + c_1 W_1$ with Z_1 and W_1 independent of c_1. A closer look shows

$$\begin{aligned} o(Z_1 - 1) &= o(S(c_1) - 1) = (q-1) \cdot o(W_0) = (q-1)(q^{2s} - 1); \qquad (5.2.3) \\ o(W_1) &= q^{2s+1} - q^{s+1}, \end{aligned}$$

as only the coefficients $m_s, m_{s+1}, \ldots$ contribute to W_1. Now

$$\Pi(2) = \prod_{c_1 \in \mathbb{F}} S(c_1) = Z_1^q - Z_1 W_1^{q-1},$$

so

$$o(\Pi(2)-1) \geq \min\{q \cdot o(Z_1 - 1), (q-1) \cdot o(W_1)\} = (q-1) \cdot o(W_1) = (q-1)q(q^{2s} - q^s),$$

and we have in fact equality since the two arguments of min are different. Hence we have shown:

Proposition 5.3 *The product* $\Pi(2) = \prod_{a \text{ monic}, \deg a = 2} S_a(X)$ *satisfies*

$$o(\Pi(2) - 1) = (q-1)q(q^{2s} - q^s) = (q-1)(q^{2r-1} - q^r).$$

Remark 5.4 Obviously, $o(\Pi(d) - 1)$ for $d \geq 3$ is much larger; each factor $S_a(X)$ satisfies $o(S_a - 1) = q^{(r-1)d} - q^{(r-1)d-1}$, and already the first step of the procedure 5.2 gives

$$o\Big(\prod_{c_0} S_a(X) - 1\Big) = (q-1)(q^{(r-1)d} - 1)$$

for the product over all $a = T^d + c_{d-1}T^{d-1} + \cdots + c_1 T + c_0$ with $c_{d-1}, \ldots, c_1$ fixed. Hence such $\Pi(d)$ don't contribute to the first $\approx q^{3r-3}$ coefficients of the complete product $\prod_{a \text{ monic}} S_a(X)$.

Theorem 5.5 *The h-function as a power series in t satisfies*

$$h = h'^q t \Pi(1)^{-1} + o(t^N),$$

where $N = (q-1)(q^{2r-1} - q^r) + 1$ and $\Pi(1) = \Pi(1)(t)$ is the polynomial defined by

$$\begin{aligned} \Pi(1)(t) &= S_T(t)^q - S_T(t) W(t)^{q-1}, \\ S_T(t) &= 1 + \frac{g'_{r-2}}{\Delta'} t^{q^{r-1}-q^{r-2}} + \frac{g'_{r-3}}{\Delta'} t^{q^{r-1}-q^{r-3}} + \cdots + \frac{g'_1}{\Delta'} t^{q^{r-1}-q} + \frac{T}{\Delta'} t^{q^{r-1}-1}, \\ W(t) &= \Delta'^{-1} t^{q^{r-1}-1}. \end{aligned}$$

$\Pi(1)$ may be inverted as a power series by $\Pi(1)(t)^{-1} = 1 - U + U^2 - \ldots$ with $U = \Pi(1)(t) - 1$.

Proof We replace the product $\prod_{a \text{ monic}} S_a(t)^{q^r-1}$ in (5.1.1) with $\prod_{a \text{ monic}, \deg a = 1} S_a(t)^{-1} = \Pi(1)(t)^{-1}$. By Proposition 5.3 and Remark 5.4, the error is $o(t^N)$ with $N = q^r \cdot o(\Pi(1) - 1) + 1$ as stated, since $o(\Pi(d) - 1) + 1 \geq N$ for $d \geq 2$. □

We note that by (5.1.7) the expansion for h starts as

$$h = h'^q t\bigl(1 + (\Delta')^{1-q} t^{(q^{r-1}-1)(q-1)} + \text{higher terms}\bigr). \tag{5.5.1}$$

The reader may verify that Theorem 5.5 is compatible with Theorem 4.1 for $k = r$, where $g_r = \Delta = (-1)^{r-1} h^{q-1}$.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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