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On L-equivalence for K3 Surfaces and Hyperkähler Manifolds

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Abstract

This paper explores the relationship between L-equivalence and D-equivalence for K3 surfaces and hyperkähler manifolds. Building on Efimov's approach using Hodge theory, we prove that very general L-equivalent K3 surfaces are D-equivalent, leveraging the Derived Torelli Theorem for K3 surfaces. Our main technical contribution is that two distinct lattice structures on an integral, irreducible Hodge structure are related by a rational endomorphism of the Hodge structure. We partially extend our results to hyperkähler fourfolds and moduli spaces of sheaves on K3 surfaces.

Contents

1 Introduction

Two varieties X, Y over a field k are L-equivalent if there exists an integer $n \in \mathbb{Z}$ such that

$$\mathbb{L}^n \cdot ([X] - [Y]) = 0 \in K_0(\mathrm{Var}_k).$$

Here, $K_0(\mathrm{Var}_k)$ denotes the Grothendieck ring of algebraic varieties over k. That is, $K_0(\mathrm{Var}_k)$ is generated, as an abelian group, by isomorphism classes $[X]$ of schemes of finite type over

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k, subject to the relations of the form $[X] = [X \setminus Z] + [Z]$, where $Z \subset X$ is any closed subset. The multiplication in $K_0(\mathrm{Var}_k)$ is given by the fibre product. The class of the affine line, denoted $\mathbb{L} := [\mathbb{A}^1_k]$, was proved to be a zero-divisor in $K_0(\mathrm{Var}_k)$ by Borisov [6]. This led to the notion of L-equivalence by Kuznetsov–Shinder [17].

Two varieties X, Y are called D-equivalent if there is a linear, exact equivalence of their bounded derived categories of coherent sheaves $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$.

This work was originally motivated by a conjecture of Kuznetsov–Shinder predicting that D-equivalence should imply L-equivalence for simply connected varieties, c.f. [17, Conjecture 1.6]. This conjecture has since been disproved (see [19]). In light of this, a revised conjectural relationship is proposed in [19]:

Conjecture 1.1 [19, Conjecture 1.5]. *Let X and Y be projective hyperkähler manifolds. If X and Y are L-equivalent, they are also D-equivalent, i.e.*

$$X \sim_{\mathbb{L}} Y \implies \mathcal{D}^b(X) \simeq \mathcal{D}^b(Y).$$

In this paper, we prove Conjecture 1.1 in the case where X and Y are sufficiently general K3 surfaces. Throughout this paper, we work over the complex numbers.

Theorem 1.2 (See Theorem 4.2) *Let X be a K3 surface with $\rho \neq 18$ and* $\mathrm{End}(T(X)_{\mathbb{Q}}) \simeq \mathbb{Q}$. *Let Y be a K3 surface. If Y is L-equivalent to X, then X and Y are D-equivalent.*

Theorem 1.2 shows that L-equivalence implies D-equivalence for sufficiently general K3 surfaces of Picard rank $\rho \leq 17$, as well as for all K3 surfaces for which $\mathrm{rk}(T(X)) = 22 - \rho$ is an odd prime. Indeed, the assumption that $\mathrm{End}(T(X)_{\mathbb{Q}}) \simeq \mathbb{Q}$ is satisfied by these K3 surfaces by [10, 24, 27] (see also Lemma 3.1) and [13, Remark 3.3.14].

Conjecture 1.1 is still open for special K3 surfaces, as our methods rely on the assumption that $\mathrm{End}(T(X)_{\mathbb{Q}}) \simeq \mathbb{Q}$. However, the currently known examples of L-equivalent K3 surfaces have been constructed without relying on this assumption. Therefore, examples of L-equivalent special K3 surfaces are already known.

We also study L-equivalence for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type. The proof of Theorem 1.2 relies on the Derived Torelli Theorem. A higher-dimensional generalisation of the Derived Torelli Theorem for hyperkähler manifolds is currently an open problem. However, for special cases of hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, positive results can be found in the literature. Using these results, we partially extend Theorem 1.2 to hyperkähler manifolds. In particular, we study L-equivalence for hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type with Picard rank 1 using the results from [16] (see Theorem 5.2), and for moduli spaces of sheaves on K3 surfaces with unimodular Néron–Severi lattices using results of [18] (see Theorem 5.3).

We note that L-equivalence does not imply D-equivalence in general, even for simply connected varieties. For example, if X and Y are Fano varieties with the property that $[X] = [Y] \in K_0(\mathrm{Var}_k)$, then X and Y are trivially L-equivalent but not D-equivalent unless they are isomorphic, by [5, Theorem 2.5]. As a simple example of two such varieties, one may take $X = \mathrm{Bl}_p\mathbb{P}^3$ for any point $p \in \mathbb{P}^3$, and $Y = \mathbb{P}^1 \times \mathbb{P}^2$. In this case, one computes that $[X] = [Y] = \mathbb{L}^3 + 2\mathbb{L}^2 + 2\mathbb{L} + 1$.

Examples of L-equivalent K3 surfaces can be found in [12, 14, 15, 17, 27], and examples of L-equivalent hyperkähler manifolds were constructed in [25]. In all these cases, the L-equivalent K3 surfaces and hyperkähler manifolds are also D-equivalent.

1.1 Proof strategy

Efimov approached L-equivalence using Hodge theory [7], because L-equivalent complex smooth projective varieties X, Y satisfy $[H^\bullet(X, \mathbb{Z})] = [H^\bullet(Y, \mathbb{Z})]$ in the Grothendieck group $K_0^\oplus(\mathrm{HS}_\mathbb{Z})$ of integral, polarisable Hodge structures. With this approach, Efimov was able to show that D-equivalent abelian varieties need not be L-equivalent, and that L-equivalence classes of K3 surfaces are finite.

Finally, Efimov showed that for L-equivalent K3 surfaces X, Y which are sufficiently general, there exists an *isomorphism* of integral Hodge structures $T(X) \simeq T(Y)$ [7, Lemma 3.7]. Here, $T(X)$ denotes the transcendental sublattice of $H^2(X, \mathbb{Z})$.

Recall that the Derived Torelli Theorem for K3 surfaces, due to Mukai and Orlov, asserts that two K3 surfaces X, Y are D-equivalent if and only if there is a Hodge *isometry* $T(X) \simeq T(Y)$ [22, 26]. A Hodge isometry is an isomorphism of integral Hodge structures that respects the bilinear forms on $T(X)$ and $T(Y)$ given by the cup-product.

Not every isomorphism of integral Hodge structures $T(X) \simeq T(Y)$ is a Hodge isometry. In Example 3.3, we give examples of K3 surfaces X, Y with isomorphic Hodge structures $T(X) \simeq T(Y)$, yet no Hodge isometries exist between them. Therefore, [7, Lemma 3.7] by itself is insufficient to establish that L-equivalent K3 surfaces are D-equivalent.

We study the relationship between Hodge isomorphisms and Hodge isometries for integral, irreducible Hodge structures with a bilinear form, which we call *Hodge lattices*. Our main technical result is the following:

Proposition 1.3 (see Proposition 3.8 and Lemma 3.9) *Let T_1 and T_2 be two irreducible Hodge lattices of K3 type. If T_1 and T_2 are isomorphic as Hodge structures, then there exists a rational Hodge automorphism $\phi\colon (T_1)_\mathbb{Q} \simeq (T_1)_\mathbb{Q}$ with the property that T_2 is Hodge isometric to $(T_1)_\phi$, which is the Hodge lattice with underlying Hodge structure T_1 and lattice structure given by $(x \cdot y)_{(T_1)_\phi} = (\phi(x) \cdot y)_{T_1}$ for all $x, y \in T_1$.*

We note that for the bilinear form $(\phi(x) \cdot y)_{T_1}$ to be a symmetric bilinear form that takes values in the integers, the rational Hodge endomorphism ϕ must satisfy two conditions. Firstly, the form is symmetric if and only if ϕ is invariant under the Rosati involution, that is, we have $(\phi(x) \cdot y)_{T_1} = (x \cdot \phi(y))_{T_1}$ for all $x, y \in T_1$. Secondly, the form takes values in the integers if and only if ϕ maps T_1 into its dual $(T_1)^*$.

A sufficiently general K3 surface X has very few rational Hodge endomorphisms, that is, we have $\mathrm{End}(T(X)_\mathbb{Q}) \simeq \mathbb{Q}$. Therefore, if X is L-equivalent to a K3 surface Y, Proposition 1.3 combined with [7, Lemma 3.7] asserts that there is a Hodge isometry $T(Y) \simeq T(X)_q$ for some $q \in \mathbb{Q}$. Therefore, we can compute the discriminant of $T(Y)$ via

$$\mathrm{disc}(T(Y)) = \mathrm{disc}(T(X)_q) = q^{\mathrm{rk}(T(X))}\,\mathrm{disc}(T(X))$$

(see Lemma 2.5). Theorem 1.2 follows from the above observations combined with the following lemma.

Lemma 1.4 (See Lemma 4.1) *If X and Y are L-equivalent K3 surfaces, then we have* $\mathrm{disc}(T(X)) = \mathrm{disc}(T(Y))$.

1.2 Structure of the paper

In Sect. 2, we recall the necessary lattice and Hodge theory. The main goal of this section is to explain Definition 2.3, which is the definition of a Hodge lattice twisted by a rational Hodge endomorphism.

In Sect. 3, we introduce the notion of T-equivalence: we say that K3 surfaces X and Y are T-equivalent if the Hodge structures $T(X)$ and $T(Y)$ are isomorphic, but not necessarily isometric. We investigate the relationship between T-equivalence and D-equivalence for K3 surfaces. There are two main results in this section, namely Corollary 3.10 and Proposition 3.11. Corollary 3.10 is a restatement of Proposition 1.3 above, and Proposition 3.11 shows that T-equivalence implies D-equivalence under the assumption that $\operatorname{disc}(T(X)) = \operatorname{disc}(T(Y))$.

In Sect. 4, the main goal is to use Proposition 3.11 to prove that sufficiently general L-equivalent K3 surfaces are D-equivalent. This proof consists of verifying the condition of Proposition 3.11 that L-equivalent K3 surfaces X and Y satisfy $\operatorname{disc}(T(X)) = \operatorname{disc}(T(Y))$. We do this by defining a group homomorphism $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Q}^{\times}$, which factorises through $K_0(\mathrm{Var}_{\mathbb{C}})[L^{-1}]$ and sends the class of a K3 surface $[X]$ to $\operatorname{disc}(T(X))$.

Finally, in Sect. 5, we use the techniques developed in this paper to study hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, and we discuss the challenge that our methods face in the higher-dimensional setting: the question of whether Lemma 1.4 generalises to the higher-dimensional setting.

2 Lattices and Hodge structures

2.1 Lattices

Our main reference for lattice theory is [23]. A lattice is a free, finitely generated abelian group L together with a non-degenerate symmetric bilinear form $b\colon L \times L \to \mathbb{Z}$. For $v, w \in L$, we usually denote $v \cdot w := b(v, w)$ and sometimes $(v \cdot w)_L := b(v, w)$. A group isomorphism of lattices respecting the bilinear forms is called an *isometry*. The group of isometries of L is denoted $O(L)$. A lattice L is called *even* if $v^2 := v \cdot v$ is even for all $v \in L$. We assume all our lattices to be even. We define the *dual lattice* to be $L^* := \mathrm{Hom}(L, \mathbb{Z})$. The dual lattice inherits a (usually not integral) symmetric bilinear form from L. There is a natural embedding $L^* \hookrightarrow L_{\mathbb{Q}} := L \otimes \mathbb{Q}$ that respects the bilinear forms and whose image is

$$\left\{x \in L_{\mathbb{Q}} \mid x \cdot v \in \mathbb{Z} \text{ for all } v \in L\right\}. \tag{2.1}$$

Since we assume L to be non-degenerate, the natural map $L \hookrightarrow L^*$ is injective. The *discriminant lattice* of L is the quotient

$$A_L := L^*/L.$$

It has a natural quadratic form $q\colon A_L \to \mathbb{Q}/2\mathbb{Z}$ which it inherits from L^*. The *discriminant* of L is

$$\operatorname{disc}(L) := |A_L|.$$

If we choose a $\mathbb{Z}$-basis for L, and let M be the integral matrix representing b in this basis, then we have

$$\operatorname{disc}(L) = |\det(M)|.$$

For an integer $n \in \mathbb{Z}$, we write $L(n)$ for the lattice whose underlying group structure is that of L, and whose bilinear form is given by $(v \cdot w)_{L(n)} = (v \cdot w)_L$. Note that we have

$$\operatorname{disc}(L(n)) = |n|^{\operatorname{rk}(L)} \operatorname{disc}(L).$$

We say that L is *unimodular* if $\mathrm{disc}(L) = 1$. An important example of a unimodular lattice is the hyperbolic plane U, which is the lattice of rank 2 whose bilinear form is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Another important example is the lattice E_8, which is the unique even unimodular positive-definite lattice of rank 8.

2.2 K3 surfaces and hyperkähler manifolds

Our basic reference on K3 surfaces is [13]. We assume all our K3 surfaces and hyperkähler manifolds to be projective. Let X be a K3 surface. The integral cohomology group $H^2(X, \mathbb{Z})$ is a free abelian group of rank 22. The cup-product is a non-degenerate symmetric bilinear form $H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$ which turns $H^2(X, \mathbb{Z})$ into a unimodular lattice. There is an isometry $H^2(X, \mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$, where

$$\Lambda_{\mathrm{K3}} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

is the *K3-lattice*.

The cohomology group $H^2(X, \mathbb{Z})$ has a natural Hodge structure of weight 2:

$$H^2(X, \mathbb{C}) \simeq H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X),$$

and $H^{2,0}(X)$ is generated by an everywhere non-degenerate symplectic form $\sigma \in H^{2,0}(X)$. The *Néron–Severi lattice* of X is the sublattice

$$\mathrm{NS}(X) := H^{1,1}(X) \cap H^2(X, \mathbb{Z}).$$

The rank of $\mathrm{NS}(X)$ is called the *Picard rank* of X, usually denoted ρ. The *transcendental lattice* of X is

$$T(X) := \mathrm{NS}(X)^{\perp} \subset H^2(X, \mathbb{Z}).$$

Note that we have $H^{2,0}(X) \subset T(X)_{\mathbb{C}}$, and in fact $T(X)$ is the smallest primitive sublattice of $H^2(X, \mathbb{Z})$ with this property. The rank of $T(X)$ is $22 - \rho$.

Definition 2.1 i) A lattice with a Hodge structure is called a *Hodge lattice*.
ii) A Hodge structure H of weight 2 is said to be of K3-type if $H^{2,0}$ is a 1-dimensional $\mathbb{C}$-vector space.
iii) A Hodge structure of K3-type T is irreducible if T admits no proper, primitive sub-Hodge structure of K3-type.

By the above, the transcendental lattice of a K3 surface is an irreducible Hodge lattice of K3-type.

More generally, if X is a complex projective hyperkähler manifold, the cohomology group $H^2(X, \mathbb{Z})$ admits a natural integral symmetric bilinear form called the *Beauville–Bogomolov–Fujiki* (BBF) form [2, 4, 8]. The BBF form turns $H^2(X, \mathbb{Z})$ into a non-degenerate integral Hodge lattice of K3-type. Similarly to the case of K3 surfaces, we write $\mathrm{NS}(X) := H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ for the Néron–Severi lattice, and we let $T(X) := \mathrm{NS}(X)^{\perp} \subset H^2(X, \mathbb{Z})$ be the transcendental lattice of X. The transcendental lattice is an irreducible Hodge lattice of K3-type.

Remark 2.2 Let T be a Hodge lattice. Then T^* inherits a Hodge structure from $T_{\mathbb{Q}}$ via the natural inclusion $T^* \subset T_{\mathbb{Q}}$ (see (2.1)). With this Hodge structure, the natural embedding $T \hookrightarrow T^*$ is a morphism of Hodge structures.

Suppose T_1 and T_2 are Hodge lattices, and that $f: T_1 \simeq T_2$ is an isomorphism of Hodge structures. Then f induces an isomorphism of Hodge structures $f^*: T_2^* \simeq T_1^*$. However, the diagram

$$\begin{tikzcd} T_1 \arrow[r, "f", "\simeq"'] \arrow[d, hook, "i_1"'] & T_2 \arrow[d, hook, "i_2"] \\ T_1^* & T_2^* \arrow[l, "\simeq"', "f^*"] \end{tikzcd} \tag{2.2}$$

is not necessarily commutative. In fact, (2.2) commutes if and only if f is an isometry.

2.3 Irreducible Hodge lattices

Let T be an integral irreducible Hodge lattice of K3-type. Consider the Hodge endomorphism algebra $E := \mathrm{End}(T_{\mathbb{Q}})$. Elements of E will be called *rational Hodge endomorphisms*. Note that an isomorphism of integral Hodge structures $\phi: T \simeq T$ induces a rational Hodge endomorphism $\phi_{\mathbb{Q}} \in E$. The converse does not hold: a rational Hodge endomorphism does not necessarily preserve $T \subset T_{\mathbb{Q}}$.

Since T is irreducible, it follows that E is a number field. Moreover, it is well-known that E is either totally real or a CM field, see for example [13, Theorem 3.3.7] for details. Recall that a totally real number field is a number field k for which all embeddings $k \hookrightarrow \mathbb{C}$ have images contained in $\mathbb{R} \subset \mathbb{C}$, and a CM field is a number field of the form $k(\sqrt{a})$, where k is totally real and $\iota(a) < 0$ for all embeddings $\iota: k \hookrightarrow \mathbb{R}$.

For a Hodge endomorphism $\phi \in E$, we define $\overline{\phi} \in E$ to be the unique Hodge endomorphism with the property that

$$\phi(x) \cdot y = x \cdot \overline{\phi}(y)$$

for all $x, y \in T_{\mathbb{Q}}$. The group homomorphism

$$\begin{aligned} E &\longrightarrow E \\ \phi &\longmapsto \overline{\phi} \end{aligned}$$

is known as the *Rosati involution*, and we consider the invariant subfield

$$K(T) := \left\{\phi \in E \mid \phi = \overline{\phi}\right\}.$$

We note that $E = K(T)$ if E is totally real, and $E = K(T)(\sqrt{a})$ for some $a \in K(T)$ otherwise.

Definition 2.3 Let T be an integral irreducible Hodge lattice of K3-type, and let $\phi \in K(T)$ be a rational Hodge endomorphism which maps T into T^*. We write T_ϕ for the integral irreducible Hodge lattice of K3-type whose underlying Hodge structure is equal to that of T and whose lattice structure is given by

$$(x \cdot y)_{T_\phi} = (\phi(x) \cdot y)_T \tag{2.3}$$

for all $x, y \in T$.

Remark 2.4 i) Note that (2.3) is a symmetric bilinear form by our assumption that $\phi \in K(T)$, since we have

$$(x \cdot y)_{T_\phi} = (\phi(x) \cdot y)_T = (x \cdot \phi(y))_T = (y \cdot x)_{T_\phi}$$

for all $x, y \in T$.

ii) By (2.1), the assumption that ϕ maps T into T^* is equivalent to (2.3) taking values in the integers.

Recall that for a number field k and an element $a \in k$, the *norm* of a is defined to be

$$N(a) := \prod_\iota \iota(a),$$

where the product runs over all embeddings $\iota \colon k \hookrightarrow \mathbb{C}$.

Lemma 2.5 [9, Lemma 4.5(2)] *Let T be an integral irreducible Hodge lattice of K3-type. Let $E := \mathrm{End}(T_\mathbb{Q})$ be the rational endomorphism field of T, and let $\phi \in K(T)$ be a rational Hodge endomorphism which maps T into T^*. Then we have*

$$\mathrm{disc}(T_\phi) = N(\phi)^m \,\mathrm{disc}(T),$$

where $m := \dim_E(T_\mathbb{Q})$.

3 Hodge isomorphisms and Hodge isometries

For a Hodge lattice T, we denote the group of Hodge isometries of T by $O_{\mathrm{Hodge}}(T)$. Similarly, we write $O_{\mathrm{Hodge}}(T_\mathbb{Q})$ for the group of rational Hodge automorphisms which respect the natural bilinear form on $T_\mathbb{Q}$, which we call *rational Hodge isometries*.

Lemma 3.1 *Let X be a K3 surface. Consider the four statements:*

i) $\mathrm{End}(T(X)) \simeq \mathbb{Z}$,
ii) $\mathrm{End}(T(X)_\mathbb{Q}) \simeq \mathbb{Q}$,
iii) $O_{\mathrm{Hodge}}(T(X)_\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$,
iv) $O_{\mathrm{Hodge}}(T(X)) \simeq \mathbb{Z}/2\mathbb{Z}$.

Then we have

$$i) \iff ii) \implies iii) \implies iv).$$

Moreover, if X has Picard rank $\rho \leq 19$ and X is very general in a moduli space of lattice-polarised K3 surfaces, then X satisfies each of these statements.

Proof The equivalence of $i)$ and $ii)$ follows from the observation that $\mathrm{End}(T(X)) \otimes \mathbb{Q} \simeq \mathrm{End}(T(X)_\mathbb{Q})$.

$ii) \implies iii)$: If we assume $\mathrm{End}(T(X)_\mathbb{Q}) \simeq \mathbb{Q}$, then $K(T(X)) = \mathrm{End}(T(X))$. Therefore, for any rational endomorphism $\phi \in \mathrm{End}(T(X)_\mathbb{Q})$, we have $\phi(x) \cdot \phi(y) = x \cdot \phi^2(y)$. If ϕ is an isometry, this means that we have $\phi^2(y) = y$ for all $y \in T(X)$, therefore $\phi^2 = \mathrm{id}_{T(X)_\mathbb{Q}}$. As $\mathrm{End}(T(X)_\mathbb{Q}) \simeq \mathbb{Q}$, this implies $\phi = \pm\,\mathrm{id}_{T(X)_\mathbb{Q}}$, hence $O_{\mathrm{Hodge}}(T(X)_\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$.

$iii) \implies iv)$: For any integral Hodge isometry $\phi \in O_{\mathrm{Hodge}}(T(X))$, the rational Hodge endomorphism $\phi_\mathbb{Q}$ is a rational Hodge isometry. Moreover, for two integral Hodge isometries $\phi, \psi \in O_{\mathrm{Hodge}}(T(X))$, we have $\phi_\mathbb{Q} = \psi_\mathbb{Q}$ if and only if $\phi = \psi$. Thus, we have an injective group homomorphism

$$O_{\mathrm{Hodge}}(T(X)) \hookrightarrow O_{\mathrm{Hodge}}(T(X)_\mathbb{Q}) \tag{3.1}$$

Since $O_{\mathrm{Hodge}}(T(X)_{\mathbb{Q}}) \simeq \mathbb{Z}/2\mathbb{Z}$, it follows that $O_{\mathrm{Hodge}}(T(X))$ is either trivial, or (3.1) is an isomorphism. Since $\pm\,\mathrm{id}_{T(X)} \in O_{\mathrm{Hodge}}(T(X))$ are two distinct integral Hodge isometries, (3.1) must be an isomorphism, as required.

The final claim is well-known, see [27, Lemma 3.9] and [24, Lemma 4.1] for a proof that a very general K3 surface satisfies iii) and iv), and [10, Lemma 9] for i) and ii). □

Definition 3.2 1. Two hyperkähler manifolds X, Y, of the same dimension, are *T-equivalent* if there is an isomorphism of Hodge structures $T(X) \simeq T(Y)$.
2. Two hyperkähler manifolds X and Y are *D-equivalent* if there is an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$.

For hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ induces a Hodge isometry $T(X) \simeq T(Y)$ [3, Corollary 9.3]. In particular, for these, D-equivalence implies T-equivalence. In this section, we investigate the converse of this statement.

Example 3.3 The following two examples show that T-equivalence does not imply D-equivalence.

i) Let X be a K3 surface admitting a Shioda–Inose structure [21]. Then there is a K3 surface Y for which there is a Hodge isometry $T(X) \simeq T(Y)(2)$. This Hodge isometry gives an isomorphism of Hodge structures $T(X) \simeq T(Y)$. However, we have

$$\mathrm{disc}(T(X)) = \mathrm{disc}(T(Y)(2)) \neq \mathrm{disc}(T(Y)),$$

hence there is no Hodge isometry between $T(X)$ and $T(Y)$.

ii) Let X be a K3 surface with Picard rank $\rho \geq 12$ and let $n > 0$ be any positive integer. Then there exists a K3 surface Y with $T(Y) \simeq T(X)(n)$. Indeed, since $\mathrm{rk}(T(X)) \leq 10$, there exists a primitive embedding into the K3-lattice $T(X)(n) \hookrightarrow \Lambda_{\mathrm{K3}}$ by [23, Corollary 1.12.3]. By the surjectivity of the period map [28], there exists a K3 surface Y for which there exists a Hodge isometry $T(Y) \simeq T(X)(n)$.

Example 3.3 ii) shows that T-equivalence classes are not necessarily finite. In our counterexamples, the discriminants of T-equivalent K3 surfaces can get arbitrarily large. Once we fix the discriminant, however, we have the following result of Efimov.

Proposition 3.4 [7, Proposition 3.3] *Let X be a K3 surface, and let $d \in \mathbb{Z}$ be an integer. Then the set of isomorphism classes of K3 surfaces Y, that are T-equivalent to X and satisfy* $\mathrm{disc}(T(Y)) = d$*, is finite.*

Definition 3.5 Let T be an integral irreducible Hodge lattice of K3-type.

i) We denote by

$$F(T) \subset K(T)^{\times}$$

the set of rational Hodge automorphisms $\phi\colon T_{\mathbb{Q}} \simeq T_{\mathbb{Q}}$ that are contained in $K(T)^{\times}$ and that map T into T^*. In other words, by Remark 2.4, $F(T)$ is the set of rational Hodge automorphisms ϕ for which T_{ϕ} is well-defined (see Definition 2.3).

ii) Let T' and T'' be integral irreducible Hodge lattices of K3-type, and suppose that $f\colon T \simeq T'$, $g\colon T \simeq T''$ are two isomorphisms of Hodge structures. Then we say that f and g are *equivalent*, denoted $f \sim g$, if the composition $g \circ f^{-1}\colon T' \simeq T''$ is a Hodge isometry. It is easy to check that $\sim$ is an equivalence relation.

iii) We denote
$$H(T) := \left(\bigcup_{T'} \mathrm{Isom}(T, T') \right) / \sim,$$
where the union runs over all integral, irreducible Hodge lattices of K3-type, and the set $\mathrm{Isom}(T, T')$ denotes the set of isomorphisms of Hodge structures $T \simeq T'$.

iv) We write $I(T)$ for the set of irreducible Hodge lattices T', that are isomorphic to T as Hodge structures, considered up to Hodge isometries.

Remark 3.6 Keeping the notation of Definition 3.5, the group of Hodge automorphisms $\mathrm{Aut}(T)$ acts naturally on $F(T)$ and $H(T)$ as follows:

i) For $h \in \mathrm{Aut}(T)$ and $\phi \in F(T)$, we define $h * \phi := h^* \circ \phi \circ g$.

ii) For $h \in \mathrm{Aut}(T)$ and $f \in H(T)$, we define $h * f := f \circ h$. Note that if $f : T \simeq T'$ and $g : T \simeq T''$ are equivalent, then $h * f$ and $h * g$ are also equivalent. Indeed,
$$h * g \circ (h * f)^{-1} = g \circ h \circ h^{-1} \circ f^{-1} = g \circ f^{-1}$$
is a Hodge isometry by assumption.

Remark 3.7 For an integral, irreducible Hodge lattice T of K3-type, it is easy to see that the map
$$\begin{aligned} H(T) &\longrightarrow I(T) \\ \left(f : T \simeq T'\right) &\longmapsto T' \end{aligned}$$
is $\mathrm{Aut}(T)$-invariant and induces a bijection $H(T)/\mathrm{Aut}(T) \simeq I(T)$.

The main technical result of this section is the following.

Proposition 3.8 *Let T be an integral, irreducible Hodge lattice of K3-type. Then we have a bijection*
$$\begin{aligned} F(T) &\xrightarrow{\sim} H(T) \\ \phi &\longmapsto \left(\mathrm{id}_T : T \simeq T_\phi\right). \end{aligned} \tag{3.2}$$
Moreover, (3.2) *is equivariant for the* $\mathrm{Aut}(T)$*-actions on* $F(T)$ *and* $H(T)$ *of Remark 3.6.*

Before we proceed to the proof of Proposition 3.8, we need one further technical result, which essentially shows how to construct the inverse to (3.2).

Lemma 3.9 *Let T_1 and T_2 be two integral irreducible Hodge lattices of K3-type that are isomorphic as Hodge structures. Let $f : T_1 \simeq T_2$ be an isomorphism of Hodge structures, and denote*
$$\phi := i_{1,\mathbb{Q}}^{-1} \circ f_{\mathbb{Q}}^* \circ i_{2,\mathbb{Q}} \circ f_{\mathbb{Q}} \in \mathrm{End}((T_1)_{\mathbb{Q}}),$$
where $i_1 : T_1 \hookrightarrow T_1^*$ and $i_2 : T_2 \hookrightarrow T_2^*$ are the natural inclusions. Then ϕ is a rational Hodge endomorphism contained in $K(T_1)$, which maps T_1 into T_1^*. Moreover, T_2 is Hodge isometric to the Hodge lattice $(T_1)_\phi$ of Definition 2.3.

Proof As discussed in Remark 2.2, the corresponding diagram
$$\begin{tikzcd} T_1 \arrow[r, "f", "\simeq"'] \arrow[d, hook, "i_1"'] & T_2 \arrow[d, hook, "i_2"] \\ T_1^* & T_2^* \arrow[l, "\simeq"', "f^*"] \end{tikzcd}$$

is not necessarily commutative. However, for $\phi := i_{1,\mathbb{Q}}^{-1} \circ f_{\mathbb{Q}}^* \circ i_{2,\mathbb{Q}} \circ f_{\mathbb{Q}}$, we have

$$(\phi(x), -)_{T_1} = (f(x), f(-))_{T_2} \tag{3.3}$$

for all $x \in (T_1)_{\mathbb{Q}}$. By symmetry, we have $(x, \phi(-))_{T_1} = (\phi(x), -)_{T_1}$, hence $\phi \in K(T_1)$. Finally, since $(\phi(x), y)_{T_1} = (f(x), f(y))_{T_2} \in \mathbb{Z}$ for all $x, y \in T_1$, it follows that ϕ maps T_1 into T_1^* by Remark 2.4 ii). By construction, the diagram

$$\begin{tikzcd} T_1 \arrow[r, "f", "\simeq"'] \arrow[d, hook, "\phi|_{T_1}"'] & T_2 \arrow[d, hook, "i_2"] \\ T_1^* & T_2^* \arrow[l, "\simeq"', "f^*"] \end{tikzcd}$$

commutes. Therefore, by Remark 2.2, T_2 is Hodge isometric to $(T_1)_\phi$. □

Proof of Proposition 3.8 We first check the surjectivity of (3.2). Let $f : T \simeq T'$ be an isomorphism of Hodge structures, where T' is some integral, irreducible Hodge lattice of K3-type. It follows from the proof of Lemma 3.9 that f is equivalent to $\mathrm{id}_T : T \simeq T_\phi$, where $\phi := i_{T,\mathbb{Q}}^{-1} \circ f_{\mathbb{Q}}^* \circ i_{T',\mathbb{Q}} \circ f_{\mathbb{Q}}$, as in the proof of Lemma 3.9. Therefore (3.2) is surjective.

For injectivity, suppose $\mathrm{id}_T : T \simeq T_\phi$ is equivalent to $\mathrm{id}_T : T \simeq T_\psi$ for some $\phi, \psi \in F(T)$. By assumption, the diagram

$$\begin{tikzcd} T_\phi \arrow[r, "\mathrm{id}_T"] \arrow[d, "\phi"'] & T_\psi \arrow[d, "\psi"] \\ T_\phi^* & T_\psi^* \arrow[l, "\mathrm{id}_T^*"] \end{tikzcd}$$

commutes, hence we have $\phi = \psi$, as required.

Finally, to check the $\mathrm{Aut}(T)$-equivariance, let $h \in \mathrm{Aut}(T)$ be a Hodge automorphism of T and let $\phi \in F(T)$. Note that we have the commutative diagram

$$\begin{tikzcd} T_{h*\phi} \arrow[r, "h"] \arrow[d, "(h*\phi)"'] & T_\phi \arrow[d, "\phi"] \\ T_{h*\phi}^* & T_\phi^*. \arrow[l, "h^*"] \end{tikzcd} \tag{3.4}$$

This shows that $\mathrm{id}_T : T \simeq T_{h*\phi}$ is equivalent to $h : T \simeq T_\phi$, hence (3.2) is equivariant. □

Corollary 3.10 *Let T be an integral, irreducible Hodge lattice of K3-type. Let $\phi, \psi \in F(T)$. Then T_ϕ is Hodge isometric to T_ψ if and only if there is a Hodge automorphism $h \in \mathrm{Aut}(T)$ such that $\phi = h * \psi$. In particular, we have a commutative diagram*

$$\begin{tikzcd} F(T) \arrow[r, "\simeq"] \arrow[d] & H(T) \arrow[d] & \\ F(T)/\mathrm{Aut}(T) \arrow[r, "\simeq"] & H(T)/\mathrm{Aut}(T) \arrow[r, "\simeq"] & I(T). \end{tikzcd}$$

Proof This follows immediately from Proposition 3.8 combined with Remark 3.7. □

The main application of Proposition 3.8 is the following, which clarifies the relationship between T-equivalence and D-equivalence for sufficiently general K3 surfaces:

Proposition 3.11 *Let X be a K3 surface with $\rho \neq 18$ and such that* $\mathrm{End}(T(X)) \simeq \mathbb{Z}$ *(see Lemma 3.1). Let Y be a K3 surface which is T-equivalent to X and which satisfies* $\mathrm{disc}(T(X)) = \mathrm{disc}(T(Y))$*. Then X and Y are D-equivalent. If $\rho = 18$, then $T(Y)$ is Hodge isometric to either $T(X)$ or $T(X)_{-1}$.*

Proof Fix an isomorphism of Hodge structures $f\colon T(X) \simeq T(Y)$. By Proposition 3.8, there is a rational endomorphism $\phi\colon T(X)_{\mathbb{Q}} \simeq T(X)_{\mathbb{Q}}$ such that $T(Y)$ is Hodge isometric to $T(X)_{\phi}$. By our assumption on X, ϕ is given by multiplication by some rational number $q \in \mathbb{Q}$. In particular, we have

$$\mathrm{disc}(T(X)) = \mathrm{disc}(T(Y)) = \mathrm{disc}(T(X)_{\phi}) = q^{\mathrm{rk}\,T(X)} \cdot \mathrm{disc}(T(X)),$$

where the third equality follows from Lemma 2.5. This shows that $q = \pm 1$, hence $T(Y)$ is Hodge isometric to $T(X)$ or to $T(X)_{-1}$. If we assume $\rho \neq 18$, the signature of $T(X)$ is $(2, \mathrm{rk}(T(X)) - 2) \neq (2, 2)$. Therefore, the signature of $T(X)_{-1}$ is not equal to that of $T(Y)$, hence $q = 1$, and $T(X)$ is Hodge isometric to $T(Y)$. □

Remark 3.12 It is possible to find a K3 surface X of Picard rank 18, and a K3 surface Y which is T-equivalent to X and which satisfies $\mathrm{disc}(T(X)) = \mathrm{disc}(T(Y))$, yet $T(X)$ and $T(Y)$ are not Hodge isometric. For example, let X be a K3 surface whose transcendental lattice is isometric to the lattice

$$L := A_2 \oplus \langle -2 \rangle \oplus \langle -2 \rangle,$$

where A_2 is the positive-definite lattice of rank 2 whose matrix is given by

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We claim that $L \not\simeq L(-1)$. Firstly, it is not difficult to check that $A_{A_2}(-1) \not\simeq A_{A_2}$. Since $A_L \simeq A_{A_2} \oplus A_{\langle -2 \rangle}^{\oplus 2}$, the 3-primary part of A_L is precisely $A_L^{(3)} \simeq A_{A_2}$. If we assume that $L \simeq L(-1)$, then it follows that $A_{A_2} \simeq A_L^{(3)} \simeq A_L(-1)^{(3)} \simeq A_{A_2}(-1)$, a contradiction.

Since there exists a primitive embedding $T(X)(-1) \hookrightarrow \Lambda_{\mathrm{K3}}$ by [23, Corollary 1.12.3] (see also Example 3.3(ii)), it follows from the surjectivity of the period map [28] that there exists a K3 surface Y with a Hodge isometry $T(Y) \simeq T(X)(-1)$. The K3 surface Y is clearly T-equivalent to X, and we also have $\mathrm{disc}(T(Y)) = \mathrm{disc}(L(-1)) = \mathrm{disc}(L) = \mathrm{disc}(T(X))$, but $T(Y)$ is not isometric to $T(X)$, let alone Hodge isometric to $T(X)$.

Of course, we can replace L by any even lattice N of rank 4 and signature $(2, 2)$ to produce more examples of this phenomenon, provided that we have $N \not\simeq N(-1)$.

4 Hodge realisation and discriminants

Following Efimov [7], we denote by $\mathrm{HS}_{\mathbb{Z},2}$ the additive category of integral, polarisable Hodge structures of weight 2. The *Grothendieck group of polarisable Hodge structures of weight 2*, denoted $K_0^{\oplus}(\mathrm{HS}_{\mathbb{Z},2})$, is defined to be the free abelian group generated by isomorphism classes of objects in $\mathrm{HS}_{\mathbb{Z},2}$, modulo the relations of the form

$$[H_1 \oplus H_2] = [H_1] + [H_2] \qquad \text{for } H_1, H_2 \in \mathrm{HS}_{\mathbb{Z},2}\,.$$

There is a group homomorphism, called the *Hodge realisation map*

$$\begin{aligned} \mathrm{Hdg}_{\mathbb{Z}} : K_0(\mathrm{Var}_{\mathbb{C}}) &\longrightarrow K_0^{\oplus}(\mathrm{HS}_{\mathbb{Z},2}) \\ [X] &\longmapsto [H^2(X,\mathbb{Z})] \end{aligned}$$

that factors through the localisation $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$. In other words, for any two L-equivalent complex, smooth, projective varieties X, Y, we have

$$[H^2(X,\mathbb{Z})] = [H^2(Y,\mathbb{Z})] \in K_0^{\oplus}(\mathrm{HS}_{\mathbb{Z},2}).$$

For $H \in \mathrm{HS}_{\mathbb{Z},2}$ an integral, polarisable Hodge structure of weight 2, we denote $\mathrm{NS}(H) := H^{1,1} \cap H$. We denote by $T(H)$ the minimal integral sub-Hodge structure of H for which $H^{2,0} \subset T(H)_{\mathbb{C}}$, and we call $T(H)$ the *transcendental sub-Hodge structure* of H. Explicitly, $T(H)$ can be constructed as follows. The rational Hodge structure $H_{\mathbb{Q}}$ can be written as a direct sum of simple rational Hodge structures $H_{\mathbb{Q}} \simeq \bigoplus_{i=1}^{n} V_i$, for example by [29, §II.7, Lemma 7.26]. Then we define $T(H)_{\mathbb{Q}}$ to be the direct sum of those V_i which intersect $H^{2,0}$ non-trivially, and then we define the transcendental sub-Hodge structure of H to be $T(H) := T(H)_{\mathbb{Q}} \cap H$. For any two Hodge structures $H_1, H_2 \in \mathrm{HS}_{\mathbb{Z},2}$, it follows from the construction of $T(H)$ that we have $T(H_1 \oplus H_2) = T(H_1) \oplus T(H_2)$. Similarly, we have $\mathrm{NS}(H_1 \oplus H_2) = \mathrm{NS}(H_1) \oplus \mathrm{NS}(H_2)$.

Finally, we denote the *gluing group* of H by

$$G(H) := \frac{H}{\mathrm{NS}(H) \oplus T(H)}.$$

By the above discussion, it follows that we have $G(H_1 \oplus H_2) \simeq G(H_1) \oplus G(H_2)$ for any $H_1, H_2 \in \mathrm{HS}_{\mathbb{Z},2}$. Therefore, we obtain a group homomorphism

$$\begin{aligned} D : K_0^{\oplus}(\mathrm{HS}_{\mathbb{Z},2}) &\longrightarrow \mathbb{Q}^{\times} \\ [H] &\longmapsto |G(H)|. \end{aligned}$$

If X is a K3 surface, then $H^2(X,\mathbb{Z})$ is unimodular, hence the natural map

$$H^2(X,\mathbb{Z})/\big(\mathrm{NS}(X) \oplus T(X)\big) \simeq A_{T(X)}$$

is an isomorphism, where $A_{T(X)} = T(X)^*/T(X)$ is the discriminant group of T [23, Proposition 1.5.1].

As an immediate consequence of the above discussion, we obtain:

Lemma 4.1 *Let X and Y be L-equivalent K3 surfaces. Then* $\mathrm{disc}(T(X)) = \mathrm{disc}(T(Y))$.

Our main result is the following.

Theorem 4.2 *Let X and Y be K3 surfaces of Picard rank $\rho \neq 18$. Assume* $\mathrm{End}(T(X)) \simeq \mathbb{Z}$. *If X and Y are L-equivalent, then they are D-equivalent.*

Proof By [7, Lemma 3.7], X and Y are T-equivalent. Moreover, we have $\mathrm{disc}(T(X)) = \mathrm{disc}(T(Y))$ by Lemma 4.1. It follows from Proposition 3.11 that X and Y are D-equivalent, as required. □

Theorem 4.2 applies to very general K3 surfaces of Picard rank $\rho \leq 17$ by Lemma 3.1.

5 Applications and challenges

We now apply the results of the previous sections to study specific examples of K3 surfaces. Following this, we study higher-dimensional hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type.

Corollary 5.1 *Let X be a K3 surface of Picard rank 2 which admits an elliptic fibration $X \to \mathbb{P}^1$ of multisection index 5. Assume that $\mathrm{End}(T(X)) \simeq \mathbb{Z}$. Let Y be another K3 surface. Then X and Y are L-equivalent if and only if they are D-equivalent.*

Proof By [20, Corollary 5.13(i)], every K3 surface Y such that $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ admits an elliptic fibration $Y \to \mathbb{P}^1$ such that $X \simeq \mathrm{J}^k(Y)$ for some $k = 1, \dots, 4$. By [27, Theorem 3.2], X and Y are then L-equivalent. Conversely, if X and Y are L-equivalent, it follows from Theorem 4.2 that X and Y are D-equivalent. □

Finally, we turn our attention to higher-dimensional hyperkähler manifolds. If X is a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type for some $n \geq 2$, then $H^2(X, \mathbb{Z})$ is not unimodular and the natural map

$$H^2(X, \mathbb{Z})/\big(\mathrm{NS}(X) \oplus T(X)\big) \hookrightarrow A_{T(X)}$$

is injective but not necessarily surjective. This means that our proof of Lemma 4.1 does not generalise to this setting. Nevertheless, Conjecture 1.1 predicts that Lemma 4.1 also holds for higher-dimensional hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type. In fact, proving the analogue of Lemma 4.1 for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type would be a big step towards proving Conjecture 1.1. We refer to [19] for details.

Theorem 5.2 *Let X be a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type, and suppose $\mathrm{End}(T(X)) \simeq \mathbb{Z}$. Assume that X has Picard rank 1, and let $H \in \mathrm{NS}(X)$ be an ample generator of $\mathrm{NS}(X)$ with $H^2 = 2g$. Let Y be a hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type which is L-equivalent to X and such that $\mathrm{disc}(T(Y)) = \mathrm{disc}(T(X))$.*

i) Suppose $g \equiv 1 \pmod 4$, or $8 \mid d$, or $\mathrm{div}(H) = 2$. Then Y is D-equivalent to X.
ii) If $g \not\equiv 1 \pmod 4$ and $8 \nmid d$ and $\mathrm{div}(H) \neq 2$, then X and Y are twisted derived equivalent.

Proof Analogously to the proof of Proposition 3.11, one may use Proposition 3.8 to show that there is a Hodge isometry $T(X) \simeq T(Y)$. The result now follows from [16, Theorem 5.1]. □

Theorem 5.3 *Let S be a K3 surface with $\rho \neq 18$, such that $\mathrm{NS}(S)$ is unimodular, and such that $\mathrm{End}(T(S)) \simeq \mathbb{Z}$. Let M be a smooth moduli space of sheaves on S of dimension $2n \geq 2$. Let X be a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type that is L-equivalent to M and such that $\mathrm{disc}(T(X)) = \mathrm{disc}(T(M))$. Then X is birational to M. In particular, X and M are D-equivalent.*

Proof We may again use Proposition 3.8 to conclude that there is a Hodge isometry $T(X) \simeq T(M)$. In particular, X is birational to a moduli space of sheaves on S by [1, Proposition 4]. Therefore, by [18, Proposition 5.11], it follows from the unimodularity of $\mathrm{NS}(S)$ that X and M are birational. Thus X and M are D-equivalent by [11, Theorem 4.5.1]. □

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