TYPICAL SPECTRA AND NON-COMMUTATIVE CHOQUET THEORY

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Abstract

The thesis addresses two main topics, spectra of typical Hilbert space operators and non-commutative Choquet theory.
The former is discussed in Chapter 2, where we show that a typical operator has non-empty point spectrum as well as non-empty continuous spectrum are typical properties. We conclude the chapter with the new result that the set of operators with empty continuous spectrum is dense.
In Chapter 3, we study the lattice of C^*-covers of operator algebras. Among other things, we show that an operator algebra with more than one C^*-cover already has continuum many C^*-covers.
Chapter 4 is devoted to Arveson's Hyperrigidity Conjecture. We first show that all operator systems of the form $A(K)$, where $K \subset \mathbb{R}^2$ is compact and convex, are hyperrigid in $C(\mathrm{ex}(K))$. After that, we look at the known counterexample by Bilich and Dor-On and present a new counterexample, new in the sense that the operator system is generated by only finitely many selfadjoint operators.
We conclude the thesis in Chapter 5 with so far unpublished results. This includes, among others, the result that the maximal unital completely positive maps form a dense G_δ-set.

Zusammenfassung

Die Doktorarbeit befasst sich mit zwei zentralen Themen, typischen Spektren von Operatoren auf Hilberträumen und nicht-kommutative Choquet Theorie.
Ersteres wird in Chapter 2 behandelt. Dort zeigen wir zuerst, dass nicht-leeres Punktspektrum sowie nicht-leeres stetiges Spektrum typische Eigenschaften sind. Das Kapitel beenden wir mit dem neuen Resultat, dass die Menge der Operatoren mit leerem stetigem Spektrum dicht liegt.
In Chapter 3 befassen wir uns mit dem Verband von C^*-Überdeckungen von Operatoralgebren. Wir zeigen unter anderem, dass eine Operatoralgebra mit mehr als einer C^*-Überdeckung bereits Kontinuum viele C^*-Überdeckungen hat.
Chapter 4 widmet sich Arveson's Hyperrigidity Vermutung. Zuerst zeigen wir, dass alle Operatorsysteme der Form $A(K)$, wobei $K \subset \mathbb{R}^2$ eine kompakte konvexe Menge ist, hyperrigid in $C(\mathrm{ex}(K))$ sind. Danach befassen wir uns mit dem bekannten Gegenbeispiel von Bilich und Dor-On und geben ein neues Gegenbeispiel, neu im Sinne, dass das Operatorsystem nur von endlich vielen selbstadjungierten Elementen erzeugt ist.
Wir schließen die Thesis mit noch nicht veröffentlichten Resultaten in Chapter 5. Dies enthält unter anderem das Resultat, dass die maximalen unitalen vollständig positiven Abbildungen eine dichte G_δ-Menge sind.

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Contents

Chapter 1

Introduction

This thesis addresses two main topics. The first concerns typical properties of Hilbert space operators (see Chapter 2), and the second is non-commutative Choquet theory, which is further subdivided into two subtopics: C^*-covers of operator algebras (see Chapter 3) and hyperrigidity of operator systems (see Chapter 4). The thesis concludes with Chapter 5, which presents a collection of smaller results. Each chapter begins with a more detailed introduction to its specific subject. The present chapter provides a brief overview of the three aforementioned areas, along with a summary of the thesis's contributions to each.

In 1914, Felix Hausdorff published his book “Grundzüge der Mengenlehre“ (see [36] for the 1949 edition), in which, among other things, he introduced G_δ and F_σ sets. The former are countable intersections of open sets, while the latter are countable unions of closed sets. In the same work, Hausdorff also established the general version of what is now known as the Baire category theorem, a theorem that was first proven by Osgood for the real line $\mathbb{R}$ and later generalized to $\mathbb{R}^n$.

Theorem (Baire): *Let (X, d) be a complete metric space and $(U_n)_{n\in\mathbb{N}}$ be a sequence of open dense subsets in X. Then, $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.*

The theorem is of importance in geometry and analysis and appears in many proofs of fundamental results.
Furthermore, in his book, Hausdorff introduced sets of first category, nowadays called meager sets, which are subsets of countable unions of nowhere dense sets. He referred to their complements as “sets of second category“, we refer to them as comeager sets. The terminology “first“ and “second category“ originates from Baire's Ph.D. thesis, while the terms “meager“ and “comeager“ were introduced later by Bourbaki.
In the Scottish book, Stanisław Mazur noted a topological game, played alternately by two players, I and II, selecting decreasing open sets of $(\mathbb{R}, |\cdot|)$. Player II is said to win the game for a set $A \subset \mathbb{R}$ if there is a strategy such that, for each possible run of the game, the intersection of the chosen sets is contained in A. Apparently, in 1935, Stefan Banach showed that Player II has a winning strategy if and only if A is comeager. Later, in 1957, John Oxtoby generalized this game

to arbitrary topological spaces (X, τ), a version now known as the Banach-Mazur game, and proved the following theorem in [54].

Theorem (Banach-Mazur, Oxtoby): *Let (X, τ) be a topological space and $A \subset X$. Then, Player II has a winning strategy in the Banach-Mazur game for the set A if and only if A is comeager.*

In Chapter 2, we study typical properties of bounded linear operators on a Hilbert space H. A "typical property" means that the set of operators having the property is comeager in $\mathcal{B}(H)$. This of course depends on the topology considered on $\mathcal{B}(H)$. In [27] and [26], Tanja Eisner and Tamás Mátrai investigated typical properties in the norm, weak, and strong operator topologies, focusing on properties such as having non-empty point spectrum. Our work focuses exclusively on the norm topology and the two properties:

(i) having non-empty point spectrum,

(ii) having empty continuous spectrum.

In [27, Question 8.4], it was asked if (i) and (ii) are typical properties. We answer both of these questions in Chapter 2. The majority of the presented results were published in [59]. In Theorem 2.3.3, we show the following:

Theorem: *Let H be an infinite dimensional separable Hilbert space. Then*

$$\{T \in \mathcal{B}(H);\ \sigma_p(T) \neq \emptyset\}$$

contains an open dense subset of $\mathcal{B}(H)$ with respect to the norm topology.

This shows, in particular, that property (i) is typical.
On the other hand, in Theorem 2.3.12, we prove the following:

Theorem: *Let H be an infinite dimensional separable Hilbert space. Then the property of having non-empty continuous spectrum is typical with respect to the norm topology.*

This shows that (ii) is not a typical property. The proof uses the Banach-Mazur game and the aforementioned result by Banach-Mazur respectively Oxtoby.
These findings motivated an investigation of the closure of the set of operators with empty continuous spectrum. Similar questions were studied for other classes of operators by Domingo Herrero, see [39]. Our central tool for addressing this problem is the Similarity Orbit Theorem of Apostol, Fialkow, Herrero and Voiculescu. It provides necessary and sufficient spectral conditions for an operator to lie in the closure of the similarity orbit of another operator.
In this context, we need certain concrete examples of operators with specific spectral properties. In particular, for every non-empty compact set $K \subset \mathbb{C}$, we require an operator T_p on a separable Hilbert space such that $\sigma(T_p) = \sigma_p(T_p) = K$.

The topological properties of the spectrum and the point spectrum of an operator have a long history. Independently, Dixmier and Foiaş, and later Nikol'skaya in [53] showed that the point spectrum of a bounded operator acting on a separable reflexive Banach space is F_σ, and that every bounded F_σ subset of $\mathbb{C}$ coincides with the point spectrum of a certain bounded operator on a separable Hilbert space. A few years later, Kaufmann showed in [44, 43] that a necessary and sufficient condition for a subset of $\mathbb{C}$ to be the point spectrum of a bounded operator on some separable complex Banach space is that it be bounded and analytic (in the sense of Suslin). These results were extended by Smolyanov and Shkarin, see also [62] and [52] for further results.
The Similarity Orbit Theorem and the operator T_p are then used to prove a new result, Theorem 2.4.7:

Theorem: *Let H be a separable Hilbert space. Then*

$$\overline{\{T \in \mathcal{B}(H);\ \sigma_c(T) = \emptyset\}} = \mathcal{B}(H)$$

Chapter 3 and Chapter 4 are devoted to non-commutative Choquet theory. We begin with a discussion of classical Choquet theory. Originally developed by Gustave Choquet in the context of potential theory, it quickly evolved into an independent area of research.
Typically, one starts with a compact convex set K in a normed space and seeks to relate points in K to the extreme points of K. The most classical and probably most general result in this setting is due to Mark Krein and David Milman (1940, [48]):

Theorem (Krein-Milman): *Let X be a locally convex Hausdorff space and $K \subset X$ a non-empty, compact, convex set. Then*

$$\overline{\mathrm{conv}(\mathrm{ex}(K))} = K,$$

where $\mathrm{conv}(\cdot)$ *denotes the convex hull.*

This general result has a stronger formulation in Euclidean spaces $\mathbb{R}^d$:

Theorem (Carathéodory-Minkowski): *Let $K \subset \mathbb{R}^d$ be non-empty and compact, and let $x \in K$. Then x can be written as a convex combination of at most $d + 1$ extreme points of K.*

Choquet attempted to generalize Carathéodroy's theorem to arbitrary compact convex sets. However, in such generality, neither finite nor countably infinite convex combinations suffice. The key lies in interpreting what a convex combination is. If one views the extreme points in Carathéodory's theorem as point measures on $\mathrm{ex}(K)$, then a convex combination corresponds to a probability measure μ supported on $\mathrm{ex}(K)$. This measure then represents the point x in the sense that

$$\int_{\mathrm{ex}(K)} f d\mu = f(x)$$

for all $f \in A(K)$, the space of affine continuous functions on K.
This perspective allowed Choquet to prove the following fundamental result:

Theorem (Choquet): *Let K be a non-empty compact convex subset of a normed space and $x \in K$. Then there exists a probability measure μ concentrated on* $\mathrm{ex}(K)$ *such that*

$$\int_{\mathrm{ex}(K)} f d\mu = f(x)$$

for all $f \in A(K)$.

Most of the theorems in classical Choquet theory are formulated for function systems or function algebras. A unital self-adjoint subspace of a commutative C^*-algebra is called a *function system* and a unital closed subalgebra of a commutative C^*-algebra is called a *function algebra*. It is easy to see that $A(K)$ is a function system in $C(K)$. These definitions naturally motivate non-commutative Choquet theory by replacing the commutative C^*-algebra with a general one. This leads to the concepts of *operator systems*, unital selfadjoint subspaces of C^*-algebras and *operator algebras*, norm-closed subalgebras of C^*-algebras. Both were studied by William Arveson: operator algebras in [9], [6], and operator systems in [8], [7]. In the theory of operator algebras, Arveson investigated ideals I in the underlying C^*-algebra such that the corresponding quotient map by I is completely isometric on the operator algebra. This led to the fundamental concepts of *C^*-covers*, the *Shilov ideal*, and the *C^*-envelope*. Roughly speaking, a C^*-cover of an operator algebra is a completely isometric embedding of the operator algebra into a C^*-algebra.
Our work, while building on these ideas, is about the lattice that is obtained by defining an equivalence class on the collection of C^*-covers of an operator algebra. This was recently studied by Adam Humeniuk and Christopher Ramsey in [41]. Although their work yielded many interesting results, the following question remained open: [41, Question 3.1]: Does there exist a non-self-adjoint operator algebra that has, up to equivalence, only one C^*-cover? This question is answered in the affirmative in Theorem 3.2.4. The solution led to a fruitful joint work with Humeniuk and Ramsey, and to the surprising result Theorem 3.2.7:

Theorem: *Let A be an operator algebra with more than one C^*-cover up to equivalence. Then the cardinality of the lattice of C^*-covers of A is at least the continuum.*

This and various other small results of this joint work are presented in Chapter 3.
For operator systems, Arveson aimed to generalize the Choquet boundary of a function system. For a function system S in a commutative C^*-algebra $C(M)$, the *Choquet boundary* $B(S)$ consists of all points $y \in M$ such that the point evaluation $e_y : S \to \mathbb{C}, f \mapsto f(y)$ is an extreme point of the state space

$$\mathbf{S}(S) = \{\phi : S \to \mathbb{C};\ \phi \text{ unital and positive}\}.$$

It is a well-known result (see [56]) that

$$\|f\|_{\infty,K} = \sup\{|f(y)|;\ y \in B(M)\}$$

and that $B(A(K)) = \mathrm{ex}(K)$ when $A(K)$ is viewed as a function system in $C(K)$. In the setting of operator systems S, the state space is replaced by the set of unital completely positive maps from S into $\mathcal{B}(H)$ for some Hilbert space H. In 1969, Arveson conjectured that an appropriate analogue of the Choquet boundary is given by the set ∂_S of unitary equivalence classes of *boundary representations*, that is, unital, irreducible $*$-representations of $C^*(S)$ whose restriction to S has a unique unital completely positive extension.[1] He proposed that

$$\|x\| = \sup\{\|\phi(x)\|;\ \phi \in \partial_S\}$$

for every $n \in \mathbb{N}$ and $n \times n$-matrix x with entries in S. This remained an open problem for nearly 40 years, until Arveson himself resolved it in the separable case in [7], and Davidson and Kennedy settled it in full generality in [22]. For further reading on non-commutative Choquet theory, we refer to [21].
During this work, Arveson encountered a related but still unresolved question: under which conditions does every unital $*$-representation of $C^*(S)$ restricted to S have a unique unital completely positive extension to $C^*(S)$?
He called operator systems with this property *hyperrigid* and conjectured:

Conjecture: *A separable operator system S is hyperrigid if and only if the restriction of every boundary representation of $C^*(S)$ to S has a unique unital completely positive extension.*

A detailed discussion of this conjecture will be given at the beginning of Chapter 4. For now, let us mention that the conjecture is false: a counterexample was constructed by Boris Bilich and Adam Dor-On in [10]. However, the conjecture remains open when restricted to function systems. This is where our main result of Chapter 4 becomes relevant. Using a technique by Lawrence Brown [14], we show that:

Theorem: *Let $\emptyset \neq K \subset \mathbb{R}^2$ be compact and convex. Then $A(K)$ is hyperrigid in $C(\mathrm{ex}(K))$.*

This result was published in [60].
The chapter concludes with a new counterexample to Arveson's Hyperrigidity Conjecture - new in the sense that it is generated by finitely many operators. This result was published in [58].
The thesis concludes with several smaller results. The first noteworthy result in

[1]Some authors define the Choquet boundary of an operator system S in a commutative C^*-algebra $C(M)$ to be all point evaluations for which the restriction to S has a unique positive extension. This is actually equivalent to our definition, see [57, Lemma 6.2], and shows the relation to boundary representation.

Chapter 5, developed in collaboration with Michael Hartz and stated in Theorem 5.1.5, establishes a connection between non-commutative Choquet theory and G_δ-sets:

Theorem: *Let S be a separable operator system and H an infinite dimensional separable Hilbert space. Then*

$$\{\pi : S \to \mathcal{B}(H);\ \pi \text{ maximal}\}$$

is a dense G_δ-subset of the set of all $\mathcal{B}(H)$-valued u.c.p. maps on S, with respect to pointwise convergence in the weak operator topology.

The last noteworthy theorem of this thesis concerns matrix convex sets and was also developed in collaboration with Michael Hartz. These are graded sets that can be viewed as a generalization of convex sets, for which analog concepts of extreme points and convex hulls exist. In particular, one can consider the *matrix state space* of an operator system S, defined as

$$\mathcal{W}(S) = \bigcup_{n\in\mathbb{N}} \{\phi : S \to M_n;\ \phi \text{ u.c.p.}\},$$

as a compact matrix convex set.
There are several notions of extreme points in this context. Our main focus lies on the Arveson extreme points, which, in the case of the matrix state space of a finite-dimensional operator system S, are given by the restrictions of finite-dimensional boundary representations. Such extreme points do not always exist, and it remains an open problem to identify conditions that ensure their existence. In Theorem 5.2.6, we will show that as long as S generates a FDI C^*-algebra, the matrix convex hull of the Arveson extreme points coincides with the entire matrix state space of S:

Theorem: *Let S be a finite-dimensional operator system generating a FDI C^*-algebra. Then*

$$\operatorname{mconv}(\operatorname{arvex}(\mathcal{W}(S))) = \mathcal{W}(S).$$

Finally, we compare this result with a related theorem by Evert and Helton, [29, Theorem 1.1], which concerns spectrahedra, and observe that the two results apply, in general, to different classes of operator systems and, respectively, matrix convex sets.

Chapter 2

Spectra of Typical Hilbert Space Operators

This chapter begins with an introduction to G_δ-sets, comeager sets, and their relation to topological games. We then provide an overview of Fredholm theory and collect several lemmas needed for the first two main results. These answer two questions posed by Tanja Eisner and Tamás Mátrai (see [27, Question 8.4]):

A) Is the set of operators with non-empty point spectrum comeager with respect to the operator norm topology?

B) Is the set of operators with empty continuous spectrum comeager with respect to the operator norm topology?

We answer the first question affirmatively in Theorem 2.3.3 and the second negatively in Theorem 2.3.12.
The latter result raises the question of how large the closure of operators with empty continuous spectrum actually is. In Theorem 2.4.7, we show that the closure is $\mathcal{B}(H)$. For the proof, we need Herrero's Similarity Orbit theorem, stated in Section 2.4, and the existence of operators on separable Hilbert spaces with certain point spectra. Dixmier and Foaiş demonstrated the existence of such operators using Sobolev spaces, which will also be introduced in the first section of this chapter.
All of the new results were published in [59].

2.1 G_δ and Comeager Sets

Let (X, τ) be a topological space. A subset $U \subset X$ is called G_δ if there exists a countable collection $(U_n)_{n\in\mathbb{N}}$ of elements in τ such that $U = \bigcap_{n\in\mathbb{N}} U_n$. G_δ-sets play an important role in descriptive set theory, where their collection is denoted by $\prod_2^0$. Here are some of the most iconic examples.

Example 2.1.1:

(i) Let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of the rational numbers. Then,

$$\mathbb{R}\setminus\mathbb{Q} = \bigcap_{n\in\mathbb{N}} (\mathbb{R}\setminus\{q_n\})$$

shows that the irrational numbers form a G_δ-set in $\mathbb{R}$ equipped with the Euclidean topology.

(ii) Let $C([0,1])$ be the space of continuous functions on $[0,1]$ equipped with the uniform norm $\|\cdot\|_\infty$. Then,

$$\bigcap_{n\in\mathbb{N}} \left\{ f\in C([0,1]);\ \forall\, t\in[0,1]:\ \sup\nolimits_{0<|h|<1/(n+1)} \left|\frac{f(t+h)-f(t)}{h}\right| > n+1 \right\}$$

is a G_δ-set consisting of nowhere differentiable functions. It is not-trivial that this set is non-empty. However, it is a consequence of the next theorem that the set is dense in $C([0,1])$.

(iii) Let K be a metrizable, compact convex subset of a topological vector space, let d be a metric inducing the topology on K, and let $A(K)$ denote the continuous affine functions on K. Then

$$F_n = \{x\in K;\ \exists\, y,z\in K \text{ such that } x=\tfrac{1}{2}(y+z) \text{ and } d(y,z)\geq \tfrac{1}{n}\}$$

is closed for each $1\leq n\in\mathbb{N}$. Moreover, it is easy to see that for every $x\in K\setminus \operatorname{ex}(K)$ there exists an $n\in\mathbb{N}$ such that $x\in F_n$. Hence,

$$\operatorname{ex}(K) = \bigcap_{n=1}^{\infty} K\setminus F_n,$$

which shows that the set of extreme points forms a G_δ-subset of K. (see also [57, Proposition 1.3])

One of the most useful applications of G_δ-sets arises from the following version of the classical Baire category theorem. A proof can be found in [45, Theorem 8.4].

Theorem 2.1.2 (The Baire Category Theorem): *Let (X,d) be a complete metric space. Then the intersection of countably many open dense sets of X is dense.*

Applying the above theorem to Example 2.1.1, we see that the set of irrational numbers is dense in $\mathbb{R}$ and that the set of nowhere differentiable functions contains a dense G_δ-subset of $C([0,1])$, and is therefore non-empty.
The property of containing a dense G_δ-set appears frequently, so we call a subset U of a topological space *comeager* (or *residual*) if it contains a countable intersection of dense open sets. The complement of a comeager set is called *meager* (or *of the first category*).
Determining comeager sets in general topological spaces can be problematic, as many intuitively expected properties do not hold. At the very least, one should

expect the Baire Category Theorem to be valid. Therefore, we define a topological space to be a *Baire space* if the countable intersection of open dense subsets is again dense. By the above theorem, complete metric spaces are Baire spaces.

To better understand the differences between general topological spaces and Baire spaces, consider the following: in a Baire space, the Baire Category Theorem implies that the intersection of two comeager sets is again comeager, and a comeager set cannot be meager. To see the latter, let U be both comeager and meager in a Baire space X. Then, $\emptyset = (X \setminus U) \cap U$ would be comeager, contradicting the assumption that the space is Baire.
This property fails drastically in $(\mathbb{Q}, |\cdot|)$, since $\mathbb{Q}$ is the countable union of its elements, all of which are nowhere dense.
The following proposition is well known, see [45, Proposition 8.1], and characterizes Baire spaces. The proof is straightforward.

Proposition 2.1.3: *Let (X, τ) be a topological space. The following statements are equivalent:*

i) Every non-empty open set in X is non-meager.

ii) Every comeager set in X is dense.

iii) The space (X, τ) is Baire.

Proof:
i) $\Rightarrow$ *ii)* Let $A \subset X$ be comeager. Then $X \setminus \overline{A}$ is an open meager set, which is empty by *i)*.
ii) $\Rightarrow$ *iii)* This follows from the observation that the intersection of countably many dense open sets is comeager.
iii) $\Rightarrow$ *i)* Being Baire implies that the interior of every meager set is empty. Therefore, the only open meager set is the empty set. □

Probably the easiest way to show that a given set is comeager is to write down a countable family of dense open sets and show that their intersection is contained in the set, as in Example 2.1.1. However, in Section 2.3 we will encounter a set for witch this procedure does not seem to work, and therefore we need other tools to determine whether a given set is comeager.

A powerful tool to determine whether a topological space is Baire or a set is comeager is provided by a topological game. We first define the *Choquet game*, which characterizes Baire spaces, and then the *Banach-Mazur game*, which characterizes comeager sets.
Both games are played in the same manner, the only difference lies in the winning condition. In general, two players take turns choosing open sets in a topological

space (X, τ):

$$\begin{array}{llllll} I: & U_0 & & U_2 & \dots & \\ II: & & U_1 & & U_3 & \dots \end{array}$$

with the condition that $\emptyset \neq U_{n+1} \subset U_n$ and $U_n \in \tau$ for all $n \in \mathbb{N}$. In the Choquet game G_X, Player II wins if

$$\bigcap_{n\in\mathbb{N}} U_{2n+1} \neq \emptyset.$$

In the Banach-Mazur game $G^{**}(A)$, which is played with respect to a subset $A \subset X$, Player II wins if

$$\bigcap_{n\in\mathbb{N}} U_{2n+1} \subset A.$$

Note, that by construction of the U_n, we always have $\bigcap_{n\in\mathbb{N}} U_{2n+1} = \bigcap_{n\in\mathbb{N}} U_{2n}$.
The connection between these games and Baire spaces respectively comeager sets is as follows: a topological space is Baire if and only if Player I has no winning strategy in the Choquet game, and a set A is comeager if and only if Player II has a winning strategy in the Banach-Mazur game $G^{**}(A)$.
However, we have omitted an important definition here. What exactly is a *winning strategy*? While this is intuitively clear, a precise mathematical definition is needed. So what follows now is the cumbersome definition of a winning strategy.
Let A be a set and $n \in \mathbb{N}$. Then A^n denotes the set of all finite sequences of length n with entries in A, and the set of all finite sequences with entries in A is denoted by

$$A^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} A^{n+1}.$$

Let $x = (x_i)_{i=0}^n \in A^{n+1}$ and $y = (y_i)_{i=0}^m \in A^{m+1}$. We say that x is an *initial segment* of y and y is an *extension* of x if $n \leq m$ and $(x_i)_{i=0}^n = (y_i)_{i=0}^n$.
A *tree* (in the sense of descriptive set theory) on A is a subset $T \subset A^{<\mathbb{N}}$ such that for every $x \in T$ and every $y \in A$ with x an initial segment of y, we have that $y \in T$. The tree is called *pruned* if every $x \in T$ has an extension $y \in T$ with $x \neq y$.
The *body* of a tree T is defined by

$$[T] = \{(x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}};\ (x_i)_{i=0}^m \in T \text{ for all } m \in \mathbb{N}\}.$$

We now define a *strategy* for Player II as a subtree of the pruned tree

$$\bigcup_{n\in\mathbb{N}} \{(U_0, \dots, U_n) \in \tau^n;\ \emptyset \neq U_{n+1} \subset U_n \subset \dots \subset U_0\}$$

on X such that, for all $n \in \mathbb{N}$:

i) $U \in \sigma$ for all $U \in \tau$.

ii) For every $(U_0, \dots, U_{2n+1}) \in \sigma$ and $\emptyset \neq V \in \tau$ with $V \subset U_{2n+1}$, we have that $(U_0, \dots, U_{2n+1}, V) \in \sigma$.

iii) Every $(U_0, \dots, U_{2n}) \in \sigma$ has a unique extension $(U_0, \dots, U_{2n+1}) \in \sigma$.

Given a strategy σ for Player II, the game proceeds as follows: if Player I plays $U_0 \in \tau$, then Player II plays the unique $U_1 \in \tau$ such that $(U_0, U_1) \in \sigma$. If the sets $(U_0, \dots, U_{2n})$ have been played, then $(U_0, \dots, U_{2n}) \in \sigma$ by the definition of strategy, and thus Player II plays the unique set $U_{2n+1} \in \sigma$ such that $(U_0, \dots, U_{2n}, U_{2n+1}) \in \sigma$. A run of the game is given by a sequence of open sets $(U_n)_n \in \mathbb{N}$ such that $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. Let σ be a strategy for Player II. Then a run $(U_n)_n$ is called *compatible* with σ if $(U_n)_n \in [\sigma]$.
The strategy σ is a *winning strategy* for Player II in the Choquet game if $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$ for every compatible run $(U_n)_n$, and in the Banach-Mazur game $G^{**}(A)$ if $\bigcap_{n\in\mathbb{N}} U_{2n+1} \subset A$ for every compatible run.
Similarly, a strategy σ for Player I is defined as a subtree of the pruned tree

$$\bigcup_{n\in\mathbb{N}} \{(U_0, \dots, U_n) \in \tau^n;\ \emptyset \neq U_{n+1} \subset U_n \subset \dots \subset U_0\}$$

such that, for all $n \in \mathbb{N}$:

i) $\sigma \neq \emptyset$.

ii) For every $(U_0, \dots, U_{2n}) \in \sigma$ and $\emptyset \neq V \in \tau$ with $V \subset U_{2n}$, we have that $(U_0, \dots, U_{2n}, V) \in \sigma$.

iii) Every $(U_0, \dots, U_{2n+1}) \in \sigma$ has a unique extension $(U_0, \dots, U_{2n+1}, U_{2n+2}) \in \sigma$.

Player I plays according to a strategy σ as follows: Player I plays an arbitrary $U_0 \in \sigma$. Then Player II plays an open non-empty $U_1 \subset U_0$. By the definition of our strategy, there is a unique non-empty $U_2 \in \tau$ such that $(U_0, U_1, U_2) \in \sigma$. Player I plays U_2 and the proceeding rounds are played recursively.
Let σ be a strategy for Player I. Then, a run $(U_n)_n$ is called *compatible* with σ, if $(U_n)_n \in [\sigma]$. The strategy σ is a *winning strategy* for Player I in G_X, if $\bigcap_{n\in\mathbb{N}} U_{2n} = \emptyset$ for every compatible run $(U_n)_n$, and a *winning strategy* for Player I in $G^{**}(A)$, if $\bigcap_{n\in\mathbb{N}} U_{2n} \not\subset A$.
The connection between the games and the respective topological concepts is established in the following two theorems (see [45, Chapter 8]). We state both but only prove the second, as we will use it in the proof of question B.

Theorem 2.1.4 (Oxtoby): *Let (X, τ) be a topological space. Then, Player I has no winning strategy in the Choquet game G_X if and only if (X, τ) is a Baire space.*

Theorem 2.1.5 (Banach-Mazur, Oxtoby): *Let (X, τ) be a topological space and $\emptyset \neq A \subset X$. Player II has a winning strategy in the Banach-Mazur game $G^{**}(A)$ if and only if the set A is comeager.*

Proof:
$\Leftarrow$: Let $A \subset X$ be comeager. Then, there exists a collection $(V_n)_{n\in\mathbb{N}}$ of open dense

subsets of X such that $\bigcap_{n\in\mathbb{N}} V_n \subset A$. Define a strategy σ for Player II as follows: if $(U_0,\dots,U_{2n})$ is played, then play $U_{2n}\cap V_n$. Note that $U_{2n}\cap V_n \neq \emptyset$ as V_n is dense. Hence, for a run $(U_n)_{n\in\mathbb{N}}$ compatible with σ,

$$\bigcap_{n\in\mathbb{N}} U_{2n+1} = \bigcap_{n\in\mathbb{N}} U_{2n}\cap V_n \subset \bigcap_{n\in\mathbb{N}} V_n \subset A,$$

so σ is a winning strategy for Player II.
$\Rightarrow$: Let σ be a winning strategy for Player II. Define

$$\tau_0 = \{U\in\tau;\ \text{there exists } V\in\tau \text{ such that } (V,U)\in\sigma\}.$$

We first claim that there exists a family $(U_i)_{i\in I}$ in τ_0 of pairwise disjoint elements such that $\bigcup_{i\in I} U_i$ is dense in X. Define

$$M_0 = \{(V_j)_{j\in J};\ V_j\in\tau_0,\ V_j\cap V_i=\emptyset \text{ for all } i\neq j\in J\},$$

and an ordering on M_0 by

$$(V_j)_{j\in J} \le (U_i)_{i\in I} \Leftrightarrow \forall\, j\in J\,\exists\, i\in I : V_j = U_i.$$

Let $((U_i)_{i\in I_j})_{j\in J}$ be a chain in M_0 and define $I=\bigcup_{j\in J} I_j$. For $i,j\in I$, define $i\sim j$ if and only if $U_i=U_j$. Denote by $\tilde{I}$ the set of equivalence classes of I with repsect to $\sim$. Then $(U_i)_{i\in\tilde{I}}$ is an element of M_0 and an upper bound of the chain $((U_i)_{i\in I_j})_{j\in J}$. By Zorn's lemma, there exists a maximal element in M_0, which we denote by $(U_i)_{i\in G}$.
We claim that $\bigcup_{i\in G} U_i$ is dense in X. Assume the contrary, that is, that

$$\emptyset \neq X\setminus \overline{\bigcup_{i\in G} U_i} = U$$

is open. By the definition of strategy, there exists a $U_g\in\tau$ such that $(U,U_g)\in\sigma$. Thus, $U_g\in\tau_1$, and

$$(U_i)_{i\in G} \le (U_i)_{i\in G\cup\{g\}}$$

which contradicts the maximality of $(U_i)_{i\in G}$. Hence, $\bigcup_{i\in G} U_i$ is dense in X.
Denote the family $(U_i)_{i\in G}$ by $(U_i^{(0)})_{i\in I_0}$ and define recursively for $1\le n\in\mathbb{N}$:

$$\tau_n = \{U\in\tau;\ \exists\ V_0,\dots,V_n\in\tau, U_0\in\tau_0,\dots,U_{n-1}\in\tau_{n-1} :\ (V_0,U_0,\dots,V_n,U)\in\sigma\}.$$

Analogously to the above, there exists a family $(U_i^{(n)})_{i\in I_n}$ of pairwise disjoint elements in τ_n such that $\bigcup_{i\in I_n} U_i^{(n)}$ is dense in X.
Define, for $n\in\mathbb{N}$,

$$U_n = \bigcup_{i\in I_n} U_i^{(n)}.$$

Then U_n is open and dense by construction. To prove that A is comeager, it remains to show that

$$\bigcap_{n\in\mathbb{N}} U_n \subset A.$$

Let $x \in \bigcap_{n\in\mathbb{N}} U_n$. Then, for every $n \in \mathbb{N}$, there exists precisely one $i_n \in I_n$ such that $x \in U_{i_n}^{(n)}$, and $V_n \in \tau$ such that

$$(V_0, U_{i_1}^{(0)}, V_2, \dots, V_n, U_{i_n}^{(n)}) \in \sigma.$$

This yields a run of the Banach-Mazur game, and since σ is a winning strategy, we obtain

$$\bigcap_{n\in\mathbb{N}} U_n \subset A.$$

Hence, $x \in A$, and the proof is complete. □

The next example illustrates a winning strategy and provides additional background on strategies, included solely for better understanding. For the proof of Question B, only the aforementioned definition of a winning strategy is required.

Example 2.1.6: In Example 2.1.1, we saw that $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ-set and dense in $\mathbb{R}$, thus it is comeager. By the previous theorem, Player II has a winning strategy. The proof shows us what a possible strategy looks like. Let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$. Then, one possible winning strategy is:

$$\begin{array}{lllll} I: & U_0 & & U_2 & \dots \\ II: & & U_0 \setminus \{q_0\} & & U_2 \setminus \{q_1\} \quad \dots \end{array}$$

More generally, given open dense sets $(V_n)_{n\in\mathbb{N}}$ of X, a winning strategy for Player II in $G^{**}(A)$ for any $A \subset X$ with $\bigcap_n V_n \subset A$ is:

$$\begin{array}{lllll} I: & U_0 & & U_2 & \dots \\ II: & & U_0 \cap V_0 & & U_2 \cap V_1 \quad \dots \end{array}$$

The above strategy actually depends only on the last move of Player I and the round number. Such a strategy is called a *Markov winning strategy* for Player II. If the strategy depends solely on the last move of Player I, it is called a *stationary winning strategy*. By definition, every stationary winning strategy is a Markov winning strategy, and every Markov winning strategy is a winning strategy. For the Banach-Mazur game, Theorem 2.1.5 and the discussion above show that the existence of a Markov winning strategy for Player II is equivalent to the existence of a winning strategy.

The relationship between winning, Markov winning and stationary winning strategies has been studied by Oxtoby, Galvin and Telgársky, yielding surprising results for the Choquet game. If there exists a Markov winning strategy, then there already exists a stationary one [33, Corollary 9]. However, there are topological spaces where Player II has a winning strategy but no stationary winning strategy [23].

Nevertheless, if Player II has a winning strategy, then there exists one that depends only on the last moves of both players, that is, if $(U_0, \dots, U_{2n})$ have been played, then the next move of Player II depends only on (U_{2n-1}, U_{2n}) [33, Corollary 14]. Interestingly, it is still an open question whether a winning strategy for Player II implies the existence of a winning strategy that depends only on the last two moves of Player I.

We end this chapter with an observation that we will need later.

Remark 2.1.7: Let (X, d) be a metric space, and suppose we want to show that Player II has a winning strategy in the Banach-Mazur game $G^{**}(A)$. Then, without loss of generality, we may assume that Player I only plays open balls $B_\epsilon(x)$. This is justified by the fact that all sets played by Player I must be open, and thus every open set Player I can possibly play contains an open ball.

2.2 Fredholm and Spectral Theory

Let H be a Hilbert space, and denote by $\mathcal{B}(H)$ the set of all bounded linear operators on H, equipped with the operator norm. An operator $T \in \mathcal{B}(H)$ is called *semi-Fredholm* if either $\dim(H/\mathrm{Im}(T)) < \infty$ or T has closed image and $\dim(\ker(T)) < \infty$. The possibly infinite number

$$\mathrm{ind}(T) = \dim(\ker(T)) - \dim(\mathrm{Im}(T)) \in \mathbb{Z} \cup \{-\infty, \infty\}$$

is then well defined and called the *index of* T. The operator T is called *Fredholm* if $\mathrm{ind}(T) \in \mathbb{Z}$. One of the results we need, and perhaps the most famous in Fredholm theory, is the following.

Lemma 2.2.1: *Let $c \in \mathbb{Z} \cup \{-\infty, \infty\}$. Then the set*

$$\{T \in \mathcal{B}(H);\ T \text{ is semi-Fredholm with } \mathrm{ind}(T) = c\}$$

is open with respect to the operator norm.

Proof:
See, for example, [51, Chapter 18, Corollary 2].

The spectrum $\sigma(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda id_H$ is not invertible. It is well known that the spectrum is always non-empty and compact. Let $\mathcal{K}(H)$ denote the compact operators on H. Then the *essential spectrum* $\sigma_e(T)$ is defined as the spectrum of the image of T under the quotient map in the *Calkin algebra* $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$. If H is infinite dimensional, then $\sigma_e(T)$ is also always non-empty and compact. Moreover, $\sigma_e(T)$ is invariant under compact perturbations, that is, $\sigma_e(T) = \sigma_e(T + K)$ for all $K \in \mathcal{K}(H)$. This follows directly from the definition of the essential spectrum. The following lemma relates these notions to Fredholm theory:

Lemma 2.2.2: *Let $T \in \mathcal{B}(H)$. Then:*

(i) *If T is semi-Fredholm, then $T + K$ is also semi-Fredholm for all $K \in \mathcal{K}(H)$.*

(ii) *$T - \lambda id_H$ is not semi-Fredholm for all $\lambda \in \partial\sigma_e(T)$.*

(iii) *If T is not semi-Fredholm, then $T + \mathcal{K}(H)$ is not left-invertible in $\mathcal{C}(H)$.*

(iv) *If T is not semi-Fredholm, then for every $\epsilon > 0$, there exists a $K \in \mathcal{K}(H)$ such that $\dim(\ker(T - K)) = \infty$ and $\|K\| < \epsilon$.*

In particular, if $\dim(H) = \infty$, then there exists a $\lambda \in \mathbb{C}$ such that $T - \lambda id_H$ is not semi-Fredholm.

Proof:
Part (i) is contained in [51, Chapter 19, Proposition 1], part (ii) follows from the same proposition, part (iii) follows from (i) and [51, Chapter 19, Theorem 7], and part (iv) is contained in [51, Chapter 16, Theorem 18].
The final remark follows from (ii) and the fact that $\sigma_e(T)$ is compact and non-empty whenever $\dim(H) = \infty$.

We now introduce the necessary definitions from spectral theory, starting with the *polar decomposition* of an operator T. For $x \in H$, we have:

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \left\langle \sqrt{T^*T}x, \sqrt{T^*T}x\right\rangle = \left\|\sqrt{T^*T}x\right\|^2.$$

Thus, there exists a partial isometry $P \in \mathcal{B}(H)$, uniquely defined by

$$Tx = P\left(\sqrt{T^*T}x\right) \text{ for all } x \in H,$$

and $Py = 0$ for all $y \in \mathrm{Im}(T^*)^\perp$. In particular, $T = P\sqrt{T^*T}$. This shows that for every operator T, there exists a positive operator A and a partial isometry P such that $T = PA$ and $\mathrm{Im}(A)^\perp = \ker(P) = \ker(T)$. This is known as the *polar decomposition of T*. A proof of the uniqueness can be found in [35, Problem 134]. The next lemma is the starting point for a series of lemmas which we will use to answer Questions A) and B).

Lemma 2.2.3: *Let H be a separable Hilbert space, $T \in \mathcal{B}(H)$ be not semi-Fredholm, and $\|T\| > \epsilon > 0$. If $T = PA$ is the polar decomposition of T, and ν the spectral measure of A, then*

$$S = P\int_\epsilon^{\|T\|} t\ d\nu(t) \in \mathcal{B}(H)$$

has the following properties:

a) *The image of S is closed, more precisely, $\|Sx\| \geq \epsilon\|x\|$ for all $x \in \ker(S)^\perp$.*

b) $\dim(\ker(S)) = \dim(\mathrm{Im}(S)^\perp) = \infty$.

c) $\|T - S\| \leq \epsilon$.

Proof:
Let T, S, P, A, ν be as above. Since P is a partial isometry and $\nu([\epsilon, \|T\|])(H) \subset \nu(\{0\})(H)^\perp = \ker(A)^\perp = \overline{\mathrm{Im}(A)} = \ker(P)^\perp$, it follows that

$$\left\| \int_\epsilon^{\|T\|} t \; d\nu(t)x \right\| = \|Sx\|$$

for every $x \in \nu([\epsilon, \|T\|])(H)$. Therefore,

$$\|Sx\|^2 = \left\| \int_\epsilon^{\|T\|} t \; d\nu(t)x \right\|^2 = \left\langle \int_\epsilon^{\|T\|} t^2 d\nu(t)x, x \right\rangle = \int_\epsilon^{\|T\|} t^2 \; d\nu_{x,x}(t) \geq \epsilon^2 \|x\|^2,$$

where $\nu_{x,x}(\cdot)$ denotes the scalar valued measure $\langle \nu(\cdot)x, x\rangle$. Together with $\ker(S)^\perp \subset \nu([\epsilon, \|T\|])(H)$, this proves *a*).
For clarity, let $[T]$ denote the equivalence class of $T + \mathcal{K}(H)$ in $\mathcal{C}(H)$. Since T is not semi-Fredholm, $[T]$ is not left-invertible by Lemma 2.2.2. Consequently, $[T^*T] = [A]^2$ is not invertible, so $[A]$ is not invertible and in particular, A is not Fredholm. We can write A as

$$A = \int_0^{\|T\|} \max(\epsilon, t) d\nu(t) - \int_0^\epsilon (\epsilon - t) d\nu(t).$$

The first operator in the sum is clearly invertible. If we assume $\dim(\nu([0, \epsilon))(H)) < \infty$, then the second operator in the sum has finite rank, contradicting the fact that A is not Fredholm. Therefore, $\dim(\nu([0, \epsilon))(H)) = \infty$. Moreover

$$\nu([0, \epsilon))(H) \subset \ker(S) \text{ and } P\nu([0, \epsilon))(H) \subset \mathrm{Im}(S)^\perp,$$

where the second inclusion follows from $\nu([0, \epsilon))(H) \subset \ker(P)^\perp$. Hence, $\dim(\ker(S)) = \dim(\mathrm{Im}(S)^\perp) = \infty$, which proves part *b*).
To prove *c*), observe that

$$\|(T - S)x\|^2 = \| \int t\chi_{[0,\epsilon)}(t) \; d\nu(t)x\|^2 = \int t^2 \chi_{[0,\epsilon)}(t) d\nu_{x,x}(t) \leq \epsilon^2 \|x\|^2$$

for all $x \in H$. □

This immediately implies the following corollary.

Corollary 2.2.4: *Let* $T \in \mathcal{B}(H)$ *and* $\epsilon > 0$. *Then there exist* $\lambda \in \mathbb{C}$ *and* $S \in B_\epsilon(T - \lambda id_H)$ *such that:*

i) the image of S *is closed.*

ii) $\dim(\ker(S)) = \dim(\mathrm{Im}(S)^\perp) = \infty$.

Proof:
First apply Lemma 2.2.2 and then Lemma 2.2.3. □

In addition to the polar decomposition, we will need the notion of a *spectral distribution* in the sense of Foiaş (see, for example, [50], [32], [61]). A spectral distribution is a map μ from the space of test functions $C_c^\infty(\mathbb{C})$, that is, infinitely differentiable functions with compact support, into $\mathcal{B}(H)$ equipped with the operator norm, such that:

(i) $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in C_c^\infty(\mathbb{C})$.

(ii) μ has compact support, that is, there is a compact set $K \subset \mathbb{C}$ such that

$$\mu(f) = 0 \text{ for all } f \in C_c^\infty(\mathbb{C}) \text{ satisfying } f|_{\mathbb{C}\setminus K} = 0.$$

(iii) $\mu(f) = id_H$ for all $f \in C_c^\infty(\mathbb{C})$ which are identically 1 near the support of μ.

(iv) μ is continuous with respect to the seminorms given by

$$\sup_{x\in K} |\partial^\alpha f(x)|, \ \alpha \in \mathbb{N}^2, \ K \subset U \text{ compact}^1.$$

We say that an operator T has the spectral distribution μ if there exists a test function f such that $\mu(f) = T$ and $f(\lambda) = \lambda$ near the support of μ.
Operators admitting a spectral distribution were studied by Foiaş in [32], where he proved the following lemma:

Lemma 2.2.5: *Let T have a spectral distribution. Then for every $x \in H$, there exists a non-empty maximal open set U_x such that there exists an analytic function $g : U_x \to H$ satisfying*

$$(\lambda id_H - T)g(\lambda) = x \text{ for all } \lambda \in U_x.$$

Here, *maximal* means that for any other set U with this property, we have $U \subset U_x$.
In the above context, we define $\sigma(T, x) = \mathbb{C} \setminus U_x$ and for a closed subset $F \subset \mathbb{C}$, we define

$$H(T, F) = \{x \in H;\ \sigma(T, x) \subset F\}.$$

It is easy to verify that this is a sub-Hilbert space of H.
The following lemma was also proven in [32].

Lemma 2.2.6: *Let T have a spectral distribution and let $F \subset \mathbb{C}$ be closed. Then $H(T, F)$ is invariant under T, and $\sigma(T|_{H(T,F)}) \subset F$. Conversely, if $\tilde{H} \subset H$ is a subspace invariant under T with $\sigma(T|_{\tilde{H}}) \subset F$, then $\tilde{H} \subset H(T, F)$.*

[1] $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ is a multiindex and we use the standard conventions: $|\alpha| = \alpha_1 + \alpha_2$, and $\alpha! = \alpha_1!\alpha_2!$

Later, we will apply this lemma to an operator defined on a Sobolev space. Since not every reader may be familiar with Sobolev spaces, we will conclude this section with their definition and the key properties for our work.
Let $U \subset \mathbb{C}$ be a non-empty open set. For a function $u \in L^2(U)$ and multi-index $(\alpha_1, \alpha_2) \in \mathbb{N}^2$, we say that a function $D^\alpha u \in L^1(U)$ is the α-th *weak derivative* of u if $D^\alpha u$ is locally integrable and

$$(-1)^{\alpha_1+\alpha_2} \int_U D^\alpha u(t)\phi(t)d(t) = \int_U u(t) \frac{\partial^\alpha}{\partial^{\alpha_1} t_1 \partial^{\alpha_2} t_2} \phi(t) d(t)$$

for all locally integrable, infinitely often differentiable functions $\phi \in C^\infty(U)$. We are not going into detail here, but it can be shown using the Du-Bois-Reymond lemma (see [16, page 314]) that the weak derivative is unique up to a set of measure zero.
Now the Sobolev space $W^{2,2}(U)$ is defined as

$$W^{2,2}(U) = \left\{ u \in L^2(U); \begin{matrix} \text{for all } \alpha \in \mathbb{N}^2 \text{ with } |\alpha| \leq 2, \\ D^\alpha u \text{ exists and lies in } L^2(U) \end{matrix} \right\}$$

together with the inner product

$$\langle u, v \rangle = \sum_{|\alpha| \leq 2} \langle D^\alpha u, D^\alpha v \rangle_{L^2(U)}.$$

The norm induced by this inner product is equal to the sum of the L^2-norms of all weak derivations of order less than or equal to 2. With this, one easily verifies that $W^{k,2}(U)$ is a closed subspace of $L^2(U)$ and hence forms a Hilbert space.
The following lemma is a special case of a classic result in the theory of Sobolev spaces. It follows from [13, Corollary 9.15].

Lemma 2.2.7: *Let U be an open disc in $\mathbb{C}$. Then for every $[u] \in W^{2,2}(U)$, there exists a continuous function $f_u \in [u]$. Moreover, there exists a constant $c > 0$, independent of $[u]$, such that*

$$\|f_u\|_\infty \leq c\|[u]\|_{W^{2,2}(U)}.$$

We finish this section by combining the theory of Sobolev spaces with the theory of spectral distributions.

Theorem 2.2.8: *Let U be an open disc in $\mathbb{C}$. Then*

$$T : W^{2,2}(U) \to W^{2,2}(U), f \mapsto zf$$

is well defined, linear, continuous, and admits a spectral distribution.

Proof:
Let $f \in W^{2,2}(U)$. Then for all multi-indices $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ with $\alpha_1 + \alpha_2 \leq 2$, the Leibniz product rule (see [28, Section 5.2.3 Theorem 1]) yields:

$$D^\alpha T(f) = \sum_{\beta_1 \leq \alpha_1} \sum_{\beta_2 \leq \alpha_2} \frac{\alpha_1!\alpha_2!}{\beta_1!\beta_2!(\alpha_1 - \beta_1)!(\alpha_2 - \beta_2)!} D^{(\beta_1,\beta_2)} z D^{(\alpha_1-\beta_1,\alpha_2-\beta_2)} f.$$

Since the norm on the Sobolev spaces is equal to the sum of the L^2-norms of the weak derivations of order up to 2, this proves that T is well defined and continuous. Since linearity is clear, it remains to show that T admits a spectral distribution. Define the map

$$\mu : C_c^\infty(\mathbb{C}) \to \mathcal{B}(W^{2,2}(U)), g \mapsto [f \mapsto gf].$$

By the Leibniz formula, this map is well-defined. Moreover, μ is supported on $\overline{U}$, multiplicative, and satisfies $\mu(f) = id_H$ for all $f \in C_c^\infty(\mathbb{C})$ which are identically 1 near $\overline{U}$. We claim that μ is a spectral distribution of T.
Since $\mu(f) = T$ for every function $f \in C_c^\infty(\mathbb{C})$ satisfying $f(\lambda) = \lambda$ on $\overline{U}$, it remains to show that μ is continuous. For this, it suffices to prove that for every compact set $K \subset U$, the map

$$C_c^\infty(K) \to B(H), f \mapsto \mu(f)$$

is continuous with respect to the topology induced by the seminorms

$$\sup_{x \in K} |\partial^p f(x)|, \; p \in \mathbb{N}^2.$$

This again follows from the Leibniz formula. □

The reader might wonder why we use Sobolev spaces rather than simply working in $L^2(\overline{U})$. A detailed explanation is given in the proof of Theorem 2.4.5. The short answer is the following observation:
By Lemma 2.2.7, point evaluations are continuous on $W^{2,2}(U)$, and therefore, by the Riesz representation theorem, for each $\lambda \in U$ there exists vectors $k_\lambda \in W^{2,2}(U)$ such that

$$\langle [u], k_\lambda \rangle = f_u(\lambda).$$

Hence, the adjoint of M_z has eigenvalues at every point $\lambda \in U$ when acting on $W^{2,2}(U)$, which is not the case for M_z^* considered as an operator on $L^2(\overline{U})$.

2.3 Typical Properties of Spectra

Given a topological space (X, τ) and a property (P) on the points of X, we say that the property is *typical* (or *generic*) if the set $\{x \in X;\; x \text{ fulfills } (P)\}$ is comeager in X. In the following, the topological space will be the C^*-algebra of bounded linear operators on a Hilbert space equipped with the operator norm, and (P) will refer to a property related to the spectrum. Note that this space is a Baire space by the Baire Category Theorem.

2.3.1 The Point Spectrum

Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be a linear bounded operator. We call $\lambda \in \mathbb{C}$ an *eigenvalue* if there is a non-zero vector $x \in H$ with $Tx = \lambda x$. The set of all eigenvalues of T is called the *point spectrum* $\sigma_p(T)$ of T. It is clear that $\sigma_p(T) \subset \sigma(T)$, since $\lambda \in \sigma_p(T)$ if and only if $\ker(T - \lambda id_H) \neq \{0\}$.
It is well known that $\sigma_p(T) \neq \emptyset$ if $\dim(H) < \infty$. Hence, having non-empty point spectrum is obviously a typical property in the finite-dimensional case. Therefore, throughout this section, we assume that H is an infinite-dimensional separable Hilbert space. We begin with an example of an operator with empty point spectrum.

Example 2.3.1: Let $\ell^2(\mathbb{Z}) = \{(a_n)_{n\in\mathbb{Z}};\ \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty\}$. Then the *bilateral shift*

$$U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), (x_n)_n \mapsto (x_{n+1})_n$$

is well-known to be a bounded linear operator on $\ell^2(\mathbb{Z})$ with $\sigma(U) = \mathbb{T}$ and $\sigma_p(U) = \emptyset$.

This example shows that there are operators with no eigenvalues and that such operators occur quite naturally. This leads to the question: Is having non-empty point spectrum a typical property? It turns out that the set of operators with non-empty point spectrum actually contains an open dense subset of $\mathcal{B}(H)$, an even stronger property than being comeager. Recall that a set is comeager in a Baire space if and only if it contains a dense G_δ-set. The proof follows easily from the following lemma.

Lemma 2.3.2: *Let $T \in \mathcal{B}(H)$ and $\epsilon > 0$. Then there exist $\delta > 0$ and $\tilde{T} \in \mathcal{B}(H)$ such that*

i) $\|T - \tilde{T}\| < \epsilon$,

ii) $B_\delta(\tilde{T}) \subset \{A \in \mathcal{B}(H);\ \sigma_p(A) \neq \emptyset\}$.

Proof:
Let $T \in \mathcal{B}(H)$. Apply Corollary 2.2.4 to T and $\epsilon/2$ to obtain an operator $S \in \mathcal{B}(H)$, a scalar $\lambda \in \mathbb{C}$ and $\delta > 0$ such that:

i) the image of S is closed,

ii) $\dim(\ker(S)) = \dim(\ker(S^*)) = \infty$,

iii) $\|S - (T - \lambda id_H)\| \leq \epsilon/2$.

Since both $\ker(S)$ and $\ker(S^*)$ are separable infinite-dimensional Hilbert spaces, there exists an isometric operator

$$j : \ker(S^*) \to \ker(S)$$

such that $\dim(\ker(S) \ominus \mathrm{Im}(j)) < \infty$ and j is not surjective. Let I denote the extension of j to H by defining $I(x) = 0$ for $x \in \ker(S^*)^\perp = \mathrm{Im}(S)$. Then the operator

$$\tilde{S} = S + \frac{\epsilon}{3} I^*$$

is a Fredholm operator with $\mathrm{index}(\tilde{S}) = c < 0$. This follows from the fact that $\mathrm{Im}(\tilde{S}) = \mathrm{Im}(S) \oplus \mathrm{Im}(I^*) = H$ and $\ker(\tilde{S}) = \ker(j^*) \neq \{0\}$. By Lemma 2.2.1, the set of Fredholm operators with index c is open. Hence, there exists a $\delta > 0$ such that

$$B_\delta(\tilde{S}) \subset \{A \in \mathcal{B}(H);\ \mathrm{ind}(A) < 0\} \subset \{A \in \mathcal{B}(H);\ \sigma_p(A) \neq \emptyset\}.$$

The operator $\tilde{T} = \tilde{S} + \lambda id$ satisfies (i), because

$$\|T - \tilde{T}\| = \|T - \lambda id - \tilde{S}\| \leq \|T - \lambda id - S\| + \|S - \tilde{S}\| < \epsilon,$$

and it satisfies (ii) as well, since for $A \in B_\delta(\tilde{T})$, it holds that $A - \lambda id \in B_\delta(\tilde{S})$, hence $\emptyset \neq \sigma_p(A - \lambda id)$, which implies $\emptyset \neq \sigma_p(A)$. □

With this lemma in hand, it is straightforward to show that having non-empty point spectrum is a typical property.

Theorem 2.3.3: *The set*

$$\{T \in \mathcal{B}(H);\ \sigma_p(T) = \emptyset\}$$

is nowhere dense with respect to the operator norm, i. e., its closure has empty interior. In particular, having non-empty point spectrum is a typical property.

Proof:
For every $T \in \mathcal{B}(H)$ and $n \in \mathbb{N}$, let $\tilde{T}_n, \delta_n$ be the operator and radius from Lemma 2.3.2 with $\|T - \tilde{T}_n\| < 1/n$ and $B_{\delta_n}(\tilde{T}_n) \subset \{A \in \mathcal{B}(H);\ \sigma_p(A) \neq \emptyset\}$. Then, the set

$$\bigcup_{n\in\mathbb{N},\ T\in\mathcal{B}(H)} B_{\delta_n}(\tilde{T}_n)$$

is open and dense in $\mathcal{B}(H)$ with respect to the operator norm and is contained within the set of operators with non-empty point spectrum. This shows that

$$\{T \in \mathcal{B}(H);\ \sigma_p(T) = \emptyset\}$$

is nowhere dense. Since the complement of the closure of this set contains an open dense set, having non-empty point spectrum is typical. □

We can actually replace the open dense subset from the previous proof with a more convenient open dense set, as shown in the following corollary. This result is similar in spirit to a theorem by Bouldin, [12],which characterizes the closure of the invertible operators. It is worth noting that Bouldin also makes use of the polar decomposition, as we do.
This following corollary and theorem are new in the sense that they are not included in the results of [59].

Corollary 2.3.4: *Let $c \in \mathbb{Z} \cup \{-\infty\} \cup \{\infty\}$. Then,*

$$\{T \in \mathcal{B}(H);\ \exists\, \lambda \in \sigma(T) :\ T - \lambda id_H \text{ is semi-Fredholm and } \operatorname{ind}(T - \lambda id_H) = c\}$$

is open and dense in $\mathcal{B}(H)$.

Proof:
The above set is open by Lemma 2.2.1. The density of the above set now follows similarly to the proof of Lemma 2.3.2, with the difference that one chooses the map

$$j : \ker(S^*) \to \ker(S)$$

such that j has closed image, and either $\dim(\ker(j)) < \infty$ or $\dim(\operatorname{Im}(j)^\perp) < \infty$, and $\dim(\ker(j)) - \dim(\operatorname{Im}(j)^\perp)) = c$. □

It is now natural to ask whether the set of operators with non-empty point spectrum is G_δ or even open. The answer to the latter is straightforward: the set is not open, since every neighborhood of the zero operator contains every operator scaled by a non-zero constant.
To see that the set is also not G_δ, we construct, for a given countable family of open dense sets that are contained in the set of operators with non-empty point spectrum, a weighted shift operator which has empty point spectrum but still belongs to the intersection of those dense open sets.

Theorem 2.3.5: *The set*

$$\{T \in \mathcal{B}(H);\ \sigma_p(T) \neq \emptyset\}$$

is not G_δ.

Proof:
Let $X = \{T \in \mathcal{B}(H);\ \sigma_p(T) \neq \emptyset\}$ and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open dense subsets of $\mathcal{B}(H)$ containing X. We aim to show that $X \neq \bigcap_{n\in\mathbb{N}} U_n$. To this end, fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H. Since $0 \in X$, there exists a $\epsilon_0 > 0$ such that $B_{\epsilon_0}(0) \subset U_0$. Define the operator

$$F_1 : H \to H, \sum_{n\in\mathbb{N}} x_n e_n \mapsto \epsilon_0/3x_0 e_1.$$

This is a finite-rank operator and thus $F_1 \in U_1$. Hence, there exists a $\epsilon_0 > \epsilon_1 > 0$ such that $B_{\epsilon_1}(F_1) \subset U_1$. Define

$$F_2 : H \to H, \sum_{n\in\mathbb{N}} x_n e_n \mapsto \epsilon_0/3x_0 e_1 + \epsilon_1/3x_1 e_2.$$

Then $F_2 \in U_2$.
Now assume that for a fixed $1 \leq n \in \mathbb{N}$, we have constructed $\epsilon_0, \dots, \epsilon_n$ and $F_0, F_1, \dots, F_{n+1}$ such that: $F_0 = 0$, $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > 0$, $B_{\epsilon_k}(F_k) \subset U_k$ and

$$F_{k+1}(\sum_{n\in\mathbb{N}} x_n e_n) = \sum_{j=0}^{k} \epsilon_j/3x_j e_{j+1}$$

for $0 \leq k \leq n$. Then F_{n+1} is again a finite-rank operator and lies in U_{n+1}. Thus, there exists $\epsilon_n > \epsilon_{n+1} > 0$ such that $B_{\epsilon_{n+1}}(F_{n+1}) \subset U_{n+1}$. Define

$$F_{n+2}(\sum_{n\in\mathbb{N}} x_n e_n) = \sum_{j=0}^{n+1} \epsilon_j/3 x_j e_{j+1}.$$

By induction, we get a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ such that $\epsilon_n > \epsilon_{n+1} > 0$ and $B_{\epsilon_n}(F_n) \subset U_n$, where

$$F_n(\sum_{n\in\mathbb{N}} x_n e_n) = \sum_{j=1}^{n} \epsilon_j/3 x_j e_{j+1}.$$

Define the operator

$$F(\sum_{n\in\mathbb{N}} x_n e_n) = \sum_{j=0}^{\infty} \epsilon_j/3 x_j e_{j+1}.$$

This operator is well-defined, since the sequence $(\epsilon_n)_n$ is bounded by ϵ_0. Clearly, $F \in B_{\epsilon_n}(F_n) \subset U_n$ for every n, hence $F \in \bigcap_{n\in\mathbb{N}} U_n$.
We claim that $\sigma_p(F) = \emptyset$. Suppose for contradiction, that there exists $\lambda \in \mathbb{C}$ and $0 \neq \sum_{n\in\mathbb{N}} x_n e_n \in H$, such that $Fx = \lambda x$. Then $x_0 = 0$ and $\epsilon_n/3x_n = \lambda x_{n+1}$ for every $n \geq 1$. We deduce recursively that $x_n = 0$ for all n, contradicting $0 \neq x$. □

We conclude the study of operators with non-empty point spectrum with a short discussion of remaining open problems.

Remark 2.3.6: It is not known whether the set of operators with non-empty point spectrum is even a Borel set. From the previous theorem, we only know that it is not a $\prod_2^0$ set. However, one can show that the set is *analytic*, in the sense that it is the continuous image of a closed set, by considering

$$\{(x,T) \in H \times \mathcal{B}(H);\ \|x\| = 1,\ T(x) = 0\} \to \mathcal{B}(H), (x,T) \mapsto T.$$

One should note, however, that in the standard literature, analytic sets are only defined in *Polish spaces*, that is, separable completely metrizable topological spaces. In our case, however, $\mathcal{B}(H)$ is not separable, and thus not Polish.

2.3.2 The Continuous Spectrum

Let H be a Hilbert space and $T \in \mathcal{B}(H)$ a bounded linear operator. The *continuous spectrum* $\sigma_c(T)$ is defined by

$$\sigma_c(T) = \sigma(T) \setminus \{\lambda \in \mathbb{C};\ \lambda \in \sigma_p(T) \text{ or } \bar{\lambda} \in \sigma_p(T^*)\}.$$

It is easy to see that $\lambda \in \sigma_c(T)$ if and only if $\ker(T - \lambda id_H) = \{0\}$ and $\operatorname{Im}(T - \lambda id_H)$ is dense but not closed. This characterization implies that $\sigma_c(T) = \emptyset$ whenever $\dim(H) < \infty$. Thus having empty continuous spectrum is a typical property in the finite-dimensional setting. So as in the previous section, let H be infinite

dimensional and separable.
In fact, we have already encountered an operator with non-empty continuous spectrum in Example 2.3.1.

Example 2.3.7: Let $U \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be the bilateral shift. It is well known that $\sigma_p(U) = \sigma_p(U^*) = \emptyset$. Hence, $\sigma(U) = \sigma_c(U) = \mathbb{T}$.

We are now led to the Question B: is having empty continuous spectrum a typical property? We will answer this question in the negative. Unfortunately, the proof is considerably more intricate than the argument for Question A. We begin with a lemma whose proof is somewhat similar to that of Theorem 2.3.3.

Lemma 2.3.8: *Let $T \in \mathcal{B}(H)$ be a non-semi-Fredholm operator. Then, for every $\epsilon > 0$, there exists $S \in B_\epsilon(T)$ such that*

i) $\sigma_e(S) = \sigma_e(T)$,

ii) $\ker(S) = \ker(S^*) = \{0\}$.

Proof:
The idea is to choose S as a compact perturbation of T, since the essential spectrum is invariant under compact perturbations. Assume there exists a compact operator K_1 with $\|K_1\| \leq \epsilon/2$ and $\dim(\ker(T - K_1)) = \dim(\ker(T^* - K_1^*))$. Since H is separable, there exists an injective, compact operator

$$K_2 : \ker(T - K_1) \to \ker(T^* - K_1^*)$$

with dense image and $\|K_2\| < \epsilon/2$. Extend K_2 to H by defining it to be 0 on $\ker(T - K_1)^\perp$. The operator $S = T - K_1 + K_2$ fulfills the conditions (i), (ii) and $S \in B_\epsilon(T)$. It remains to show that such an operator K_1 exists.
By Lemma 2.2.2, there exists a compact operator $K \in \mathcal{B}(H)$ such that $\dim(\ker(T - K)) = \infty$ and $\|K\| < \epsilon/4$. If $\dim(\ker(T^* - K^*)) = \infty$, we are done. Otherwise, assume that $\dim(\ker(T^* - K^*)) < \infty$. Since $T^* - K^*$ is not semi-Fredholm, its image is not closed. Therefore, we may apply the same theorem to the operator

$$H \to \ker(T - K)^\perp, x \mapsto (T^* - K^*)x.$$

This yields another compact operator $\tilde{K} : H \to \ker(T - K)^\perp$ such that $\dim(\ker(T^* - K^* - \tilde{K})) = \infty$ and $\|\tilde{K}\| < \epsilon/4$.
Consider $\tilde{K}$ as an operator from H to H. Then the operator $K_1 = K + \tilde{K}^* \in \mathcal{B}(H)$ is compact as sum of two compact operators, satisfies $\|K_1\| < \epsilon/2$, and fulfills $\dim(\ker(T - K_1)) = \dim(\ker(T^* - K_1^*)) = \infty$. □

Lemma 2.3.9: *Let $T \in \mathcal{B}(H)$ be not semi-Fredholm with* $\ker(T) = \{0\}$ *and let $T = PA$ be its polar decomposition with spectral measure v. Then*

$$v([\delta, ||T||]) \to_{SOT} id_H \text{ for } \delta \to 0.$$

In particular, for every finite-dimensional subspace $F \subset H$, we have

$$||(v([\delta, ||T||]) - id_H)|_F|| \to 0 \tag{2.1}$$

for $\delta \to 0$.

Proof:
Let T, P, A, ν be as above. We have $v((0, \|T\|]) = id_H$, since $v((0, ||T||])$ is the projection onto $\ker(A)^{\perp}$. Therefore, for each $x \in H$,

$$\lim_{\delta \to 0} \|(id_H - v([\delta, \|T\|]))x\|^2 = \lim_{\delta \to 0} \|v([0, \delta))x\|^2 = \lim_{\delta \to 0} v_{x,x}([0, \delta)) = v_{x,x}(\{0\}) = 0,$$

where $v_{x,x}(\cdot)$ denotes the scalar valued measure $\langle v(\cdot)x, x\rangle$. Hence, $v([\delta, ||T||])$ converges to id_H in the strong operator topology as $\delta \to 0$.
To proof the additional statement, let $F \subset H$ be a non-empty finite-dimensional subspace, and let $(\delta_n)_{n\in\mathbb{N}}$ be a null sequence of positive numbers. Since F is finite-dimensional, for each $n \in \mathbb{N}$ there exists $x_n \in F$ with $\|x_n\| = 1$ such that

$$\|(v(\delta_n, \|T\|) - id_H)|_F\| = \|(v(\delta_n, \|T\|) - id_H)(x_n)\|.$$

Because F is finite-dimensional, the sequence (x_n) has a convergent subsequence with limit $x \in F$ such that

$$\text{limsup}_{n\in\mathbb{N}} \|(v(\delta_n, \|T\|) - id_H)|_F\| = \text{limsup}_{n\in\mathbb{N}} \|(v(\delta_n, \|T\|) - id_H)(x)\|.$$

By the first part of the lemma,

$$\|(v(\delta_n, \|T\|) - id_H)(x)\| \to 0 \text{ for } n \to \infty,$$

which implies that

$$\|(v(\delta_n, \|T\|) - id_H)|_F\| \to 0.$$

This completes the proof. □

A simple application of the triangle inequality shows that a sequence of norm bounded operators that converges pointwise on a dense subspace to an operator already converges pointwise on the entire space. This proves the next lemma.

Lemma 2.3.10: *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of spectral measures and $(r_n)_{n\in\mathbb{N}}$ a sequence in $[0, \infty)$ such that $v_n((0, r_n]) = id_H$. Assume that there exists an increasing sequence of finite-dimensional subspaces $F_n \subset H$ whose union is dense in H, and a null sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive numbers such that*

$$||(v_n([\delta_n, r_n]) - id_H)|_{F_n}|| < 1/n.$$

Then,

$$v_n([\delta_n, r_n])x \to x$$

for every $x \in H$.

Lemma 2.3.11: *Let $\epsilon > 0, T \in \mathcal{B}(H)$ and $\lambda \in \partial\sigma_e(T)$. Then there exists $S \in B_\epsilon(T)$ and $\delta > 0$ such that*

$$B_\epsilon(\lambda) \cap \partial\sigma_e(A) \neq \emptyset$$

for every $A \in B_\delta(S)$.

Proof:
Let $\epsilon > 0$, $T \in \mathcal{B}(H)$, and $\lambda \in \partial\sigma_e(T)$. Since λ lies in the boundary of $\sigma_e(T)$, the operator $T - \lambda id_H$ is not semi-Fredholm by Lemma 2.2.2. Furthermore, there exists $\tilde{\lambda} \in B_\epsilon(\lambda)$ such that $T - \tilde{\lambda} id_H$ is Fredholm. Denote $\text{ind}(T - \tilde{\lambda} id_H) = c$. By Lemma 2.2.1, the set of Fredholm operators with index c is open, so there exists $\tilde{\delta} > 0$ such that

$$\text{ind}(A - \tilde{\lambda} id_H) = c \ \forall \ A \in B_{\tilde{\delta}}(T). \tag{2.2}$$

Without loss of generality, we may assume $\tilde{\delta} < \epsilon$. Applying Lemma 2.2.3 to $T - \lambda id_H$ and $\tilde{\delta}/2$, we obtain an operator $\tilde{S} \in \mathcal{B}(H)$ such that:

i) $\text{Im}(\tilde{S})$ is closed,

ii) $\dim(\ker(\tilde{S})) = \dim(\text{Im}(\tilde{S})^\perp) = \infty$,

iii) $||T - \lambda id_H - \tilde{S}|| \leq \tilde{\delta}/2$.

Let $j : \ker(\tilde{S}) \to \text{Im}(\tilde{S})^\perp$ be an isometric operator such that $\dim(\ker(j)) - \dim(\ker(j^*)) = \tilde{c} \in \mathbb{Z}\backslash\{c\}$, and extend it to H by defining $j(x) = 0$ for $x \in \ker(\tilde{S})^\perp$. Define

$$S = \tilde{S} + \lambda id_H + \tilde{\delta}/3j.$$

Then:

a) $||T - S|| \leq ||T - \lambda id_H - \tilde{S}|| + \tilde{\delta}/3||j|| < \tilde{\delta}$,

b) $S - \lambda id_H$ is Fredholm with $\text{ind}(S - \lambda id_H) = \tilde{c}$,

c) $S - \tilde{\lambda} id_H$ is semi-Fredholm with $\text{ind}(S - \tilde{\lambda} id_H) = c$,

where (c) follows from $S \in B_{\tilde{\delta}}(T)$ and Eq. (2.2). By Lemma 2.2.1, there exists $\delta > 0$ such that for all $A \in B_\delta(S)$,

$$\text{ind}(A - \lambda id_H) = \tilde{c},$$
$$\text{ind}(A - \tilde{\lambda} id_H) = c.$$

It remains to show that $B_\epsilon(\lambda) \cap \partial\sigma_e(A) \neq \emptyset$ for all $A \in B_\delta(S)$. Define

$$t_0 = \min\{t \in [0,1];\ A - \lambda id_H + t(\lambda - \tilde{\lambda}) id_H \text{ is not semi-Fredholm}\},$$

and set $\lambda_0 = \lambda - t_0(\lambda - \tilde{\lambda})$. Note that t_0 exists because $\text{ind}(A - \lambda id_H) \neq \text{ind}(A - \tilde{\lambda} id_h)$. By definition, $A - \lambda_0 id_H$ is not semi-Fredholm, hence $\lambda_0 \in \sigma_e(A)$.

Since every neighborhood of λ_0 contains a point λ_1 for which $A-\lambda_1 id_H$ is Fredholm, we conclude $\lambda_0 \in \partial\sigma_e(A)$. Finally, we observe that

$$|\lambda - \lambda_0| \leq |\lambda - \tilde{\lambda}| < \epsilon,$$

which implies $\lambda_0 \in B_\epsilon(\lambda)$. □

We are now ready to answer Question B.

Theorem 2.3.12: *The set*

$$\{T \in \mathcal{B}(H); \sigma_c(T) \neq \emptyset\}$$

is comeager with respect to the operator norm.

Proof:
Recall that we aim to use the Banach-Mazur game. By Remark 2.1.7, we may assume that Player I chooses the set $B_{\epsilon_1}(T_1) \subset \mathcal{B}(H)$. Let $\lambda_1 \in \partial\sigma_e(T_1)$. Then, by Lemma 2.3.8, we may assume that

$$\ker(T_1 - \lambda_1 id_H) = \ker(T_1^* - \bar{\lambda}_1 id_H) = \{0\}.$$

Let $T_1 - \lambda_1 id_H = P_1 A_1$ and $T_1^* - \bar{\lambda}_1 id_H = Q_1 B_1$ be the polar decompositions and ν_1, μ_1 be the spectral measures corresponding toA_1 and B_1. By Lemma 2.3.9, there exists a $1 > \tilde{\delta_1} > 0$ such that

$$||(\nu_1([\tilde{\delta_1}, ||A_1||]) - id_H)|_{F_1}|| < 1,$$
$$||(\mu_1([\tilde{\delta_1}, ||B_1||]) - id_H)|_{F_1}|| < 1.$$

Moreover, by Lemma 2.3.11, there exists $\delta_1 > 0$ and $S_1 \in B_{\tilde{\delta}_1/4}(T_1)$ such that

$$B_{\tilde{\delta}_1/4}(\lambda_1) \cap \partial\sigma_e(A) \neq \emptyset \text{ for all } A \in B_{\delta_1}(S_1).$$

Without loss of generality, assume $\delta_1 < 1$. Now Player II plays the set $B_{\delta_1}(S_1)$. Let $n > 1$. We construct the strategy of Player II inductively. Assume the sets $B_{\epsilon_i}(T_i)$ and $B_{\delta_i}(S_i), i = 1, \ldots, n-1$ have been played. Set $\lambda_0 = 0$ and $\tilde{\delta_0} = \infty$. The induction hypothesis is as follows:
There exists $\lambda_i \in \partial\sigma_e(T_i) \cap B_{\tilde{\delta}_{i-1}/4}(\lambda_{i-1})$ with polar decompositions

$$T_i - \lambda_i id_H = P_i A_i, \quad T_i^* - \bar{\lambda}_i id_H = Q_i B_i$$

and corresponding spectral measures ν_i, μ_i such that

$$||(\nu_i([\tilde{\delta_i}, ||A_i||]) - id_H)|_{F_i}|| < 1/i,$$
$$||(\mu_i([\tilde{\delta_i}, ||B_i||]) - id_H)|_{F_i}|| < 1/i$$

for some $\tilde{\delta}_i > 0$ with $1/i > \tilde{\delta}_i$ and

$$B_{\tilde{\delta}_i/4}(\lambda_i) \subset B_{\tilde{\delta}_{i-1}/4}(\lambda_{i-1}).$$

Also, $\delta_i < \min\{1/i, \tilde{\delta}_i/4\}$ and

$$B_{\tilde{\delta}_i/4}(\lambda_i) \cap \partial\sigma_e(A) \neq \emptyset \text{ for all } A \in B_{\delta_i}(S_i)$$

Now assume Player I plays $B_{\epsilon_n}(T_n)$. By the construction, there exists $\lambda_n \in \partial\sigma_e(T_n) \cap B_{\tilde{\delta}_{n-1}/4}(\lambda_{n-1})$, and by Lemma 2.3.8, we can assume that

$$\ker(T_n - \lambda_n) = \ker(T_n^* - \bar{\lambda}_n) = \{0\}.$$

Let $T_n - \lambda_n id_H = P_n A_n$ and $T_n^* - \bar{\lambda}_n id_H = Q_n B_n$ be the polar decompositions and denote by v_n and μ_n the corresponding spectral measures. Then, by Lemma 2.3.9, there exits $0 < \tilde{\delta}_n < 1/n$ such that

$$||(v_n([\tilde{\delta}_n, ||A_n||]) - id_H)|_{F_n}|| < 1/n,$$
$$||(\mu_n([\tilde{\delta}_n, ||B_n||]) - id_H)|_{F_n}|| < 1/n.$$

Without loss of generality, we may assume that $\tilde{\delta}_n$ is so small that

$$B_{\tilde{\delta}_n/4}(\lambda_n) \subset B_{\tilde{\delta}_{n-1}/4}(\lambda_{n-1}).$$

By Lemma 2.3.11, there exists $\delta_n > 0$ and $S_n \in B_{\tilde{\delta}_n/4}(T_n)$ such that

$$B_{\tilde{\delta}_n/4}(\lambda_n) \cap \partial\sigma_e(A) \neq \emptyset \text{ for all } A \in B_{\delta_n}(S_n)$$

Without loss of generality, we can assume $\delta_n < \min\{\tilde{\delta}_n/4, 1/n\}$.
Now, Player II plays $B_{\delta_n}(S_n)$.
It remains to show that any $T \in \bigcap_{n\in\mathbb{N}} B_{\delta_n}(S_n) = \bigcap_{n\in\mathbb{N}} B_{\epsilon_n}(T_n)$ satisfies $\sigma_c(T) \neq \emptyset$.
Let $\lambda_n, \tilde{\delta}_n, A_n$ and v_n be as above. Since

$$B_{\tilde{\delta}_i/4}(\lambda_i) \subset B_{\tilde{\delta}_{i-1}/4}(\lambda_{i-1}) \text{ for every } i \in \mathbb{N},$$

the sequence $(\lambda_n)_{n\in\mathbb{N}}$ converges to some $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_n| \leq \tilde{\delta}_n/4$.
On the other hand, $T_n \to T$ because $||T_n - T|| < \epsilon_n \leq \delta_{n-1} < 1/(n-1)$. Hence, $T - \lambda id_H$ is the limit of non-semi-Fredholm operators and thus itself not semi-Fredholm.
It remains to show that $\ker(T - \lambda id_H) = \ker(T^* - \bar{\lambda} id_H) = \{0\}$. Let $x \in \ker(T - \lambda id_H)$ and define $x_n = v_n([\tilde{\delta}_n, ||A_n||])x$. Then

$$\begin{aligned}
\tilde{\delta}_n(||x_n|| - ||x - x_n||) &\leq ||(T_n - \lambda_n id_H)x|||\\
&= ||(T_n - S_n)x + (S_n - T)x - (\lambda_n - \lambda)x||\\
&\leq \delta_n||x|| + \delta_n||x|| + \tilde{\delta}_n/4||x||\\
&\leq \tilde{\delta}_n/4||x|| + \tilde{\delta}_n/4||x|| + \tilde{\delta}_n/4||x||\\
&= 3/4\tilde{\delta}_n||x||.
\end{aligned}$$

Here, the first inequality uses $||(T_n - \lambda_n id_H)x_n|| \geq \tilde{\delta}_n||x_n||$ and $||(T_n - \lambda_n id_H)(x - x_n)|| \leq \tilde{\delta}_n||x - x_n||$. By Lemma 2.3.8, $x_n \to x$, so the inequality above implies $x = 0$.
A similar argument applies to $x \in \ker(T^*)$, using $x_n = \mu_n([\tilde{\delta}_n, \|A_n\|])x$, and shows that

$$\ker(T - \lambda id_H) = \ker(T^* - \bar{\lambda} id_H) = \{0\}.$$

Thus, $\lambda \in \sigma_c(T)$. □

We can now combine Theorem 2.3.3 and Theorem 2.3.12 to conclude that having both non-empty point and non-empty continuous spectrum is a typical property.

Corollary 2.3.13: *A typical operator has non-empty point and non-empty continuous spectrum.*

Proof:
By Theorem 2.1.2, the intersection of two dense G_δ-sets is again a dense G_δ-set. Thus the intersection of two comeager sets is again comeager. The corollary now follows from Theorem 2.3.3 and Theorem 2.3.12. □

As in the case of the set of operators with non-empty point spectrum, it is not known whether the set of operators with non-empty continuous spectrum is Borel. It is also unclear whether this set is a G_δ-set. A technique similar to the proof of Theorem 2.3.5 appears conceivable, however, one must overcome the problem that finite-rank operators have empty continuous spectrum, and that the point spectrum of the adjoint a weight shift strongly depends on the choice of weights (see [35, Problem 93]).

2.4 Density of Operators with Empty Continuous Spectrum

The previous chapter raises the question of how common operators with empty continuous spectrum actually are. This, of course, can be is related to the question: Is the set of operators with empty continuous spectrum dense in $\mathcal{B}(H)$. In Theorem 2.4.7, we answer this question in the affirmative. To demonstrate this, we will invoke the restricted Similarity Orbit Theorem from [3]. We begin by introducing the basic definitions needed to understand this theorem.

For an operator $T \in \mathcal{B}(H)$, define the *similarity orbit* of T by

$$\mathrm{Sim}(T) = \{S^{-1}TS;\ S \in \mathcal{B}(H) \text{ invertible}\}.$$

In [38], the closure of $\mathrm{Sim}(T)$ is characterized in terms of properties of the spectrum of T. The following three types of points in the spectrum are preserved under

similarity and thus will play an important role in the proof of Theorem 2.4.7.
The *normal spectrum* $\sigma_0(T)$ consists of the isolated points in $\sigma(T)$ which are not in the essential spectrum $\sigma_e(T)$.
The set of points $\lambda \in \mathbb{C}$ such that $T - \lambda id_H$ is semi-Fredholm is denoted by $\rho_{sF}(T)$. It follows immediately from Lemma 2.2.1 that this set is open.
The third type of spectral points we consider are the isolated point in $\sigma_e(T)$. We do not introduce a separate notation for them, but we characterize them further: Let $\lambda \in \sigma_e(T)$ be isolated, and let Φ be a unital faithful *-representation of the Calkin algebra $\mathcal{C}(H)$. Let $f(z)$ be a holomorphic function that equals $z - \lambda$ in a neighborhood of λ, and is zero in a neighborhood of $\sigma_e(T) \setminus \{\lambda\}$. Applying the holomorphic functional calculus to $\Phi(T + \mathcal{K}(H))$, we obtain the quasinilpotent element

$$f(\Phi(T + \mathcal{K}(H))) = Q_\lambda.$$

For $\lambda \in \mathbb{C}$, we define

$$k(\lambda; T) = \begin{cases} 0 & \text{if } \lambda \notin \sigma_e(T), \\ n & \text{if } \lambda \text{ is isolated in } \sigma_e(T) \text{ and } Q_\lambda \text{ is nilpotent of order } n, \\ \infty & \text{otherwise.} \end{cases}$$

In the following remark, we show that $k(\lambda; T)$ is independent of the chosen representation Φ.

Remark 2.4.1: For convenience, we write $[T]$ for $T + \mathcal{K}(H)$. Let Φ_1, Φ_2 be two unital faithful *-representations of the Calkin algebra $\mathcal{C}(H)$, and $k_i(\cdot; T)$ denote the corresponding functions as defined above. Since for $i = 1, 2$, we have

$$\sigma_e(T) = \sigma([T]) = \sigma(\Phi_i([T])),$$

it follows that $k_1(\lambda; T) = 0$ if and only if $k_2(\lambda; T) = 0$. It remains to show that $k_1(\lambda; T) = n$ if and only if $k_2(\lambda; T) = n$ for $1 \leq n$.
Let λ be isolated in $\sigma_e(T)$, and f be as above. Suppose $f(\Phi_1([T]))$ is nilpotent of order n. Let U be an open neighborhood of $\sigma_e(T)$. Then, for $i = 1, 2$,

$$\mathcal{O}(U) \to \mathcal{C}(H), g \mapsto \Phi_i(g([T]))$$

extends the polynomial functional calculus, is an algebra homomorphism, and preserves uniform convergence. By the uniqueness of the holomorphic functional calculus, this map coincide with the holomorphic functional calculus. In particular,

$$f(\Phi_i([T])) = \Phi_i(f([T])).$$

Thus, $f([T])$ is nilpotent of order n since Φ_1 is faithful, and hence $k_2(\lambda; T) = n$. Reversing the roles of Φ_1 and Φ_2 completes the proof.

The reader might rightly ask why we defined $k(\cdot;T)$ via a unital faithful $*$-representation of $\mathcal{C}(H)$ rather than directly through $f([T])$. The reason for this choice becomes clear in the proof of the next lemma.

Lemma 2.4.2: *Let $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(\tilde{H})$ be operators on Hilbert spaces H and $\tilde{H}$. If $\lambda \in \sigma_e(A \oplus B)$ is isolated, then*

$$k(\lambda; A \oplus B) = \max(k(\lambda; A), k(\lambda; B)).$$

Proof:

Let $H, \tilde{H}, A, B$ be as above and $\lambda \in \sigma_e(A \oplus B)$ be isolated. For clarity, denote the equivalence class of an operator T in the Calkin algebra by $[T]$. Define the projection P by

$$P : H \oplus \tilde{H} \to H \oplus \tilde{H}, x \oplus y \mapsto x \oplus 0.$$

Let $\Phi : \mathcal{B}(H \oplus \tilde{H})/\mathcal{K}(H \oplus \tilde{H}) \to \mathcal{B}(K)$ be a unital faithful $*$-representation. Then $\Phi([P])$ is a projection, since Φ is a *-homomorphism. Denote this projection by P_1, and let $H_1 = P_1(K)$. Set $P_2 = id_K - P_1$ and $H_2 = P_2(K)$, yielding the decomposition of $K = H_1 \oplus H_2$.

Since P_1 commutes with $\Phi([A \oplus 0])$ and P_2 with $\Phi([0 \oplus B])$, the subspaces H_1 and H_2 reduce $\Phi([A \oplus B])$. Denote by A_0 the restriction $\Phi([A \oplus B])|_{H_1}$, and B_0 the restriction to H_2.

Now define

$$\Phi_1 : \mathcal{B}(H)/\mathcal{K}(H) \to \mathcal{B}(H_1), [T] \mapsto \Phi([T \oplus 0])|_{H_1}.$$

This map is well defined since P_1 commutes with $\Phi([T \oplus 0])$ for all $T \in \mathcal{B}(H)$, and it is again a unital faithful $*$-representation. Thus $\Phi_1([A]) = A_0$, and $\sigma([A]) = \sigma(A_0)$. Analogously, for

$$\Phi_2 : \mathcal{B}(H)/\mathcal{K}(H) \to \mathcal{B}(H_2), [T] \mapsto \Phi([0 \oplus T])|_{H_2},$$

we have $\Phi_2([B]) = B_0$ and $\sigma([B]) = \sigma(B_0)$.

Let $f(z)$ be a function that is equal to $z - \lambda$ on a neighborhood of λ, and zero in a neighborhood of $\sigma_e(A \oplus B) \setminus \{\lambda\}$. Since $\sigma_e(A \oplus B) = \sigma_e(A) \cup \sigma_e(B)$, we can apply f to obtain

$$f(\Phi([A \oplus B])) = f(A_0 \oplus B_0) = f(A_0) \oplus f(B_0).$$

Because Φ_1 and Φ_2 are unital faithful $*$-representations, and $k(\lambda; \cdot)$ is independent of the representation by Remark 2.4.1, we conclude: $f(A_0)$ is nilpotent of order n if and only if $k(\lambda; A) = n$, and likewise $f(B_0)$ is nilpotent of order n if and only if $k(\lambda; B) = n$.

Hence, $f(\Phi([A \oplus B]))$ is nilpotent of order n if and only if

$$n = \max(k(\lambda; A), k(\lambda; B)),$$

and it is not nilpotent if and only if either $k(\lambda; A) = \infty$ or $k(\lambda; B) = \infty$. This proves the lemma. $\square$

Let $A \in \mathcal{B}(H)$. We say that A satisfies property $(S), (F)$ or (A) with respect to T if:

(S) $\sigma_0(A) \subset \sigma_0(T)$ and every component of $\mathbb{C} \setminus \rho_{sF}(A)$ intersects $\sigma_e(T)$,

(F) $\rho_{sF}(A) \subset \rho_{sF}(T)$, $\operatorname{ind}(\lambda id_H - A) = \operatorname{ind}(\lambda id_H - T)$, and

$$\min \operatorname{ind}(\lambda id_H - T)^k \leq \min \operatorname{ind}(\lambda id_H - A)^k$$

for all $\lambda \in \rho_{sF}(T)$ and $k \geq 1$,

(A) $\dim(\ker(A - \lambda id_H)) = \dim(\ker(T - \lambda id_H))$ for all $\lambda \in \sigma_0(A)$.

Here, $\min \operatorname{ind}(x)$ denotes

$$\min\{\dim(\ker(x)), \dim(\ker(x^*))\}.$$

We are ready to state the restricted Similarity Orbit Theorem [38, Corollary 1.6].

Theorem 2.4.3 (restricted Similarity Orbit Theorem): *Let $T \in \mathcal{B}(H)$ and suppose $k(\lambda; T) = \infty$ for every isolated point $\lambda \in \sigma_e(T)$. Then*

$$\overline{\operatorname{Sim}(T)} = \{X \in \mathcal{B}(H); X \text{ satisfies property } (S), (F) \text{ and } (A) \text{ with respect to } T\}.$$

Now that the reader is familiar with the Similarity Orbit Theorem, we can outline our strategy for proving Theorem 2.4.7. Given an arbitrary operator $X \in \mathcal{B}(H)$, we will construct an operator T such that $\sigma_c(T) = \emptyset$ and $X \in \overline{\operatorname{Sim}(T)}$. According to the theorem above, it suffices to verify that X satisfies property $(S), (F)$ and (A) with respect to T. The operator T will be unitarily equivalent to a direct sum of three operators. One of them being X itself, and the others arising from the spectral properties of X. The first one is given in the next example, where we later replace the sequence $(\lambda_n)_{n\in\mathbb{N}}$ with the isolated points of $\sigma_e(X)$.

Example 2.4.4: Let $T \in \mathcal{B}(H)$ be a quasinilpotent operator such that $k(0; T_0) = \infty$, and let $a = (\lambda_n)_{n\in\mathbb{N}}$ be a bounded sequence of isolated points in $\mathbb{C}$. Then the operator

$$T_a : \bigoplus_{n\in\mathbb{N}} H \to \bigoplus_{n\in\mathbb{N}} H, (x_n)_n \mapsto ((\lambda_n id_H - T_0)(x_n))_n$$

is bounded, since the sequence a is bounded. Furthermore, if $\lambda \notin \overline{\{\lambda_n,\ n \in \mathbb{N}\}}$, then, by continuity of the inverse map, we have

$$\sup_{n\in\mathbb{N}} \|((\lambda - \lambda_n)id_H - T_0)^{-1}\| < \infty,$$

and hence $\lambda \notin \sigma(T_a)$. We conclude that

$$\overline{\{\lambda_n;\ n \in \mathbb{N}\}} = \sigma(T_a) = \sigma_e(T_a),$$

and each λ_n is an isolated point in $\sigma_e(T_a)$. Fix $m \in \mathbb{N}$. We can decompose T_a as a direct sum of $\lambda_m id_H - T_0$ and

$$\bigoplus_{n \in \mathbb{N}\setminus\{m\}} H \to \bigoplus_{n \in \mathbb{N}\setminus\{m\}} H, (x_n)_n \mapsto ((\lambda_n id_H - T_0)(x_n))_n.$$

By assumption, $k(\lambda_m; \lambda_m id_H - T_0) = \infty$ and since λ_m is isolated in $\sigma_e(T_a)$, we obtain $k(\lambda_m; T_a) = \infty$ by Lemma 2.4.2. Since $m \in \mathbb{N}$ was arbitrary, it follows that $k(\lambda_n; T_a) = \infty$ for all $n \in \mathbb{N}$.

The second operator will ensure that $\sigma_c(T) = \emptyset$. For this, we need an operator T_p such that its spectrum and point spectrum coincide with the spectrum of T: $\sigma(T) = \sigma_p(T_p) = \sigma(T_p)$. This would not be a problem if T_p could be chosen on a Hilbert space of arbitrary cardinality, but for separable Hilbert spaces, it becomes a nontrivial problem. Fortunately, this was already shown by Dixmier and Foiaş [24]. The general theorem states that for every non-empty compact set $K \subset \mathbb{C}$ and every $F \subset K$ that is F_σ, there exists an operator T_p on a separable Hilbert space such that $K = \sigma(T_p)$ and $F = \sigma_p(T_p)$. In our case, we only need this for $K = F$, and since the original source [24] is difficult to access (and in French), we provide their proof for this special case.

Theorem 2.4.5: *Let $K \subset \mathbb{C}$ be a non-empty compact set. Then there exists an operator T on a separable Hilbert space such that*

$$\sigma(T) = \sigma_p(T) = K.$$

Proof:
Let $\emptyset \neq K \subset \mathbb{C}$ be compact, and let $U \subset \mathbb{C}$ be an open disc containing K. Recall that $W^{2,2}(U)$ denotes the Sobolev space. Let M_z be the multiplication operator by the identity. By Theorem 2.2.8, this defines a bounded operator with a spectral distribution, and it is straightforward to verify that M_z^* also has a spectral distribution.
By Lemma 2.2.7 and the Riesz representation theorem, for every $\lambda \in U$, there exists a function k_λ such that

$$\langle [u], k_\lambda \rangle = f_u(\lambda)$$

for all $[u] \in W^{2,2}(U)$, where f_u is the unique continuous representative of the equivalence class $[u]$. Moreover,

$$\langle M_z[u], k_\lambda \rangle = \lambda f_u(\lambda) = \langle [u], \lambda k_\lambda \rangle = \langle [u], M_z^*(k_\lambda) \rangle$$

for all $[u] \in W^{2,2}(U)$ and $\lambda \in U$. Thus, $M_z^*(k_\lambda) = \lambda k_\lambda$, which implies $U \subset \sigma_p(M_z^*)$. Let $H = H(M_z^*, K)$ (see Lemma 2.2.6 for the definition) and define $T = M_z^*|_H$. Then, by Lemma 2.2.6, we have $\sigma(T) \subset K$ and $k_\lambda \in H$ for all $\lambda \in K$. Hence, $\sigma_p(T) = \sigma(T) = K$. □

Lemma 2.4.6: *Let $\tilde{H}$ be a Hilbert space and $A \in \mathcal{B}(H), B \in \mathcal{B}(\tilde{H})$. Assume that $\sigma(B) = \sigma_p(B) = \mathbb{C} \setminus \rho_{sF}(A)$. Then:*

i) $\sigma(A \oplus B) = \sigma(A)$,

ii) $\sigma_c(A \oplus B) = \emptyset$,

iii) $\sigma_e(A \oplus B) = \sigma_e(A)$,

iv) $\sigma_0(A \oplus B) = \sigma_0(A)$ *and* $\dim(\ker(A \oplus B - \lambda id_{H\oplus\tilde{H}})) = \dim(\ker(A - \lambda id_H))$ *for all* $\lambda \in \sigma_0(A)$,

v) $\rho_{sF}(A \oplus B) = \rho_{sF}(A)$,

vi) $\operatorname{ind}(A \oplus B - \lambda id_{H\oplus\tilde{H}}) = \operatorname{ind}(A - \lambda id_H)$ *for all* $\lambda \in \rho_{sF}(A \oplus B)$,

vii) $\min\operatorname{ind}(A \oplus B - \lambda id_{H\oplus\tilde{H}})^k = \min\operatorname{ind}(A - \lambda id_H)^k$ *for all* $\lambda \in \rho_{sF}(A \oplus B)$ *and* $k \geq 1$.

If U is a linear, invertible operator from $H \oplus \tilde{H} \to H$, then A has property $(S), (A)$, and (F) with respect to $U(A \oplus B)U^{-1}$.

Proof:
Let $A, B, \tilde{H}$ and U be as above.
(i) follows from $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$ and $\sigma(B) = \mathbb{C} \setminus \rho_{sF}(A) \subset \sigma(A)$.
Let $\lambda \in \sigma_c(A \oplus B)$. Then λ lies in $\sigma(A)$ but not in $\rho_{sF}(A)$, hence $\lambda \in \sigma(B)$. However, $\sigma(B) = \sigma_p(B) \subset \sigma_p(A \oplus B)$, which contradicts $\lambda \in \sigma_c(A \oplus B)$. Thus, (ii) holds.
For (iii), observe that $\sigma_e(A \oplus B) = \sigma_e(A) \cup \sigma_e(B)$ and $\sigma_e(B) \subset \sigma(B) \subset \sigma_e(A)$.
From (i) and (iii), we deduce that $\sigma_0(A \oplus B) = \sigma_0(A)$. Furthermore, we have:

$$\dim(\ker(A \oplus B - \lambda id_{H\oplus\tilde{H}})) = \dim(\ker(A - \lambda id_H)) + \dim(\ker(B - \lambda id_{\tilde{H}})) \quad (2.3)$$

for all $\lambda \in \mathbb{C}$. Since $\dim(\ker(B - \lambda id_{\tilde{H}})) = 0$ for all $\lambda \in \sigma_0(A) \subset \mathbb{C} \setminus \sigma(B)$, we obtain (iv).
Assertions (v) and (vi) follow essentially from $\sigma(B) \subset \mathbb{C} \setminus \rho_{sF}(A)$ and $\operatorname{ind}(B - \lambda id_{\tilde{H}}) = 0$ for all $\lambda \in \rho_{sF}(A)$. More precisely, $\operatorname{Im}(A \oplus B - \lambda id_{H\oplus\tilde{H}})$ is closed if and only if the image of $A - \lambda id_H$ and $B - \lambda id_{\tilde{H}}$ is closed. Using Eq. (2.3), if $A - \lambda id_H$ is semi-Fredholm and $B - \lambda id_{\tilde{H}}$ is Fredholm, then $A \oplus B - \lambda id_{H\oplus\tilde{H}}$ is semi-Fredholm and satisfies:

$$\operatorname{ind}(A \oplus B - \lambda id_{H\oplus\tilde{H}}) = \operatorname{ind}(A - \lambda id_H) + \operatorname{ind}(B - \lambda id_{\tilde{H}})$$

for all $\lambda \in \rho_{sF}(A \oplus B)$. Since $B - \lambda id_{\tilde{H}}$ is invertible for all $\lambda \in \rho_{sF}(A)$, it follows that $\text{ind}(B - \lambda id_{\tilde{H}}) = 0$ and $\rho_{sF}(A) \subset \rho_{sF}(A \oplus B)$. Conversly, if $A - \lambda id_H$ is not semi-Fredholm, then neither is $A \oplus B - \lambda id_{H\oplus\tilde{H}}$. This proves the reverse inclusion $\rho_{sF}(A \oplus B) \subset \rho_{sF}(A)$.
Part (vii) follows from

$$\ker(A \oplus B - \lambda id_{H\oplus\tilde{H}}) = \ker(A - \lambda id_H) \oplus \ker(B - \lambda id_{\tilde{H}}),$$
$$(A \oplus B - \lambda id_{H\oplus\tilde{H}})^k = (A - \lambda id_H)^k \oplus (B - \lambda id_{\tilde{H}})^k$$

and from $\ker((B - \lambda id_{\tilde{H}})^*)^k = \ker(B - \lambda id_{\tilde{H}})^k = \{0\}$ for all $\lambda \in \rho_{sF}(A) = \rho_{sF}(A \oplus B)$.
The final statement follows from $(i), \ldots, (viii)$ and the observation that each of the sets

$$\sigma(\cdot),\ \sigma_e(\cdot),\ \sigma_c(\cdot),\ \rho_{sF}(\cdot),\ \sigma_0(\cdot)$$

and each of the numbers

$$\dim(\ker(\cdot)),\ \text{ind}(\cdot),\ \min\text{ind}(\cdot)$$

are invariant under similarity. □

Theorem 2.4.7: *Let H be a separable Hilbert space. Then*

$$\overline{\{T \in \mathcal{B}(H);\ \sigma_c(T) = \emptyset\}} = \mathcal{B}(H)$$

with respect to the norm topology.

Proof:
We already know that the theorem is true for $\dim(H) < \infty$, so we may assume $\dim(H) = \infty$. Let $A \in \mathcal{B}(H)$ and $\{\lambda_n; n \in \mathbb{N}\}$ be the set of isolated points in $\sigma_e(A)$. If the set is finite, repeat one λ infinitely often. Let T_a be the operator from Example 2.4.4 with respect to the sequence $(\lambda_n)_{n\in\mathbb{N}}$. Recall that $\sigma(T_a) = \overline{\{\lambda_n;\ n \in \mathbb{N}\}}$ and $k(\lambda_n; T_a) = \infty$ for all $n \in \mathbb{N}$.
By Theorem 2.4.5 and since the set $\mathbb{C} \setminus \rho_{sF}(A)$ is compact and non-empty, there exists an operator $T_p \in \mathcal{B}(H)$ such that

$$\sigma(T_p) = \sigma_p(T_p) = \mathbb{C} \setminus \rho_{sF}(A).$$

Define the operators $B = T_p \oplus T_a \in \mathcal{B}(H \oplus \bigoplus_{n\in\mathbb{N}} H)$ and $\tilde{T} = A \oplus B$. Fix a linear, invertible operator $U : H \oplus H \oplus \bigoplus_{n\in\mathbb{N}} H \to H$ and define $T = U\tilde{T}U^{-1}$.
The set

$$\{\lambda \in \rho_{sF}(A);\ \text{ind}(A - \lambda id_H) = \infty \text{ or } \text{ind}(A - \lambda id_H) = -\infty\}$$

is open by Lemma 2.2.1 and equal to $\sigma_e(A) \cap \rho_{sF}(A)$. Hence no isolated point of $\sigma_e(A)$ lies in $\rho_{sF}(A)$, so $\{\lambda_n,\ n \in \mathbb{N}\} \subset \mathbb{C} \setminus \rho_{sF}(A)$. Moreover, we have

$\sigma(B) = \sigma(T_a) \cup \sigma(T_p) = \mathbb{C} \setminus \rho_{sF}(A)$ and $\sigma_p(B) = \sigma_p(T_a) \cup \sigma_p(T_p) = \sigma(B)$.
Thus, we may apply Lemma 2.4.6 to conclude that A has property $(A), (S)$ and (F) with respect to T. Furthermore, by Lemma 2.4.2 and the invariance of $k(\lambda; \cdot)$ under similarity, we have $k(\lambda_n; T) = k(\lambda_n; T_a) = \infty$ for all $n \in \mathbb{N}$. Hence the requirements for the restricted Similarity Orbit Theorem Theorem 2.4.3 are fulfilled, and we obtain $A \in \overline{\mathrm{Sim}(T)}$. Since T has empty continuous spectrum, it follows that

$$A \in \overline{\mathrm{Sim}(T)} \subset \overline{\{X \in \mathcal{B}(H);\ \sigma_c(X) = \emptyset\}}.$$

□

The idea of constructing an operator with certain spectral properties via direct sums is well known and appears, for example, in the proof of [62, Theorem 2]. We conclude this chapter with a theorem that characterises the closure of all operators whose spectrum coincides with their point spectrum. The proof is a slight modification of that of Theorem 2.4.7.

Theorem 2.4.8: *Let H be a separable Hilbert space. We have*

$$\overline{\{T \in \mathcal{B}(H);\ \sigma_p(T) = \sigma(T)\}} = \left\{ T \in \mathcal{B}(H);\ \begin{array}{c} \dim(\ker(T - \lambda id_H)) \neq 0 \\ \text{for all } \lambda \in \rho_{sF}(T) \cap \sigma(T) \end{array} \right\}.$$

Proof:
The statement is clear if H is a finite-dimensional Hilbert space, so assume that H is a separable infinite dimensional Hilbert space. Let $T \in \mathcal{B}(H)$ and suppose that there exists $\lambda \in \rho_{sF}(T) \cap \sigma(T)$ such that $\dim(\ker(T - \lambda id_H)) = 0$. Since the operator $T - \lambda id_H$ is semi-Fredholm with trivial kernel, it is bounded below by some constant $c > 0$. By Lemma 2.2.1, there exists $\epsilon < c/2$ such that $\mathrm{ind}(S) = \mathrm{ind}(T - \lambda id_H) \neq 0$ for all $S \in B_\epsilon(T - \lambda id_H)$. Every operator $S \in B_\epsilon(T)$ then satisfies that $S - \lambda id_H$ is bounded below and $\lambda \in \sigma(S)$. Thus,

$$B_\epsilon(T) \subset \mathcal{B}(H) \setminus \{X \in \mathcal{B}(H);\ \sigma_p(X) = \sigma(X)\},$$

and in particular, $T \notin \overline{\{X \in \mathcal{B}(H);\ \sigma_p(X) = \sigma(X)\}}$.
Conversely, if $A \in \mathcal{B}(H)$ and $\dim(\ker(A - \lambda id_H)) \neq 0$ for all $\lambda \in \rho_{sF}(A) \cap \sigma(A)$, we only need to verify that the operator T from the proof of Theorem 2.4.7 lies in $\{X \in \mathcal{B}(H);\ \sigma_p(X) = \sigma(X)\}$. By construction, we have

$$\mathbb{C} \setminus \rho_{sF}(T) \subset \sigma_p(T),\ \sigma(T) = \sigma(A),\ \rho_{sF}(T) = \rho_{sF}(A)$$

and by the assumption, $\rho_{sF}(A) \cap \sigma(A) \subset \sigma_p(A) \subset \sigma_p(T)$. Therefore, $\sigma_p(T) = \sigma(T)$ and the proof is complete. □

Chapter 3

The Lattice of C^*-covers

We begin this chapter with the basic definitions and well known results of non-commutative Choquet theory. Most of these definitions will also be relevant for the remaining chapters, and many of the well known results will be used later without explicit mentioning.
The new results presented in this chapter are a joint work with Adam Humeniuk and Christopher Ramsey and provide new insights into the structure of the C^*-cover lattice of operator algebras. The first new result, Theorem 3.2.4, answers [41, Question 3.1] by Ramsey and Humeniuk, who asked whether there exists a non-selfadjoint operator algebra with a unique C^*-cover. Here, "unique" means that all C^*-covers are equivalent.
Motivated by this example, we study the lattice of operator algebras with more than one C^*-cover, which leads to the surprising result Theorem 3.2.7:

Theorem 3.0.1: *Let A be an operator algebra that has more than one C^*-cover. Then A has uncountably many C^*-covers.*

For the proof, we invoke a lemma by Katznelson that has largely fallen into obscurity. This technique also allows us to study the C^*-cover lattice of Dirichlet operator algebras and to answer [40, Question 3.13]: When is the maximal semi-Dirichlet C^*-cover equal to the maximal C^*-cover? This is resolved by Theorem 3.3.1:

Theorem 3.0.2: *If A is a semi-Dirichlet operator algebra, then its maximal semi-Dirichlet C^*-cover of A is equal to the maximal C^*-cover of A if and only if A is selfadjoint.*

We conclude this chapter with an example of a residual finite-dimensional operator algebra for which the collection of residual finite-dimensional C^*-covers does not form a lattice.

3.1 Preliminaries

This section introduces the definitions of operator spaces and operator systems and everything that we need related to this topic. Most of the definitions and several well-known results can be found in [55]. However, we will also cover topics like the existence of maximal dilations, the equivalence between the unique extension property and maximality, Sarason's lemma and more. Ergo incipiamus.

3.1.1 Operator Spaces and Operator Systems

An *operator space* is a subspace of a C^*-algebra that contains 1. If the space is also closed under the adjoint operation, it is called an *operator system*.
Let M be an operator space in a C^*-algebra $\mathcal{A}$ and $\phi : M \to \mathcal{B}$ a linear map into another C^*-algebra $\mathcal{B}$. The map ϕ is called *contractive* if

$$\|\phi(a)\| \leq 1$$

for all $a \in \mathcal{A}$ with $\|a\| \leq 1$. For $1 \leq n \in \mathbb{N}$, we write $M_n(M)$ for the space of $n \times n$ matrices with entries in M, which we regard as an operator space within the C^*-algebra $M_n(\mathcal{A})$. The map ϕ is called *completely contractive* if

$$\|(\phi(a_{i,j}))_{1\leq i,j\leq n}\| \leq 1$$

for all $1 \leq n \in \mathbb{N}$ and $(a_{i,j})_{1\leq i,j\leq n} \in M_n(M)$ with $\|(a_{i,j})_{1\leq i,j\leq n}\| \leq 1$. A map that is both completely contractive and unital is called *u.c.c.*.
Given an operator system S in a C^*-algebra $\mathcal{A}$, a linear map $\phi : S \to \mathcal{B}$ into a C^*-algebra $\mathcal{B}$ is called *positive* if $\phi(a) \geq 0$ for all $0 \leq a \in \mathcal{A}$, and *completely positive* if

$$0 \leq (\phi(a_{i,j}))_{1\leq i,j\leq n}$$

for all $0 \leq (a_{i,j})_{1\leq i,j\leq n} \in M_n(S)$, where we view $M_n(S)$ as an operator system in $M_n(\mathcal{A})$. If ϕ is both completely positive and unital, we write that ϕ is *u.c.p.*.
Given an operator space (or system) M, a matrix $A = (a_{i,j})_{1\leq i,j\leq n}$ with entries in M, and a map ϕ on M, we write $\phi(A)$ for the matrix $(\phi(a_{i,j}))_{1\leq i,j\leq n}$.
One of the most fundamental results about positive maps, that is even the first exercise in [55, Chapter 2], is given in the following proposition. We will use it frequently without explicitly mentioning it. Since the book provides no proof, we include one here.

Proposition 3.1.1: *Let S be an operator system and ϕ be a positive map on S. Then,*

$$\phi(a^*) = \phi(a)^*$$

for all $a \in S$.

Proof:
First, suppose that $a \in S$ is selfadjoint. Since $\phi(1_{\mathcal{A}}) \geq 0$, it follows that $\phi(1)^* =$

$\phi(1)$. Moreover, as $\|a\|1_{\mathcal{A}} - a \geq 0$, we have $\|a\|\phi(1_{\mathcal{A}}) - \phi(a) \geq 0$, and hence $\phi(a)$ is selfadjoint. Now, let $a \in S$ be arbitrary. We have the decomposition

$$a = \frac{a+a^*}{2} - i\frac{i(a-a^*)}{2},$$

note that both $\frac{a+a^*}{2}$ and $\frac{i(a-a^*)}{2}$ are selfadjoint. Therefore, by the first step,

$$\phi\left(\frac{a+a^*}{2}\right) = \phi\left(\frac{a+a^*}{2}\right)^*, \; \phi\left(\frac{i(a-a^*)}{2}\right) = \phi\left(\frac{i(a-a^*)}{2}\right)^*.$$

Hence,

$$\begin{aligned}\phi(a^*) &= \phi\left(\frac{a+a^*}{2} + i\frac{i(a-a^*)}{2}\right) = \phi\left(\frac{a+a^*}{2}\right)^* - i\phi\left(\frac{i(a-a^*)}{2}\right)^* \\ &= \phi\left(\frac{a+a^*}{2} - i\frac{a+a^*}{2}\right)^* = \phi(a)^*.\end{aligned}$$

This completes the proof. □

The next proposition is another basic result that we will often use without explicitly mentioning it. It combines [55, Proposition 3.5 and Proposition 3.6].

Proposition 3.1.2: *Let M be an operator space, S an operator system, $\mathcal{B}$ a C^*-algebra. Given a u.c.c. map $\phi : M \to \mathcal{B}$ and a u.c.p. map $\rho : S \to \mathcal{B}$, then:*

(i) $M + M^* = \{a + b^*;\ a \in M, b \in M\}$ *is an operator system,*

(ii) $\tilde{\phi} : M + M^* \to \mathcal{B}$*, defined by* $a + b^* \mapsto \phi(a) + \phi(b^*)$*, is well defined and u.c.p.,*

(iii) ρ *is u.c.c..*

This proposition allows us to apply results about operator system to operator spaces. We illustrate this with the following results by Arveson.

Theorem 3.1.3 (Arveson's extension theorem): *Let S be an operator system in a C^*-algebra $\mathcal{A}$ and let $\phi : S \to \mathcal{B}(H)$ be a u.c.p. map, where H is a Hilbert space. Then there exists a u.c.p. map $\psi : \mathcal{A} \to \mathcal{B}(H)$ such that $\phi(a) = \psi(a)$ for all $a \in S$.*

Corollary 3.1.4 (Arveson): *Let M be an operator space in a C^*-algebra $\mathcal{A}$ and let $\phi : M \to \mathcal{B}(H)$ be a u.c.c. map, where H is a Hilbert space. Then there exists a u.c.c. map on $\psi : \mathcal{A} \to \mathcal{B}(H)$, and hence also a u.c.p. map, such that $\phi(a) = \psi(a)$ for all $a \in M$.*

At this point, we briefly introduce the concept of *Schur complents* and a related lemma, which we will use in Section 4.2.3.
Let H and K be Hilbert spaces, and let $A \in \mathcal{B}(H), C \in \mathcal{B}(K)$ and $B : K \to H$ be a linear bounded operator. If

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in \mathcal{B}(H \oplus K)$$

and C is invertible, then the operator

$$A - BC^{-1}B^*$$

is called the *Schur complement* of C in M. The following lemma relates the positivity of the Schur complement to the positivity of the operator M. The statement and proof can, for instance, be found in [1, Lemma 7.27].

Lemma 3.1.5: *Let A, B, C and M be as above, C invertible. Then M is positive if and only if C and the Schur complement of C in M are positive.*

Proof:
The proof immediately follows from the identity

$$M = \begin{pmatrix} id_H & BC^{-1} \\ 0 & id_K \end{pmatrix} \begin{pmatrix} A - BC^{-1}B^* & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} id_H & 0 \\ C^{-1}B^* & id_K \end{pmatrix}.$$

□

Using this lemma, we can now prove the well-known Schwarz inequality.

Proposition 3.1.6 (Schwarz inequality): *Let A and B be C^*-algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a u.c.p. map. Then*

$$\phi(a)^*\phi(a) \leq \phi(a^*a)$$

for all $a \in A$.

Proof:
Let $a \in \mathcal{A}$. By scaling, it suffices to prove the proposition for all a with $\|a\| \leq 1$. Embedding $\mathcal{A}$ into $\mathcal{B}(H)$ for some Hilbert space H, and applying Lemma 3.1.5 with $A = a^*a$, $B = a$, and $C = 1_{\mathcal{A}}$ shows that

$$\begin{pmatrix} a^*a & a^* \\ a & 1_{\mathcal{A}} \end{pmatrix} \in \mathcal{B}(H \oplus K)$$

is a positive operator. Since ϕ is completely positive, it follows that

$$\begin{pmatrix} \phi(a^*a) & \phi(a^*) \\ \phi(a) & 1_{\mathcal{B}} \end{pmatrix} \in \mathcal{B}(H \oplus K)$$

is also positive. Applying Lemma 3.1.5 again yields

$$\phi(a^*a) \geq \phi(a^*)\phi(a) = \phi(a)^*\phi(a),$$

which completes the proof. □

Next, we present examples of u.c.p. maps.

Example 3.1.7: Let $\mathcal{A}$ be a C^*-algebra.
(i) Every unital $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ is u.c.p.. Indeed, let $n \in \mathbb{N}$ and $0 \leq A \in M_n(\mathcal{A})$. Since A is positive, there is an element $B \in M_n(\mathcal{A})$ such that $A = B^*B$. Since π is a $*-$homomorphism, it is easy to verify that

$$\pi(A) = \pi(B^*)\pi(B) = \pi(B)^*\pi(B),$$

and thus, π is u.c.p..
(ii) If, additionally, $V : H \to K$ is an isometry into another Hilbert space K. We claim that the map $\phi : \mathcal{A} \to \mathcal{B}(K), a \mapsto V^*\pi(a)V$ is also u.c.p.. It should be clear that ϕ is unital and linear. So let $n \in \mathbb{N}$ and $0 \leq A \in M_n(\mathcal{A})$. Then, it is again easy to verify that

$$\phi(A) = V_n^*\pi(A)V_n,$$

where $V_n : H^n \to K^n, x_1 \oplus \ldots x_n \mapsto V(x_1) \oplus \ldots V(x_n)$. Thus, by (i), $\phi(A) \geq 0$, and we obtain that ϕ is u.c.p..
(iii) Let B be a commutative C^*-algebra. Then, every unital positive map $\phi : A \to B$ is u.c.p.. The proof is too lengthy and not relevant for our purposes, so we refer to [55, Theorem 3.9]. However, we will later use the special case where $B = \mathbb{C}$, which yields the result: Every state on a C^*-algebra is u.c.p..

That actually all u.c.p. maps on a C^*-algebra are a compression of a unital $*-$homomorphism was shown by Stinespring and is stated in the next theorem.

Theorem 3.1.8 (Stinespring's dilation theorem): *Let $\mathcal{A}$ be a C^*-algebra, H a Hilbert space and $\phi : \mathcal{A} \to \mathcal{B}(H)$ be a u.c.p. map. Then there exists a Hilbert space K, an isometry $V : H \to K$ and a unital $*-$homomorphism $\pi : \mathcal{A} \to \mathcal{B}(K)$ such that*

$$\phi(a) = V^*\pi(a)V$$

for all $a \in \mathcal{A}$.

In general, if S is an operator system (respectively operator space), H a Hilbert space and $\phi : S \to \mathcal{B}(H)$ a u.c.p. (respectively a u.c.c.) map, then we say that a u.c.p. (respectively u.c.c.) map $\Phi : S \to \mathcal{B}(K)$, for a Hilbert space K, is a *dilation* of ϕ, if there exists an isometry $V : H \to K$ such that

$$\phi(a) = V^*\Phi(a)V$$

for all $a \in S$. The dilation Φ is called *trivial* if $V(H)$ is a reducing subspace for Φ. If every dilation of ϕ to a u.c.p. map is trivial, then we call the map *maximal*. Combing Stinespring's dilation theorem with Arveson's extension theorem shows that every u.c.p. (respectively u.c.c.) map admits a dilation to a restriction of a unital $*$-homomorphism.

Corollary 3.1.9: *Let S be an operator system (respectively operator space) in a C^*-algebra $\mathcal{A}$, H a Hilbert space, and $\phi : S \to \mathcal{B}(H)$ be a u.c.p. (respectively u.c.c.) map. Then there exists a Hilbert space K, an isometry $V : H \to K$ and a unital $*-$homomorphism $\pi : \mathcal{A} \to \mathcal{B}(K)$ such that*

$$\phi(a) = V^*\pi(a)V$$

for all $a \in S$.

It is a non trivial result that every u.c.p. map admits a maximal dilation. Before stating this theorem, we present an example to illustrate the concept of maximality for a u.c.p. map.

Example 3.1.10: Let z be the identity function on $\mathbb{C}$, and let S be the operator system generated by z in $C(\mathbb{T})$. Let H and K be Hilbert spaces. Any u.c.p. map $\phi : S \to \mathcal{B}(H)$ is uniquely determined by $\phi(z)$, which must be a contraction since $\|\phi(z)\| \leq \|z\| = 1$.
Suppose that $\Phi : S \to \mathcal{B}(K)$ is a u.c.p. dilation of ϕ with respect to an isometry $V : H \to K$, and assume that $\phi(z)$ is unitary. Then:

$$\begin{aligned} 1 \geq& \|V^*\Phi(z)^*\Phi(z)V\| = \|V^*\Phi(z)^*VV^*\Phi(z)V + V^*\Phi(z)^*(id_K - VV^*)\Phi(z)V\| \\ &= \|\phi(z)^*\phi(z) + V^*\Phi(z)^*(id_K - VV^*)\Phi(z)V\| \\ &= \|id_H + V^*\Phi(z)^*(id_K - VV^*)\Phi(z)V\|, \end{aligned}$$

which implies $V^*\Phi(z)^*(id_K - VV^*)\Phi(z)V = 0$, so $V(H)$ is reducing for $\Phi(z)$. Therefore, any u.c.p. map ϕ, such that $\phi(z)$ is a unitary operator, is maximal.
Conversely, let T be a contraction on H. Then

$$U = \begin{pmatrix} T & \sqrt{id_H - TT^*} \\ \sqrt{id_H - T^*T} & -T^* \end{pmatrix} \in \mathcal{B}(H \oplus H)$$

is called a dilation of T. Using the series expansion of the square root function on $[0, \infty)$, one verifies that

$$T^*\sqrt{id_H - TT^*} = \sqrt{id_H - T^*T}T^* \text{ and } T\sqrt{id_H - T^*T} = \sqrt{id_H - TT^*}T.$$

Hence, U is a unitary. Define the unital $*$-homomorphism

$$\pi : C(\mathbb{T}) \to \mathcal{B}(H \oplus H), z \mapsto U,$$

and let

$$V : H \to H \oplus H, x \mapsto x \oplus 0.$$

Clearly, V is an isometry. Thus, by Example 3.1.7, the map

$$\phi(\cdot) = V^*\pi(\cdot)V$$

is a u.c.p. map on $C(\mathbb{T})$, and by construction, $\phi(z) = V^*UV = T$. Note that π is a trivial dilation of ϕ if and only if $T^*T = TT^* = id_H$.
Altogether, we have shown that the maps

$$\{\phi : S \to \mathcal{B}(H);\ \phi \text{ u.c.p.}\} \to \{T \in \mathcal{B}(H);\ \|T\| \leq 1\}, \phi \mapsto \phi(z)$$

and

$$\{\phi : S \to \mathcal{B}(H);\ \phi \text{ maximal}\} \to \{T \in \mathcal{B}(H);\ T \text{ unitary}\}, \phi \mapsto \phi(z)$$

are well defined and bijective.

We now state the result that every u.c.p. map admits a maximal dilation. For a proof, we refer to Dritschel-McCullough [25] respectively Arveson [7, Theorem 2.5].

Theorem 3.1.11: *Let S be an operator system (respectively operator space), H a Hilbert space, and let $\phi : S \to \mathcal{B}(H)$ be a u.c.p. (respectively u.c.c.) map. Then there exists a dilation $\Phi : S \to \mathcal{B}(K)$ of ϕ for some Hilbert space K such that Φ is maximal. If both S and H are separable, then K can be chosen to be separable.*

Hence, there always exists many maximal u.c.p. maps on an operator system, and understanding these maps provides deeper insight into the structure of all u.c.p. maps. Let us now collect some fundamental results about maximal u.c.p. maps. A well-known and frequently used, and often not explicitly stated fact throughout this thesis is the close connection between maximality and the *unique extension property* (*u.e.p.*).[1] A u.c.p. map ϕ on an operator system S (respectively space) in an C^*-algebra $\mathcal{A}$ is said to have the u.e.p. if there exists a unique u.c.p. (respectively u.c.c.) extension of ρ to $C^*(S)$ such that $\phi|_S = \rho$.
Before we explore the relationship between maximal and the unique extension property, let us collect some useful basic facts about maximal maps.

Lemma 3.1.12: *Let S be a operator system in a C^*-algebra $\mathcal{A}$. Then:*

[1] We emphasize that our definition of the unique extension property differs from the usual definition, as given, for instance, in [5] or [22]. Typically, the unique extension is required to be a $*$-homomorphism.

(i) *Every unital $*$-homomorphism on $\mathcal{A}$ is maximal (seen as a u.c.p. map on the operator system $\mathcal{A}$).*

(ii) *Let $n \in \mathbb{N}$, and for each n, let H_n be a Hilbert space and $\phi_n : S \to \mathcal{B}(H_n)$ be u.c.p. maps. Then the map*

$$\phi : S \to \mathcal{B}(\bigoplus_{n\in\mathbb{N}} H_n), s \mapsto \bigoplus_{n\in\mathbb{N}} \phi_n(s)$$

is maximal if and only if each ϕ_n is maximal.

Proof:
(i): Let H, K be Hilbert spaces, $\pi : \mathcal{A} \to \mathcal{B}(H)$ be a unital $*$-homomorphism, and $\rho : \mathcal{A} \to \mathcal{B}(K)$ a dilation of π with respect to an isometry $V : H \to K$. Then for every $a \in \mathcal{A}$,

$$\begin{aligned} V^*\rho(a^*)(id_K - VV^*)\rho(a)V &= V^*\rho(a^*a)V - V^*\rho(a^*)VV^*\rho(a)V \\ &= \pi(a^*a) - \pi(a^*)\pi(a) = 0. \end{aligned}$$

It follows that $V(H)$ is a reducing subspace for ρ.
(ii): Let $(\phi_n)_{n\in\mathbb{N}}$ and ϕ be as above. For every $n \in \mathbb{N}$ and every dilation ρ of ϕ_n, one can construct a dilation of ϕ by replacing ϕ_n in the direct sum $\oplus_{n\in\mathbb{N}}\phi_n$ with ρ. Hence, if ϕ is maximal, then each ϕ_n must also be maximal.
Conversly, since $H_n \subset H$, every dilation of ϕ restricts to a dilation of each ϕ_n for all n. So if ρ is a dilation of ϕ with respect to an isometry V, then $V(H_n)$ is reducing for ρ for all n, and thus the entire space $\bigoplus_{n\in\mathbb{N}} H_n$ is reducing for ρ, showing that ϕ is maximal. $\square$.

Theorem 3.1.13: *Let S be an operator system (respectively space) and ϕ a u.c.p. (respectively u.c.c.) on S. Then:*

(i) *If ϕ is maximal, then it has the unique extension property.*

(ii) *If ϕ has the unique extension property and the unique extension is a unital $*$-homomorphism, then ϕ is maximal.*

(iii) *If ϕ is maximal and ρ is its unique u.c.p. extension on $C^*(S)$, then ρ is a unital $*$-homomorphism.*

Proof:
Let S be an operator system in a C^*-algebra and $C^*(S)$ the C^*-algebra generated by S. Let $\phi : S \to \mathcal{B}(H)$ be a u.c.p. map for some Hilbert space H.
Assume first that ϕ is maximal, and let ρ be any u.c.p. extension to $C^*(S)$.

By Stinespring's dilation theorem, there exists a Hilbert space K, a unital $*$-homomorphism $\pi : C^*(S) \to \mathcal{B}(K)$, and an isometry $V : H \to K$ such that

$$\phi(s) = V^*\pi(S)V \text{ for all } s \in S.$$

Then $\pi|_S$ is a dilation of ϕ, and since ϕ is maximal, $V(H)$ is reducing for $\pi|_S$. Because S is selfadjoint, $V(H)$ is reducing for π as well. Hence, the compression $\pi|_{V(H)} = \rho$ is a unital $*$-homomorphism. Thus, we have shown that any u.c.p. extension of ϕ is a unital $*$-homomorphism. Since all such extensions agree on S, they must also agree on $C^*(S)$. This proves both (i) and (iii).
Now assume that ϕ has a unique u.c.p. extension $\tilde{\phi}$, which is a $*$-homomorphism. Let $\rho : S \to \mathcal{B}(K)$ be a maximal dilation of ϕ with respect to an isometry $V : H \to K$. By the first part, ρ has a unique u.c.p. extension $\tilde{\rho}$ to $C^*(S)$, which is a unital $*$-homomorphism. The map

$$V^*\tilde{\rho}(\cdot)V$$

is a u.c.p. extension of ϕ, and thus equals $\tilde{\phi}$. In particular, $\tilde{\rho}$ is a dilation of $\tilde{\phi}$. Since $\tilde{\phi}$ is a unital $*$-homomorphism, it is maximal by Lemma 3.1.12. Therefore, $V(H)$ is reducing for $\tilde{\rho}$ and consequently for ρ, proving that ϕ is maximal. $\square$.

Given two operator system S_1 and S_2 in C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, we say that S_1 and S_2 are *completely order isomorphic* if there is a bijective u.c.p. map $\phi : S_1 \to S_2$ such that ϕ^{-1} is also u.c.p.. In this case, ϕ is called *complete order isomorphism*.
Completely order isomorphic operator systems share the property that every u.c.p. map on one of them induces a u.c.p. map on the other one via composition with the complete order isomorphism. Given an operator system S and a C^*-algebra $\mathcal{A}$, we say that a pair $(\mathcal{A}, \phi)$ is a C^*-cover of S if $\phi : S \to \phi(S)$ is a complete order isomorphism and $C^*(\phi(S)) = \mathcal{A}$.
It is easy to see that the composition with a complete order isomorphism preserves trivial dilations and thus also maximality. This leads to an interesting observation: whether a u.c.p. on an operator system S is the restriction of a unital $*$-homomorphism depends strongly on the underlying C^*-algebra and is not preserved under complete order isomorphism, as the next example illustrates. Nevertheless, since maximality is preserved under complete order isomorphisms, and since maximal u.c.p. maps are restrictions of unital $*$-homomorphism on $C^*(S)$, knowing that a map is maximal on S yields information about every C^*-cover of S.

Example 3.1.14: Let $C^*(1, x)$ denote the *universal* C^*-algebra in a contraction. That is, up to isometric isomorphism, the unique unital C^*-algebra $\mathcal{A}$ generated by a contraction x such that for every Hilbert space and contraction $T \in \mathcal{B}(H)$, there exists a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ with $\pi(x) = T$. It follows from the definition that every C^*-algebra with this universal property is unique

up to isometric isomorphism.
Let S be the operator system generated by x, and $\tilde{S}$ be the operator system consider in Example 3.1.10, the operator system generated by z in $C(\mathbb{T})$. Since x is a contraction, we saw that

$$\phi : \tilde{S} \to S, a1 + bz + c\bar{z} \mapsto a1 + bx + cx^*$$

is well defined u.c.p. map. Since z is a contraction, there is a unital $*$-homomorphism $\pi : C^*(1, x) \to C(\mathbb{T})$ with $\pi(x) = z$. Now, it is easy to verify that $\pi|_S$ is the inverse to ϕ and thus ϕ is a complete order isomorphism. To come back to the comment preceding this example, the universal property of $C^*(1, x)$ shows that every u.c.p. map on S is a restriction of a unital $*$-homomorphism. In contrast, the only u.c.p. maps on $\tilde{S}$ that are restrictions of $*$-homomorphism are those such that $\phi(z)$ is unitary.

For an operator system S, we say that two C^*-covers $(\mathcal{A}_1, i_1)$ and $(\mathcal{A}_2, i_2)$ are equivalent if there exists a unital $*$-isomorphism $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ such that

$$i_2 = \pi \circ i_1.$$

We denote the collection of equivalent classes of C^*-covers of S by $C^*\text{-Lat}(S)$. We equip this set with a partial order as follows. For $[(\mathcal{A}_1, i_1)], [(\mathcal{A}_2, i_2)] \in C^*\text{-Lat}(S)$, we write $[(\mathcal{A}_2, i_1)] \le [(\mathcal{A}_1, i_1)]$ if there exists a unital $*$-homomorphism $\pi : \mathcal{A}_2 \to \mathcal{A}_1$ such that

$$i_2 = \pi \circ i_1.$$

An partially ordered set $(I, \le)$ is called lattice if for all $a, b \in I$, the *meet* $a \wedge b$ (greatest lower bound) and the *join* $a \vee b$ (least upper bound) exist. It is called *complete lattice* if the meet and join exist for every subset $J \subset I$, that is, for every such J, there exists an element $c \in I$ such that $c \le a$ (respectively $a \le c$) for all $a \in J$ and for every other element $b \in I$ with $b \le a$ (respectively $a \le b$) for all $a \in J$, we have $b \le c$ (respectively $c \le b$).
One can show that $(C^*\text{-Lat}(S), \le)$ is a complete lattice, however, the proof is similar to the proof for $(C^*\text{-Lat}(A), \le)$, where A is an operator algebra. So we will only prove the latter later in Proposition 3.1.25.

Proposition 3.1.15: *Let S be an operator system. Then $(C^*\text{-Lat}(S), \le)$ is a complete lattice.*

The smallest and largest elements in $(C^*\text{-Lat}(S), \le)$ have special names. The smallest element, $(C^*_e(S), i_e)$, is called the C^*-*envelope* of S and is uniquely characterized by the property that for every C^*-cover $(\mathcal{A}, i)$ of S, there exists a unital $*$-homomorphism $\pi : A \to C^*_e(S)$ such that

$$i_e = \pi \circ i.$$

The largest element, $(C^*_{max}(S), i_{max})$, is called the *universal* C^*-algebra of S and is uniquely characterized by the property that for every C^*-cover $(\mathcal{A}, i)$ of S, there exists a unital $*$-homomorphism $\pi : C^*_{max}(S) \to \mathcal{A}$ such that

$$i = \pi \circ i_{max}.$$

The existence of the C^*-envelope was an open problem for some time. One of the earliest contributions came from Arveson, who proved existence in certain particular cases. The problem was later resolved in full generality by Hamana in [36].

The last topic concerning operator systems we will address in this section is that of irreducible maximal maps. Given an operator system S in a C^*-algebra $\mathcal{A}$ such that S generates $\mathcal{A}$, and an irreducible unital $*$-homomorphism π on $\mathcal{A}$, we say that π is a boundary representation if its restriction $\pi|_S$ is maximal.
It is worth noting that maximality and irreducibility are preserved under complete order isomorphisms. Therefore, when considering only the restriction to S, the particular C^*-algebra does not matter.
Arveson conjectured that irreducible representations, and in particular the boundary representations, play a central role in understanding operator system. One famous conjecture, known as *Arveson's Hyperrigidity Conjecture*, states that the restrictions of all unital $*$-homomorphisms are maximal if and only if all irreducible representations are boundary representations. In this case, the operator system is called *hyperrigid*.
A counterexample to this conjecture was constructed by Boris Bilich and Adam Dor-On in [10], and it will be presented in Chapter 4. Their example involves an infinite-dimensional operator system, and this infinite-dimensionality plays a crucial role. Therefore, we will construct a new counterexample in which the operator system is finite-dimensional in Chapter 4.
For this purpose, we need several preliminary lemmas and theorems. The first three results are due to Arveson [4] and form the foundation for both counterexamples. The first lemma is a remark preceding [4, Theorem1.3.4].

Lemma 3.1.16: *Let $\mathcal{A}$ be a C^*-algebra, $I \subset \mathcal{A}$ a closed ideal, and $\pi : \mathcal{A} \to \mathcal{B}(H)$ a unital $*$-homomorphism. Then $K = \overline{\pi(I)(H)}$ is a reducing subspace for π, and we have the decomposition*

$$\pi = \pi|_{K^\perp} \oplus \pi|_K.$$

Moreover, $\pi(a)|_{K^\perp} = 0$ for all $a \in I$, and $\pi|_K$ is uniquely determined by its action on I.

<u>Proof</u>:
In the above setting, note that for all $a \in \mathcal{A}, i \in I$, and $x \in H$, we have that

$$\pi(a)\pi(i)(x) = \pi(ai)(x) \in \pi(I)(H),$$

since I is an ideal. Thus, K is reducing for π, and the stated decomposition follows. Since $\pi(a)(H) \subset K$ for all $a \in I$, we have that $\pi(a)|_{K^\perp} = 0$ for all $a \in I$. So it remains to show that $\pi|_K$ is uniquely determined by its action on I.
Let ρ be another unital $*$-homomorphism such that $\pi|_K = \rho$ on I. Then, for all $a \in A, i \in I$, and $x \in H$, we have

$$\rho(a)\pi(i)(x) = \rho(a)\rho(i)(x) = \rho(ia)(x) = \pi(ia)(x) = \pi(a)\pi(i)(x),$$

which shows that $\rho(a)$ and $\pi|_K(a)$ agree on $\pi(I)(H)$. Since both maps are continuous and $\pi(I)(H)$ is dense in K, it follows that $\rho = \pi|_K$. □

The next lemma is [4, Corollary 1.3, Section 1.4].

Lemma 3.1.17: *Let H be a Hilbert space. Then every irreducible representation of $K(H)$ is unitarily equivalent to the identity representation.*

Thus, if we take $I = \mathcal{K}(H)$ in Lemma 3.1.16 and know the irreducible representations of $\mathcal{A}/I$, for instance, if $\mathcal{A}/I$ is a commutative C^*-algebra, then the previous lemma implies that the irreducible representations of $\mathcal{A}$ consist precisely of the irreducible representations of $\mathcal{A}/I$ and the identity representation of $\mathcal{A}$. We will use the case in which $\mathcal{A}/I$ is commutative, and record this fact in the next theorem. The additional statement is known as Arveson's boundary theorem, see for example, [18].

Theorem 3.1.18: *Let H be a Hilbert space and let $\mathcal{A} \subset \mathcal{B}(H)$ be a C^*-algebra such that $\mathcal{K}(H) \subset \mathcal{A}$. If $\mathcal{A}/\mathcal{K}(H)$ is $*$-isomorphic to $C(K)$ for some compact set K, then every irreducible representation of $\mathcal{A}$ is either given by a point evaluation at some $z \in K$,*

$$\mathcal{A} \to \mathbb{C}, a \mapsto [a + \mathcal{K}(H)](z),$$

where we identify $[a+\mathcal{K}(H)]$ with the corresponding element in $C(K)$, or unitarily equivalent to the identity representation.
Moreover, if $S \subset \mathcal{A}$ is an operator system such that $C^*(S) = \mathcal{A}$, and the quotient map by $\mathcal{K}(H)$ restricted to S is not completely isometric, then the identity map on S already has the unique extension property.

Proof:
In the setting above, let $\pi : \mathcal{A} \to \mathcal{B}(H)$ be a unital irreducible representation. By Lemma 3.1.16, $\overline{\pi(\mathcal{K}(H))}$ is a reducing subspace for π, and since π is irreducible, we must have either $\overline{\pi(\mathcal{K}(H))} = 0$ or $\overline{\pi(\mathcal{K}(H))} = H$.
In the first case, π induces a irreducible representation on $C(K)$ and is therefore given by point evaluation at some $z \in K$. In the second case, there exists, by Lemma 3.1.17, a unitary operator U such that $U^*\pi(a)U = a$ for all $a \in \mathcal{K}(H)$. By Lemma 3.1.16, $U^*\pi U$ is uniquely determined by its action on $\mathcal{K}(H)$, and thus equals the identity map on $\mathcal{A}$, proving the first part of the theorem.

Now let $S \subset \mathcal{A}$ be an operator system generating $\mathcal{A}$. If S were reducible, then so would be $\mathcal{A}$, which contradicts $\mathcal{K}(H) \subset \mathcal{A}$. Hence, S is irreducible. By [18], the identity map on S has a unique extension to a u.c.p. map on $\mathcal{B}(H)$, and thus has the unique extension property. □

The last tool needed for the constructing the finite-dimensional counterexample is a dilation theorem by Davidson and Kennedy concerning pure u.c.p. maps [22, Theorem 2.4]. A u.c.p. map ϕ is called *pure* if the only completely positive maps ψ such that

$$\phi - \psi$$

is completely positive, are scalar multiples of ϕ.
Scalar-valued u.c.p. maps are closely related to extreme points of the state space, as captured in the following proposition.
For an operator system S, we define its state space by

$$\mathbf{S}(S) = \{\phi : S \to \mathbb{C};\ \phi \text{ u.c.p.}\}.$$

Note that, by Example 3.1.7, states are already u.c.p.maps. Therefore, our definition of the state space coincides with the standard definition via states.

Proposition 3.1.19: *Let S be an operator system. Then $\boldsymbol{S}(S)$ is convex and compact with respect to the topology of pointwise convergence in norm. Furthermore,*

$$\text{ex}(\boldsymbol{S}(S)) = \{\phi : S \to \mathbb{C};\ \phi \text{ pure}\}.$$

Proof:
Let S be an operator system. It is straightforward to verify that a convex combinations of u.c.p. maps are again u.c.p.. The compactness of $\mathbf{S}(S)$ follows from the Banach-Alaoglu theorem and the fact the pointwise limit of positive maps remains positive. It remains to show that

$$\text{ex}(\mathbf{S}(S)) = \{\phi : S \to \mathbb{C};\ \phi \text{ pure}\}.$$

First, suppose that $\phi \in \mathbf{S}(S)$ is not an extreme point. Then there exist $\phi_1, \phi_2 \in \mathbf{S}(S)$ and $t \in (0,1)$ such that $\phi = t\phi_1 + (1-t)\phi_2$ with $\phi_2 \neq \phi \neq \phi_1$. The maps $t\phi_1, (1-t)\phi_2$ are still completely positive. and

$$\phi - t\phi_1 = (1-t)\phi_2.$$

Since $\phi \neq \phi_1$, we conclude that ϕ is not pure.
Conversely, assume that $\phi \in \text{ex}(\mathbf{S}(S))$, and let $\psi : S \to \mathbb{C}$ be a completely positive map such that $\phi - \psi$ is also completely positive. It is easy to see that if $\psi(1) = 0$, then $\psi = 0$, because $\|x\|1 - x \geq 0$ for all selfadjoint $x \in S$. Thus, assume

$t = \psi(1) > 0$. Then $1 - t = (\phi - \psi)(1) > 0$, and so both ψ/t and $(\phi - \psi)/(1 - t)$ are unital completely positive. Hence,

$$\phi = t(\frac{\psi}{t}) + (1 - t)\frac{\phi - \psi}{1 - t}.$$

is a convex combination of two u.c.p. maps. Since ϕ was assumed to be an extreme point of $\mathbf{S}(S)$, this implies that $\phi/t = (\phi - \psi)/(1 - t) = \psi$, hence ϕ is pure. □

Theorem 3.1.20 (Davidson-Kennedy): *Let S be an operator system contained in a C^*-algebra $\mathcal{A}$ such that $C^*(S) = \mathcal{A}$, and let ϕ be a pure u.c.p. map on S. Then there exists a dilation ρ of ϕ that is both maximal and pure. In particular, the unique extension of ρ to $\mathcal{A}$ is a boundary representation.*

This result was used to resolve another conjecture by Arveson, namely that the boundary representations of S are completely norming for S, that is,

$$\|s\| = \sup\{\|\pi(s)\|;\ \pi \text{ boundary representation of } S\}$$

for all $n \in \mathbb{N}$ and $s \in M_n(S)$.
Although we do not need this result here, we include some historical context for interested readers. This problem remained open for nearly 40 years and was settled in the separable case by Arveson himself using disintegration techniques for representations of C^*-algebras, a method he expressed regret to appear necessary. Sic fata ferunt: Arveson passed away in 2011, two years before the publication of [22].

3.1.2 Operator Algebras

An operator algebra is essentially an algebra that is an operator space. More precisely, an operator algebra A is a closed unital subalgebra of a C^*-algebra. Since such a subalgebra is automatically an operator space, all results from the previous section on operator spaces also apply to operator algebras.
A major advantage of operator algebras is that maximal u.c.c. maps are always homomorphisms. Since every u.c.c. map dilates to a maximal one, it follows that every non-maximal u.c.c. homomorphism dilates non-trivially to a u.c.c. homomorphism. However, this can only occur in a specific form, described in the following lemma, known as Sarason's lemma.

Lemma 3.1.21 (Sarason): *Let A be an operator algebra, $\pi : A \to \mathcal{B}(H)$ be a u.c.c. homomorphism for a Hilbert space H, and $\rho : A \to \mathcal{B}(K)$ a u.c.c. homomorphism that dilates π. Identifying H with its image under the isometry from the dilation,* *we may assume $H \subset K$. Then H is a semi-invariant subspace of ρ, that is, there exists an orthogonal decomposition of $H^{\perp} = H_1 \oplus H_3$ such that*

$$P_H \rho|_{H_1} = P_{H_3}\rho|_{H_1} = P_{H_3}\rho|_H = 0,$$

or equivalently, we have a matrix decomposition of ρ:

$$\rho = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} \\ 0 & \pi & \pi_{2,3} \\ 0 & 0 & \pi_{3,3} \end{pmatrix},$$

for maps $\pi_{i,j} : A \to \mathcal{B}(H_i, H_j)$, where $H_2 = H$ and $\pi_{2,2} = \pi$.

Proof:
In the setting of the lemma, define $H_1 = \overline{\rho(A)(H)} \ominus H$. Note that $H \subset \overline{\rho(A)(H)}$ since $1 \in A$ and $\rho(1) = id_K$. Set $H_3 = H^{\perp} \ominus H_1$. Then $H^{\perp} = H_1 \oplus H_3$. We now verify H_1 and H_3 satisfy the semi-invariance conditions.
Let $x \in H_1, y \in H, z \in H_3$ and $a \in A$ be arbitrary. Then, since ρ is a homomorphism and by the definition of H_1, we have $\rho(a)(x), \rho(a)(y) \in H_1 \oplus H$, and so

$$\langle \rho(a)(x), z\rangle = \langle \rho(a)(y), z\rangle = 0.$$

As $x \in H_1$, there exist sequences $(a_n)_{n\in\mathbb{N}}$ in A and $(x_n)_{n\in\mathbb{N}}$ in H such that $\rho(a_n)x_n \to x$. Then

$$\begin{aligned} \langle \rho(a)(x), y\rangle &= \lim_{n\to\infty} \langle \rho(aa_n)(x_n), y\rangle \\ &= \lim_{n\to\infty} \langle \pi(a)\pi(a_n)(x_n), y\rangle = \langle x, \pi(a)^*(y)\rangle = 0, \end{aligned}$$

since $\pi(a)^*(y) \in H$, and the proof is complete. □

Although not relevant for our purposes, it is still worth noting that the proof only uses that A is a unital algebra, not a subalgebra of a C^*-algebra, and that π and ρ are unital homomorphism. Thus, the lemma holds in a more general setting. Later, we will use Sarason's lemma both to control u.c.c. homomorphism on a given operator algebra and and to construct new u.c.c. homomorphisms from a non-maximal one. The latter will be achieved by multiplying the off diagonal entries in the above matrix decomposition in a specific way. This idea was first observed and written down by Kaznelson in [42]. We include a proof here, as the original paper is written in Russian.

Theorem 3.1.22 (Kaznelson): *Let A be an operator algebra and $\pi : A \to \mathcal{B}(H)$ a u.c.c. homomorphism that admits a matrix decomposition of the form*

$$\pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} \\ 0 & \pi_{2,2} & \pi_{2,3} \\ 0 & 0 & \pi_{3,3} \end{pmatrix}.$$

Then, for every $z \in \overline{\mathbb{D}}$, the map

$$\pi_z = \begin{pmatrix} \pi_{1,1} & z\pi_{1,2} & z^2\pi_{1,3} \\ 0 & \pi_{2,2} & z\pi_{2,3} \\ 0 & 0 & \pi_{3,3} \end{pmatrix}.$$

is again a u.c.c. homomorphism.

Proof:
If $z = 0$, then $f(z)$ is a direct sum of u.c.c. homomorphisms, which is itself a u.c.c. homomorphism. Now assume $z \neq 0$. Then, the map $f(z)$ is obtained via the similarity transformation

$$f(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} \begin{pmatrix} \pi_1 & \pi_{1,2} & \pi_{1,3} \\ 0 & \pi & \pi_{2,3} \\ 0 & 0 & \pi_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix}.$$

This shows that each $f(z)$ is a homomorphism. Moreover, for every $x \in M_n(\mathcal{A})$, the map

$$\begin{aligned} \mathbb{C} &\to \mathcal{B}(K^n) \\ z &\mapsto f(z)(x) \end{aligned}$$

is analytic. Thus, by the maximum modulus principle,

$$\sup_{z \in \overline{\mathbb{D}}} \|f(z)(x)\| = \sup_{z \in \mathbb{T}} \|f(z)(x)\| = \|\pi(x)\| \leq \|x\|$$

for every $x \in M_n(\mathcal{A})$. Therefore, $f(z)$ is u.c.c for all $z \in \overline{\mathbb{D}}$. $\square$

We will later apply this theorem in the context of C^*-covers. A C^*-cover of an operator algebra A is a pair $(\mathcal{A}, i)$, where $\mathcal{A}$ is a C^*-algebra and $i : A \to \mathcal{A}$ a unital completely isometric homomorphism, that is, a injective u.c.c. homomorphism such that $i^{-1} : i(A) \to A$ is also u.c.c. with $C^*(i(A)) = \mathcal{A}$.
Given two C^*-covers $(\mathcal{A}_1, i_1)$ and $(\mathcal{A}_2, i_2)$ of A, we write

$$(\mathcal{A}_1, i_1) \precsim (\mathcal{A}_2, i_2)$$

if there exists a unital $*$-homomorphism $\pi : \mathcal{A}_1 \to \mathcal{A}_2$ such that

$$i_2 \circ \pi = i_1.$$

If π is additionally bijective, we say that the two C^*-covers are *equivalent*, denoted by

$$(\mathcal{A}_1, i_1) \sim (\mathcal{A}_2, i_2).$$

We denote the collection of equivalence classes of C^*-covers of A by C^*-Lat(A). It is well known that $(C^*\text{-Lat}(A), \precsim)$ forms a complete lattice. As announced in

the previous section, we will now provide a proof of this fact. We split the proof in two steps.
In the first step, we show that the smallest and largest elements in C^*-Lat(A) exists, even without first proving that this collection is actually a set. The largest element is called the *universal* C^*-algebra of A, denoted by $(C^*_{max}(A), i_{max})$, the smallest is called the C^*-*envelope* of A, denoted by $(C^*_e(A), i_e)$. Both are uniquely defined up to equivalence, with the property that for every C^*-cover $(\mathcal{A}, i)$ of A, we have

$$(C^*_e(A), i_e) \precsim (\mathcal{A}, i) \precsim (C^*_{max}(A), i_{max}).$$

The intuitive idea behind the construction of the maximal C^*-cover is take the sum over all possible u.c.c. homomorphisms on A. However, this collection does not form a set. One approach is to restrict to u.c.c. homomorphism acting on a sufficiently large Hilbert space, but this leads to unsightly handling with cardinalities.
The proof we present is, to the best of our knowledge, the first written proof of the existence of the maximal C^*-cover given by Blecher in [11]. In fact, this was one of the earliest works to study the maximal C^*-cover in depth.
We will use the maximal C^*-cover to construct the C^*-envelope. This is, in fact, not necessary, as can be seen from a proof by Dritschel and McCullough, see [25]. Our proof is not simpler, however, the construction used here will reappear in the proof of Theorem 3.2.7.

Lemma 3.1.23: *Let A be an operator algebra. Then $(C^*_{max}(A), i_{max})$ and $(C^*_e(A), i_e)$ exist.*

Proof:
We begin with the existence of $(C^*_{max}(A), i_{max})$. Let A be an operator algebra in $C^*(A)$. Let $\mathcal{E}$ be the $*$-algebra given by the algebraic free product of A and A^*. Let $\pi : A \to \mathcal{B}(H)$ be a u.c.c. homomorphism for some Hilbert space H. It is easy to check that the map

$$\pi^* : A^* \to \mathcal{B}(H), \ x \mapsto \pi(x^*)$$

is still u.c.c.. These two homomorphisms give rise to a $*$-representation

$$\pi \star \pi^* : \mathcal{E} \to \mathcal{B}(H).$$

Similar to the usual construction of universal C^*-algebra, $\mathcal{E}$ gives rise to a C^*-algebra $C^*_{max}(A)$ by taking the supremum over all $\pi \star \pi^*$. We claim that $(C^*_{max}(A), i_{max})$ is a maximal C^*-cover of A where i_{max} is the canonical embedding. Indeed, this is a C^*-cover, since the supremum runs over all possible u.c.c. homomorphism, and taking one that is completely isometric shows that i_{max} is completely isometric. It is also clear that $C^*_{max}(A) = C^*(i_{max}(A))$. So it remains to shows this C^*-cover is maximal in our partial order. Let $(\mathcal{A}, i)$ be an arbitrary C^*-cover of A. By construction of $C^*_{max}(A)$, there exists a unital $*$-homomorphism

$$\pi : C^*_{max}(A) \to \mathcal{A}$$

extending $i \star i^*$. Hence, $\pi \circ i_{max} = i$, and thus

$$(\mathcal{A}, i) \precsim (C^*_{max}(A), i_{max}).$$

Now consider the C^*-envelope. Define

$$I_e = \bigcap \left\{ \ker(\pi);\ {\scriptstyle \pi \text{ unital } *\text{-homomorphism on } C^*_{max}(A), \atop \pi|_{i_{max}(A)} \text{ maximal}} \right\}.$$

Since I_e is an intersection of closed ideals, it is a closed ideal in $C^*_{max}(A)$, and we define

$$(C^*_e(A), i_e) = (C^*_{max}(A)/I_e, i_e), \text{ with } i_e(a) = i_{max}(a) + I_e.$$

To see that $(C^*_e(A), i_e)$ is a C^*-cover of A, the only non-trivial thing to check is that i_e is completely isometric. For this, let π be the unital completely isometric homomorphism obtained by restricting the identity representation on $C^*_{max}(A)$ to A, and let ρ be a maximal dilation of π. Then ρ is still a unital completely isometric homomorphism, which extends to a unital $*$-homomorphism Ψ on $C^*_{max}(A)$.
Since $I_e \subset \ker(\Psi)$, we have for all $a = (a_{i,j})_{1 \le i,j \le n} \in M_n(A)$,

$$\begin{aligned}\|a\| &= \|i_{max}(a)\| = \|\rho(i_{max}(a))\| \\ &= \|\Psi(i_{max}(a))\| \le \|(i_{max}(a_{i,j}) + \ker(\Psi))_{1 \le i,j \le n}\| \le \|i_e(a)\| \le \|a\|,\end{aligned}$$

proving that i_e is completely isometric.
Now suppose that $(\mathcal{A}, i)$ is a C^*-cover of A. From the existence of the maximal C^*-cover, we know that there exists an ideal $I \subset C^*_{max}(A)$ such that $(\mathcal{A}, i) \sim (C^*_{max}(A)/I, q)$, where $q(a) = i_{max}(a) + I$. So it remains to show that

$$(C^*_{max}(A)/I_e, i_e) \precsim (C^*_{max}(A)/I, q),$$

which will immediately follow as soon as we have shown that $I \subset I_e$.
Let π be a unital $*$-homomorphism on $C^*_{max}(A)$ such that $\pi|_{i_{max}(A)}$ is maximal. Define a u.c.c. homomorphism on $q(A)$ via $\pi \circ q^{-1}$, which is maximal since $\pi|_{i_{max}(A)}$ is maximal. Hence, $\pi \circ q^{-1}$ admits a unique u.c.p. extension ϕ to $C^*_{max}(A)/I$, which is necessarily a unital $*$-homomorphism.
Since $\phi(a + I) = \pi(a)$ for all $a \in i_{max}(A)$, and both ϕ and π are unital $*$-homomorphisms, this identity extends to all of $C^*_{max}(A)$, that is, $\phi(a + I) = \pi(a)$ for all $a \in C^*_{max}(A)$, yielding that $I \subset \ker(\pi)$.
It follows that $I \subset I_e$, completing the proof of the existence of the C^*-envelope of A. □

Remark 3.1.24: There are several remarks to be made regarding $C^*_{max}(A)$, $C^*_e(A)$ and C^*-covers in general. First, whether the term "C^*-cover" refers to an equivalence class or a specific representative of the class depends on the context.
Second, a C^*-cover is the maximal C^*-cover if and only if every u.c.c. homomorphism on A extends to a unital $*$-homomorphism on the C^*-cover.
And finally, every bijective unital $*$-isomorphism between C^*-algebras is automatically unital completely isometric.

Proposition 3.1.25: *Let A be an operator algebra. Then C^*-Lat(A) is a set and forms a complete lattice.*

Proof:
Let A be an operator algebra. The proof of the previous lemma showed that for every C^*-cover $(\mathcal{A}, i)$ of A, there exists a closed ideal $I \subset C^*_{max}(A)$ such that

$$(C^*_{max}(A)/I, q) \sim (\mathcal{A}, i), \text{ where } q(a) = i_{max}(a) + I.$$

Moreover, every such I contains

$$I_e = \bigcap\{\ker(\pi);\ \pi : C^*_{max}(A) \to \mathcal{B}(H) \text{ unital } *\text{-homomorphism, } \pi|_{i_{max}(A)} \text{maximal}\}.$$

Conversely, for every closed ideal $I \subset C^*_{max}(A)$ with $I \subset I_e$, the pair $(C^*_{max}(A)/I, q)$, where $q : A \to C^*_{max}(A)/I, a \mapsto i_{max}(a) + I$, is a C^*-cover of A.
Thus, there is order isomorphism between C^*-Lat(A) and

$$\{I \subset C^*_{max}(A);\ I \text{ closed ideal, } I \subset I_e\},$$

ordered by inclusion. This is clearly a complete lattice, and in particular a set. Therefore, C^*-Lat(A) is a complete lattice. □

Let us now consider two of the most iconic examples of operator algebras and their C^*-envelopes and maximal C^*-algebra.

Example 3.1.26:

(i) Let $\mathcal{T}_2$ be the algebra of upper triangular matrices in M_2. This obviously a closed subalgebra and thus an operator algebra. The only u.c.c. extension of $id_{\mathcal{T}_2}$ is id_{M_2}, since u.c.c. maps on operator systems are u.c.p and $\mathcal{T}_2 + \mathcal{T}_2^* = M_2$. Therefore, $id_{\mathcal{T}_2}$ is maximal, and the construction of the C^*-envelope yields $C^*_e(\mathcal{T}_2) = M_2$, so

$$C^*_e(\mathcal{T}_2, i_e) = (M_2, id_{\mathcal{T}_2}).$$

Although we will not compute $C^*_{max}(\mathcal{T}_2)$ in detail, for completeness we mention that it is given by the M_2-valued continuous function on $[0, 1]$ such that $f(0)$ is a diagonal matrix with the embedding

$$i_{max}\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \left[t \mapsto \begin{pmatrix} a & tx \\ 0 & c \end{pmatrix}\right],$$

(see [11, Example 2.4]).

(ii) Let $A(\mathbb{D})$ be the *disk algebra*, the algebra of holomorphic functions on $\mathbb{D}$ that extend continuously to $\overline{\mathbb{D}}$. By the maximum principle, we can view $A(\mathbb{D})$ as a subalgebra of $C(\mathbb{T})$. Using Example 3.1.10, we can characterize the maximal maps on $A(\mathbb{D})$.
First, note that the operator algebra $A(\mathbb{D})$ is generated by z, the identity function. Hence, every u.c.c. homomorphism is uniquely determined by its value at z. Now, let S be the operator system generated by z. Since $z^*z = 1$ on $\mathbb{T}$, the restrictions of unital $*$-homomorphism π on $C(\mathbb{T})$ are precisely those such that $\pi(z)$ is unitary. Any u.c.c. extension ρ of $\pi|_{A(\mathbb{D})}$ to $C(\mathbb{T})$ is u.c.p., and hence also a u.c.p. extension of the u.c.p. map $\pi|_S$. By Example 3.1.10, the only u.c.p. extension of $\pi|_S$ is π itself. Therefore, the maximal maps on $A(\mathbb{D})$ are exactly the u.c.c. homomorphism which send z to a unitary element.
In particular, $id_{A(\mathbb{D})}$ is maximal, and thus

$$(C_e^*(A(\mathbb{D})), i_e) = (C(\mathbb{T}), id_{A(\mathbb{D})}).$$

On the other hand, the maximal C^*-algebra is given by the universal C^*-algebra $C^*(1, x)$ generated in a contraction x with the embedding $i_{max}(z) = x$. To see that this is indeed a C^*-cover of $A(\mathbb{D})$, one can apply the Sz.-Nagy dilation theorem [55, Theorem 1.1]. To verify that this is also the maximal C^*-cover, observe that any u.c.c. homomorphism π on $A(\mathbb{D})$ extends to a unital $*$-homomorphism on $C^*(1, x)$ via $x \mapsto \pi(z)$.

We conclude the preliminaries with two special types of operator algebras. The first are *(semi-)Dirichlet* operator algebras (see [20, Definition 4.1]). An operator algebra A is called Dirichlet if

$$\overline{i_e(A) + i_e(A)^*} = C_e^*(A)$$

and semi-Dirichlet if

$$i_e(A)^* i_e(A) \subset \overline{i_e(A) + i_e(A)^*}.$$

A general C^*-cover $(\mathcal{A}, i)$ is called *semi-Dirichlet* C^*-cover of A if

$$i(A)^* i(A) \subset \overline{i(A) + i(A)^*}.$$

The second type are *residual finite dimensional* (RFD) operator algebras. These are operator algebras A for which there exists a C^*-cover $(\mathcal{A}, i)$ with a RFD C^*-algebra A, that is,

$$\|a\| = \sup\{\|\pi(a)\|;\ \pi : \mathcal{A} \to M_n \text{ unital *-homomorphism},\ 1 \leq n \in \mathbb{N}\}$$

for all $a \in \mathcal{A}$. In this case, $(\mathcal{A}, i)$ is called a RFD C^*-cover, and any such π is called a *finite-dimensional representation* of $\mathcal{A}$.
Let us list some examples of RFD and (semi-)Dirichlet operator algebras. Some of these example will reappear later.

Example 3.1.27:

(i) Commutative C^*-algebras are RFD, since their norm is given by the supremum over point evaluations, which are finite-dimensional representations.

(ii) The unital C^*-algebra generated by a universal contraction is RFD by [17, Theorem5.1].

(iii) The disk algebra $A(\mathbb{D})$ is a Dirichlet operator algebra since every real-valued continuous function $f \in C(\mathbb{T})$ can be approximated via convolutions with the Poisson kernel and these convolutions are the real parts of functions in $A(\mathbb{D})$.

(iv) $\mathcal{T}_2$ is clearly a Dirichlet operator algebra.

(v) If A is a Dirichlet operator algebra, then

$$B = \begin{pmatrix} \frac{A}{A + A^*} & 0 \\ & A^* \end{pmatrix}$$

is a semi-Dirichlet operator algebra. For a proof see [40, Proposition 2.15].

3.2 The Lattice of C^*-covers

The main goal of this section is to characterize those C^*-cover lattices that have finite cardinality. It is evident that if A is a C^*-algebra, then every unital completely isometric homomorphism $\pi : A \to B$ for some C^*-algebra B is already a unital $*$-isomorphism. Thus, $|C^*\text{-Lat}(A)| = 1$ for every C^*-algebra A. However, the question of whether there exist an operator algebra A with $|C^*\text{-Lat}(A)| = 1$ becomes non-trivial if we require A to be non-selfadjoint. This question was posed by Humeniuk and Ramsey in [41, Question 3.1], and we will answer it affirmatively in Theorem 3.2.4.

We will also address the follow-up question concerning the existence of an operator algebra with $|C^*\text{-Lat}(A)| = n$ for $2 \leq n \in \mathbb{N}$. Surprisingly, we will show that no such an operator algebra exist - not even one with $|C^*\text{-Lat}(A)| = \aleph_0$, see Theorem 3.2.7.

We then answer [40, Question 3.1.13] in Theorem 3.3.1 by showing that the maximal C^*-cover of a non-selfadjoint operator algebra is never a semi-Dirichlet C^*-cover.

All results presented in this chapter are joint work with Adam Humeniuk and Christopher Ramsey.

3.2.1 One-Point Lattice

To construct a non-selfadjoint operator algebra with $|C^*\text{-Lat}(A)| = 1$, we require the following lemma, for which we introduce one more piece of notation.
Given u.c.c. maps $(\pi_n)_{n\in\mathbb{N}}$ from an operator algebra A to $\mathcal{B}(H)$, we say that $(\pi_n)_n$ converges pointwise in the $*$-SOT if both $(\pi_n(a))_n$ and $\pi_n(a)^*)_n$ converge in the SOT for all $a \in A$.

Lemma 3.2.1: *Let A be an operator algebra in a C^*-algebra $\mathcal{A}$. For $n \in \mathbb{N}$, let $\pi, \pi_n : \mathcal{A} \to \mathcal{B}(H)$ be unital $*$-homomorphisms such that $\pi_n \to \pi$ pointwise in the *-SOT. If every restriction $\pi_n|_A$ is maximal, then there exists a unital $*$-homomorphism ψ on the C^*-envelope $C_e^*(A)$ such that $\psi \circ \sigma = \pi$, where $\sigma : \mathcal{A} \to C_e^*(A)$ is the unique morphism satisfying $\sigma|_A = i_e$.*

Proof:
Let $A, (\pi_n)_{n\in\mathbb{N}}$ and π be as above. Since each $\pi_n|_A$ is maximal, there exist $*$-homomorphisms ψ_n defined on $C_e^*(A)$ such that $\psi_n \circ \sigma = \pi_n$. Let $a \in C_e^*(A)$. As σ is surjective, there exists $\tilde{a} \in \mathcal{A}$ such that $\sigma(\tilde{a}) = a$. Then

$$\|(\psi_n - \psi_m)(a)x\| = \|(\pi_n - \pi_m)(\tilde{a})x\|$$

for all $x \in H$. Hence, the sequence (ψ_n) has a *-SOT-limit ψ, which is again a $*$-homomorphism. This map also satisfies $\psi \circ \sigma = \pi$. □

Now we construct the desired example. Let $C^*(1, x)$ be the unital universal C*-algebra generated by a contraction x, for a definition see Example 3.1.14. Let A be the closed operator algebra generated by $1, x, (x^2)^*, (x^3)^*$. We will show that the equivalence class of both the C^*-envelope and the maximal C^*-algebra of A are given by $C^*(1, x)$ with the identity embedding, and that A is non-selfadjoint. Note that A is RFD, since $C^*(1, x)$ is a RFD C*-algebra.

Remark 3.2.2: Let $\pi : A \to \mathcal{B}(H)$ be a u.c.c. homomorphism. Since π extends to a completely positive map on $A + A^*$, it follows that if $a \in A$ and $a^* \in A$, then $\pi(a^*) = \pi(a)^*$. Therefore, $\pi((x^2)^*) = \pi(x^2)^* = (\pi(x)^2)^*$, and $\pi((x^3)^*) = (\pi(x)^3)^*$. In particular, every u.c.c. homomorphism on A is uniquely determined by its value at x. Consequently, for every contraction T, there exists a unique u.c.c. homomorphism π on A such that $\pi(x) = T$.

Theorem 3.2.3: *Let H be a Hilbert space, $T \in \mathcal{B}(H)$ be an invertible contraction, and let $\pi : C^*(1, x) \to \mathcal{B}(H)$ be the unital $*$-homomorphism with $\pi(x) = T$. Then $\pi|_A$ is maximal.*

Proof:
Let π be as above, and let $\Pi : C^*(1, x) \to \mathcal{B}(K)$ be a unital *-homomorpism with $H \subset K$, such that $P_H \Pi|_H|_A = \pi|_A$. By Sarason's Lemma (Lemma 3.1.21), the space H is semi-invariant for Π. Therefore, there exist closed subspaces

$M \subseteq L \subseteq K$ with $L \ominus M = H$, such that both M and L are invariant under $\Pi|_A$. Since the C*-algebra $C^*(1, x^2, x^3) \subset A$, the subspaces M and L already reduce $\Pi|_{C^*(1,x^2,x^3)}$.
We write $\Pi(x)$ as a matrix with respect to the orthogonal decomposition $K = M \oplus H \oplus (K \ominus L)$:

$$\Pi(x) = \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ 0 & T & T_{2,3} \\ 0 & 0 & T_{3,3} \end{pmatrix}$$

Now, we can compute $\Pi(x^3)$ in three different ways. First, since $x^3 \in C^*(1, x^2, x^3)$, M and L reduce Π, and we have

$$\Pi(x^3) = \begin{pmatrix} T_{1,1}^3 & 0 & 0 \\ 0 & T^3 & 0 \\ 0 & 0 & T_{3,3}^3 \end{pmatrix},$$

On the other hand, using $\Pi(x^3) = \Pi(x)\Pi(x^2)$ and the fact that $x^2 \in C^*(1, x^2, x^3)$, we obtain:

$$\begin{aligned} \Pi(x^3)) &= \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ 0 & T & T_{2,3} \\ 0 & 0 & T_{3,3} \end{pmatrix} \begin{pmatrix} T_{1,1}^2 & 0 & 0 \\ 0 & T^2 & 0 \\ 0 & 0 & T_{3,3}^2 \end{pmatrix} \\ &= \begin{pmatrix} T_{1,1}^3 & T_{1,2}T^2 & T_{1,3}T_{3,3}^2 \\ 0 & T^3 & T_{2,3}T_{3,3}^2 \\ 0 & 0 & T_{3,3}^3. \end{pmatrix} \end{aligned}$$

Comparing with the first expression for $\Pi(x^3)$, we find $T_{1,2}T^2 = 0$. Since T is invertible by assumption, it follows $T_{1,2} = 0$.
Similarly, using $\Pi(x^3) = \Pi(x^2)\Pi(x)$, we compute:

$$\begin{aligned} \Pi(x^3) &= \begin{pmatrix} T_{1,1}^2 & 0 & 0 \\ 0 & T^2 & 0 \\ 0 & 0 & T_{3,3}^2 \end{pmatrix} \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ 0 & T & T_{2,3} \\ 0 & 0 & T_{3,3} \end{pmatrix} \\ &= \begin{pmatrix} T_{1,1}^3 & T_{1,1}^2T_{1,2} & T_{1,1}^2T_{1,3} \\ 0 & T^3 & T^2T_{2,3} \\ 0 & 0 & T_{3,3}^3. \end{pmatrix} \end{aligned}$$

Comparing with the first expression again, we see that $T_{2,3} = 0$. This shows that H is reducing for Π, and thus π is maximal. □

Theorem 3.2.4: *The C^*-envelope and the maximal C^*-algebra of A are both given by $C^*(1, x)$, with the identity embedding. However, A is non-selfadjoint.*

Proof:
It is clear that $(C^*_{max}(A), i_{max})$ is given by $(C^*(1, x), id)$, since every contractive

operator $T \in \mathcal{B}(H)$ defines a u.c.c. homomorphism $\pi : A \to \mathcal{B}(H), x \mapsto T$. By Remark 3.2.2 every u.c.c. homomorphism $\pi : A \to \mathcal{B}(H)$ is uniquely determined by the contraction $\pi(x)$, and thus extends uniquely to a $*$-homomorphism of the universal C*-algebra $C^*(1, x)$.
Let $\iota : A \to C_e^*(A)$ be the embedding of A into its C*-envelope, and let $\sigma : C^*(1, x) \to C_e^*(A)$ be the unique morphism of C*-covers of A satisfying $\sigma|_A = \iota$. For any $1 \leq n \in \mathbb{N}$, any invertible contraction $T \in M_n$ determines a maximal representation of A by Theorem Theorem 3.2.3, and hence the $*$-homomorphism $C^*(1, x) \to M_n$, $x \mapsto T$, factors through σ. Since the invertible contractions are dense in the set of all contractions in M_n with respect to the operator norm, such homomorphisms $C^*(x) \to M_n$ are in particular point $*$-SOT dense in all homomorphisms $C^*(x) \to M_n$. By Lemma Lemma 3.2.1, it follows that *every* $*$-homomorphism $C^*(x) \to M_n$ factors through σ. Because $C^*(1, x)$ is RFD, the set of finite-dimensional representations norms $C^*(1, x)$, and so σ is isometric.
It remains to show that A is non-selfadjoint. Since u.c.c. homomorphisms on selfadjoint operator algebras have the unique extension property, and therefore are maximal, it suffices to show that there exist at least one u.c.c. homomorphism on A that is not maximal. Consider the following u.c.c. homomorphisms $\pi : A \to \mathbb{C}, x \mapsto 0$ and

$$\psi : A \to M_2, \quad x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $\psi((x^*)^2) = \psi((x^*)^3) = 0$, the subspace $\mathbb{C} \oplus 0$ is invariant under ψ. The compression of ψ onto $\mathbb{C} \oplus 0 \cong \mathbb{C}$ yields π. This shows that π is not maximal, and hence A is non-selfadjoint. □

3.2.2 No Two-Point Lattice

To study operator algebras with more than one equivalence class of C^*-covers, we employ Sarason's lemma in combination with Kaznelson's lemma. We illustrate the technique with the following example.

Example 3.2.5: Let $\pi : A(\mathbb{D}) \to C^*(S)$ be the completely isometric representation that sends the generator f to the unilateral shift S. A maximal dilation of this representation sends f to the Sz. Nagy dilation of S, i. e., to a unitary

$$U = \begin{pmatrix} S & I - SS^* \\ 0 & S^* \end{pmatrix}.$$

The process in Lemma Theorem 3.1.22 applied to this dilation (note that because U extends S, we can take $H_1 = 0$ in Sarason's Lemma) corresponds to the family of completely isometric representations of $A(\mathbb{D})$ that send z to

$$V_z = \begin{pmatrix} S & z(I - SS^*) \\ 0 & S^* \end{pmatrix}$$

for $z \in \overline{\mathbb{D}}$. We will see that for $|z| \neq |w|$, the representations determined by V_z and V_w generate C*-covers that are not even comparable in the ordering of C*-covers, unless $|z| = 1$ or $|w| = 1$.

Suppose that there is a morphism of C*-covers $\pi : C^*(V_z) \to C^*(V_w)$, meaning a (unital) $*$-homomorphism that satisfies $\pi(V_z) = V_w$. We have

$$V_z^* V_z = \begin{pmatrix} I & 0 \\ 0 & |z|^2(I - SS^*) + SS^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I - SS^* \end{pmatrix} + |z|^2 \begin{pmatrix} 0 & 0 \\ 0 & SS^* \end{pmatrix}.$$

Since the two matrices on the right-hand-side are orthogonal projections summing to the identity, the spectrum is

$$\sigma(V_z^* V_z) = \{|z|^2, 1\}.$$

Similarly, $\sigma(V_w^* V_w) = \{|w|^2, 1\}$. Since a $*$-homomorphism shrinks spectra, we have

$$\sigma(V_w^* V_w) = \sigma(\pi(V_z^* V_z)) \subseteq \sigma(V_z^* V_z),$$

and so $\{|w|^2, 1\} \subseteq \{|z|^2, 1\}$. Therefore, we must have $|w| = |z|$–in which case V_z and V_w are unitarily equivalent, or else $|w| = 1$–in which case $C^*(V_w) = C^*(U)$ is the C*-envelope.

Example 3.2.5 demonstrates that even though the definiton of $f(z)$ in Theorem 3.1.22 is a nice point-SOT continuous map, the resulting C*-covers produced along the path from $z = 0$ to $z = 1$ can be "badly discontinuous". In Example 3.2.5, the C*-covers produced start from $f(0)$, which generates the Toeplitz algebra, end at $f(1)$ giving the C*-envelope, but along the way the representations $f(z)$ for $0 \leq z < 1$ are all mutually incomparable in the ordering of the C*-cover lattice.

Example 3.2.6: Now, consider the completely isometric representation $\pi : A(\mathbb{D}) \to C(\overline{\mathbb{D}})$ given by inclusion. This is the universal C*-algebra generated by a normal contraction $f \in C(\overline{\mathbb{D}})$ with $f(z) = z$. A maximal dilation to a unitary is given by the Schaeffer dilation

$$f \mapsto U := \begin{pmatrix} \ddots & & & & & & \\ & 0 & 1 & & & & \\ & & 0 & \sqrt{1-|f|^2} & -\overline{f} & & \\ & & 0 & f & \sqrt{1-|f|^2} & 0 & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ & & & & & & \ddots \end{pmatrix} \in C(\overline{\mathbb{D}}, B(\ell^2))$$

For $z \in \overline{\mathbb{D}}$, the dilation produced by the process in Theorem 3.1.22 sends the generator $f \in A(\mathbb{D})$ to

$$f \mapsto V_z := \begin{pmatrix} \ddots & & & & & & & \\ & 0 & 1 & & & & & \\ & & 0 & z\sqrt{1-|f|^2} & -z^2\overline{f} & & & \\ & & 0 & f & z\sqrt{1-|f|^2} & 0 & & \\ & & & & 0 & 1 & & \\ & & & & & 0 & 1 & \\ & & & & & & & \ddots \end{pmatrix}$$

It is straightforward to check that when $|z| \neq 0$ and $|z| \neq 1$, the operator V_z is not normal. Therefore, even though $C^*(V_0) = C(\overline{\mathbb{D}})$, and $C^*(V_1) = C(\mathbb{T})$, for $z \in (0,1)$, none of the C*-covers $C^*(V_z)$ sit between $C(\overline{\mathbb{D}})$ and $C(\mathbb{T})$ in the ordering of C*-covers.

It is time to show that every operator algebra with more than one equivalence class of C^*-covers already has uncountably many such classes. For this, we introduce the following notation. For an operator algebra A in a C^*-algebra $\mathcal{A}$, the universal property of the maximal C^*-cover guarantees that for every u.c.c. homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ there exists a unique unital *-homomorphism, denoted by $\pi^* : C^*_{max}(\mathcal{A}) \to \mathcal{B}(H)$, such that $\pi^* \circ i_{\max} = \pi$.

Theorem 3.2.7: *Let A be an operator algebra with $|C^*\text{-Lat}(A)| \neq 1$. Then $|C^*\text{-Lat}(A)|$ is at least that of the continuum $\mathfrak{c}$.*

Proof:
Let A be as above. Since $(C^*_{max}(A), i_{\max}) \neq (C^*_e(A), i_e)$, the map $i_{\max}$ is not maximal. Let Φ be a maximal dilation of $i_{\max}$, and let

$$\Phi = \begin{pmatrix} \pi_1 & \phi_{1,2} & \phi_{1,3} \\ 0 & i_{\max} & \phi_{2,3} \\ 0 & 0 & \pi_3 \end{pmatrix}$$

be the decomposition from Sarason's Lemma (Lemma 3.1.21), and let $f(z)$ the family of u.c.c. homomorphisms from Theorem 3.1.22. The ideal[2]

$$I = \bigcap \{\ker(\rho^*);\ \rho : A \to \mathcal{B}(H) \text{ u.c.c. and maximal}\}$$

is not trivial since A has more than one C*-cover. Thus, there exists a $x \in I$ with $x \neq 0$. By construction, $\Phi = f(1)$ is maximal, hence

$$f(1)^*(x) = \Phi^*(x) = 0.$$

[2]This ideal is actually the *Shilov ideal* of A seen as a subalgebra of $C^*_{max}(A)$. For more information about the Shilov ideal, we refer to [5]

On the other hand, $i^*_{\max}$ is an isometric representation of $C^*_{max}(A)$, so $\ker(i^*_{\max}) = \{0\}$. Since $f(0)$ is a trivial dilation of $i_{\max}$, we have

$$f(0)^*(x) \neq 0.$$

We now show that for every $y \in C^*_{max}(A)$, the function

$$\delta_y : \overline{\mathbb{D}} \to \mathbb{R}, z \mapsto \|f(z)^*(y)\|$$

is continuous. It is clear that δ_y is continuous for every $y \in i_{\max}(A)$, and since $f(z)^*$ is a *-homomorphism, δ_y is continuous for every y in the $*$-algebra generated by $i_{\max}(A)$. By contractivity of $f(z)^*$ and the triangle inequality, we have

$$|\delta_y(z) - \delta_y(\omega)| \leq 2\|y - v\| + |\delta_v(z) - \delta_v(\omega)|$$

for every $y, v \in C^*_{max}(A)$. Hence, δ_y is continuous for every $y \in C^*_{max}(A)$.
Thus, for every $s \in [0, \|f(0)^*(x)\|]$, there exists a $z_s \in [0,1]$ such that $\|f(z_s)^*(x)\| = s$. Define $i_s = i_e \oplus f(z_s)$ and let $C^*(i_s(A)) = C^*_s(A)$. It remains to show that the C*-covers $(C^*_s(A), i_s)$ are pairwise non-equivalent.
Suppose there exists a *-isomorphism $\rho : C^*_s(\mathcal{A}) \to C^*_t(\mathcal{A})$ such that $i_t = \rho \circ i_s$ on $\mathcal{A}$. Since $x \in \ker(i^*_e)$ and $\rho \circ i^*_s = (\rho \circ i_s)^*$, comparing the norms of the image of x in both C*-cover yields:

$$\begin{aligned} s &= \|(i_e \oplus f(z_s))^*(x)\| = \|i^*_s(x)\| = \|\rho \circ i^*_s(x)\| \\ &= \|(\rho \circ i_s)^*(x)\| = \|i^*_t(x)\| = \|(i_e \oplus f(z_t))^*(x)\| \\ &= \|f(z_t)^*(x)\| = t. \end{aligned}$$

Therefore, $(C^*_s(\mathcal{A}), i_s) \ncong (C^*_t(\mathcal{A}), i_t)$ for every $s \neq t$ in $[0, \|f(0)^*(x)\|]$. □

Corollary 3.2.8: *If the operator algebra in the previous theorem is separable, then the cardinality of the set of C^*-covers is $\mathfrak{c}$.*

Proof:
We already know that the cardinality is at least $\mathfrak{c}$. To see that it is not larger, note that every C*-cover corresponds to a norm-closed ideal in $C_{max}(\mathcal{A})$. Since $\mathcal{A}$ is separable, so is $C^*_{max}(\mathcal{A})$. Hence, $C^*_{max}(\mathcal{A})$ is a second-countable space with respect to the norm topology. It is well known that the cardinality of the set of closed sets in a secound-countable space is at most $\mathfrak{c}$. Indeed, let $(U_n)_{n\in\mathbb{N}}$ be a basis for the norm topology. Then,

$$\mathcal{P}(\mathbb{N}) \to \{U \subset C^*_{max}(A);\ U \text{ open}\}, F \mapsto \bigcup_{n\in F} U_n$$

is surjective, and therefore the cardinality of open sets in $C^*_{max}(A)$ is at most $\mathfrak{c}$. Since the closed sets are precisely the complements of open sets, we obtain that also the cardinality of closed sets is at most $\mathfrak{c}$. □

3.3 Semi-Dirichlet C*-covers

The following argument answers [40, Question 3.13], which asks when $C^*_{max}(A)$ is a semi-Dirichlet C*-cover.

Theorem 3.3.1: *If A is a non-selfadjoint, semi-Dirichlet operator algebra, then $C^*_{max}(A)$ is not a semi-Dirichlet C*-cover.*

Proof:
By contradiction, assume that $[C^*_{max}(A), \iota_{max}]$ is semi-Dirichlet. Suppose $\rho : A \to B(H)$ is any completely contractive representation. By universality, there exists $\pi : C^*_{max}(A) \to C^*(\rho(A))$ such that $\pi \cdot \iota_{max} = \rho$. Then ρ must be a semi-Dirichlet representation:

$$\begin{aligned}\rho(a)^*\rho(b) &= \pi(\iota_{\max}(a)^*\iota_{\max}(b)) \\ &= \pi\left(\lim_{n\to\infty} \iota_{\max}(c_n) + \iota_{\max}(d_n)^*\right) \\ &= \lim_{n\to\infty} \rho(c_n) + \rho(d_n)^* .\end{aligned}$$

We will see that it is impossible that all representations are semi-Dirichlet.
Now, by [41, Proposition 3.4], since A is non-selfadjoint, there exists a non-maximal, completely contractive representation $\varphi : A \to B(H)$. Non-maximality implies that φ either extends or coextends non-trivially by Sarason's Lemma (see Lemma 3.1.21).
First, assume that φ extends non-trivially to a completely contractive representation $\Phi : A \to B(K)$ with $K = H \oplus H^\perp$ and block structure

$$\Phi = \begin{bmatrix} \varphi & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix}$$

where $\Phi_{12} \neq 0$. Define $\Phi' : A \to B(K)$

$$\Phi' = \begin{bmatrix} \varphi & \frac{1}{2}\Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix} .$$

Both Φ and Φ' are semi-Dirichlet representations by assumption.
Now, there exists $a \in A$ such that $\Phi_{12}(a) \neq 0$. By the semi-Dirichlet property for $\iota_{\max}$ there exist $b_n \in A$ such that

$$\iota_{\max}(a)^*\iota_{\max}(a) = \lim_{n\to\infty} \iota_{\max}(b_n) + \iota_{\max}(b_n)^* .$$

By universality, there exists $*$-homomorphisms $\pi, \pi' : C^*_{max}(A) \to B(K)$ such that $\pi \circ \iota_{\max} = \Phi$ and $\pi' \circ \iota_{\max} = \Phi'$. Following the argument at the start of this proof,

$$\begin{aligned}\Phi(a)^*\Phi(a) &= \lim_{n\to\infty} \Phi(b_n) + \Phi(b_n)^*, \quad \text{and} \\ \Phi'(a)^*\Phi'(a) &= \lim_{n\to\infty} \Phi'(b_n) + \Phi'(b_n)^* .\end{aligned}$$

But this implies that

$$\begin{bmatrix} \varphi(a)^*\varphi(a) & \varphi(a)^*\Phi_{12}(a) \\ \Phi_{12}(a)^*\varphi(a) & \Phi_{12}(a)^*\Phi_{12}(a) + \Phi_{22}(a)^*\Phi_{22}(a) \end{bmatrix}$$
$$= \lim_{n\to\infty} \begin{bmatrix} \varphi(b_n) + \varphi(b_n)^* & \Phi_{12}(b_n) \\ \Phi_{12}(b_n)^* & \Phi_{22}(b_n) + \Phi_{22}(b_n)^* \end{bmatrix}$$

and

$$\begin{bmatrix} \varphi(a)^*\varphi(a) & \frac{1}{2}\varphi(a)^*\Phi_{12}(a) \\ \frac{1}{2}\Phi_{12}(a)^*\varphi(a) & \frac{1}{4}\Phi_{12}(a)^*\Phi_{12}(a) + \Phi_{22}(a)^*\Phi_{22}(a) \end{bmatrix}$$
$$= \lim_{n\to\infty} \begin{bmatrix} \varphi(b_n) + \varphi(b_n)^* & \frac{1}{2}\Phi_{12}(b_n) \\ \frac{1}{2}\Phi_{12}(b_n)^* & \Phi_{22}(b_n) + \Phi_{22}(b_n)^* \end{bmatrix}.$$

Looking at the (2,2)-entries of both equations we see that $\frac{3}{4}\Phi_{12}(a)^*\Phi_{12}(a) = 0$ and so $\Phi_{12}(a) = 0$, a contradiction.
Essentially the same argument works in the case that φ has a nontrivial coextension, in which case the operator matrices involved are lower triangular, so we will omit those details.
Therefore, both Φ and Φ' cannot be semi-Dirichlet representations which implies that $[C^*_{max}(A), \iota_{\max}]$ is not a semi-Dirichlet C*-cover. □

Corollary 3.3.2: *If A is a non-selfadjoint, semi-Dirichlet operator algebra, then it has an infinite lattice of C*-covers.*

Proof:
By definition $[C^*_e(A), \iota_e]$ is a semi-Dirichlet C^*-cover and $[C^*_{max}(A), \iota_{\max}]$ is not a semi-Dirichlet C^*-cover. Since there are two points in the lattice then there are infinitely many by Theorem 3.2.7. □

3.4 RFD C^*-Covers

We conclude this chapter with the existence of a RFD operator algebra A such that

$$\{[(\mathcal{A}, i)];\ (\mathcal{A}, i) \text{ RFD } C^*\text{-cover of } A\},$$

equipped with the order induced by C^*-Lat(A), is not a lattice. The main idea behind the proof is to construct two C^*-covers for which the finite-dimensional representations admit a good characterization. This is achieved via an infinite product of matrix algebras M_n for increasing numbers n.
Let us begin with the construction of the operator algebra and the RFD C^*-covers.

Let $(e_n)_n$ be the standard orthonormal basis of $\ell^2(\mathbb{N})$. For $(i,j) \in \mathbb{N}^2$, denote by $E_{i,j}$ the operator in $\mathcal{B}(\ell^2(\mathbb{N}))$ defined by $E_{i,j}(e_n) = \delta_{j,n} e_i$ for all $n \in \mathbb{N}$. Let A be the operator algebra obtained as the norm closure of the algebra generated by $\{E_{i,j},\ j \geq i\}$ and $id_{\ell^2(\mathbb{N})}$. Thus, every element in A can be written as a sum of a compact upper triangular operator and $\lambda id_{\ell^2(\mathbb{N})}$ for some $\lambda \in \mathbb{C}$.
Furthermore, for $m \in \mathbb{N}$, define the projection $P_m : \ell^2(\mathbb{N}) \to \mathbb{C}^{m+1}, (x)_n \mapsto (x_0, \ldots, x_m)$. These projections induce u.c.c. homomorphisms on A via

$$\pi_{m+1} : A \to \mathcal{B}(\mathbb{C}^{m+1}), x \mapsto P_m x P_m^*.$$

We now define three representations of A:

$$\begin{aligned} i_0 &= \bigoplus_{n\in\mathbb{N}} \pi_n, \\ i_1 &= \bigoplus_{n\in\mathbb{N}} \pi_{2n+1}, \\ i_2 &= \bigoplus_{n\in\mathbb{N}} \pi_{2n}. \end{aligned}$$

Let

$$\begin{aligned} B &= C^*(i_0(A)) \subset \prod_{n\in\mathbb{N}} M_{n+1} \subset \mathcal{B}(\bigoplus_{n\in\mathbb{N}} \mathbb{C}^{n+1}), \\ B_1 &= C^*(i_1(A)) \subset \prod_{n\in\mathbb{N}} M_{2n+1} \subset \mathcal{B}(\bigoplus_{n\in\mathbb{N}} \mathbb{C}^{2n+1}), \\ B_2 &= C^*(i_2(A)) \subset \prod_{n\in\mathbb{N}} M_{2n+2} \subset \mathcal{B}(\bigoplus_{n\in\mathbb{N}} \mathbb{C}^{2n+2}). \end{aligned}$$

The first thing to check is that these are indeed RFD C^*-covers.

Proposition 3.4.1: *The pairs $(B, i_0), (B_1, i_1)$ and (B_2, i_2) are RFD C^*-covers of A.*

Proof:
For every $m \in \mathbb{N}$,

$$H_m = \left(\bigoplus_{k=1}^{m+1} \mathbb{C}^k \right) \oplus \bigoplus_{k=m+1}^{\infty} 0$$

is a reducing subspace for B, and thus induces a finite-dimensional representation of B by compression. Hence, B is a RFD C^*-algebra, since

$$\|a\| = \sup_{m\in\mathbb{N}} \|P_{H_m} a|_{H_m}\|$$

for all $a \in B$. To see that (B, i) is a C^*-cover of A, note that for all $a \in A$,

$$\|a\| = \sup_{n\in\mathbb{N}} \|\pi_n(a)\| = \|i_0(a)\|,$$

as the π_n are compressions to an increasing sequence of finite-dimensional Hilbert spaces whose union is dense in $\ell^2(\mathbb{N})$. For $a \in M_m(A) \subset \mathcal{B}(\ell^2(\mathbb{N})^m)$, the same argument applies, yielding

$$\|a\| = \sup_{n\in\mathbb{N}} \|\pi_n(a)\| = \|i_0(a)\|.$$

In the same way, one shows that (B_1, i_1) and (B_2, i_2) are RFD C^*-covers. □

The C^*-cover (B, i) is only needed in the following lemma and serves as a tool for the subsequent corollary.

Lemma 3.4.2: *For every $N \in \mathbb{N}$, it holds that*

$$\prod_{i=1}^{n} M_i \times \prod_{i=n+1}^{\infty} \{0\} \subset B.$$

Proof:

For $n \in \mathbb{N}$ and $(i, j) \in \mathbb{N}^2$, let $E_{i,j}^{(n)}$ denote the element in B whose n-th entry is $P_n E_{i,j} P_n^*$ and is 0 elsewhere. Then

$$i_0(E_{0,0}) - i_0(E_{0,1}) i_0(E_{0,1})^* = E_{0,0}^{(0)} \in B.$$

Hence,

$$M_1 \times \prod_{i=2}^{\infty} \{0\} \subset B.$$

Assume inductively that

$$\prod_{i=1}^{m} M_i \times \prod_{i=m+1}^{\infty} \{0\} \subset B.$$

for some fixed $1 \leq m \in \mathbb{N}$. For $0 \leq k \leq m$, we then have

$$i_0(E_{k,k}) - i_0(E_{k,m+1}) i_0(E_{k,m+1})^* = \sum_{i=k}^{m} E_{k,k}^{(i)}.$$

The induction hypothesis guarantees that $E_{k,k}^{(i)} \in B$ for $1 \leq i \leq m-1$ and $k \in \mathbb{N}$, hence $E_{k,k}^{(m)} \in B$ for all $0 \leq k \leq m$. Moreover, for $1 \leq i \leq j \leq m$, we have

$$E_{i,j}^{(m)} = E_{i,i}^{(m)} i_0(E_{i,j}) E_{j,j}^{(m)} \in B,$$

and since $(E_{i,j}^{(m)})^* = E_{j,i}^{(m)} \in B$, it follows that

$$\prod_{i=1}^{m+1} M_i \times \prod_{i=m+2}^{\infty} \{0\} \subset B.$$

The lemma then follows by induction. □

Corollary 3.4.3: *For every $1 \leq n \in \mathbb{N}$, it holds that*

$$\prod_{i=1}^{n} M_{2i-1} \times \prod_{i=2n}^{\infty} \{0\} \subset B_1$$
$$\prod_{i=1}^{n} M_{2i} \times \prod_{i=2n+1}^{\infty} \{0\} \subset B_2.$$

<u>Proof</u>:
For $m \in \mathbb{N}$, define $H_m = \{0\}$ if m is odd, and $H_m = \mathbb{C}^{2m}$ if m is even. Set $H = \bigoplus_{m\in\mathbb{N}} H_m$. It is clear that H is a reducing subspace for B. Identifying H with $\bigoplus_{m\in\mathbb{N}} \mathbb{C}^{2m}$ yields a unital $*$-homomorphism

$$\sigma : B \to B_2, a \mapsto P_H a|_H$$

which satisfies $\sigma(i(a)) = i_2(a)$ for all $a \in A$. By the previous lemma, we conclude for all $1 \leq n \in \mathbb{N}$ that

$$\prod_{i=1}^{n} M_{2i} \times \prod_{i=2n}^{\infty} \{0\} = \sigma\left(\prod_{i=1}^{2n} M_i \times \prod_{i=2n+1}^{\infty} \{0\}\right) \subset B_2.$$

The claim for B_1 follows analogously. □

Theorem 3.4.4: *Let $\rho : B_1 \to M_l$ be a unital *-homomorphism. Then there exists odd numbers $a_1, \ldots, a_m$ and a unitary operator $U : \mathbb{C}^l \to \bigoplus_{k=1}^{m} \mathbb{C}^{a_k}$ such that*

$$\rho = U^*(\oplus_{k=1}^{m} id_{a_k})U,$$

where $id_{a_k} : B_1 \to M_{a_k}, (x_{2n})_n \mapsto x_{a_k}$.

<u>Proof</u>:
Let $E_{i,j}^{(2n+1)}$ be the element in B_1 with $P_{2n+1}E_{i,j}P_{2n+1}$ in the $(2n+1)$-th entry and 0 elsewhere. For each odd $n \in \mathbb{N}$, the algebra J_n, generated by

$$\{E_{i,j}^{(n)}, (i,j) \in \mathbb{N}^2\},$$

is a closed two-sided ideal in B_1. By Lemma 3.1.16, the subspace $H_n = \overline{\rho(J_n)(\mathbb{C}^l)}$ is reducing for ρ. Since ρ is a finite-dimensional representation, there exist only finitely many $n_1, \ldots, n_{\tilde{m}}$ such that $H_{n_i} \neq 0$. Furthermore, Lemma 3.1.16 implies that the compression of ρ to H_{n_i} is uniquely determined by its action on J_{n_i}. Identifying $J_{n_i} \simeq M_{n_i}$, we find unitary operators U_i such that the compression of ρ to H_{n_i} is a direct sum of identity representations of M_{n_i}. Letting $U = \oplus_{i=1}^{\tilde{m}} U_i$ and $H = (\oplus_{i=1}^{\tilde{m}} H_{n_i})^{\perp}$, we obtain

$$\rho = (U^*(\oplus_{k=1}^{m} id_{a_k})U) \oplus P_H \rho|_H$$

for some $m \in \mathbb{N}$ and odd integers $a_1, \ldots, a_m \in \mathbb{N}$. It remains to show that $\rho_0 = P_H\rho|_H = 0$. For this, it suffices to show that $\rho_0(i_1(E_{i,j})) = 0$ for all $i \leq j \in \mathbb{N}$.
Fix $i \in \mathbb{N}$. The elements $i_1(E_{n,n})$ are orthogonal projections in B_1, hence only finitely many $\rho_0(i_1(E_{n,n}))$ are nonzero. Thus, there exists an odd $m \in \mathbb{N}$ such that $\rho_0(i_1(E_{m,m})) = 0$ and $i \leq m$. If i is odd, then

$$\begin{aligned}\rho_0(i_1(E_{i,i})) &= \rho_0(i_1(E_{i,i})) - \rho_0(i_1(E_{i,m}))\rho_0(i_1(E_{m,m}))\rho_0(i_1(E_{i,m})^*)\\ &= \rho_0(i_1(E_{i,i}) - i_1(E_{i,m})i_1(E_{m,m})i_1(E_{i,m})^*)\\ &= \rho_0(i_1(E_{i,i}) - i_1(E_{i,m})i_1(E_{i,m})^*)\\ &= \rho_0\left(\sum_{k=(i+1)/2}^{(m-1)/2} E_{i,i}^{(2k-1)}\right) = 0,\end{aligned}$$

since $\rho_0(J_n) = 0$ for all $n \in \mathbb{N}$. If i is even, then

$$\begin{aligned}\rho_0(i_1(E_{i,i})) &= \rho_0(i_1(E_{i,i}) - i_1(E_{i,m})i_1(E_{i,m})^*)\\ &= \rho_0\left(\sum_{k=(i+2)/2}^{(m-1)/2} E_{i,i}^{(2k-1)}\right) = 0.\end{aligned}$$

Hence, $\rho_0(i_1(E_{i,i})) = 0$, and since $i_1(E_{i,j}) = i_1(E_{i,i})i_1(E_{i,j})$, it follows that $\rho_0(i_1(E_{i,j})) = 0$ for $i \leq j$. This completes the proof. $\square$

We obtain a similar result for B_2 with an analogous proof.

Theorem 3.4.5: *Let $\rho : B_2 \to M_l$ be a unital *-homomorphism. Then there exist even numbers $b_1, \ldots, b_m$ and a unitary operator $U : \mathbb{C}^l \to \bigoplus_{k=1}^m \mathbb{C}^{b_k}$ such that*

$$\rho = U^* \oplus_{k=1}^m id_{b_k} U \oplus \rho_0,$$

*where $id_{b_k} : B_2 \to M_{b_k}, (x_{2n})_n \mapsto x_{b_k}$, and ρ_0 is a unital *-homomorphism such that $\rho_0(i_2(E_{i,j})) = 0$ for all $i \leq j$.*

Theorem 3.4.6: *The pair of RFD C^*-covers (B_1, i_1) and (B_2, i_2) has no meet in the partially ordered set of RFD C^*-covers.*

Proof:
Let (C, i_c) be a C*-cover such that $C \leq B_1$ and $C \leq B_2$. We will show that there exists no unital $*$-homomorphism $\rho : C \to M_n$ for any $1 \leq n \in \mathbb{N}$.
Assume to the contrary that such homomorphism exists. Then, there are unital $*$-homomorphisms $\rho_1 : B_1 \to M_n$ and $\rho_2 : B_2 \to M_n$ such that

$$\rho \circ i_c = \rho_1 \circ i_1 = \rho_2 \circ i_2.$$

By the two previous theorems, we obtain decompositions

$$\begin{aligned} \rho_1 &= U_1^* \left(\oplus_{k=1}^{m_1} id_{a_k}\right) U_1, \\ \rho_2 &= U_2^* \left(\oplus_{k=1}^{m_2} id_{b_k}\right) U_2, \end{aligned}$$

where each a_k odd numbers, b_k even numbers, and U_1, U_2 unitary operators.
Let a_p be the smallest of the $a_1, \ldots, a_{m_1}$, and let $k = |\{n;\ a_n = a_p\}|$. Assume that $a_p < b_n$ for all $1 \leq n \leq m_2$. Then,

$$\begin{aligned} m_2 &= \dim(\rho_2(i_2(E_{1,1}))(\mathbb{C}^n)) = \dim(\rho_1(i_1(E_{1,1}))(\mathbb{C}^n)) = m_1, \\ m_2 - k &= \dim(\rho_2(i_2(E_{a_p+1,a_p+1}))((\mathbb{C}^n)) = \dim(\rho_1(i_1(E_{a_p+1,a_p+1}))(\mathbb{C}^n)) = m_2. \end{aligned}$$

This leads to the contradiction $k = 0$.
If there is a n such that $b_n \leq a_p$, then the smallest b_p satisfies $b_p < a_p$ (since the b_n are even and the a_n are odd). In this case, we argue analogously using b_p in place of a_p.
Therefore, C has no finite-dimensional representation and hence is not RFD. $\square$

Chapter 4

Hyperrigidity

This chapter is devoted entirely to Arveson's Hyperrigidity Conjecture. Although it was already introduced in the preliminaries of Chapter 3, we will recall the conjecture here and provide a detailed background on the known partial results in the case of commutative C^*-algebras. It should be noted that there are many more papers about the conjecture concerning the case of operator systems in non-commutative C^*-algebras.
The starting point for the original definition of a hyperrigid operator system was a result by Korovkin. In [47], he showed that if a sequence of positive linear maps $\phi_n : C([0,1]) \to C([0,1])$ satisfies

$$\lim_{n\to\infty} \|\phi_n(f) - f\| = 0$$

for all $f \in \operatorname{span}(1, t, t^2)$, then

$$\lim_{n\to\infty} \|\phi_n(f) - f\| = 0$$

also holds for all $f \in C([0,1])$. In [8], Arveson defined a separable operator system S, generating a C^*-algebra $\mathcal{A}$, to be *hyperrigid* if, for every faithful representation $\mathcal{A} \subset \mathcal{B}(H)$ on a Hilbert space H and every sequence of u.c.p. maps $\phi_n : \mathcal{B}(H) \to \mathcal{B}(H)$, $n \in \mathbb{N}$, we have the implication

$$\lim_{n\to\infty} \|\phi_n(a) - a\| = 0 \text{ for all } a \in S \;\Rightarrow\; \lim_{n\to\infty} \|\phi(a) - a\| = 0 \text{ for all } a \in \mathcal{A}. \quad (4.1)$$

In [8, Theorem 2.1], Arveson showed that the above definition is equivalent to the one given in Chapter 3, namely that every restriction of a unital $*$-homomorphism on $\mathcal{A}$ has the unique extension property. One of his goals in that paper was to find good criteria implying hyperrigidity. He proved hyperrigidity of some basic examples using dilation theory and conjectured in [8, Conjecture 4.3] that

Conjecture (Arveson): *Let S be a separable operator system generating a C^*-algebra $\mathcal{A}$. If the restriction of every irreducible representation of $\mathcal{A}$ to S has the unique extension property, then S is hyperrigid.*

He showed that the conjecture holds if $\mathcal{A}$ has a countable spectrum and also proved a local version of it. He also proved that operator systems generated by $1, t, f$ in $C([0,1])$ satisfy the assumption of his conjecture if and only if f is strictly convex or strictly concave. Although he did not formally state it as a theorem, he mentioned in the introduction that an inequality by Petz [56] can be used to deduce that $\operatorname{span}(1, t, f)$ is hyperrigid for operator convex functions f. His paper concludes with a proof of a local version of the Hyperrigidity Conjecture which, up to today, remains the strongest known result for general operator systems in commutative C^*-algebras.
Eight years later, in 2015, Kleski published his work [46] about operator systems in Type I C^*-algebras. He obtained a result about u.c.p. maps with range A'', the bicommutant of A, and showed that a u.c.p. map is maximal if and only if every sequence of u.c.p. maps that converges pointwise in the weak operator topology also converges pointwise in the strong operator topology.
One year later, Brown [14] showed that the operator systems $\operatorname{span}(1, t, f) \subset C([0,1])$ considered by Arveson are hyperrigid for all strictly convex functions f. His approach was to separate two disjoint intervals in $[0,1]$ with a linear combination of $1, t, f$, and then use the fact that convex functions have left and right derivatives to establish certain upper and lower bounds for the separating function.
The next major contribution came in 2021, when Kennedy and Davidson published their work [22]. They tried to generalize the Choquet order to what they called the dilation order, and obtained a generalization of a theorem by Saskin [22, Theorm 5.3]. Of particular importance for us is that they were the first to shift perspective away from abstract operator systems: instead, they used the fact that every operator system in a commutative C^*-algebra is completely order isomorphic to the space of continuous affine functions on a compact convex set, and worked directly with the convex set.
In 2024, a remarkable paper by Bilich and Dor-On [10] demonstrated that Arveson's Hyperrigidity Conjecture is false, using a relatively simple operator system in a non-commutative Type I C^*-algebra.

In this chapter, we combine the ideas of Kennedy and Davidson, approaching operator systems in terms of compact convex sets, with those of Brown, separating disjoint sets by functions in the operator system. The one-sided derivatives in Brown's approach will be translated into supporting hyperplanes, and the upper and lower bounds will be linked to geometric properties of the compact convex set. This will lead to the main result of this chapter:

Theorem: *Let $K \subset \mathbb{R}^2$ be a non-empty compact convex set. Then the continuous affine functions $A(K) \subset C(\operatorname{ex}(K))$ are hyperrigid.*

We conclude the chapter with Bilich and Dor-On's counter example, followed by a new counter example that differs by being generated by finitely many elements. We begin with the basics concepts necessary to understand the proofs.

4.1 Preliminaries

If not further specified, K is a non-empty compact convex subset of $\mathbb{R}^2$ such that $0 \in \mathrm{Int}(K)$. This is not really a big restriction since $\mathrm{Int}(K) = \emptyset$ implies that K is a line segment, and $\mathrm{ex}(K)$ contains either one or two points, and if $\mathrm{Int}(K) \neq \emptyset$, there is a translation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $0 \in \mathrm{Int}(T(K))$.
We begin with a well-known fact: the extreme points of K are closed. Therefore, $C(\mathrm{ex}(K))$ is a C^*-algebra, and we may view $A(K)$ as an operator system within $C(\mathrm{ex}(K))$.

Proposition 4.1.1: *For every non-empty compact convex set $K \subset \mathbb{R}^2$, the set of extreme points $\mathrm{ex}(K)$ is closed, and ∂K is a rectifiable curve.*

Proof:
Let $x \in \overline{\mathrm{ex}(K)}$, and assume there exist $w \neq z \in K$ such that

$$x \in \{tw + (1-t)z;\ t \in (0,1)\}.$$

Since $\mathrm{ex}(K) \subset \partial K$, it follows that $x \in \partial K$. Denote by $[w : z]$ the unique line through w and z, and let H_1 and H_2 be the two closed half spaces bounded by $[w : z]$. We claim that for $i = 1, 2$, there exist points $z_i \in (\mathbb{R}^2 \setminus H_i) \cap \mathrm{ex}(K)$.
To construct z_1, choose $z_1 \in \mathrm{ex}(K)$ such that $|x - z_1| < \min(|x - z|, |w - z|) = \delta$. This z_1 cannot lie on $[w : z]$, because

$$[w : z] \cap B_\delta(x) \subset \{tw + (1-t)z;\ t \in (0,1)\} \subset K \setminus \mathrm{ex}(K).$$

By change of notation, we may assume $z_1 \in \mathbb{R}^2 \setminus H_1$. For the second point, choose $\epsilon > 0$ such that $B_\epsilon(x) \cap H_2 \subset \mathrm{conv}(w, z, z_1)$. Then, select $z_2 \in \mathrm{ex}(K)$ with $|x - z_2| < \epsilon$. This z_2 is not in H_2, otherwise it would be in the interior of $\mathrm{conv}(w, z, z_1)$.
It now follows that $x \in \mathrm{Int}(\mathrm{conv}(w, z, z_1, z_2))$, contradicting the assumption that $x \in \overline{\mathrm{ex}(K)}$.
The fact that ∂K is rectifiable can be found in [63]. $\square$

Next, we parametrize ∂K. Before doing so, we collect some well-known facts about convex functions and their one-sided derivatives. Parts of the following lemma and its proof can be found in [19] and other classical books on convex analysis.

Lemma 4.1.2: *Let $I \subset \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be a convex function. Then:*

(i) *The Secant lemma holds for all $a < x < b \in I$:*

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}.$$

(ii) *The right-hand derivative* $f'_+(t)$ *and left-hand derivative* $f'_-(t)$ *exist for every* $t \in I$.

(iii) f *is continuous.*

(iv) f'_+ *is right-continuous and* f'_- *is left-continuous.*

(v) $\lim_{s \to t, s<t \in I} f'_+(s) = f'_-(t)$ *and* $\lim_{s \to t, s>t \in I} f'_-(s) = f'_+(t)$.

Proof:
Let I and f be as above, and $a < x < b \in I$. Then $\lambda = \frac{b-x}{b-a} \in (0,1)$ and by convexity of f, $f(x) \leq \lambda f(a) + (1-\lambda)f(b)$. This implies that

$$f(x) - f(a) \leq (1-\lambda)(f(b) - f(a))$$

and dividing both sides by $x - a = (1-\lambda)(b-a)$ yields

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}.$$

The second inequality of part (i) follows analogously.
Part (ii) follows directly from the Secant Lemma, since

$$\frac{f(x) - f(a)}{x - a}$$

is monotone decreasing as $x \to a$ from above, and is bounded below by

$$\frac{f(z) - f(a)}{z - a}$$

for some $z < a \in I$ with $z \in I$. Such a z exists because I is open. This implies the existence of $f'_+(a)$, and a similar argument shows the existence of $f'_-(a)$.
With the existence of the one-sided derivatives, we can conclude that

$$\lim_{x \to a, x>a} f(x) - f(a) = \lim_{x \to a, x>a} (x-a)\frac{f(x) - f(a)}{x - a} \lim_{x \to a, x>a} (x-a) f'_+(a) = 0$$

and similarly,

$$\lim_{x \to a, x<a} f(x) - f(a) = \lim_{x \to a, x<a} (x-a)\frac{f(x) - f(a)}{x - a} \lim_{x \to a, x>a} (x-a) f'_-(a) = 0.$$

Therefore, f is continuous.
For part (iv), let $\epsilon > 0$ and let $(s_n)_n$ be a sequence in I such that $s_n \to t \in I$ with $t < s_n$. Let $\tilde{s} \in I$ such that

$$0 \leq \frac{f(s) - f(t)}{s - t} - f'_+(t) < \epsilon$$

for every $t < s \leq \tilde{s}$. From the Secant Lemma, we conclude that

$$0 \leq \limsup_{n\to\infty} f'_+(s_n) - f'_+(t) \leq \limsup_{n\to\infty} \frac{f(s_n) - f(\tilde{s})}{s_n - \tilde{s}} - f'_+(t) = \frac{f(t) - f(\tilde{s})}{t - \tilde{s}} - f'_+(t) < \epsilon,$$

where we also used the continuity of f from part (iii). Therefore, f'_+ is right-continuous. A similar argument shows that f'_- is left-continuous.
For part (v), let $s_n < t$ with $s_n \to t$. The Secant Lemma implies that

$$f'_-(a) \leq f'_+(a) \leq f'_-(b)$$

for $a < b \in I$. Using the left-continuity of f'_-, we obtain

$$\lim_{n\to\infty} f'_-(s_n) = \lim_{n\to\infty} f'_+(s_n) = f'_-(t).$$

The second statement in part (v) follows analogously. □

To define the parametrization, we also need the *Minkowski functional* f_K of K, defined by

$$f_K : \mathbb{R}^2 \to \mathbb{R}, x \mapsto \inf\{r \in \mathbb{R};\ r > 0 \text{ and } x \in rK\}.$$

Note that this function is well-defined since, by assumption, $0 \in \operatorname{Int}(K)$. The following properties of f_K are well knwon.

Proposition 4.1.3: *The Minkowski functional f_K is convex, continuous, and non-negative homogeneous.*

Proof:
First, note that $x \in f_K(x)K$ for all $x \in \mathbb{R}^2$, since K is closed. Let $x, y \in \mathbb{R}^2$, $t \in [0, 1]$, and let $0 < r_1, r_2 \in \mathbb{R}$ be such that $x \in r_1 K$ and $y \in r_2 K$. Then, since K is convex,

$$\frac{tr_1}{tr_1 + (1-t)r_2} x/r_1 + \frac{(1-t)r_2}{tr_1 + (1-t)r_2} y/r_2 \in K,$$

which implies

$$tx + (1-t)y \in (tr_1 + (1-t)r_2)K.$$

Taking the infimum over all such r_1, r_2 yields the inequality

$$f_K(tx + (1-t)y) \leq t f_K(x) + (1-t) f_K(y),$$

thus proving the convexity of f_K.
Next, let $0 < t \in \mathbb{R}$ and $x \in \mathbb{R}^2$. Then,

$$\begin{aligned} f_K(tx) &= \inf\{r \in \mathbb{R};\ r > 0 \text{ and } tx \in rK\} \\ &= \inf\{r \in \mathbb{R};\ r > 0 \text{ and } x \in r/tK\} \\ &= \inf\{r/t \in \mathbb{R};\ r > 0 \text{ and } tx \in rK\} = f_K(x)/t, \end{aligned}$$

which shows nonnegative homogeneity.
Finally, since $0 \in \operatorname{Int}(K)$, there exists $s > 0$ such that $B_s(0) \subset K$. Then, for any $x \in \mathbb{R}^2$, we have $\frac{s}{\|x\|}x \in K$, so

$$f_K(x) \leq \frac{\|x\|}{s},$$

which implies that f_K is continuous in 0. For general $x, y \in \mathbb{R}^2$, we can use convexity and nonnegative homogeneity to conclude:

$$f_K(x) \leq 1/2 f_K(2(x-y)) + 1/2 f_K(2y) = f_K(x-y) + f_K(y),$$

thus,

$$|f_K(x) - f_K(y)| \leq \max\{f_K(x-y), f_K(y-x)\},$$

and so the continuity of f_K follows form its continuity at 0. □

The Minkowski functional allows us to define the *polar parameterization* of the boundary of K as

$$p : \mathbb{R} \to \partial K, t \mapsto \frac{(\cos(t), \sin(t))}{f_K((\cos(t), \sin(t)))}.$$

It is clear that this is a 2π-periodic function. In the following lemma, we show that p shares properties similar to those established in Lemma 4.1.2.
Throughout this chapter, we identify $\mathbb{R}^2$ with $\mathbb{C}$ to improve the readability of our arguments. However, we emphasize that the canonical scalar product we use remains $\mathbb{R}$-linear and $\mathbb{R}$-valued.

Lemma 4.1.4: *The following statements hold:*

(i) *The left- and right-hand derivatives* $p'_-(\cdot)$ *and* $p'_+(\cdot)$ *exist.*

(ii) p'_- *is left-continuous and* p'_+ *is right-continuous.*

(iii) $\lim_{s \to t, s<t} p'_+(s) = p'_-(t)$ *and* $\lim_{s \to t, s>t} p'_-(s) = p'_+(t)$.

Proof:
We begin by proving the lemma for the restriction of p to the interval $I = (-\frac{\pi}{2}, \frac{\pi}{2})$.
Define $h(t) = 1/\cos(t)$. Then the function

$$\tilde{p}(t) = f_K(h(t)e^{it}) = f_K(1 + i\tan(t)) = f_K(1 + i(\cdot)) \circ \tan(t)$$

is the composition of the convex function $f_K(1 + i(\cdot))$ with the strictly increasing, continuously differentiable function $\tan(\cdot)$. From Lemma 4.1.2, we know that $f_K(1 + i(\cdot))$ satisfies properties $(ii), (iv)$ and (v) from Lemma 4.1.2, and hence $\tilde{p}$

also inherits these properties.
Using the nonnegative homogeneity of f_K, we observe that

$$\begin{aligned}
&h(a)\frac{f_K(e^{ia})-f_K(e^{is})}{a-s}-\frac{\tilde{p}(a)-\tilde{p}(s)}{a-s}\\
&=\frac{h(a)(f_K(e^{ia})-f_K(e^{is}))-f_K(h(a)e^{ia})+f_K(h(s)e^{is})}{a-s}\\
&=f_K(e^{is})\frac{h(s)-h(a)}{a-s}
\end{aligned}$$

for all $a\in I$, showing that $f_K(e^{i\cdot})$ is both right- and left-differentiable at a with

$$\begin{aligned}
f'_{K+}(e^{ia}) &= \frac{1}{h(a)}(\tilde{p}'_+(a)-f_K(e^{ia})h'(a))\\
f'_{K-}(e^{ia}) &= \frac{1}{h(a)}(\tilde{p}'_-(a)-f_K(e^{ia})h'(a)).
\end{aligned}$$

Therefore, f_K has properties (ii) and (iii) from Lemma 4.1.2, and we can immediately conclude that p, when restricted to $(-\frac{\pi}{2},\frac{\pi}{2})$, satisfies properties (1), (2) and (3).
To complete the proof, we repeat the same argument on the following intervals:

- $I=(0,\pi)$ with $h(t)=1/\sin(t)$,
- $I=(-\pi,0)$ with $h(t)=-1/\sin(t)$,
- $I=(\frac{\pi}{2},\frac{3\pi}{2})$ with $h(t)=-1/\cos(t)$.

□

Example 4.1.5: Let K be the unit disc. Then the polar parametrization is simply given by

$$p(t)=(\cos(t),\sin(t)).$$

The derivative is, of course, $p'(t)=(-\sin(t),\cos(t))$. Figure 4.1 visualizes the boundary ∂K as well as the vectors $p'(0)$ and $p'(\pi)$. It should be noted at this point that the direction of the vectors $p'(t)$ will become important later on.

For $t\in\mathbb{R}$, we define the subderivatives by

$$\partial^-_{p(t)}=\{p(t)+sp'_-(t),s\in\mathbb{R}\};\qquad \partial^+_{p(t)}=\{p(t)+sp'_+(t);s\in\mathbb{R}\}.$$

In the next theorem, we will show that both $\partial^-_{p(t)}$ and $\partial^+_{p(t)}$ define supporting hyperplanes. Recall that a supporting hyperplane of K is a hyperplane that contains a boundary point of K and such that K is entirely contained in one of the two closed half-spaces it determines.

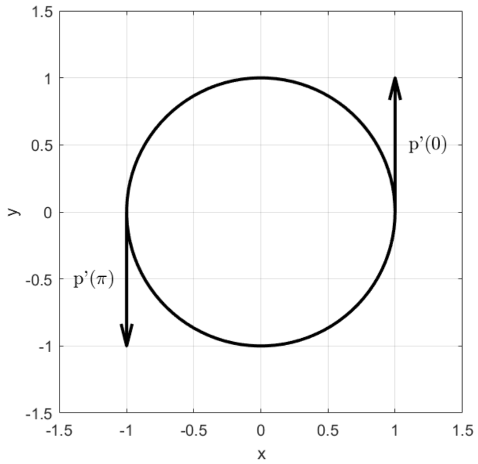

Figure 4.1: $\partial\mathbb{D}$ with $p'(0)$ and $p'(\pi)$

Example 4.1.6: Let K be the convex hull of the points $\{1, -1, i, -i\}$. Then the polar parametrization p is given by

$$p(t) = (0, 1) + \frac{1}{1 + \tan(t)}(1, -1)$$

on the interval $(0, \frac{\pi}{2})$, and

$$p(t) = (1, 0) + \frac{1}{1 - \tan(t)}(1, 1)$$

on the interval $(\frac{3\pi}{2}, 2\pi)$. Therefore,

$$p'_+(0) = (-1, 1), \quad p'_-(0) = (1, 1),$$

and

$$\begin{aligned} \partial^+_{p(0)} &= \{(0, 1) + s(-1, 1);\ s \in \mathbb{R}\}, \\ \partial^-_{p(0)} &= \{(0, 1) + s(1, 1);\ s \in \mathbb{R}\}. \end{aligned}$$

For better visualization, this example is illustrated in Figure 4.2.

Theorem 4.1.7: *For every $x \in \partial K$, the lines ∂^-_x and ∂^+_x are supporting hyperplanes of K.*

Proof:
We begin by verifying that ∂^-_x and ∂^+_x are indeed hyperplanes and not just points. It suffices to show that $p'_-(t) \neq 0$ and $p'_+(t) \neq 0$.

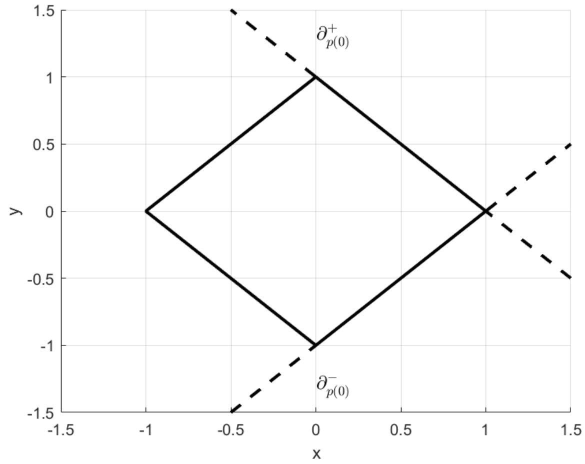

Figure 4.2: An example of ∂K with $\partial^+_{p(0)}$ and $\partial^-_{p(0)}$

Recall from the proof of Lemma 4.1.4 that $f_K(e^{i\cdot})$ is differentiable from both left and the right. From this, we can derive the following inequality:

$$\begin{aligned}
\lim_{s\to t, s<t}\left|\frac{p(t)-p(s)}{t-s}\right| &= \lim_{s\to t, s<t}\frac{1}{f_K(e^{is})f_K(e^{it})}\left|\frac{f_K(e^{is})e^{it}-f_K(e^{it})e^{is}}{t-s}\right| \\
&= \frac{1}{f_K(e^{it})^2}\lim_{s\to t, s<t}\left|\frac{f_K(e^{is})e^{it}-f_K(e^{is})e^{is}+f_K(e^{is})e^{is}-f_K(e^{it})e^{is})}{t-s}\right| \\
&= \frac{1}{f_K(e^{it})^2}\lim_{s\to t, s<t}\left|f_K(e^{is})\frac{e^{it}-e^{is}}{t-s}+e^{is}\frac{f_K(e^{is})-f_K(e^{it})}{t-s}\right| \\
&= \frac{1}{f_K(e^{it})^2}\left|ie^{it}f_K(e^{it})+e^{it}f_K(e^{i\cdot})'_-(t)\right| \\
&= \frac{1}{f_K(e^{it})^2}\left|if_K(e^{it})+f_K(e^{i\cdot})'_-(t)\right| \\
&\geq \frac{1}{f_K(e^{it})} > 0.
\end{aligned}$$

An analogous estimate shows that $|p'_+(t)| \geq \frac{1}{f_K(e^{it})^2}|if_K(e^{it}) + f_K(e^{i\cdot})'_+(t)| \geq \frac{1}{f_K(e^{it})} > 0$. Hence, both derivatives are nonzero, and ∂^-_x, ∂^+_x are indeed hyperplanes.

We now prove that these hyperplanes are supporting hyperplanes of K. Let $x = p(t) \in \partial K$ with $t \in [0, 2\pi)$, and let $(t_n)_n$ be a sequence in $(0, 2\pi)$ such that $t_n \to t$ and $t_n > t$. The line through the points $p(t)$ and $p(t_n)$ divides $\mathbb{R}^2$ into two closed half planes:

$$H_1^n = \left\{ z \in \mathbb{R}^2;\ \left\langle z - p(t), \frac{i(p(t) - p(t_n))}{t - t_n} \right\rangle \geq 0 \right\},$$
$$H_2^n = \left\{ z \in \mathbb{R}^2;\ \left\langle z - p(t), \frac{i(p(t) - p(t_n))}{t - t_n} \right\rangle \leq 0 \right\}.$$

By the continuity of the polar parameterization, we know that

$$\{p(s);\ s \in [0, 2\pi) \setminus (t, t_n)\}$$

is entirely contained in either H_1^n or H_2^n. By the pigeonhole principle, for each n we may assume (after possibly passing to a subsequence) that

$$p([0, 2\pi) \setminus (t, t_n)) \subset H_i^n$$

for some fixed $i \in \{1, 2\}$. On the other hand, the two closed half spaces bounded by ∂_x^+ are given by

$$\tilde{H}_1 = \{z \in \mathbb{R}^2;\ \langle z - p(t), ip'_+(t)\rangle \geq 0\},$$
$$\tilde{H}_2 = \{z \in \mathbb{R}^2;\ \langle z - p(t), ip'_+(t)\rangle \leq 0\}.$$

Since $p'_+(t) = \lim_{n\to\infty} \frac{p(t)-p(t_n)}{t-t_n}$, either $\tilde{H}_1$ or $\tilde{H}_2$ contains

$$\{p(s);\ s \in [0, 2\pi)\}.$$

Therefore, ∂_x^+ is a supporting hyperplane. An analogous argument shows that ∂_x^- is also a supporting hyperplane. □

Remark 4.1.8: If there is only one supporting hyperplane ∂_x at a point $x \in \partial K$, then Theorem 4.1.7 implies that

$$\partial_x^+ = \partial_x^- = \partial_x.$$

In particular, let $[a, b] = I \subset [0, 2\pi)$ be such that $\{p(t);\ t \in [a, b]\}$ lies entirely in a face of K. Then for every $s \in (a, b)$, there exists exactly one supporting hyperplane at $p(s)$, namely the line through $p(a)$ and $p(b)$, which we denote by $[p(a) : p(b)]$. Hence,

$$\partial_{p(s)}^+ = \partial_{p(s)}^- = [p(a) : p(b)]$$

for all $s \in (a, b)$. Furthermore, by Lemma 4.1.4, we also obtain

$$\partial_{p(a)}^+ = [p(a) : p(b)] = \partial_{p(b)}^-.$$

The next step is to define a function $\sphericalangle(\cdot,\cdot) : [0,2\pi]^2 \to [0,\pi]$ that measures the angle between two subderivatives. To do this, we introduce the following two notational conventions. For three distinct points $x, y, z \in \mathbb{R}^2$, we denote by $\triangle(x,y,z)$ the triangle with vertices x, y, z, and by $\angle(x,y,z)$ the angle at the vertex y.
Now define for $s \le t$,

$$\sphericalangle(s,t) = \begin{cases} 0 & \text{if } \partial^+_{p(s)} \cap \partial^-_{p(t)} = \emptyset \\ \pi & \text{if } \partial^+_{p(s)} = \partial^-_{p(t)} \\ \cos^{-1}\left(\frac{\langle p'_-(t), -p'_+(s)\rangle}{|p'_-(t)||p'_+(s)|}\right) & \text{if } \partial^+_{p(s)} \cap \partial^-_{p(t)} = \{z\} \text{ and} \\ & \{p(l);\ l \in [s,t]\} \subset \triangle(p(s), z, p(t)) \\ 0 & \text{otherwise.} \end{cases}$$

For $s > t$, we set $\sphericalangle(s,t) = \sphericalangle(t,s)$. Note that in the third case, we have

$$\sphericalangle(s,t) = \angle(p(s), z, p(t)) \tag{4.2}$$

for $s \neq t$.

Remark 4.1.9: A careful reader might wonder why $\angle(p(s), z, p(t))$ is not given by

$$\cos^{-1}\left(\frac{\langle p'_-(t), p'_+(s)\rangle}{|p'_-(t)||p'_+(s)|}\right).$$

To justify Eq. (4.2), it suffices to show that for every $t \in [0, 2\pi)$, the set K lies to the left of the line $\{p(t) + sp'_-(t);\ s \in \mathbb{R}\}$, respectively $\{p(t) + sp'_+(t);\ s \in \mathbb{R}\}$. Since $0 \in \mathrm{Int}(K)$, this follows directly from

$$\begin{aligned} \langle 0 - p(t), ip'_+(t)\rangle &= \lim_{n\to\infty, t_n > t} \left\langle -p(t), i\frac{p(t) - p(t_n)}{t - t_n}\right\rangle \\ &= \lim_{n\to\infty, t_n > t} \frac{1}{f_K(e^{it}) f_K(e^{it_n})(t - t_n)} \langle(\cos(t), \sin(t)), (-\sin(t_n), \cos(t_n))\rangle \\ &= \lim_{n\to\infty, t_n > t} \frac{-1}{f_K(e^{it}) f_K(e^{it_n})(t_n - t)} (-\cos(t)\sin(t_n) + \sin(t)\cos(t_n)) \ge 0, \end{aligned}$$

where the last inequality holds for $t \neq \pi/2$ and $t \neq 3/2\pi$, and for t_n sufficiently close to t, due to

$$\frac{\sin(t)}{\cos(t)} = \tan(t) \le \tan(t_n) = \frac{\sin(t_n)}{\cos(t_n)},$$

and $\sin(t)\cos(t_n) \le 0$ for $t = \pi/2$ or $t = 3/2\pi$.
A similar computation shows that $\langle 0 - p(t), ip'_-(t)\rangle \ge 0$.

Lemma 4.1.10: *Let $(s_n), (t_n)$ be sequences in $[0, 2\pi]$ such that $s_n, t_n \to t$ and $s_n < t < t_n$. Then:*

(i) $\lim_{n\to\infty} \sphericalangle(s_n, t_n) = \sphericalangle(t, t)$.

If $t \leq s_n < t_n$ *and* $s_n, t_n \to t$*, then:*

(i) $\lim_{n\to\infty} \sphericalangle(s_n, t_n) = \pi$.

If $t_n < s_n \leq t$ *and* $s_n, t_n \to t$*, then:*

(i) $\lim_{n\to\infty} \sphericalangle(t_n, s_n) = \pi$,

Proof:
We first show that for every $x \in \mathbb{R}$, there exists $\epsilon > 0$ such that for all $s < t \in B_\epsilon(x)$, either $\partial^+_{p(s)} = \partial^-_{p(t)}$ or $\partial^+_{p(s)} \cap \partial^-_{p(t)} = \{z\}$ and

$$\{p(l);\ l \in [s,t]\} \subset \triangle(p(s), z, p(t)).$$

Assume the contrary. Then for every $n \in \mathbb{N}$, there exists $s_n < t_n \in B_{1/n}(x)$ such that either $\partial^+_{p(s_n)} \cap \partial^-_{p(t_n)} = \emptyset$ or $\partial^+_{p(s_n)} \cap \partial^-_{p(t_n)} = \{z_n\}$ and

$$\partial K \setminus \{p(l);\ l \in [s_n, t_n]\} \subset \triangle(p(s_n), z_n, p(t_n)).$$

In the first case, K lies between the two parallel lines $\partial^+_{p(s_n)}$ and $\partial^-_{p(t_n)}$. In the second case, we get

$$\{p(t);\ t \in [x-\pi, x+\pi] \setminus (s_n, t_n)\} \subset \triangle(p(s_n), z_n, p(t_n)).$$

Since $s_n, t_n \to x$ and p is continuous, this contradicts the assumption that $0 \in \mathrm{Int}(K)$.
The lemma now follows from Eq. (4.2) and Lemma 4.1.4. $\square$

Lemma 4.1.11: *Let* $a < b \in [0, 2\pi)$ *and* $\epsilon > 0$*. Then there exists points* $a = t_0 < t_1 < \cdots < t_m = b$ *such that*

$$\sphericalangle(t_n, t_{n+1}) \geq \pi - \epsilon.$$

Proof:
First, we prove that the set

$$\{x \in (a,b);\ \sphericalangle(x,x) \leq \pi - \epsilon\}$$

contains only finitely many points $s_1 < s_2 < \cdots < s_k$. Assume otherwise. Then, for every n, we could construct a convex polygon with n vertices, each of which has angle strictly less than $\pi - \epsilon$. But the sum of the internal angles of such a

polygon is $(n-2)\pi$, which contradicts the assumption for sufficiently large n.
Let $s_0 = a, s_{k+1} = b$. We now show that there exists $\delta > 0$ such that

$$\sphericalangle(x,y) \geq \pi - \epsilon$$

for all $x, y \in [s_n, s_{n+1}], n = 0, \ldots, k$, with $|x-y| < \delta$. If this were not the case, then there would exist $m \in \{0, \ldots, k\}$ and sequences $x_n < y_n \in [s_m, s_{m+1}]$ such that $|x_n - y_n| < 1/n$ and

$$\sphericalangle(x_n, y_n) \leq \pi - \epsilon.$$

Without loss of generality, assume that $(x_n)_n$ and $(y_n)_n$ both converge to some $z \in [s_m, s_{m+1}]$. Then by Lemma 4.1.10, we either have

$$\lim_{n\to\infty} \sphericalangle(x_n, y_n) = \pi$$

if $z \leq x_n$ or $y_n \leq z$, or

$$\lim_{n\to\infty} \sphericalangle(x_n, y_n) = \sphericalangle(z, z)$$

if $x_n < z < y_n$. but in this case, $\sphericalangle(z,z) > \pi - \epsilon$ by the choice of the points $s_1, \ldots, s_k$, contradicting the assumption $\sphericalangle(x_n, y_n) \leq \pi - \epsilon$.
Thus, we obtain the desired partition as a refinement of $s_0 < \cdots < s_{k+1}$. □

The final technique we introduce in the preliminaries will be used in the construction of our counterexample to Arveson's Hyperrigidity Conjecture and is taken from [2]. It determines the optimal constant by which a rank-one operator must be scaled to be bounded above by a multiplication operator.

Theorem 4.1.12: *Let $K \subset \mathbb{R}^d$ be compact for some $1 \leq d \in \mathbb{N}$, and let μ be a probability measure on K. Then, for every function $g \in L^2(K, \mu)$ and every positive function $f \in L^\infty(K, \mu)$, we have:*

$$\inf\{c \geq 0;\ P_g \leq cM_f\} = \int_K \frac{|g|^2}{f} d\mu,$$

where P_g denotes the rank-one operator defined by $P_g(\cdot) = \langle \cdot, g\rangle g$.

<u>Proof</u>:
Let K, g, f, μ be as above, and let $h \in L^2(K, \mu)$. Then:

$$\begin{aligned}
\langle P_g(h), h\rangle &= \left| \int_K h\bar{g} d\mu \right|^2 \\
&= \left| \int_K f^{1/2} h \bar{g} f^{-1/2} d\mu \right|^2 \\
&= \langle f^{1/2}h, g f^{-1/2}\rangle^2 \\
&\leq \|f^{1/2}h\|^2 \|g f^{-1/2}\|^2 \\
&= \langle M_f h, h\rangle \int_K \frac{|g|^2}{f} d\mu.
\end{aligned}$$

This proves $\leq$ in the statement.
For the reverse inequality $\geq$, let $h = f^{-1}g$. Then all the inequalities in the chain above become equalities, and the result follows. □

This result has a particular elegant analog in the setting of matrices.

Theorem 4.1.13: *Let $1 \leq n \in \mathbb{N}$, and let $A \in M_n$ be a positive, invertible matrix and $P \in M_n$ a projection. Then,*

$$\operatorname{tr}(PA^{-1}) \leq 1$$

if and only if $P \leq A$.

Proof:
Let A and P be as above. Then $A \geq P$ if and only if

$$id \geq A^{-1/2} P A^{-1/2}.$$

Since P is a projection, $A^{-1/2} P A^{-1/2}$ is a rank-one operator. Thus, the inequality holds if and only if the (unique) non-zero eigenvalue of $A^{-1/2} P A^{-1/2}$ is less than or equal to one. Since the trace equals the sum of the eigenvalues, we conclude that $A \geq P$ if and only if

$$\operatorname{tr}(PA^{-1}) = \operatorname{tr}(A^{-1/2} P A^{-1/2}) \leq 1.$$

□

4.2 Hyperrigidity of $A(K)$

Let $K \subset \mathbb{R}^2$ be compact and convex. If $\operatorname{Int}(K) = \emptyset$, then K is merely a line segment, and thus $A(K) = C(\operatorname{ex}(K))$. In this case, $A(K)$ is trivially hyperrigid in $C(\operatorname{ex}(K))$.
Now consider the case where $x \in \operatorname{Int}(K)$. Define $\tilde{K} = \{y - x;\ y \in K\}$, and consider the map

$$j : C(\operatorname{ex}(K)) \to C(\operatorname{ex}(\tilde{K})), f \mapsto f(\cdot + x),$$

where we do not need to take the closure of $\operatorname{ex}(K)$, since the set of extreme points of a compact convex subset of $\mathbb{R}^2$ is always closed by Proposition 4.1.1. It is clear that j is a unital *-isomorphism mapping $A(K)$ onto $A(\tilde{K})$. Therefore, showing that $A(K)$ is hyperrigid in $C(\operatorname{ex}(K))$ is equivalent to proving that $A(\tilde{K})$ is hyperrigid in $C(\operatorname{ex}(\tilde{K}))$. Since $0 \in \operatorname{Int}(\tilde{K})$ by construction, we will henceforth assume that $0 \in \operatorname{Int}(K)$, allowing us to use the polar parametrization p.
Let H be a Hilbert space, $\pi : C(\operatorname{ex}(K)) \to \mathcal{B}(H)$ a unital *-homomorphism, and

$\phi : C(\mathrm{ex}(K)) \to \mathcal{B}(H)$ be a u.c.p. map such that $\pi = \phi$ on $A(K)$. For Borel sets $F \subset \mathbb{R}^2$, define

$$\phi(\chi_F) = \mu(F \cap \mathrm{ex}(K)),$$
$$\pi(\chi_F) = \nu(F \cap \mathrm{ex}(K)),$$

where μ and ν denote the positive operator-valued measure corresponding to ϕ and π.

We now briefly outline the strategy behind the proof of the main theorem of this chapter. As in Brown's approach, we aim to show that

$$\pi(\chi_{p(I)})\phi(\chi_{p(J)}) = 0$$

for all disjoint closed intervals $I, J \subset [0, 2\pi)$. Form this, it wont be too far to see that $\pi = \phi$. To show the above equality, we will cover I with finitely many disjoint intervals I_i, small enough so that we may apply [14, Corollary 1.2]. For convenience, we restate that corollary here:

Lemma 4.2.1: *Let $H = H_1 \oplus \ldots, H_n$ be a direct sum of Hilbert spaces, A a positive operator on H and P_i the orthogonal projection onto H_i. Then*

$$\|A\| \leq \sum_{i=1}^{n} \|P_i A P_i\|.$$

In our context, we let $A = \pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)})$ and $H_i = \pi(\chi_{p(I_i)})(H)$, where $(I_i)_{i=1}^n$ is a finite family of disjoint Borel subsets such that $I = \bigcup_{i=1}^n I_i$. Then the lemma gives:

$$\|\pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)})\| \leq \sum_{i=1}^{n} \|\pi(\chi_{p(I_i)})\phi(\chi_{p(J)})\pi(\chi_{p(I_i)})\|.$$

It thus suffices to estimate the individual terms on the right-hand side. Dropping the index i for simplicity, let $f \in A(K)$ be such that $\chi_{p(J)} \leq f$. Then the positivity of ϕ implies

$$\pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)}) \leq \pi(\chi_{p(I)})\phi(f)\pi(\chi_{p(I)}) = \pi(f\chi_{p(I)}). \tag{4.3}$$

Our next goal is to construct such a function f. Let $I = [a, b] \subset [0, 2\pi)$ with $a < b$, and define g_I to be the unique composition of a translation and a rotation such that

$$\mathrm{Im}(g(p(a))) = \mathrm{Im}(g(p(b))) = 0, \quad \mathrm{Re}(g(p(a))) < \mathrm{Re}(g(p(b))) = 0,$$

and assume that

$$\inf_{p(J)} |\mathrm{Im}(g_I)| > 0.$$

Then the function
$$\tilde{g}_I = \frac{\mathrm{Im}(g_I) + \|\mathrm{Im}(g_I)\|_{p(I),\infty}}{\inf_{p(J)}|\mathrm{Im}(g_I)|}$$
is an affine function which, when used in Eq. (4.3), yields the estimate
$$\|\pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)})\| \leq \frac{\|\mathrm{Im}(g_I)\|_{p(I),\infty}}{\inf_{p(J)}|\mathrm{Im}(g_I)|},$$
since $\pi(\mathrm{Im}(g_I)\chi_{p(I)}) \leq 0$.
Therefore, we establish an upper bound for $\|\mathrm{Im}(g_I)\|_{p(I),\infty}$ in Section 4.2.1 and a lower bound for $\inf_{p(J)}|\mathrm{Im}(g_I)|$ in Section 4.2.2.

4.2.1 The upper bound for $\|\mathrm{Im}(g_I)\|_{p(I),\infty}$

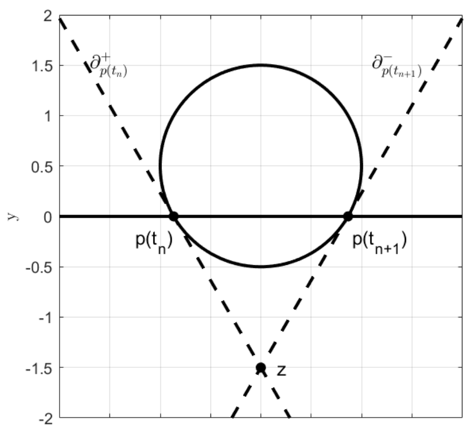

Figure 4.3: An illustration for the proof of Lemma 4.2.2

Lemma 4.2.2: *Let $I = [a,b], J \subset [0,2\pi)$ be closed disjoint intervals with $a < b$. Then for every $\epsilon > 0$, there exists a partition $a = t_0 < t_1 < \cdots < t_m = b$ such that*
$$\|\mathrm{Im}(g_{[t_n,t_{n+1}]})\|_{p([t_n,t_{n+1}]),\infty} \leq \epsilon|p(t_n) - p(t_{n+1})|$$
for every $n = 0,\ldots,m-1$.

The idea of the proof is illustrated in Figure 4.3.

Proof:
Let $\pi/2 > \epsilon > 0$. By Lemma 4.1.2, there exists a partition $a = t_0 < t_1 < \cdots < t_m = b$ such that
$$\sphericalangle(t_n, t_{n+1}) \geq \pi - \epsilon$$

for every $0 \leq n \leq m-1$. Fix such an n. For clarity, write $g = g_{[t_n,t_{n+1}]}$ and $I_n = [t_n, t_{n+1}]$. If $\partial^+_{p(t_n)} = \partial^-_{p(t_{n+1})}$, then $\|\mathrm{Im}(g)\|_{p(I_n),\infty} = 0$. Thus, we may assume that $\partial^+_{p(t_n)} \cap \partial^-_{p(t_{n+1})} = \{z\}$ for some $z \in \mathbb{R}^2$.
By the definition of $\sphericalangle(\cdot,\cdot)$ and our choice of the partition, we have

$$p(I_n) \subset \triangle(p(t_n), z, p(t_{n+1})),$$

and hence

$$\|\mathrm{Im}(g)\|_{p(I_n),\infty} \leq |\mathrm{Im}(g(z))|.$$

Moreover, since

$$\sphericalangle(t_n, t_{n+1}) \geq \pi - \epsilon > \pi/2,$$

we conclude that $\angle(p(t_{n+1}), p(t_n), z) < \pi/2$ and $\angle(z, p(t_{n+1}), p(t_n)) < \pi/2$. This implies

$$\mathrm{Re}(g(p(t_{n+1}))) \leq \mathrm{Re}(g(z)) \leq \mathrm{Re}(g(p(t_n))).$$

Putting everything together and applying elementary geometry, we obtain

$$\begin{aligned}
\frac{\|\mathrm{Im}(g)\|_{p(I_n),\infty}}{|p(t_n) - p(t_{n+1})|} &= \frac{\|\mathrm{Im}(g)\|_{p(I_n),\infty}}{|\mathrm{Re}(g(p(t_n))) - \mathrm{Re}(g(p(t_{n+1})))|} \\
&\leq \frac{|\mathrm{Im}(g(z))|}{|\mathrm{Re}(g(p(t_n))) - \mathrm{Re}(g(z))|} \\
&= \tan(\angle(g(p(t_{n+1})), g(p(t_n)), g(z)) \\
&= \tan(\angle(p(t_{n+1}), p(t_n), z) \\
&\leq \tan(\pi - \sphericalangle(t_n, t_{n+1})) \leq \tan(\epsilon).
\end{aligned}$$

Thus, the lemma follows by choosing ϵ sufficiently small. □

4.2.2 The lower bound for $\inf_{p(J)}|\mathrm{Im}(g_I)|$

For two sets $A, B \subset \mathbb{R}^2$, we define the distance between A and B as

$$\mathrm{dist}(A, B) = \inf\{|x - y|;\ x \in A, y \in B\}.$$

Lemma 4.2.3: *Let I, J be closed, disjoint intervals in $[0, 2\pi)$, and $I = [\alpha, \beta]$ with $\alpha < \beta$. Then*

$$\inf_{p(J)}|\mathrm{Im}(g_I)| = \mathrm{dist}(g_I(J), \mathbb{R} \times \{0\}) \geq \mathrm{dist}(p(J), \partial^+_{p(\alpha)} \cup \partial^-_{p(\beta)} \cup [p(\alpha) : p(\beta)]),$$

where $[p(\alpha) : p(\beta)]$ denotes the hyperplane through $p(\alpha)$ and $p(\beta)$.

Proof:
First observe that

$$\begin{aligned}
\inf_{p(J)}|\mathrm{Im}(g_I)| &= \inf\{|x - g_I(y)|;\ x \in \mathbb{R} \times \{0\}, y \in p(J)\} \\
&= \mathrm{dist}(g_I(p(J)), \mathbb{R} \times \{0\}).
\end{aligned} \tag{4.4}$$

Now, note that $g_I([p(\alpha) : p(\beta)]) = \mathbb{R} \times \{0\}$, since $g_I(p(\alpha)), g_I(p(\beta)) \in \mathbb{R} \times \{0\}$ by construction of g_I. Furthermore, since g_I is a composition of a translation and a rotation, it preserves Euclidean distances:

$$|g_I(x) - g_I(y)| = |x - y| \text{ for all } x, y \in \mathbb{R}^2.$$

Combining these observations with Eq. (4.4) yields

$$\inf_{p(J)} |\mathrm{Im}(g_I)| = \mathrm{dist}(g_I(J), \mathbb{R} \times \{0\}) \geq \mathrm{dist}(p(J), [p(\alpha) : p(\beta)]),$$

which obviously implies the claim. □

The following lemma and its proof are visualized in Figure 4.4. The lemma is

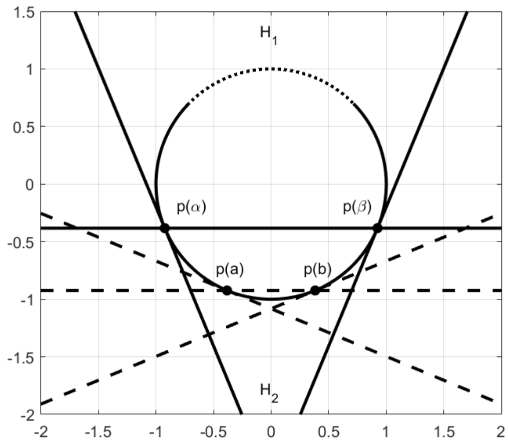

Figure 4.4: $\partial\mathbb{D}$ with $\alpha = 9/8\pi$, $a = 11/8\pi$, $b = 13/8\pi$ and $\beta = 15/8\pi$. The hyperplanes $\partial^+_{p(\alpha)}, \partial^-_{p(\beta)}$ and $[p(\alpha) : p(\beta)]$ are solid, $\partial^+_{p(a)}, \partial^-_{p(b)}$ and $[p(a) : p(b)]$ are dashed. The set $p(J)$ is given by the dotted line.

geometrically intuitive, but the rigorous argument is more involved.
Before proceeding to the proof, we highlight a key geometric fact the reader should keep in mind: if $\alpha < \beta \in [0, 2\pi)$, and a supporting hyperplane contains both $p(\alpha)$ and $p(\beta)$, then either $p([\alpha, \beta])$ or $p([0, 2\pi) \setminus (\alpha, \beta))$ is contained in a face of K.

Lemma 4.2.4: *Let $I = [\alpha, \beta]$ and $J \subset [0, 2\pi)$ be disjoint closed intervals and $[a, b] \subset I \subset [0, 2\pi)$ with $a < b$. Then*

$$\mathrm{dist}(p(J), \partial^+_{p(\alpha)} \cup \partial^-_{p(\beta)} \cup [p(\alpha) : p(\beta)]) \leq \mathrm{dist}(p(J), \partial^+_{p(a)} \cup \partial^-_{p(b)} \cup [p(a) : p(b)]).$$

Proof:
First, note that if $p([0, 2\pi) \setminus (\alpha, \beta)) \subset [p(\alpha) : p(\beta)]$, then

$$\mathrm{dist}(p(J), \partial^+_{p(\alpha)} \cup \partial^-_{p(\beta)} \cup [p(\alpha) : p(\beta)]) = 0$$

and there is nothing to prove. While this case may seem trivial, we will need to exclude it later in our arguments.
So assume that $p([0,2\pi)\setminus(\alpha,\beta))$ is not be contained in $[p(\alpha):p(\beta)]$. The following construction, illustrated in Figure 4.4, will guide us. The hyperplanes $\partial^+_{p(\alpha)}$ and $\partial^-_{p(\beta)}$ divide $\mathbb{R}^2$ into up to four convex sets. One of these sets contains K and is, as established in Remark 4.1.9, given by:

$$H = \{z \in \mathbb{R}^2;\ \langle z - p(\beta), ip'_-(\beta)\rangle \geq 0,\ \langle z - p(\alpha), ip'_+(\alpha)\rangle \geq 0\}.$$

Moreover, the hyperplane $[p(\alpha):p(\beta)]$ further subdivides H into two closed convex sets:

$$\begin{aligned} H_1 &= H \cap \{z \in \mathbb{R}^2;\ \langle z - p(\alpha), i(p(\beta) - p(\alpha))\rangle \geq 0\}, \\ H_2 &= H \cap \{z \in \mathbb{R}^2;\ \langle z - p(\alpha), i(p(\beta) - p(\alpha))\rangle \leq 0\}. \end{aligned}$$

Similarly to Remark 4.1.9, one can show that $p(J) \subset H_1$ and $p([\alpha,\beta]) \subset H_2$. Let us quickly argue why it suffices to show that

$$(\partial^+_{p(a)} \cup \partial^-_{p(b)} \cup [p(a):p(b)]) \cap \operatorname{Int}(H_1) = \emptyset. \tag{4.5}$$

AssumeEq. (4.5) holds, but the lemma is false. Then there exists $z \in \partial^+_{p(a)} \cup \partial^-_{p(b)} \cup [p(a):p(b)]$ and $t \in J$ such that

$$\|p(t) - z\| < \operatorname{dist}(p(J), \partial^+_{p(\alpha)} \cup \partial^-_{p(\beta)} \cup [p(\alpha):p(\beta)]).$$

This inequality together with Eq. (4.5) and the observation $\partial H_1 \subset \partial^+_{p(\alpha)} \cup \partial^-_{p(\beta)} \cup [p(\alpha):p(\beta)]$, imply that $z \notin H_1$, and therefore we either have

$$\langle z - p(\beta), ip'_-(\beta)\rangle < 0$$

or

$$\langle z - p(\alpha), ip'_+(\alpha)\rangle < 0$$

or

$$\langle z - p(\alpha), i(p(\beta) - p(\alpha)) < 0.$$

Since $p(t) \in p(J) \subset H_1$, in each of the three cases above, there exists an $s \in [0,1]$ such that the convex combination $sz + (1-s)p(t)$ lies on either $\partial^+_{p(\alpha)}, \partial^-_{p(\beta)}$, or $[p(\alpha):p(\beta)]$. This leads to the contradiction:

$$\begin{aligned} \|p(t) - z\| &< \operatorname{dist}(p(J), \partial^+_{p(\alpha)} \cup \partial^-_{p(\beta)} \cup [p(\alpha):p(\beta)]) \\ &\leq \|p(t) - (sz + (1-s)p(t))\| = s\|(p(t) - z)\|, \end{aligned}$$

which is impossible since $z \notin \partial^+_{p(\alpha)} \cup \partial^-_{p(\beta)} \cup [p(\alpha):p(\beta)]$ and therefore $s < 1$.
It remains to verify Eq. (4.5). Suppose there exists z in the left-hand side of Eq. (4.5). Then, since $z \in \operatorname{Int}(H_1)$, we have:

$$\begin{aligned} &\langle z - p(\alpha), i(p(\beta) - p(\alpha))\rangle > 0, \\ &\langle z - p(\alpha), ip'_+(\alpha)\rangle > 0, \\ &\langle z - p(\beta), ip'_-(\beta)\rangle > 0. \end{aligned}$$

Furthermore, since $p([a,b]) \subset p([\alpha,\beta]) \subset H_2$, we also have:

$$\langle p(a) - p(\alpha), i(p(\beta) - p(\alpha))\rangle \leq 0,$$
$$\langle p(b) - p(\alpha), i(p(\beta) - p(\alpha))\rangle \leq 0.$$

Thus, there exists a point $w \in \{tz + (1-t)p(a);\ t \in [0,1]\}$ such that

$$\langle \omega - p(\alpha), i(p(\beta) - p(\alpha))\rangle = 0.$$

We also know that $w \in H \cap [p(\alpha) : p(\beta)]$, since both z and $p(a)$ lie in H.
Next, we show that $\omega \in \{tp(\alpha) + (1-t)p(\beta);\ t \in (0,1)\}$. Let $s \in \mathbb{R}$ such that $w = sp(\alpha) + (1-s)p(\beta)$, and observe that

$$0 \leq \langle w - p(\alpha), ip'_+(\alpha)\rangle = (1-s)\langle p(\beta) - p(\alpha), ip'_+(\alpha)\rangle$$

and similarly,

$$0 \leq \langle w - p(\beta), ip'_-(\beta)\rangle = s\langle p(\alpha) - p(\beta), ip'_-(\beta)\rangle.$$

Hence, it suffices to show that

$$\langle p(\beta) - p(\alpha), ip'_+(\alpha)\rangle > 0 \text{ and}$$
$$\langle p(\alpha) - p(\beta), ip'_-(\beta)\rangle > 0.$$

Assume the contrary. Then either

$$\langle p(\beta) - p(\alpha), ip'_+(\alpha)\rangle = 0 \text{ or}$$
$$\langle p(\alpha) - p(\beta), ip'_-(\beta)\rangle = 0.$$

Since $p(\beta), p(\alpha) \in H$, it follows that both are contained in either $\partial^+_{p(\alpha)}$ or $\partial^-_{p(\beta)}$. According to the note preceding the lemma, we must then have either $p([\alpha,\beta]) \subset [p(\alpha) : p(\beta)]$ or $p([0,2\pi) \setminus (\alpha,\beta)) \subset [p(\alpha) : p(\beta)]$.
The first case implies that $\partial^+_{p(a)} = \partial^-_{p(b)} = [p(a) : p(b)] = [p(\alpha) : p(\beta)]$ by Remark 4.1.8, contradicting the existence of the point z. The second case contradicts our initial assumption, namely that $p([0,2\pi) \setminus (\alpha,\beta))$ is not contained in $[p(\alpha) : p(\beta)]$.
Thus, there exists $s \in (0,1)$ such that $\omega = sp(\alpha) + (1-s)p(\beta)$. We now conclude the proof via a case distinction:
(1) If $z \in \partial^+_{p(a)}$, then, since $z \in \operatorname{Int}(H_1)$, we must have $\partial^+_{p(a)} \neq [p(\alpha) : p(\beta)]$. Moreover, we saw that $w \in \{tz + (1-t)p(a);\ t \in [0,1]\} \subset \partial^+_{p(a)}$ and hence

$$\langle w - p(a), ip'_+(a)\rangle = 0.$$

But, since $\partial^+_{p(a)} \neq [p(\alpha) : p(\beta)]$, we have

$$\langle p(\alpha) - p(a), ip'_+(a)\rangle > 0 \text{ or } \langle p(\beta) - p(a), ip'_+(a)\rangle > 0,$$

and it follows that

$$0 = \langle w - p(a), ip'_+(a)\rangle = s\langle p(\alpha) - p(a), ip'_+(a)\rangle + (1-s)\langle p(\beta) - p(a), ip'_+(a)\rangle,$$

which implies that $p(\alpha)$ and $p(\beta)$ lie on opposite sides of the supporting hyperplane $\partial^+_{p(a)}$, contradicting the fact that $\partial^+_{p(a)}$ is a supporting hyperplane.
(2) If $z \in \partial^-_{p(b)}$, we argue analogously by considering $\tilde{\omega} \in \{tz+(1-t)p(b);\ t \in [0,1]\}$ such that

$$\langle \tilde{\omega} - p(\alpha), i(p(\alpha) - p(\beta))\rangle = 0,$$

and repeat the argument as above with $\tilde{\omega}$ instead of ω.
(3) If $z \in [p(a) : p(b)]$, then $w \in [p(a) : p(b)]$. Analogously to Remark 4.1.9, we observe that

$$\langle p(\alpha) - p(a), i(p(b) - p(a))\rangle \geq 0,$$
$$\langle p(\beta) - p(a), i(p(b) - p(a))\rangle \geq 0.$$

Thus,

$$\begin{aligned} 0 &= \langle \omega - p(a), i(p(b) - p(a))\rangle \\ &= s\langle p(\alpha) - p(a), i(p(b) - p(a))\rangle + (1-s)\langle p(\beta) - p(a), i(p(b) - p(a))\rangle \end{aligned}$$

implies that $p(\alpha), p(\beta) \in [p(a) : p(b)]$ and hence $[p(a) : p(b)] = [p(\alpha) : p(\beta)]$, contradicting $z \in \mathrm{Int}(H_1)$.
Thus, we have shown Eq. (4.5), and the lemma is proven. □

4.2.3 Hyperrigidity of $A(K)$

Let us briefly recall the setting from the beginning of this section. Let $K \subset \mathbb{R}^2$ be compact and convex with $0 \in \mathrm{Int}(K)$, and let p denote its polar parametrization. For two disjoint closed intervals $[a,b] = I, J \subset [0, 2\pi)$ with $a < b$, the map g_I was defined as the unique composition of a rotation and a translation such that

$$\mathrm{Im}(g_I(p(a))) = \mathrm{Im}(g_I(p(b))) = 0 \text{ and } \mathrm{Re}(g_I(p(a))) < \mathrm{Re}(g_I(p(b))) = 0.$$

In addition, we defined the affine function

$$\tilde{g}_I = \frac{\mathrm{Im}(g_I) + \|\mathrm{Im}(g_I)\|_{p(I),\infty}}{\inf_{p(J)} |\mathrm{Im}(g_I)|},$$

whenever $\inf_{p(J)} |\mathrm{Im}(g_I)| > 0$.
If φ is a u.c.p. map on $C(K)$, we defined $\varphi(\chi_F) = \mu(K \cap F)$, where μ is the positive operator-valued measure corresponding to φ.

Lemma 4.2.5: *Let K be as above, and let $\pi : C(\mathrm{ex}(K)) \to \mathcal{B}(H)$ be a unital *-homomorphism, and $\phi : C(\mathrm{ex}(K)) \to \mathcal{B}(H)$ a u.c.p. map. If $x \in \mathrm{ex}(K)$ and $F \subset \mathrm{ex}(K) \setminus \{x\}$ is Borel, then*

$$\pi(\chi_{\{x\}})\phi(\chi_F)\pi(\chi_{\{x\}}) = 0.$$

Proof:
Let $x \in \mathrm{ex}(K)$, $E = \mathrm{ex}(K)\setminus\{x\}$, and let $\tilde{H} = \pi(\chi_{\{x\}})(H)$. Assume that $\pi(\chi_{\{x\}}) \neq 0$. Then the map $P_{\tilde{H}}\pi|_{\tilde{H}}$ is a direct sum of copies of the point evaluation $e_x : C(\mathrm{ex}(K)) \to \mathbb{C}$, and therefore has the unique extension property by Lemma 3.1.12. It follows that

$$P_{\tilde{H}}\phi|_{\tilde{H}} = P_{\tilde{H}}\pi|_{\tilde{H}} \text{ on } C(\mathrm{ex}(K)).$$

Let μ be the positive operator-valued measure corresponding to ϕ. Then $P_{\tilde{H}}\mu|_{\tilde{H}}$ is the positive operator-valued measure corresponding to $P_{\tilde{H}}\phi|_{\tilde{H}}$, and by the previous identity, it equals the scalar valued measure $\delta_x id_{\tilde{H}}$. Thus:

$$\pi(\chi_{\{x\}})\phi(\chi_{\{x\}})\pi(\chi_{\{x\}}) = P_{\tilde{H}}$$

and

$$\pi(\chi_{\{x\}})\phi(\chi_E)\pi(\chi_{\{x\}}) = P_{\tilde{H}} - \pi(\chi_{\{x\}})\phi(\chi_{\{x\}})\pi(\chi_{\{x\}}) = 0.$$

Now for arbitrary Borel $F \subset E$, we have:

$$0 \leq \pi(\chi_{\{x\}})\phi(\chi_F)\pi(\chi_{\{x\}}) \leq \pi(\chi_{\{x\}})\phi(\chi_E)\pi(\chi_{\{x\}}) = 0,$$

and hence the claim follows. □

Theorem 4.2.6: *Let K be as above, $\pi : C(\mathrm{ex}(K)) \to \mathcal{B}(H)$ a unital *-homomorphism, and $\phi : C(\mathrm{ex}(K)) \to \mathcal{B}(H)$ a u.c.p. map such that $\pi = \phi$ on $A(K)$. Let $I, J \subset [0, 2\pi)$ be disjoint closed intervals. Then*

$$\pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)}) = 0.$$

Proof:
The case where I is a singleton has already been treated in Lemma 4.2.5. Hence, we assume that $I = [a, b]$ with $a < b$, and fix $\epsilon > 0$. By Lemma 4.2.2, there exists a partition $a = t_0 < t_1 < \cdots < t_m = b$ such that

$$\|\mathrm{Im}(g_{[t_n,t_{n+1}]})\|_{[t_n,t_{n+1}],\infty} \leq \epsilon|p(t_n) - p(t_{n+1})| \tag{4.6}$$

for every $n = 0, \ldots, m-1$. Denote $[t_n, t_{n+1}] = I_n$.
As discussed at the beginning of Section 4.2, we have:

$$\|\pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)})\| \leq \|\pi(\chi_{p([t_0,t_1])})\phi(\chi_{p(J)})\pi(\chi_{p([t_0,t_1])})\| + \sum_{n=1}^{m-1} \|\pi(\chi_{p((t_n,t_{n+1}])})\phi(\chi_{p(J)})\pi(\chi_{p((t_n,t_{n+1}])})\|.$$

In turn, for each n, the terms satisfy

$$\|\pi(\chi_{p((t_n,t_{n+1}])})\phi(\chi_{p(J)})\pi(\chi_{p((t_n,t_{n+1}])})\| \leq \|\pi(\chi_{p(I_n)})\phi(\chi_{p(J)})\pi(\chi_{p(I_n)})\|.$$

Therefore,

$$\|\pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)})\| \leq \sum_{n=0}^{m-1} \|\pi(\chi_{p(I_n)})\phi(\chi_{p(J)})\pi(\chi_{p(I_n)})\|.$$

As shown earlier, we also have:

$$\pi(\chi_{p(I_n)})\phi(\chi_{p(J)})\pi(\chi_{p(I_n)}) \leq \frac{\|\mathrm{Im}(g_{I_n})\|_{I_n,\infty}}{\inf_{p(J)} |\mathrm{Im}(g_{I_n})|} id_H,$$

provided $\inf_{p(J)} |\mathrm{Im}(g_{I_n})| > 0$.
The next step is a case distinction based on the value

$$c = \mathrm{dist}(p(J), \partial^+_{p(a)} \cup \partial^-_{p(b)} \cup [p(a) : p(b)]).$$

<u>Case 1:</u>
Assume that $c > 0$. By Lemma 4.2.3 and Lemma 4.2.4, we know that

$$\inf_J |\mathrm{Im}(g_{I_n})| \geq c \text{ for all } n = 0, \ldots, m-1. \tag{4.7}$$

Thus, the inequalities together with Eq. (4.6) yield

$$\|\pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)})\| \leq \sum_{n=0}^{m-1} \frac{\epsilon}{c} |p(t_n) - p(t_{n+1})|. \tag{4.8}$$

However, the boundary of a compact convex set has finite length by Proposition 4.1.1. Let L denote the length of ∂K. Then we conclude that

$$\|\pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)})\| \leq \epsilon \frac{L}{c}.$$

Since c and L are independent of ϵ, we obtain the claim in this case.

<u>Case 2</u>:
Assume that $c = 0$. Define

$$\tilde{I} = \mathrm{conv}\{t \in [0, 2\pi);\; p(t) \in p(I) \cap \mathrm{ex}(K)\} \subset I.$$

The definition of $\tilde{I}$ yields that $\mathrm{ex}(K) \cap p(I \setminus \tilde{I}) = \emptyset$. Thus, the theorem follows immediately from Lemma 4.2.5 if $\tilde{I} = \emptyset$ or $\tilde{I}$ is a singleton. So assume $\tilde{I} = [\tilde{a}, \tilde{b}]$ with $\tilde{a} < \tilde{b} \in [0, 2\pi)$. Note that $p(\tilde{a}), p(\tilde{b}) \in \mathrm{ex}(K)$ by definition of $\tilde{I}$.
Choose $N \in \mathbb{N}$ such that $\tilde{a} + 1/N < \tilde{b} - 1/N$, and define intervals

$$\tilde{I}_n = [\tilde{a} + 1/n, \tilde{b} - 1/n] \text{ for } n \geq N.$$

We claim that

$$\operatorname{dist}(p(J), \partial^+_{p(\tilde{a}+1/n)} \cup \partial^-_{p(\tilde{b}-1/n)} \cup [p(\tilde{a}+1/n) : p(\tilde{b}-1/n)]) > 0$$

for every $n \geq N$. Otherwise, there would exist a point $t \in J$ such that either $p(\tilde{a}+1/n)$ and $p(t)$ or $p(\tilde{b}-1/n)$ and $p(t)$ lie on a common face of K, contradicting $p(\tilde{a}), p(\tilde{b}) \in \operatorname{ex}(K)$.
Hence, Case 1 applies to each $\tilde{I}_n$, and we obtain

$$\pi(\chi_{p(\tilde{I}_n)})\phi(\chi_{p(J)})\pi(\chi_{p(\tilde{I}_n)}) = 0 \text{ for all } n \geq N.$$

Since $\pi(\chi_{p(\tilde{I}_n)}) \to_{SOT} \pi(\chi_{p((\tilde{a},\tilde{b}))})$, we also obtain

$$\pi(\chi_{p((\tilde{a},\tilde{b}))})\phi(\chi_{p(J)})\pi(\chi_{p((\tilde{a},\tilde{b}))}) = 0.$$

Now I splits into

$$I = \{\tilde{a}\} \cup \{\tilde{b}\} \cup (\tilde{a}, \tilde{b}) \cup (I \setminus \tilde{I}).$$

Note that $\pi(\chi_{p(I\setminus\tilde{I})}) = 0$, since $\operatorname{ex}(K) \cap (I \setminus \tilde{I}) = \emptyset$, and

$$\pi(\chi_{\{p(\tilde{a})\}})\phi(\chi_{\{p(J)\}})\pi(\chi_{\{p(\tilde{a})\}}) = 0$$

respectively,

$$\pi(\chi_{\{p(\tilde{b})\}})\phi(\chi_{\{p(J)\}})\pi(\chi_{\{p(\tilde{b})\}}) = 0,$$

due to Lemma 4.2.5. By applying Lemma 4.2.1 once again, we conclude that

$$\pi(\chi_{p(I)})\phi(\chi_{p(J)})\pi(\chi_{p(I)}) = 0.$$

□

Corollary 4.2.7: *Under the assumptions of the preceding theorem,*

$$\pi(\chi_{p(J)}) = \phi(\chi_{p(J)}).$$

Proof:
First, note that, since

$$\pi(\chi_{p(J)}) = \phi(\chi_{p(J)}) + \phi(\chi_{p([0,2\pi)\setminus J)})\pi(\chi_{p(J)}) - \phi(\chi_{p(J)})\pi(\chi_{p([0,2\pi)\setminus J)}),$$

it suffices to show that

$$\phi(\chi_{p(J)})\pi(\chi_{p([0,2\pi)\setminus J)}) = 0$$

and

$$\phi(\chi_{p([0,2\pi)\setminus J)})\pi(\chi_{p(J)}) = 0.$$

Using the Schwarz inequality for u.c.p. maps, we get

$$\begin{aligned}0 &\leq (\phi(\chi_{p(J)})\pi(\chi_{p([0,2\pi)\setminus J)}))^*\phi(\chi_{p(J)})\pi(\chi_{p([0,2\pi)\setminus J)}) \\ &\leq \pi(\chi_{p([0,2\pi)\setminus J)})\phi(\chi_{p(J)})\pi(\chi_{p([0,2\pi)\setminus J)}), \\ 0 &\leq (\phi(\chi_{p([0,2\pi)\setminus J)})\pi(\chi_{p(J)}))^*\phi(\chi_{p([0,2\pi)\setminus J)})\pi(\chi_{p(J)}) \\ &\leq \pi(\chi_{p(J)})\phi(\chi_{p([0,2\pi)\setminus J)})\pi(\chi_{p(J)}).\end{aligned}$$

Thus it suffices to show that the right-hand sides of the these inequalities are zero. Let $I_n \subset [0, 2\pi) \setminus J$ be an increasing sequence of sets, each being the union of at most two closed intervals, such that $\bigcup_n I_n = [0, 2\pi) \setminus J$. Then

$$\begin{aligned}\pi(\chi_{p(I_n)}) &\to_{SOT} \pi(\chi_{p([0,2\pi)\setminus J)}), \\ \phi(\chi_{p(I_n)}) &\to_{WOT} \phi(\chi_{p([0,2\pi)\setminus J)})\end{aligned}$$

By Theorem 4.2.6 and Lemma 4.2.1, it follows that

$$\pi(\chi_{p(J)})\phi(\chi_{p(I_n)})\pi(\chi_{p(J)}) = 0 \text{ and } \pi(\chi_{p(I_n)})\phi(\chi_{p(J}) \pi(\chi_{p(I_n)}) = 0.$$

Taking the limit as $n \to \infty$ yields

$$\begin{aligned}\pi(\chi_{p(J)})\phi(\chi_{p([0,2\pi)\setminus J})\pi(\chi_{p(J)}) = 0, \\ \pi(\chi_{p([0,2\pi)\setminus J)})\phi(\chi_{p(J)})\pi(\chi_{p([0,2\pi)\setminus J)}) = 0\end{aligned}$$

and the corollary follows. □.

Theorem 4.2.8: *Let $K \subset \mathbb{R}^2$ be a convex compact set. Then the operator system $A(K)$ is hyperrigid in $C(\mathrm{ex}(K))$.*

Proof:
In the discussion at the beginning of Section 4.2, we already saw that the theorem holds if $\mathrm{Int}(K) = \emptyset$, and that it suffices to prove the case where $0 \in \mathrm{Int}(K)$.
Let π be a unital *-homomorphism and ϕ be a u.c.p. map such that π on $A(K)$.
By Corollary 4.2.7, the set

$$\mathcal{E} = \{E \subset [0, 2\pi);\ E \text{ Borel }, \pi(\chi_{p(E)}) = \phi(\chi_{p(E)})\}$$

contains all closed intervals. It is clear that $\mathcal{E}$ is closed under complements and countable disjoint unions. Therefore, $\mathcal{E}$ is a Dynkin system. By the Dynkin's π-λ theorem, we conclude that $\pi(\chi_E) = \phi(\chi_E)$ for every Borel set $E \subset \mathrm{ex}(K)$, and hence $\pi = \phi$. □

To conclude the section, we present a notable application of the main theorem, which generalizes the classical result that the weak and strong operator topologies agree on the unitary operators.

Corollary 4.2.9: *For every compact convex $K \subset \mathbb{R}^2$ and Hilbert space H, the weak and strong operator topologies coincide on the set*

$$\{T \in \mathcal{B}(H);\ T \text{ normal and } \sigma(T) \subset \mathrm{ex}(K)\}.$$

Proof:
Let K and H be as above, and let $T_n, T \in \mathcal{B}(H)$ be normal operators with $\sigma(T_n), \sigma(T) \subset \mathrm{ex}(K)$, such that $T_n \to T$ in the weak operator topology. Define unital *-homomorphims $\pi, \pi_n : C(\mathrm{ex}(K)) \to \mathcal{B}(H)$ by $\pi_n(z) = T_n$ and $\pi(z) = T$. Then, $\pi_n|_{A(K)} \to \pi|_{A(K)}$ in the WOT. By Theorem 4.2.8, the restriction $\pi|_{A(K)}$ is maximal. Thus, by [46, Lemma 2.1], it follows that $\pi_n \to \pi$ in the strong operator topology, completing the proof. □

4.3 Counterexamples

We close this chapter with two counterexample to Arveson's Hyperrigidity Conjecture. We begin with the example provided in [10], which was the first known counterexample.

4.3.1 Counterexample by Bilich and Dor-On

Let $U \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be the unilateral shift, $H = \ell^2(\mathbb{Z}) \oplus \mathbb{C}$, let $(e_n)_{n\in\mathbb{Z}}$ be the canonical orthonormal basis of $\ell^2(\mathbb{N}) \oplus 0$, and define $e = 0 \oplus 1 \in H$.
For $x, y \in H$, we denote by $P_{x,y}$ the rank one operator given by

$$P_{x,y}(z) = \langle z, y\rangle x.$$

Now define the operator

$$T = (U \oplus 0) + P_{e_0,e},$$

which may be written in block matrix form as

$$T = \begin{pmatrix} U & P_{e_0,e} \\ 0 & 0 \end{pmatrix}.$$

Let S denote the operator system generated by $\{T^n, n \in \mathbb{N}\}$. The first thing we show is that $C^*(S) \subset \mathcal{B}(H)$ contains all compact operators. This allows us to use Theorem 3.1.18.

Lemma 4.3.1: *We have $\mathcal{K}(H) \subset C^*(S)$ and $C^*(S)/\mathcal{K}(H)$ is $*$-isomorphic to $C(\mathbb{T})$.*

Proof:
We compute
$$TT^* - id_H = P_{e_0,e}P_{e,e_0} - P_e = P_{e_0} - P_e.$$
Thus,
$$\frac{(TT^* - id_H)^2 - (TT^* - id_H)}{2} = \frac{P_{e_0} + P_e - P_{e_0} + P_e}{2} = P_e \in C^*(S).$$
From here, we obtain for all $n, m \in \mathbb{Z}$:
$$T^n P_e = P_{e_n,e} \in C^*(S)$$
and thus:
$$P_{e_n,e_m} = P_{e_n,e}P^*_{e_m,e} \in C^*(S).$$
Since $\text{span}(\{e_n;\ n \in \mathbb{Z}\} \cup \{e\})$ is dense in H, every rank-one operator $P_{x,y}$ can be approximated in norm by a linear combination of the operators P_{e_n,e_m}. Thus, $P_{x,y} \in C^*(S)$ for all $x, y \in H$.
We conclude that $\mathcal{K}(H) \subset C^*(S)$, since every finite-rank operator is a linear combination of such $P_{x,y}$, and the finite-rank operators are norm-dense in $\mathcal{K}(H)$.
Since T is a finite-rank perturbation of the unitary operator
$$(U \oplus 0) + P_e = T - P_{e_0,e} + P_e,$$
we see that the image of T in the Calkin algebra equals the image of $(U \oplus 0) + P_e$, and hence
$$C^*(S)/\mathcal{K}(H) = C^*([(U \oplus 0) + P_e]) \cong C(\mathbb{T}).$$
□

This places us in the setting of Theorem 3.1.18, and since
$$\|T + \mathcal{K}(H)\| \leq 1 < \|T\|,$$
we may also apply Arveson's boundary theorem. It follows that the only irreducible representations of $C^*(S)$, up to unitary equivalence, are the identity representation and the point evaluations
$$e_z : C^*(S) \to \mathbb{C},\ T \mapsto z$$
for $z \in \mathbb{T}$, and moreover, the identity map $id : S \to \mathcal{B}(H)$ has the unique extension property.
Thus, in order to contradict Arveson's Hyperrigidity Conjecture, it suffices to show that the restrictions of the point evaluations have the unique extension property and that there exists a unital $*$-homomorphism on $C^*(S)$ whose restriction to S does not have the unique extension property.
To this end, we need the following technical lemma:

Lemma 4.3.2: *For any set I and any vector $x = \oplus_{i\in I} x_i \in \bigoplus_{i\in I} H$, we have*

$$\lim_{n\to\infty} \langle (\oplus_{i\in I} T^n)(x), x\rangle = 0$$

Proof:
Let I be a set, $x = \oplus_{i\in I} x_i \in \bigoplus_{i\in I} H$, and fix $\epsilon > 0$. Then there exists a finite subset $J \subset I$ such that

$$\sum_{i\in I\setminus J} \|x_i\|^2 < \epsilon.$$

Since

$$\begin{aligned} |\langle T^n(x), x\rangle| &= |\sum_{i\in I} \langle T^n(x_i), x_i\rangle| \\ &\leq |\sum_{i\in J} \langle T^n(x_i), x_i\rangle| + \sum_{i\in I\setminus J} \|T^n\| \|x_i\|^2 \\ &< |\sum_{i\in J} \langle T^n(x_i), x_i\rangle| + 2\epsilon, \end{aligned}$$

where we used the Cauchy-Schwarz inequality and the estimate $\|T^n\| = \|(U^n \oplus 0) + P_{e,e_n}\| \leq 2$ for $1 \leq n \in \mathbb{N}$, it suffices to show that $(T^n)_{n\in\mathbb{N}} \to 0$ in the weak operator topology. But since $T^n = (U^n \oplus 0) + P_{e,e_n}$, and $(U^n)_{n\in\mathbb{N}}$ converges to zero in the weak operator topology, and the rank-one perturbation P_{e,e_n} vanishes weakly as $n \to \infty$, we conclude that $(T^n)_{n\in\mathbb{N}} \to 0$ in the weak operator topology. □

Theorem 4.3.3: *The operator system S is not hyperrigid, however, the restriction of every irreducible representation of $C^*(S)$ has the unique extension property.*

Proof:
Following on from the above observations, note that the unital $*$-homomorphism

$$\rho : C(\mathbb{T}) \to \mathcal{B}(\ell^2(\mathbb{Z})), z \mapsto U$$

induces a unital $*$-homomorphism on $C^*(S)$ by

$$\pi : C^*(S) \to \mathcal{B}(\ell^2(\mathbb{Z})), x \mapsto \rho(x + \mathcal{K}(H)),$$

where we identified $C^*(S)/\mathcal{K}(H)$ with $C(\mathbb{T})$. But $\pi|_S$ clearly dilates non-trivially to the identity representation of S. Hence, $\pi|_S$ is a restriction of a unital $*$-homomorphism that is not maximal and thus does not have the unique extension property.
It remains to show that the maps $e_z|_S$ are maximal for every $z \in \mathbb{T}$. Let $z \in \mathbb{T}$ and $\rho : S \to \mathcal{B}(K)$ be a maximal dilation of e_z. This, in particular, means that there is a unit vector $x \in K$ such that

$$e_z(s) = \langle \rho(s)(x), x\rangle$$

for all $s \in S$. Let $\pi : C^*(S) \to \mathcal{B}(K)$ be the unique extension of ρ to a unital $*$-homomorphism. By Lemma 3.1.16, there is a decomposition of $K = H_1 \oplus H_2$ into reducing subspaces of π such that $\pi_1 = P_{H_1}\pi|_{H_1}$ vanishes on $\mathcal{K}(H)$ and $\pi_2 = P_{H_2}\pi|_{H_2}$ is uniquely determined by its action on $\mathcal{K}(H)$. Therefore, π_2 is unitary equivalent to a direct sum of identity representation of $\mathcal{K}(H)$, so let $\pi_2 = U^* \oplus_{i\in I} id_{\mathcal{K}(H)} U$ for some set I and unitary operator U. Let $x_1 = P_{H_1}(x)$ and $x_2 = P_{H_2}(x)$. It holds that

$$\begin{aligned} 1 = |e_z(s)| &= |\langle \rho(s)(x), x\rangle| = |\langle \pi(s)(x), x\rangle| \\ &= |\langle \pi_1(s)(x_1), x_1\rangle + \langle \pi_2(s)(x_2), x_2\rangle| \\ &\le |\langle \pi_1(s)(x_1), x_1\rangle| + |\langle \pi_2(s)(x_2), x_2\rangle| \\ &\le \|\pi_1(s)\|\|x_1\|^2 + |\langle \pi_2(s)(x_2), x_2\rangle| \\ &= \|\pi_1(s)\|\|x_1\|^2 + |\langle (\oplus_{i\in I} s)(U(x_2)), U(x_2)\rangle| \end{aligned}$$

for all $s \in S$. Since T is a compact perturbation of $U \oplus 0$, we have $\|T^n\| \le 1$ for all $n \in \mathbb{N}$. Hence, by Lemma 4.3.2:

$$1 \le \limsup_{n\to\infty} \|\pi_1(T^n)\|\|x_1\|^2 + |\langle (\oplus_{i\in I} T^n)(U(x_2)), U(x_2)\rangle| \le \|x_1\|^2,$$

and therefore $1 = \|x_1\|^2$, which implies $x_2 = 0$.
We have thus shown that $\pi_1|_S$ is a dilation of $e_z|_S$, and $\pi|_S$ is a trivial dilation of $\pi_1|_S$. It remains to show that $\pi_1|_S$ is a trivial dilation of $e_z|_S$.
For all $n \in \mathbb{N}$, the operator $\pi(T^n)$ is unitary, since π_1 annihilates the compact operators, and T^n is a compact perturbation of $U^n \oplus 1$. Thus, as in Example 3.1.10,the subspace spanned by x_1 reduces each $\pi(T^n)$, and since S is generated by the T^n, it follows that $\pi_1|_S$ is a trivial dilation of $e_z|_S$. Hence, $e_z|S$ is maximal, and the proof is complete. □

Remark 4.3.4: There are several noteworthy aspects of the presented example. Firstly, the C^*-algebra $C^*(S)$ is of type I, since the quotient $C^*(S)/\mathcal{K}(H)$ is commutative, see [34, Theorem 1]. Secondly, the operator system is generated by the operator algebra spanned by T. This also shows that Arveson's Hyperrigidity Conjecture remains false even when "operator system"is replaced by "operator algebra".
Interestingly, this is not the case in commutative C^*-algebras. A point evaluation e_x on an operator algebra contained in a commutative C^*-algebra $C(K)$ is maximal if and only if x is a peak point, that is, there exists a function f in the operator algebra such that $|f(y)| < |f(x)|$ for all $x \ne y \in K$, see [56, Chapter 8]. However, in the above example we have

$$\|e_z(p(T))\| \le \|id(p(T))\|$$

for all $z \in \mathbb{T}$, so the peak point phenomenon does not hold in general for operator algebras in non-commutative C^*-algebras.
This observation, together with the idea of the proof of Theorem 4.2.6, might lead to a proof of the conjecture for operator algebras in commutative C^*-algebras.

4.3.2 A finite dimensional Counterexample

We finish this chapter with a new counterexample for Arveson's Hyperrigidity Conjecture, new in the sense that the operator system is generated by finitely many elements.
We start with the Euclidean open unit ball $\mathbb{B}_4$ in $\mathbb{R}^4$, and denote by S_3 be its boundary, the sphere in $\mathbb{R}^4$. Let $A(\overline{\mathbb{B}_4})$ denote the continuous affine functions on the closed unit ball, and let $t_i \in A(\overline{\mathbb{B}_4})$, $i = 1,2,3,4$, be the projection onto the i-th component. Note that $A(\overline{\mathbb{B}_4})$ is spanned by $1, t_1, t_2, t_3, t_4$.
Define $P : L^2(S_3) \to \mathbb{C}, g \mapsto \langle g, 1\rangle$. For $i = 1,2,3,4$ and $c > 0$, we define the operators

$$T_i = M_{t_i} \oplus 0,\ T_{1,c} = \begin{pmatrix} M_{t_1} & cP^* \\ cP & 0 \end{pmatrix},\ \tilde{T}_{1,c} = \begin{pmatrix} M_{t_1} & cP^* \\ 0 & 0 \end{pmatrix}$$

on $L^2(S_3) \oplus \mathbb{C}$. Here, $L^2(S_3)$ is equipped with the unique rotation-invariant probability measure m on S_3. Let

$$S_c = \operatorname{span}\{\tilde{T}_{1,c}, T_2, T_3, T_4\}.$$

In Theorem 4.3.10, we show that S_c is not hyperrigid, although the restrictions of all irreducible representations of $C^*(S_c)$ are boundary representations.
A crucial part of the proof is to show that the *joint numerical range* of the operator tuple $(T_{1,c}, T_2, T_3, T_4)$, defined as

$$\mathcal{W}((T_{1,c}, T_2, T_3, T_4)) = \{(\langle T_{1,c}x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle);\ \|x\| = 1\},$$

is contained in the open unit ball $\mathbb{B}_4$. To show this, we first establish that the joint numerical range of (T_1, T_2, T_3, T_4) lies in $\mathbb{B}_4$.

Lemma 4.3.5: *It holds that*

$$\mathcal{W}((T_1, T_2, T_3, T_4)) \subset \mathbb{B}_4.$$

<u>Proof</u>:
Let $x \in L^2(S_3) \oplus \mathbb{C}$ with $\|x\| = 1$. Using the identity $\sum_{i=1}^4 t_i^2 = 1$ on S_3 and applying the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}\|(\langle T_1 x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle)\|^2 &= \sum_{i=1}^4 |\langle T_i x, x\rangle|^2 \leq \sum_{i=1}^4 \|T_i x\|^2 \|x\|^2 \\ &= \sum_{i=1}^4 \langle T_i x, T_i x\rangle = \sum_{i=1}^4 \langle T_i^2 x, x\rangle \\ &\leq \|x\|^2 = 1.\end{aligned}$$

Furthermore, equality in the Cauchy-Schwarz inequality would require that there exists some $i \in \{1,2,3,4\}$ and a scalar $0 \neq \lambda \in \mathbb{C}$ such that $T_i x = \lambda x$. However, the operator M_{t_i} has no eigenvalues, so we conclude that:

$$\sum_{i=1}^{4} (\langle T_i x, x \rangle)^2 < 1,$$

which completes the proof. □

We also need the value of $\int_{S_3} \frac{1}{1+z_1} dm$, which we quickly calculate in the next lemma.

Lemma 4.3.6: *It holds that*

$$\int_{S_3} \frac{1}{1+z_1} dm = 2.$$

Proof:
We will use an analogue of the spherical coordinates. Define a parametrization of

$$F = S_3 \setminus \{(1,0,0,0), (-1,0,0,0), (0,1,0,0), (0,-1,0,0)\}$$

by

$$\phi : (0,\pi) \times (0,\pi) \times [0,2\pi) \to F, \ (\theta, \Phi, \Psi) \mapsto \begin{pmatrix} \cos(\theta) \\ \sin(\theta)\cos(\Phi) \\ \sin(\theta)\sin(\Phi)\cos(\Psi) \\ \sin(\theta)\sin(\Phi)\sin(\Psi) \end{pmatrix}.$$

Then the Jacobian matrix is given by:

$$D\phi = \begin{pmatrix} -\sin(\theta) & 0 & 0 \\ \cos(\theta)\cos(\Phi) & -\sin(\theta)\sin(\Phi) & 0 \\ \cos(\theta)\sin(\Phi)\cos(\Psi) & \sin(\theta)\cos(\Phi)\cos(\Psi) & -\sin(\theta)\sin(\Phi)\sin(\Psi) \\ \cos(\theta)\sin(\Phi)\sin(\Psi) & \sin(\theta)\cos(\Phi)\sin(\Psi) & \sin(\theta)\sin(\Phi)\cos(\Psi) \end{pmatrix}.$$

This gives the Gram matrix:

$$(D\phi)^* D\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin(\theta)^2 & 0 \\ 0 & 0 & \sin(\theta)^2\sin(\Phi)^2 \end{pmatrix}.$$

Since $m(S_3 \setminus F) = 0$, the integral becomes

$$\begin{aligned}\int_{S_3} \frac{1}{1+z_1} dm &= \int_F \frac{1}{1+z_1} dm \\ &= (2\pi^2)^{-1} \int_0^\pi \int_0^\pi \int_0^{2\pi} \frac{\sqrt{\det((D\phi)^* D\phi)}}{1+\cos(\theta)} d\Psi d\Phi d\theta \\ &= (2\pi^2)^{-1} \int_0^\pi \int_0^\pi \int_0^{2\pi} \frac{\sin^2(\theta)\sin(\Phi)}{1+\cos(\theta)} d\Psi d\Phi d\theta \\ &= \pi^{-1} \int_0^\pi \int_0^\pi \frac{(1-\cos(\theta)^2)\sin(\Phi)}{1+\cos(\theta)} d\Phi d\theta \\ &= \pi^{-1} \int_0^\pi \int_0^\pi (1-\cos(\theta))\sin(\Phi) d\Phi d\theta = 2\end{aligned}$$

□

Lemma 4.3.7: *Let $0 < c < 1/2$. Then, the following holds:*

$$\mathcal{W}((T_{1,c}, T_2, T_3, T_4)) \subset \mathbb{B}_4.$$

Proof:
Let $0 < c \leq 1/2$, let $\tilde{S}_c$ be the operator system generated by $T_{1,c}, T_2, T_3, T_4$, and let $f = \alpha + \beta t_1 + \gamma t_2 + \delta t_e + \epsilon t_4 \in A(\overline{\mathbb{B}_4})$. Define a map $\Phi : A(\overline{\mathbb{B}_4}) \to \tilde{S}_c$ by

$$\Phi(f) = \alpha I + \beta T_1 + \gamma T_2 + \delta T_3 + \epsilon T_4 = \begin{pmatrix} M_f & c\beta P^* \\ c\beta P & \alpha \end{pmatrix}.$$

It is clear that this map is well-defined, bijective and that its inverse is positive. The next step is to show that Φ is positive. Suppose that $f \geq 0$ and notice that this implies $\alpha \geq 0$ and $\beta, \gamma, \delta, \epsilon \in \mathbb{R}$. If $\alpha = 0$, then $f = 0$, and there is nothing to show. Thus assume $\alpha > 0$. Then by Lemma 3.1.5:

$$\Phi(f) \geq 0$$

if and only if the Schur complement

$$M_f - c^2\beta^2\alpha^{-1}P^*P$$

is positive. Therefore, the positivity of $\Phi(f)$ is equivalent to

$$c^2\beta^2 P^*P \leq \alpha M_f.$$

Evaluating f in $(1,0,0,0)$ and $(-1,0,0,0)$ shows that $|\beta| \leq \alpha$, and since there is nothing to show for $\beta = 0$, it suffices to check that

$$c^2 P^*P \leq |\beta|^{-1} M_f.$$

Recall that m is the unique rotation-invariant probability measure on S_3. Let $g \in L^2(S_3)$, and write $\tilde{f} = f/|\beta|$. By Theorem 4.1.12 and since $P^*P = P_1$, we have that

$$\inf\{a \geq 0;\ P^*P \leq aM_{\tilde{f}}\} = \int_{S_3} \tilde{f}^{-1} dm.$$

Therefore, we only have to check that

$$c^2 \int_{S_3} \tilde{f}^{-1} dm \leq 1.$$

Note that $\tilde{f}(z) = |\beta|^{-1}(\alpha + \langle z, \omega\rangle_{\mathbb{R}^4})$, where $\omega = (\beta, \gamma, \delta, \epsilon)$, and $\alpha \geq \|\omega\|$ since $f \geq 0$. Let $U \in \mathcal{B}(\mathbb{R}^4)$ be an orthogonal matrix such that $U^*\omega = (\|\omega\|, 0, 0, 0)$. Then

$$\begin{aligned}\int_{S_3} \tilde{f}^{-1} dm &= \int_{S_3} \frac{|\beta|}{\alpha + \langle z, \omega\rangle} dm(z) = \int_{S_3} \frac{|\beta|}{\alpha + \langle U(z), \omega\rangle} dm(z) \\ &= \int_{S_3} \frac{|\beta|}{\alpha + \|\omega\| t_1} dm \leq \int_{S_3} \frac{|\beta|}{\|\omega\|} \frac{1}{1 + t_1} dm \leq \int_{S_3} \frac{1}{1 + t_1} dm\end{aligned}$$

and by Lemma 4.3.6

$$\int_{S_3} \frac{1}{1 + t_1} dm = 2.$$

Therefore, $\Phi(f) \geq 0$ if $c \leq 1/2$ which is the case by choice of c.
In total, we have shown that if ϕ is a positive state on $\tilde{S}_c$, then $\phi \circ \Phi$ is a positive state on $A(\overline{\mathbb{B}_4})$, and therefore

$$(\phi(T_1), \phi(T_2), \phi(T_3), \phi(T_4)) \in \overline{\mathbb{B}_4}.$$

In particular, since $0 < c \leq 1/2$, we obtain

$$\mathcal{W}((T_{1,c}, T_2, T_3, T_4)) \subset \overline{\mathbb{B}_4}. \tag{4.9}$$

Let $x \in L^2(S_3) \oplus \mathbb{C}$ with $\|x\| = 1$ and define

$$\begin{aligned} z_1 &= (\langle T_1 x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle), \\ z_2 &= (\langle T_{1,1/2} x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle). \end{aligned}$$

Then, $z_1 \in \mathbb{B}_4$ by Lemma 4.3.5 and $z_2 \in \overline{\mathbb{B}_4}$ by Eq. (4.9). Thus, since $0 < c < 1/2$, we have that

$$(\langle T_{1,c} x, x\rangle, \langle T_2 x, x\rangle, \langle T_3 x, x\rangle, \langle T_4 x, x\rangle) = (1 - 2c) z_1 + 2c z_2 \in \mathbb{B}_4.$$

□

Lemma 4.3.8: *Let $0 < c < 1$ and $x \in L^2(S_3) \oplus \mathbb{C}$ with $\|x\| = 1$. Then, the following holds:*

$$\|(\langle \tilde{T}_{1,c}x, x\rangle, \langle T_2x, x\rangle, \langle T_3x, x\rangle, \langle T_4x, x\rangle)\| < 1.$$

Proof:
Let $x = (y, a) \in L^2(S_3) \oplus \mathbb{C}$ with $\|x\| = 1$ and define

$$z = (\langle \tilde{T}_{1,c}x, x\rangle, \langle T_2x, x\rangle, \langle T_3x, x\rangle, \langle T_4x, x\rangle).$$

Assume, for contradiction, that $\|z\| \geq 1$. The equation

$$|\langle \tilde{T}_{1,c}x, x\rangle| = |\langle M_{t_1}y, y\rangle + ca\langle 1, y\rangle| \leq |\langle M_{t_1}y, y\rangle| + |ca\langle 1, y\rangle|$$

shows, on one hand, that $\langle 1, y\rangle \neq 0$, and on the other, that for

$$\tilde{x} = \begin{cases} (y, |a|\frac{\langle y,1\rangle}{|\langle y,1\rangle|}) & \text{if } \langle M_{t_1}y, y\rangle \geq 0 \\ (y, -|a|\frac{\langle y,1\rangle}{|\langle y,1\rangle|}) & \text{else} \end{cases}$$

and

$$\tilde{z} = (\langle \tilde{T}_{1,c}\tilde{x}, \tilde{x}\rangle, \langle T_2\tilde{x}, \tilde{x}\rangle, \langle T_3\tilde{x}, \tilde{x}\rangle, \langle T_4\tilde{x}, \tilde{x}\rangle),$$

we have $\|\tilde{x}\| = \|x\| = 1$, $\langle \tilde{T}_{1,c}\tilde{x}, \tilde{x}\rangle \in \mathbb{R}$ and $1 \leq \|z\| \leq \|\tilde{z}\|$. Thus, applying the identity $\mathrm{Re}(\tilde{T}_{1,c}) = T_{1,c/2}$, we conclude that

$$\tilde{z} = (\langle T_{1,c/2}\tilde{x}, \tilde{x}\rangle, \langle T_2\tilde{x}, \tilde{x}\rangle, \langle T_3\tilde{x}, \tilde{x}\rangle, \langle T_4\tilde{x}, \tilde{x}\rangle),$$

which lies within the unit ball $\mathbb{B}_4$ by Lemma 4.3.7, thus leading to a contradiction. Hence, $\|z\| < 1$. □

Lemma 4.3.9: *Let $0 < c < 1$. Then, the compact operators $\mathcal{K}(L^2(S_3) \oplus \mathbb{C})$ are contained in $C^*(S_c)$.*

Proof:
We start by noting that

$$\tilde{T}_{1,c}\tilde{T}_{1,c}^* + \sum_{i=2}^{4} T_i^*T_i - 1 = \begin{pmatrix} c^2P^*P & 0 \\ 0 & -PP^* \end{pmatrix} \in C^*(S_c).$$

This implies that

$$0 \oplus PP^* = \frac{1}{c^{-2}+1}\left(c^{-2}\begin{pmatrix} c^2P^*P & 0 \\ 0 & -PP^* \end{pmatrix}^2 - \begin{pmatrix} c^2P^*P & 0 \\ 0 & -PP^* \end{pmatrix}\right) \in C^*(S_c),$$

$$id \oplus 0 = 1 - 0 \oplus PP^* \in C^*(S_c).$$

Therefore,

$$P^*P \oplus 0 = \frac{1}{c^2}(id \oplus 0)(\tilde{T}_{1,c}\tilde{T}_{1,c}^* + \sum_{i=2}^{4} T_i^* T_i - 1) \in C^*(S_c).$$

Additionally, $M_{t_1} \oplus 0 = \tilde{T}_{1,c}(id \oplus 0) \in C^*(S_c)$ and

$$\begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix} = \tilde{T}_{1,c}^* - M_{t_1} \oplus 0 \in C^*(S_c).$$

Further multiplying by $M_{t_i} \oplus 0$, $i = 1, 2, 3, 4$, from the left and right to $P^*P \oplus 0$ and $\begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix}$ shows that

$$\begin{pmatrix} p\langle \cdot, q \rangle & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \langle \cdot, q \rangle & 0 \end{pmatrix} \in C^*(S_c)$$

for every $p, q \in \mathbb{C}[t_1, t_2, t_3, t_4]$. Finally, since the polynomials are dense in $L^2(S_3)$, any compact operator can be approximated by elements of $C^*(S_c)$, completing the proof. □

The previous lemma places us in the setting of Theorem 3.1.18. Since $(T_{1,c}, T_2, T_3, T_4)$ is a compact perturbation of $(M_{t_1} \oplus 1, M_{t_2} \oplus 1, M_{t_3} \oplus 1, M_{t_4} \oplus 1)$, by the previous lemma, we have the following split short exact sequence:

$$0 \to \mathcal{K}(L^2(S_3 \oplus \mathbb{C})) \to C^*(S_c) \to C(S_3) \to 0.$$

Consequently, the only irreducible representations of $C^*(S_c)$ are given by the identity representation and the evaluations e_z, defined by

$$C^*(S_c) \to \mathbb{C}, (T_{1,c}, T_2, T_3, T_4) \mapsto z$$

for $z \in S_3$, see Theorem 3.1.18.

Theorem 4.3.10: *Let $0 < c < 1$. The operator system S_c is not hyperrigid. However, the restrictions of all irreducible representations of $C^*(S_c)$ to S_c have the unique extension property.*

Proof:
To show that S_c is not hyperrigid, we begin by considering the $*$-homomorphisms

$$\pi : C^*(S_c) \to \mathcal{B}(L^2(S_3)), (\tilde{T}_{1,c}, T_2, T_3, T_4) \mapsto (M_{t_1}, M_{t_2}, M_{t_3}, M_{t_4})$$

and

$$\Phi : C^*(S_c) \to \mathcal{B}(L^2(S_3)), A \mapsto P_{L^2(S_3)} A|_{L^2(S_3)}.$$

Clearly, $\pi|_{S_c} = \phi|_{S_c}$, but $\pi \neq \phi$ because the range of π is commutative, whereas the range of Φ contains all compact operators by Lemma 4.3.9. Hence, $\pi|_{S_c}$ does not have the unqiue extension property, and thus S_c is not hyperirigid.
It remains to show that the irreducible representations of $C^*(S_c)$ are boundary representations. By Arveson's boundary theorem (see Theorem 3.1.18), we observe that since

$$1 = \| \sum_{i=1}^{4} T_i^* T_i \| < \| \tilde{T}_{1,c}^* \tilde{T}_{1,c} + \sum_{i=2}^{4} T_i^* T_i \|,$$

the identity representation of $C^*(S_c)$ is a boundary representation. Thus, it remains to verify that the point evaluations e_z, restricted to S_c, are maximal for all $z \in S_3$.
We begin by showing that

$$\|(\phi(\tilde{T}_{1,c}), \phi(T_2), \phi(T_3), \phi(T_4))\| \leq 1 \tag{4.10}$$

for every $\phi \in S(S_c)$ and that

$$\|(\phi(\tilde{T}_{1,c}), \phi(T_2), \phi(T_3), \phi(T_4))\| < 1 \tag{4.11}$$

for every pure state ϕ that is not maximal.
Let $\phi \in S(S_c)$. If ϕ is pure and maximal, we have

$$\|(\phi(\tilde{T}_{1,c}), \phi(T_2), \phi(T_3), \phi(T_4))\| = 1,$$

since the only representations of $C^*(S_c)$ with image in $\mathbb{C}$ are given by the point evaluations. If ϕ is pure and not maximal, then by Theorem 3.1.20, ϕ dilates non-trivially to a maximal irreducible u.c.p. map, which must be the identity representation, since this is the only irreducible representation besides the point evaluations. Thus, there exists $x \in L^2(S_3) \oplus \mathbb{C}$ with $\|x\| = 1$ such that $\phi(\cdot) = \langle \cdot x, x \rangle$. Since $c < 1$, we can apply Lemma 4.3.8 to obtain Eq. (4.11). Moreover, since the convex hull of the extreme points of $S(S_c)$ is the entire state space $S(S_c)$, and the pure states are precisely the extreme points by Proposition 3.1.19, we also obtain Eq. (4.10) via Carathédory's theorem.
It follows from Eq. (4.10) that the restrictions of the maps e_z to S_c are extreme points of $S(S_c)$. By equation Eq. (4.11), these maps must also be maximal, completing the proof. □

Remark 4.3.11: Perhaps the most interesting aspect of this example is that it is necessary to work with the sphere in $\mathbb{R}^4$, since 4 is the smallest dimension d for which

$$\int_{S_d} \frac{1}{1+z_1} < \infty.$$

At present, it is not known whether there exists operator systems generated by fewer than four selfadjoint operators that provide an counterexample to Arveson's Hyperrigidity Conjecture.

Chapter 5

Miscellaneous Results on u.c.p. Maps and Matrix Convex Sets

This chapter contains so far unpublished results. The first one, Theorem 5.1.5, connects G_δ-sets with maximal u.c.p. maps. The second one, Theorem 5.2.6, is about matrix convex sets and Arveson extreme points. The two main results are joint work with Michael Hartz.

5.1 Maximal u.c.p. Maps are Dense and G_δ

For an operator system S and a Hilbert space H, we define

$$\mathrm{UCP}(S, \mathcal{B}(H)) = \{\phi : S \to \mathcal{B}(H);\ \phi \text{ u.c.p.}\}$$

and if not stated otherwise, we equip the set with the topology induced by pointwise WOT convergence. The following lemma is well-known.

Proposition 5.1.1: *Let S be an operator system and H a Hilbert space. Then,*

$$\mathrm{UCP}(S, \mathcal{B}(H))$$

is convex and compact.

Proof:
Convexity is clear. For compactness, one first considers the set

$$\{L \in \mathcal{B}(S, \mathcal{B}(H));\ \|L\| \leq 1\}.$$

The topology of pointwise WOT convergence on the above set coincides with a weak* topology, see, for instance, [55, Lemma 7.1+ Proposition 7.3]. Thus, the above set is compact by the Banach-Alaoglu theorem. Moreover, this set contains $\mathrm{UCP}(S, \mathcal{B}(H))$, because u.c.p. maps are already completely contractive.
It remains to show that

$$\mathrm{UCP}(S, \mathcal{B}(H))$$

is closed. Let $(\phi_\lambda)_{\lambda\in I}$ be a net of u.c.p. maps converging pointwise in the WOT to a map $\phi \in \mathcal{B}(S, \mathcal{B}(H))$. Let $1 \leq n \in \mathbb{N}$ and $(a_{i,j})_{1\leq i,j\leq n} \in M_n(S)$ be positive. Since we can identify $M_n(\mathcal{B}(H))$ with $\mathcal{B}(H^n)$, it suffices to verify that

$$\langle \phi((a_{i,j})_{1\leq i,j\leq n})(x_1 \oplus \cdots \oplus x_n), x_1 \oplus \cdots \oplus x_n\rangle \geq 0$$

for all $x_1 \oplus \cdots \oplus x_n \in H^n$. However, for all $x = x_1 \oplus \cdots \oplus x_n \in H^n$, we have

$$\langle \phi((a_{i,j})_{1\leq i,j\leq n})(x), x\rangle = \lim_\lambda \langle \phi_\lambda((a_{i,j})_{1\leq i,j\leq n})(x), x\rangle \geq 0,$$

because ϕ_λ is completely positive for all $\lambda \in I$. Therefore, ϕ is completely positive. The proof is completed by noticing that $\phi(1) = id_H$ since $\phi_\lambda(1) = id_H$ for all $\lambda \in I$. □

Lemma 5.1.2: *Let S be a separable operator system and H an infinite dimensional separable Hilbert space. Then,*

$$\{\phi : S \to \mathcal{B}(H);\ \phi \text{ maximal}\}$$

is dense in $\mathrm{UCP}(S, \mathcal{B}(H))$.

Proof:
Let S and H be as above and define

$$\Lambda = \{F \subset H;\ F \text{ finite-dimensional subspace}\}.$$

Let $\phi \in \mathrm{UCP}(S, \mathcal{B}(H))$. By Theorem 3.1.11, there is a maximal dilation $\psi : S \to \mathcal{B}(K)$ of ϕ such that K is still separable. For simplicity, we identify $V(H)$ with H. Then, for every $F \in \Lambda$, since K and H are separable and infinite dimensional, there is a unitary operator $U_F : H \to K$ such that $U(x) = x$ for all $x \in F$. It is easy to check that $(U_F^* \psi U_F)_{F\in\Lambda}$ is a net of maximal maps. Let $x, y \in H$ and choose $F \in \Lambda$ such that $x, y \in F$. Then,

$$\langle U_G^* \psi(s) U_G(x), y\rangle = \langle \phi(s)(x), y\rangle$$

for all $F \leq G \in \Lambda$ and $s \in S$. Hence $(U_F^* \psi U_F)_{F\in\Lambda}$ is a net of maximal maps, converging to ϕ. □

Lemma 5.1.3: *Let K, L be compact Hausdorff spaces, $\pi : K \to L$ be surjective and continuous, and $f : K \to \mathbb{R}$ upper semi-continuous. Then*

$$g : L \to \mathbb{R}, x \mapsto \sup\{f(y);\ y \in \pi^{-1}(x)\}$$

is upper semi-continuous.

Proof:
Let K, L, π and f be as above. For every $t \in \mathbb{R}$, we have to show that

$$\{x \in L;\ g(x) \geq t\}$$

is closed. But

$$\{x \in L;\ g(x) \geq t\} = \pi(\{y \in K;\ f(y) \geq t\}).$$

Since K is compact, f upper semi-continuous, and π continuous,

$$\{y \in K;\ f(y) \geq t\}$$

and therefore also

$$\pi(\{y \in K;\ f(y) \geq t\})$$

are closed. □

Given a $s \in S, x \in H$ and a u.c.p. map $\phi : S \to \mathcal{B}(H)$, we say that ϕ is *maximal* in (s, x) if for every dilation $\psi : S \to \mathcal{B}(K)$, it holds that $\psi(V(x)) \in V(H)$, where V is the isometry belonging to the dilation (see [5]). We also define $\beta_{s,x} : \mathrm{UCP}(S, \mathcal{B}(H)) \to [0, \infty)$ by

$$\beta_{s,x}(\phi) = \sup\{\langle \psi(s^*s)x, x\rangle,\ \psi \text{ u.c.p. extension of } \phi \text{ on } C^*(S)\}.$$

Lemma 5.1.4: *Let S be an operator system, H a Hilbert space, $s \in S$ and $x \in H$. Then, the function*

$$\mathrm{UCP}(S, \mathcal{B}(H)) \to [0, \infty), \phi \mapsto \|\phi(s)(x)\|^2$$

is WOT lower semi-continuous and the function $\beta_{s,x}$ WOT upper semi-continuous.

Proof:
Let S, H, s and x be as above. The lower semi-continuity of the first function follows from the lower semi-continuity of the map $H \to [0, \infty), y \mapsto \|y\|$, see for example [36, Problem 21].
For the upper semi-continuity of the functions $\beta_{s,x}$, we will use Lemma 5.1.3. Define $K = \mathrm{UCP}(C^*(S), \mathcal{B}(H))$, $L = \mathrm{UCP}(S, \mathcal{B}(H))$,

$$\pi : K \to L, \rho \mapsto \rho|_S$$

and

$$f : K \to \mathbb{R}, \rho \mapsto \langle \rho(s^*s)x, x\rangle.$$

The maps π and F are clearly continuous. Additionally, π is also surjective, and by Proposition 5.1.1, K and L are compact. Thus, by Lemma 5.1.2,

$$g : L \to \mathbb{R}, \rho \mapsto \sup\{f(y);\ y \in \pi^{-1}(\rho)\}$$

is upper semi-continuous. The proof is complete with the observation that $g = \beta_{s,x}$.
□

Theorem 5.1.5: *Let S be a separable operator system and H an infinite dimensional separable Hilbert space. Then,*

$$\{\phi : S \to \mathcal{B}(H);\ \phi \text{ maximal}\} \subset \mathrm{UCP}(S, \mathcal{B}(H))$$

is a dense G_δ-set.

Proof:
Let S and H be as above and $S_0 \subset S, H_0 \subset H$ be dense countable subsets. We claim that

$$\{\phi \in UCP(S, \mathcal{B}(H));\ \phi \text{ maximal}\}$$

equals

$$\bigcap_{s\in S_0} \bigcap_{x\in H_0} \bigcap_{0<\epsilon\in\mathbb{Q}} \{\phi \in UCP(S, \mathcal{B}(H));\ \beta_{s,x}(\phi) - \|\phi(s)(x)\|^2 < \epsilon\}$$

and that the sets on the rights are WOT open. First assume that $\phi \in \mathrm{UCP}(S, \mathcal{B}(H))$ is maximal. Then, ϕ has a unique u.c.p. extension ψ to $C^*(S)$ and ψ is a unital $*$-homomorphism. Therefore, $\beta_{s,x}(\phi) = \|\phi(s)(x)\|^2$ for all $s \in S$ and $x \in H$ and we obtain „ $\subset$ “.
For the other direction, let $\phi \in \mathrm{UCP}(S, \mathcal{B}(H))$ such that $\beta_{s,x}(\phi) = \|\phi(s)(x)\|^2$ for all $s \in S_0, x \in H_0$, and $\psi : C^*(S) \to \mathcal{B}(K)$ be a u.c.p. dilation of ϕ. For clarity, we identify H with $V(H)$. Since $\beta_{s,x}(\phi) = \|\phi(s)(x)\|^2$, we have that

$$\|\psi(s)(x)\|^2 = \|\phi(s)(x)\|^2,$$

and since $P_H\psi|_H = \phi$, it follows that $\psi(s)(x) = \phi(s)(x)$ for all $s \in S_0, x \in H_0$. By density of these two spaces, we obtain that $\psi|_H = \phi$. Hence, ψ is a trivial dilation of ϕ, and ϕ is maximal.
Finally, the maps $\beta_{s,x}$ and $-\|(\cdot)(s)(x)\|^2$ are upper semi-continuous by Lemma 5.1.4 and thus

$$\{\phi \in UCP(S, \mathcal{B}(H));\ \beta_{s,x}(\phi) - \|\phi(s)(x)\|^2 < \epsilon\}$$

is WOT open for all $s \in S_0, x \in H_0$. □

5.2 Matrix Convex Sets and Arveson Extreme Points

We have already mentioned that every commutative operator system is completely order isomorphic to the affine functions on some compact convex set. This result extends to arbitrary operator system, but one must replace affine functions and compact convex set with suitable non-commutative analogues. In the following, we present the fundamental definitions associated with this concept.

A *matrix convex set* over a topological vector space V is a graded set $(K_n)_{n=1}^{\infty}$, where each $K_n \subset M_n(V)$, that it is closed under *matrix convex combinations*, that is, a combination of the form

$$\sum_{i=1}^{n} V_i^* X_i V_i \in M_m(V),$$

where $1 \leq n, m \in \mathbb{N}, X_i \in K_{n_i}, V_i \in M_{m,n_i}(\mathbb{C})$, and $\sum_{i=1}^{n} V_i^* V_i = id$.
A matrix convex set $(K_n)_n$ is called *compact* if each K_n is compact.
A point $X \in (K_n)_n$ is called *matrix extreme* if for every matrix convex combination

$$X = \sum_{i=1}^{n} V_i^* X_i V_i$$

with each V_i surjective, it follows that every X_i is unitarily equivalent to X. The set of matrix extreme points is denoted by $\operatorname{mex}(K)$.

A proper generalization of compact set and extreme points should, of course, satisfy analogues version of the Krein-Milman and Carathéodory theorems. Indeed, such analogues exist. To state them, we first have to introduce the concept of the *matrix convex hull* of a graded set, defined as

$$\operatorname{mconv}(L) = \left\{ \sum_{i=1}^{n} V_i^* X_i V_i;\ X_i \in L_{n_i}, V_i \in M_{m,n_i}, \sum_{i=1}^{n} V_i^* V_i = id \right\},$$

where $L = (L_n)_{n\in\mathbb{N}} \subset (M_n(V))_{n\in\mathbb{N}}$.
The generalization of the Krein-Milman theorem is due to Webster and Winkler and can be found in [64].

Theorem 5.2.1 (Webster-Winkler): *Let $K = (K_n)_n$ be a compact matrix convex set. Then*

$$\overline{\operatorname{mconv}(K)} = K.$$

One generalization of Carathéodory's theorem is due to Hartz and Lupini and can be found in [37]. We also note that there is a generalization by Kriel concerning free spectrahedra (see [49]), which we will introduce later. We state only the first generalization, as it is the one we will need.

Theorem 5.2.2 (Hartz-Lupini): *Let V be a finite-dimensional topological vector space and K a compact matrix convex set. Then every point in K can be written as a matrix convex combinations of matrix extreme points.*

We now turn to the connection between matrix convex sets and operator systems. Given an operator system S, the *matrix state space* of S is defined by

$$\mathcal{W}(S) = \bigcup_{n=1}^{\infty} \{\phi : S \to M_n(\mathbb{C});\ \phi \text{ u.c.p.}\}.$$

We view this set as a graded set over the dual space of S. It is easy to verify that matrix convex combinations of u.c.p. maps are again u.c.p., and thus, by Proposition 5.1.1, $\mathcal{W}(S)$ is a compact matrix convex set.
This shows that every operator system gives rise to a compact matrix convex set. To see that this correspondence is one-to-one, we introduce a non-commutative analogue of affine functions:
Let V, W be topological vector spaces, and let $K = (K_n)_n$ be a compact matrix convex set. A sequence $\theta = (\theta_n)_{n=1}^{\infty}$ with $\theta : K_n \to M_n(W)$, is called *matrix affine* if

$$\theta_m\left(\sum_{i=1}^{n} V_i^* X_i V_i\right) = \sum_{i=1}^{n} V_i^* \theta_{n_i}(X_i) V_i$$

for all $m \in \mathbb{N}, X_i \in K_{n_i}, V_i \in M_{m,n_i}(\mathbb{C})$ with $\sum_{i=1}^{n} V_i^* V_i = id$. If in addition each θ_n is a homeomorphism, we call θ a *matrix affine homeomorphism*. In the case $W = \mathbb{C}$, we write $A(K)$ for the graded set of all matrix affine maps θ such that $\theta_1 : K_1 \to \mathbb{C}$ is continuous.

We now state the general relationship between $A(K)$ and operator systems. The following theorem was proved by Webster and Winkler [64, Proposition 3.5]. It is not immediately obvious why $A(K)$ can be seen as an operator system. There are two ways to do this: One can define a suitable C^*-algebra of non-commutative functions into which $A(K)$ embeds (see [21]). Alternatively, one can characterize operator systems via the Choi-Effors axioms and verify that $A(K)$ satisfies them. For the axiomatic characterization of operator systems, see [55, Chapter 13].

Theorem 5.2.3:

(i) *If S is an operator system, then there exists a completely order isomorphic map between $A(\mathcal{W}(S))$ and S.*

(ii) *If K is a matrix convex set, then the sequence $(\theta_n)_n$, defined by*

$$\theta_n : K_n \to \mathcal{W}(A(K))_n, \; x \mapsto [f \mapsto f(x)]\,,$$

is a matrix affine homeomorphism between K and $\mathcal{W}(A(K))$.

Besides matrix extreme points, there exists a second, more restrictive notion of extremality. A point X in a matrix convex set K is called an *Arveson extreme point* (also known as *absolute matrix extreme point*) if for every matrix convex combination

$$X = \sum_{i=1}^{n} V_i^* X_i V_i$$

with $0 \neq V_i \in M_{m,n_i}$, the following holds: For each $i \in \{1, \ldots, n\}$, either X_i is unitarily equivalent to X, or $n_i > m$ and there exits $Z_i \in K$ such that X_i is unitarily equivalent to $X \oplus Z_i$. The set of all Arveson extreme points is denoted

by arvex(K).
Note that $\operatorname{arvex}(K) \subset \operatorname{mex}(K)$. This inclusion follows from the fact that if all V_i are surjective, then the case $n_i > m$ cannot occur.

The following theorem relates matrix extreme points to pure u.c.p. maps and the Arveson extreme points to boundary representations. It will enable us to determine matrix extreme points and Arveson extreme point more easily. Part (i) is due to Farenick [31, Theorem B], and part (ii) can be found in [49, Corollary 6.27] and [30, Theorem 3.10].

Theorem 5.2.4: *Let S be an operator system. Then:*

(i) *A map $\phi \in \mathcal{W}(S)$ is a matrix extreme point if and only if ϕ is a pure u.c.p. map.*
(ii) *If S is finite-dimensional, then $\phi \in \mathcal{W}(S)$ is an Arveson extreme point if and only if ϕ is the restriction of a boundary representation.*

The disadvantage of matrix extreme points compared to Arveson extreme points is that matrix extreme points are often more numerous than necessary to recover the matrix convex set. We did not mention this earlier but, in the classical case, for a compact convex set K and a subset F such that $\overline{\operatorname{conv}(F)} = K$, Milman's converse theorem implies that $\operatorname{ex}(K) \subset \overline{F}$. In this sense, the extreme points are minimal. However, this minimality does not hold for matrix extreme points.
On the other hand, the downside of Arveson extreme points is that they do not have to exist, as the following example demonstrates.

Example 5.2.5: Let u, v be the universal generator of the Cuntz algebra

$$\mathcal{O}_2 = C^*(u, v;\ u^*u = v^*v = uu^* + vv^* = 1)$$

and define the operator system $S = \operatorname{span}(1, u, v, u^*, v^*)$. Since $C^*(S) = \mathcal{O}_2$ and $\mathcal{O}_2$ admits no finite-dimensional representations, it follows that $\mathcal{W}(S) = \emptyset$. Nevertheless, Theorem 5.2.1 guarantees the existence of matrix extreme points.

It remains an open problem to find conditions under which the matrix convex hull of the Arveson extreme points equals the full matrix convex set. We now study two such conditions and their relationship. The first one is a new result obtained in collaboration with Michael Hartz. For its formulation, we need the following definition.
A C^*-algebra is called *finite-dimensional irreducible* (FDI) if every irreducible representation is unitarily equivalent to a finite-dimensional one.

Theorem 5.2.6: *Let S be a finite-dimensional operator system generating a FDI C^*-algebra $\mathcal{A}$. Then*

$$\operatorname{mconv}(\operatorname{arvex}(\mathcal{W}(S)) = \mathcal{W}(S).$$

Proof:
Let S and $\mathcal{A}$ be as above, and let $\phi \in \mathcal{W}(S)_m$. By Theorem 5.2.2, there exists a matrix convex combination of matrix extreme points $\phi_1, \dots, \phi_n$ such that

$$\phi = \sum_{i=1}^{n} V_i^* \phi_i V_i. \tag{5.1}$$

By Theorem 5.2.4, each ϕ_i is pure, and by Theorem 3.1.20, each $\phi_i : S \to \mathcal{B}(H_i)$ has a dilation to a maximal irreducible representation $\psi_i : S \to \mathcal{B}(K_i)$ with respect to an isometry $\gamma_i : H_i \to K_i$.
Each ψ_i has a unique u.c.p. extension to a boundary representation. Since $\mathcal{A}$ is FDI, we conclude that $\dim(K_i) < \infty$. Again, by Theorem 5.2.4, each ψ_i is an Arveson extreme point. Moreover, we can write

$$\phi = \sum_{i=1}^{n} (\gamma_i V_i)^* \psi_i (\gamma_i V_i)$$

which is a matrix convex combination, since

$$\sum_{i=1}^{n} (\gamma_i V_i)^* (\gamma_i V_i) = \sum_{i=1}^{n} V_i \gamma_i^* \gamma_i V_i = \sum_{i=1}^{n} V_i^* V_i = id.$$

Hence, $\phi \in \mathrm{mconv}(\mathrm{arvex}(\mathcal{W}(S)))$, completing the proof. □

Example 5.2.7:

(i) Every commutative C^*-algebra is FDI, since its irreducible representations are given by point evaluations.

(ii) Every C^*-algebra of continuous matrix-valued functions is FDI, as the irreducible representations are again given by matrix-valued point evaluations.

(iii) The universal C^*-algebra by two unitary operators

$$C^*(u, v;\ u^*u = uu^* = v^*v = vv^* = 1)$$

is not FDI, because there exists two unitary operators U, V on an infinite-dimensional Hilbert space that do not admit a common non-trivial reducing subspace. For example, let $H = \ell^2(\mathbb{Z})$, U be the bilateral shift, and V be the diagonal operator with respect to the canonical orthonormal basis and the sequence $(e^{iqn})_{n\in\mathbb{Z}}$ for some $q \in \mathbb{R} \setminus \mathbb{Q}$. Nevertheless, this is a RFD C^*-algebra by [15].

The last example together with the following example show that the conclusion of Theorem 5.2.6 may still hold even if the operator system does not embed completely order isomorphic into a FDI C^*-algebra.

Example 5.2.8: Let u, v be as in the previous example, and let S be the operator system generated by u and v. Similar to Example 3.1.10, one can show that

$$\theta = (\theta_n)_{n\in\mathbb{N}} : \mathcal{W}(S) \to \bigcup_{1\leq n\in\mathbb{N}} \{(T_1, T_2) \in M_n(\mathbb{C})^2;\ \|T_i\| \leq 1\}, \phi \mapsto (\phi(u), \phi(v))$$

is well-defined and bijective. Moreover, the maximal u.c.p. maps ϕ are exactly those for which $\phi(u)$ and $\phi(v)$ are unitary operators.
Given any $\phi \in \mathcal{W}(S)$, one can construct a dilation $\psi : S \to \mathcal{B}(K)$ with respect to an isometry $V : H \to K$, such that ψ is a restriction of a finite-dimensional representation of $C^*(u, v)$, using the construction in Example 3.1.10. ψ is maximal, since both $\psi(u)$ and $\psi(v)$ are unitary operators, and since ψ is a finite-dimensional representation, it is the direct sum of restrictions of boundary representations. Hence, $\psi \in \text{mconv}(\text{arvex}(\mathcal{W}(S)))$, and since

$$\phi = V^*\psi V,$$

we obtain that $\phi \in \text{mconv}(\text{arvex}(\mathcal{W}(S)))$. Thus we have proven that

$$\text{mconv}(\text{arvex}(\mathcal{W}(S))) = \mathcal{W}(S).$$

An alternative proof of this is given in Example 5.2.15. However, we claim that S is not completely order isomorphic to an operator system that generates a FDI C^*-algebra. We have already observed that maximality is preserved under completely order isomorphic maps, and it is straightforward to verify that irreducibility of u.c.p. maps is also preserved under such isomorphisms. Therefore, since $C^*(u, v)$ is not FDI by the previous example, the claim follows.

Let us now turn to a second condition which ensures that the matrix convex hull of the Arveson extreme points coincides with the entire matrix convex set. To state this condition, we introduce the concept of linear pencils.
Let $A = (A_1, \ldots, A_d) \in M_k(\mathbb{C})^d$ be a tuple of complex-valued matrices. Then the map

$$L_A((T_1, \ldots, T_d)) = id - 2\text{Re}\left(\sum_{i=1}^{d} A_i \otimes T_i\right)$$

is called *monic linear pencil*. The *free spectrahedron* of A is the matrix convex set

$$D_A = \bigcup_{1\leq n\in\mathbb{N}} \{T \in M_n(\mathbb{C})^d;\ L_A(T) \geq 0\}.$$

Note that this yields a matrix convex set, but not necessarily a compact one.
Usually, one restricts to selfadjoint tuples A, but this is actually no real restriction, since

$$\text{Re}(A_j \otimes T_j) = \text{Re}(A_j) \otimes \text{Re}(T_j) - \text{Im}(A_j) \otimes \text{Im}(T_j)$$

implies that for $B = (\mathrm{Re}(A_1), -\mathrm{Im}(A_1), \dots, \mathrm{Re}(A_d), -\mathrm{Im}(A_d))$:

$$Z = (X_1 + iY_1, \dots, X_d + iY_d) \in D_A \Leftrightarrow L_B((X_1, Y_1, \dots, X_d, Y_d)) \geq 0.$$

Thus we can identify D_A with

$$D_B = \bigcup_{n \in \mathbb{N}} \left\{ (T \in M_n(\mathbb{C})^{2d};\ T_i = T_i^* \text{ and } L_B(T) = id - \sum_{i=1}^{2d} B_i \otimes T_i \geq 0 \right\}.$$

The form D_B is called the *selfadjoint form* of D_A.
If one restricts the underlying field to $\mathbb{R}$, then we have a *reel free spectrahedron* with selfadjoint form

$$D_A^{\mathbb{R}} = \bigcup_{n \in \mathbb{N}} \{T \in M_n(\mathbb{R})^d;\ T_i = T_i^* \text{ and } L_A(T) \geq 0\}.$$

Remark 5.2.9: Let $A = (A_1, \dots, A_d) \in M_n(\mathbb{C})^d$. Then the *spectraball* is defined by

$$B_A = \bigcup_{1 \leq n \in \mathbb{N}} \left\{ X \in M_n(\mathbb{C})^d;\ \left\| \sum_{i=1}^{d} A_i \otimes X_i \right\| \leq 1 \right\}.$$

An advantage of the spectraball is that it is useful in the construction of examples. To see that this is a spectrahedron, define

$$E_j = \begin{pmatrix} 0 & A_j \\ 0 & 0 \end{pmatrix},$$

and observe that

$$L_E((T_1, \dots, T_d)) = id - 2\mathrm{Re}\left(\sum_{i=1}^{d} E_i \otimes T_i \right) = \begin{pmatrix} id & \sum_{i=1}^{d} A_i \otimes T_i \\ \sum_{i=1}^{d} A_i^* \otimes T_i^* & id \end{pmatrix}.$$

Thus,

$$L_E((T_1, \dots, T_d)) \geq 0 \ \Leftrightarrow \ \left(\sum_{i=1}^{d} A_i \otimes T_i \right)^* \left(\sum_{i=1}^{d} A_i \otimes T_i \right) \leq id,$$

from which it follows that $D_E = B_A$.

In the following, we study free spectrahedra that are closed under complex conjugation. For a matrix $X \in M_n(\mathbb{C})$, let $\overline{X}$ denote the entrywise complex conjugate, and X^T the *transpose*.

Example 5.2.10: Let

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the spectralball is given by

$$B_A = \bigcup_{n\in\mathbb{N}} \left\{(T_1, T_2) \in M_n(\mathbb{C})^2;\ T_1^*T_1 + T_2^*T_2 \le id\right\}.$$

Define

$$E_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $D_E = B_A$, and the selfadjoint form of D_A is given by

$$\bigcup_{n\in\mathbb{N}} \left\{(T_1, T_2, T_3, T_4) \in M_n(\mathbb{C})^4;\ {}_{(T_1 - iT_2)(T_1+iT_2)+(T_3-iT_4)(T_3+iT_4)\le id}^{T_i^* = T_i,}\right\}.$$

Note that for any $X_1, X_2 \in M_n(\mathbb{C})^2$, we have

$$\overline{X_1}\ \overline{X_2} = \overline{X_1 X_2} \text{ and } \overline{X_1}^* = \overline{X_1^*}.$$

Thus, the spectraball B_A is closed under complax conjugation. However,

$$T = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}\right)$$

lies in the selfadjoint form of D_E, while $\overline{T}$ does not.
Furthermore, we want to determine $\text{arvex}(B_A)$. For this, consider the Cuntz algebra $\mathcal{O}_2$ with generators u, v, and define $S = \text{span}\{1, u, u^*, v, v^*\}$. Then we can identify $\mathcal{W}(S)$ with B_A via the map $\phi \mapsto (\phi(u^*), \phi(v^*))$.
Since the Cuntz algebra $\mathcal{O}_2$ admits no finite-dimensional representations, it follows from Theorem 5.2.4 that $\text{arvex}(\mathcal{W}(S)) = \emptyset$, and hence

$$\text{arvex}(B_A) = \emptyset.$$

That a free spectrahedron is closed under complex conjugation has a surprising consequence for the Arveson extreme points. The following result is due to [29, Theorem 1.1].

Theorem 5.2.11 (Evert-Helton): *Let D_A be a free spectrahedron whose selfadjoint form is closed under complex conjugation. Then*

$$D_A = \text{mconv}(\text{arvex}(D_A)).$$

Naturally, one may ask how this theorem relates to Theorem 5.2.6. To answer this, we begin by collecting two useful lemmas. The first helps determine when the selfadjoint form of a spectrahedron is closed under complex conjugation.

Theorem 5.2.12: *Let D_A be a free spectrahedron. The following are equivalent:*

(i) *D_A is closed under transposition.*

(ii) *The selfadjoint form of D_A is closed under complex conjugation.*

Proof:
Let D_A be a free spectrahedron. The core of the proof lies in the observation:

$$X^T = \mathrm{Re}(X)^T + i\mathrm{Im}(X)^T \text{ for all } X \in M_n(\mathbb{C}).$$

Let $B = (\mathrm{Re}(A_1), -\mathrm{Im}(A_1), \ldots, \mathrm{Re}(A_d), -\mathrm{Im}(A_d))$, and suppose

$$(X_1, Y_1, \ldots, X_d, Y_d) \in D_B.$$

Then we have:

$$(\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_d, \overline{Y}_d) = (X_1^T, Y_1^T, \ldots, X_d^T, Y_d^T),$$

and hence

$$((X_1 + iY_1)^T, \ldots, (X_d + iY_d)^T) \in D_A \;\Leftrightarrow\; (\overline{X}_1, \overline{Y}_1, \ldots, \overline{X}_d, \overline{Y}_d) \in D_B.$$

This completes the proof. □

The next lemma gives some sort of twisted Cauchy-Schwartz inequality in the case that the matrix state space is closed under the transpose and the operator system is a C^*-algebra.

Lemma 5.2.13: *Let $\mathcal{A}$ be a C^*-algebra, and let $\phi \in \mathcal{W}(\mathcal{A})$ such that ϕ^T is u.c.p.. Then, for all $x \in \mathcal{A}$, it holds that*

$$\phi(x)\phi(x^*) \leq \phi(x^*x). \tag{5.2}$$

Proof:
Let $x \in \mathcal{A}$, and without loss of generality assume $\|x\| \leq 1$. Then the matrix

$$\begin{pmatrix} 1 & x \\ x^* & x^*x \end{pmatrix} \geq 0$$

and thus,

$$\phi^T\left(\begin{pmatrix} 1 & x \\ x^* & x^*x \end{pmatrix}\right) = \begin{pmatrix} 1 & \phi(x)^T \\ \phi(x^*)^T & \phi(x^*x)^T \end{pmatrix} \geq 0.$$

Since the transpose of a positive matrix is again positive, it follows that

$$\begin{pmatrix} 1 & \phi(x)^T \\ \phi(x^*)^T & \phi(x^*x)^T \end{pmatrix}^T = \begin{pmatrix} 1 & \phi(x^*) \\ \phi(x) & \phi(x^*x) \end{pmatrix} \geq 0.$$

Conjugating this matrix with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and applying the lemma about Schur complements, Lemma 3.1.5, yields the inequality $\phi(x)\phi(x^*) \leq \phi(x^*x)$. □

We conclude this chapter, and with it, the thesis, with two examples that illustrate that Theorem 5.2.6 and Theorem 5.2.11 are in general independent. Specifically, we show that:

- there exist finite-dimensional operator systems in FDI C^*-algebras whose matrix state space is not closed under transposition, and
- there exists free spectrahedra that are closed under transposition, although the corresponding operator system does not embed completely order isomorphic into a FDI C^*-algebra.

Example 5.2.14: Consider $M_2(\mathbb{C})$ as an operator system in itself. This is clearly a finite-dimensional operator system inside a FDI C^*-algebra and thus, by Theorem 5.2.6,

$$\text{mconv}(\text{arvex}(M_2(\mathbb{C}))) = \mathcal{W}(M_2(\mathbb{C})).$$

However, $\mathcal{W}(M_2(\mathbb{C}))$ is not closed under transposition. To see this, consider the identity map

$$id : M_2(\mathbb{C}) \to M_2(\mathbb{C}),\ X \mapsto X.$$

This is clearly a u.c.p. map. If id^T were also u.c.p., then by Lemma 5.2.13, it would follow that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = id\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^*\right) id\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \leq id\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^*\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which clearly is false. Therefore, $\mathcal{W}(M_2(\mathbb{C}))$ is not closed under transposition.

Example 5.2.15: Let $C^*(u, v)$ be the universal C^*-algebra generated by two unitary elements u and v, as already encountered in Example 5.2.7, and let

$$S = \text{span}(1, u, v, u^*, v^*).$$

In Example 5.2.8, we showed that S is not completely order isomorphic to an operator system contained in a FDI C^*-algebra, even though

$$\text{mconv}(\text{arvex}(\mathcal{W}(S))) = \mathcal{W}(S).$$

We now claim that $\mathcal{W}(S)$ is matrix affine homeomorphic to a free spectrahedron that is closed under transposition. Define

$$\theta : \mathcal{W}(S)_n \to \bigcup_{1 \leq n \in \mathbb{N}} \{(T_1, T_2) \in M_n(\mathbb{C})^2;\ \|T_i\| \leq 1\}, \phi \mapsto (\phi(u), \phi(v)).$$

It is easy to verify that this is a matrix affine homeomorphism. Let

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$B_A = \bigcup_{n\in\mathbb{N}} \left\{ (T_1, T_2) \in M_n(\mathbb{C})^2;\ \left\| \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \right\| \leq 1 \right\},$$

and by Remark 5.2.9, we have that

$$B_A = D_E,$$

where

$$E_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The spectraball B_A is clearly closed under transposition, therefore, so is the free spectrahedron D_E. Hence Theorem 5.2.11 applies to D_E, whereas Theorem 5.2.6 does not.

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