Classifying birational automorphisms of irreducible holomorphic symplectic manifolds

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Abstract

The subject of the thesis is the study of birational automorphisms of finite order on irreducible holomorphic symplectic (IHS) manifolds, from a lattice-theoretic point of view.

IHS manifolds are one of the building blocks of compact Kähler manifolds with trivial first Chern class, and the question of studying their symmetries arises naturally. Thanks to the Torelli-type theorems for IHS manifolds, these symmetries can be analyzed through certain isometries of even indefinite $\mathbb{Z}$-lattices. The main focus of the work is on describing computational methods for classifying finite groups of such isometries. In particular, most of the approaches presented in the thesis have an algorithmic counterpart.

The work starts with some definitions and notations about lattices and IHS manifolds. Special attention is given to exploring the implementation of some of the introduced notions, setting up a computational framework for the rest of the thesis.

The second part of the work is structured around five classification problems, which serve as a guiding line for the classification of finite groups of birational automorphisms of IHS manifolds. Two of these problems concern the classification of certain finite cyclic groups of isometries of even indefinite $\mathbb{Z}$-lattices. A theoretical approach to solve each of them is given in specific cases. For solving each of the remaining three classification problems, a general procedure is presented, along with discussions about its effectiveness and limitations. Each of the methods described in this part of the thesis is applied to concrete examples. These five classification problems allow for a global classification procedure. This procedure is practical for the deformation types $\mathrm{K3}^{[p^k+1]}$, OG6 and OG10. An extra effort is required for the other known deformation types of IHS manifolds and singular analogs.

The final part of the work is focused on geometric applications for the lattice techniques described so far. An account is also given of the theory of projective representations of finite groups, which has useful applications in determining equations that describe projective IHS manifolds. It is applied to find an explicit geometric description of some projective K3 surfaces. The rest of this part is about studying symmetries of special projective IHS manifolds, known as double EPW-cubes and LSV manifolds.

Zusammenfassung

Das Thema der Dissertation ist die Untersuchung birationaler Automorphismen endlicher Ordnung auf irreduziblen holomorphen symplektischen (IHS-)Mannigfaltigkeiten mittels $\mathbb{Z}$-Gitter.

IHS-Mannigfaltigkeiten gehören zu den Grundbausteinen kompakter Kähler-Mannigfaltigkeiten mit trivialer erster Chern-Klasse, und die Frage nach ihren Symmetrien ergibt sich auf natürliche Weise. Dank der Torelli-artigen Sätze für IHS-Mannigfaltigkeiten können diese Symmetrien durch bestimmte Isometrien gerader, indefiniter $\mathbb{Z}$-Gitter untersucht werden. Der Schwerpunkt der Arbeit liegt auf der Beschreibung rechnergestützter Methoden zur Klassifikation endlicher Gruppen solcher Isometrien. Insbesondere haben die meisten der beschriebenen Ansätze eine algorithmische Entsprechung.

Die Arbeit beginnt mit einigen Definitionen zu Gittern und IHS-Mannigfaltigkeiten. Ein besonderes Augenmerk liegt auf der Implementierung einiger der eingeführten Begriffe, um einen rechnerischen Rahmen für den weiteren Verlauf der Arbeit zu schaffen.

Der zweite Teil der Arbeit behandelt fünf Klassifikationsprobleme, die einen Leitfaden für die Klassifikation endlicher Gruppen birationaler Automorphismen von IHS-Mannigfaltigkeiten liefern. Zwei dieser Probleme betreffen die Klassifikation bestimmter zyklischer Isometriegruppen von geraden, indefiniten $\mathbb{Z}$-Gittern. Ein theoretischer Ansatz zur Lösung jedes dieser Probleme wird in spezifischen Fällen gegeben. Für die übrigen drei Klassifikationsprobleme wird ein allgemeines Verfahren präsentiert, zusammen mit einer Diskussion über dessen Effektivität und Einschränkungen. Jede der in diesem Abschnitt beschriebenen Methoden wird auf konkrete Beispiele angewendet. Diese fünf Klassifikationsprobleme ermöglichen ein globales Klassifikationsverfahren. Dieses Verfahren ist effektiv für die Deformationstypen K3$^{[p^k+1]}$, OG6 und OG10. Für die anderen bekannten Deformationstypen von IHS-Mannigfaltigkeiten und singulären Analogien ist zusätzlicher Aufwand erforderlich.

Der letzte Teil der Arbeit ist auf geometrische Anwendungen der bisher beschriebenen Gittertechniken fokussiert. Es wird auch ein Überblick über die Theorie der projektiven Darstellungen endlicher Gruppen gegeben, die nützliche Anwendungen zur Bestimmung der definierenden Gleichungen von projektiven IHS-Mannigfaltigkeiten haben. Sie wird angewendet, um eine explizite geometrische Beschreibung einiger projektiver K3-Flächen zu finden. Der Rest dieses Teils befasst sich mit der Untersuchung von Symmetrien spezieller projektiver IHS-Mannigfaltigkeiten, die als „double EPW-cubes" und „LSV manifolds" bekannt sind.

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Notation

- the symbol $\simeq$ is used to denote isomorphism between objects of a same category, and the symbol $\cong$ is exclusively used for isomorphism of groups;
- the symbols $\subset$, $\subseteq$ are used for inclusion as sets, while we use $<$, $\leq$ for subsets inheriting an algebraic structure (similarly for $\supset$, $\supseteq$, $>$, $\geq$);
- given a binary form on an abelian group A and given a subset $S \subseteq A$, we use the notation $S_A^\perp$ to refer to the orthogonal of S in A with respect to the given form (or simply $S^\perp$ if A is understood from the context);
- the symbol $\otimes$ denotes tensor products and $\oplus$ denotes direct sums — in the context of abelian groups equipped with a binary form, direct sums are supposed to be *orthogonal* with respect to such a form;
- given a group G and given a finite collection $S \subseteq G$, we denote by $\langle S \rangle$ the subgroup of G generated by the elements in S;
- given a group G acting on a set S, we denote by $\mathrm{Stab}_G(M)$ the stabilizer in G of any subset $M \subseteq S$;
- we denote by C_n the finite cyclic group of order $n \geq 1$, and by $\mu_n \leq \mathbb{C}^\times$ the cyclic group of nth roots of unity;
- for $n \geq 2$ even, we make the convention that D_n denotes the dihedral group with n elements;
- given a morphism of commutative rings with unity $\phi \colon A \to B$ and given a B-module M, we denote by $\phi^* M$ to be M endowed with an A-module structure via ϕ;
- given a finite extension of fields E/K, we denote by $\mathfrak{D}_{E/K}$, N_K^E and Tr_K^E respectively the different ideal, the norm map and the trace form associated to the extension;
- we denote by $\left(\frac{\cdot}{\cdot}\right)$ the *Kronecker symbol*, which is a generalization of Jacobi symbol to all integers (which itself is a generalization of Legendre symbol from odd prime numbers to any odd integers);
- in the context of smooth projective varieties, Bir and Aut will respectively stand for birational automorphism group and automorphism group;
- given a complex variety X, we denote by $\mathrm{Sing}(X)$ its singular locus;
- we denote ADE roots lattices by A_n, D_n, E_n, and we define $\mathbb{L}$ to be the Leech lattice;
- in the context of hermitian lattices or $\mathbb{Z}$-lattices, we denote by $g(L)$ the *genus* of a given lattice L.

Introduction

The application of the Torelli-type theorems for **irreducible holomorphic symplectic (IHS)** manifolds encourages the development of new techniques for studying isometries of indefinite integer lattices. The analysis of the symmetries of IHS manifolds is motivated by certain geometric applications, such as the construction of new examples of **primitive symplectic varieties**. This often leads to classifying birational automorphisms of IHS manifolds, from a lattice-theoretic point of view. Thanks to the continuous improvements in computational capacities, such a classification can actually be made systematic using linear algebra. In this thesis, we explore the computational aspects of the study of symmetries of IHS manifolds and we describe procedures for classifying such symmetries. These could in principle be adapted to other classes of varieties for which Torelli-type theorems are known. Our main motivation is to develop an approach for classifying the representations of finite groups of symmetries on the second integral cohomology for the known examples of IHS manifolds. As geometric applications, we construct some explicit examples of **symmetric IHS manifolds**.

Irreducible holomorphic symplectic manifolds

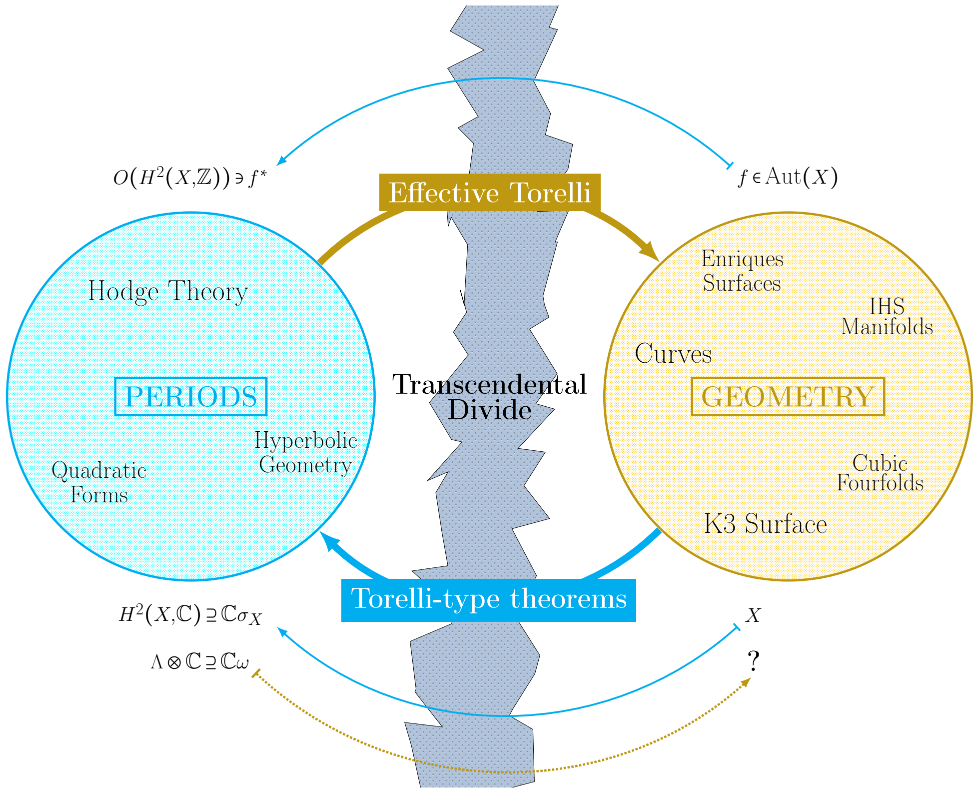
IHS manifolds are one of the building blocks for compact Kähler manifolds with trivial first Chern class, together with complex tori and strict Calabi–Yau manifolds. They can be seen as a higher-dimensional analog of K3 surfaces for various legitimate reasons. Contrary to complex tori and Calabi–Yau manifolds, constructing IHS manifolds is a rather difficult task, and up to deformation, we know very few examples per dimension. For a long time, K3 surfaces were thought to be the unique examples of IHS manifolds; a first example in dimension 4 is given by Fujiki in [Fuj83]. By generalizing Fujiki's construction, Beauville described two series of examples of IHS manifolds, providing examples for each even complex dimension $2n > 2$ [Bea83a]: the corresponding deformation families are usually denoted by $\mathrm{K3}^{[n]}$ and Kum_n. Back in around 2000, O'Grady showed the existence of two new exceptional examples in complex dimension 6 and 10 [O'G99, O'G03], whose deformation families will be denoted OG6 and OG10 respectively. And this is it. It is still an open problem to know whether the known examples are the only existing ones, up to deformation.

In view of the Minimal Model Program, the definition of IHS manifolds has been generalized to allow mild singularities, as explained for instance in [Per20]. This has proven to be a promising strategy for constructing new examples of singular analogs of IHS manifolds in complex dimension 4 [Men22]. The study of birational automorphisms plays a crucial role in the construction of such **primitive symplectic varieties**. Indeed, many recent examples were discovered by studying terminalizations of quotients of known IHS manifolds by finite groups of **symplectic automorphisms** [BGMM24].

While such groups of automorphisms might admit fixed points for their action on the associated IHS manifolds, fixed-point free actions on IHS manifolds give rise to a new class of varieties referred to as **Enriques manifolds** [OS11]. The concept of Enriques manifolds takes its name from Enriques surfaces: similarly to the latter, the universal cover of an Enriques manifold is an IHS manifold. IHS manifolds, their singular analogs and Enriques manifolds provide very interesting classes of varieties which appear to offer a good testing ground for conjectures. A remarkable example are the proofs of the Morrison–Kawamata cone conjecture for such varieties (see [AV17a] for IHS manifolds, [LMP24] for symplectic varieties and [PS23] for Enriques manifolds with universal cover of prime degree).

Torelli-type theorems

A crucial tool while working with IHS manifolds is the existence of an integral $\mathbb{Z}$-lattice structure on their second cohomology group with integer coefficients [Rap08]. The underlying nondegenerate quadratic form, usually referred to as **Beauville–Bogomolov–Fujiki (BBF)** form, is of topological nature. The natural pure Hodge structure of weight 2 on the second cohomology of any IHS manifold is *polarized* with respect to the BBF form. This actually implies that the $(2, 0)$-part of such a Hodge structure determines the full weight 2 Hodge structure. This observation gives rise to the concept of **periods**, which are at the heart of the Torelli-type theorems. Any IHS manifold X admits a **universal deformation family** over of smooth and connected base, and any small deformation of X is again an IHS manifold [Huy99]. IHS manifolds are therefore gathered into deformation families, and two elements in the same deformation family have isometric BBF forms. By fixing an abstract representative Λ for such an isometry class, X can be endowed with a **marking**, which is an isometry of $\mathbb{Z}$-lattices $\eta\colon H^2(X, \mathbb{Z}) \to \Lambda$. There is a coarse moduli space $\mathcal{M}$ parametrizing equivalence classes of **marked pairs** (X, η) for a fixed deformation family, and for a naturally defined equivalence relation. Such a space admits a **period map** $\mathcal{P}$ which associates to any class $[(X, \eta)] \in \mathcal{M}$ the period $\eta_{\mathbb{C}}(H^{2,0}(X))$. This period map is known to be surjective, even when restricted to a connected component of $\mathcal{M}$ [Huy99]. A global version of the Torelli-type theorems for IHS manifolds states that two points in the same connected component of $\mathcal{M}$ describe birational IHS manifolds if and only if they are inseparable, and have thus the same period (Proposition 5.33).



A birational map between two projective IHS manifolds induces a **parallel transport operator** which describes an isometry between their associated polarized weight 2 Hodge structures. A Hodge-theoretic version of the Torelli-type theorems for IHS manifolds determines the existence of birational maps between two IHS manifolds of the same deformation family, inducing certain isometries of the polarized Hodge structures. A typical statement of such a theorem is due to

Markman and Verbitsky (Theorems 5.35 and 5.48). Birational automorphisms of a projective IHS manifold X induce isometries of the associated BBF form which come with three flavors of properties:

(1) a complex one, by preserving a given polarized weight 2 Hodge structure;

(2) a real one, by preserving certain cones spanned by real classes;

(3) an integral one, by respecting some parallel transport conditions.

Each of these describes numerical properties for the isometries of the BBF form of X. The Hodge-theoretic version of the Torelli-type theorems states that the isometries satisfying these properties are actually induced by birational automorphisms of X.

The Torelli-type theorems for IHS manifolds allow one to study and classify IHS manifolds and their symmetries via algebraic methods, i.e. from a lattice-theoretic point of view. Moreover, by the surjectivity of the aforementioned period map, one can decide on the existence of IHS manifolds with a given group of symmetries. Nonetheless, they do not provide a systematic procedure for reconstructing *explicitly* such examples geometrically. We say that the Torelli-type theorems are *not effective*.

In order to apply the Hodge-theoretic version of the Torelli-type theorems for IHS manifolds, it is required to know the isometry class of the BBF form, the description of **monodromy operators** and certain decompositions of the **positive cone** for each IHS manifolds of a given deformation family. Those have been determined for the known IHS manifolds, but also for other classes of varieties admitting Torelli-type theorems such as cubic fourfolds and Enriques surfaces. While Torelli-type theorems are known for the singular analogs of IHS manifolds [Men20, BL21], the previous list of deformation invariants is still incomplete for the majority of the known examples.

Isometries of even indefinite $\mathbb{Z}$-lattices

For the known examples of IHS manifolds, the associated BBF quadratic forms turn out to be **even and indefinite**. Constructing and classifying isometries of even $\mathbb{Z}$-lattices is a challenging problem. The isometry group of such $\mathbb{Z}$-lattices is discrete, and it is finite in the definite case. We currently know efficient algorithmic ways to compute the generators of $O(L)$ for an even definite $\mathbb{Z}$-lattice L [PS97]. In the indefinite case, the problem is much more difficult since even indefinite $\mathbb{Z}$-lattices have, generically, infinitely many isometries. In [Mil69], Milnor studies isometries of indefinite quadratic spaces with given characteristic polynomial (see also [BF15]). His work has been carried out further by McMullen who studied isometries of even indefinite unimodular $\mathbb{Z}$-lattices related to discrete dynamical systems on K3 surfaces [McM02, McM11, McM15]. In particular, McMullen describes an *assembly procedure* to construct infinite order isometries of even indefinite unimodular $\mathbb{Z}$-lattices, which can be applied to some extent to finite order isometries.

In [GM02], Gross and McMullen give necessary conditions for the existence of certain isometries of even indefinite unimodular $\mathbb{Z}$-lattices with given characteristic polynomial, and conjecture that these are sufficient. In [BFT20], Bayer and Taelman solve this conjecture in the case of characteristic polynomials which are powers of an irreducible polynomial. Such result was further extended by Bayer in [BF25]. From this latter work, we have now a very good understanding on the existence of *semisimple isometries* of even indefinite unimodular $\mathbb{Z}$-lattices with given characteristic polynomials. However, despite the strength of Bayer's result on the existence of semisimple isometries with given minimal polynomials and McMullen's procedure to reconstruct such isometries, the actual classification of conjugacy classes of isometries with given characteristic polynomial remains still complicated and poorly understood (in general).

Hermitian lattices

The assembly procedure of McMullen consists of reconstructing isometries of even $\mathbb{Z}$-lattices by *gluing equivariantly* $\mathbb{Z}$-lattices of smaller rank equipped with an isometry with irreducible minimal polynomial. Each such elementary block, or *summand*, consists of a $\mathbb{Z}$-lattice (L, b) equipped either with $\pm\,\mathrm{id}$, or with an isometry f whose minimal polynomial χ is irreducible, **symmetric**, and of even degree. In the latter case, the action of the isometry f on L can be identified with multiplication by a primitive element of $\mathbb{Z}[\chi]$. In the case where χ is a **cyclotomic** polynomial, the previous order is maximal in the number field $\mathbb{Q}(\chi)$ and the $\mathbb{Z}$-lattice (L, b) is the **trace lattice** associated to a **hermitian** $\mathbb{Z}[\chi]$**-lattice**. Those hermitian lattices have been utilized in some of the influential works about birational automorphisms of IHS manifolds (see for instance [McM11, BCMS16, Bra19, BF24]). Nonetheless a proper local study of such lattices, in the context of studying isometries of BBF forms, has been carried out only recently in the prime cyclotomic case by Brandhorst and Cattaneo [BC23]. Their analysis relied on the work of Kirschmer in his Habilitation thesis [Kir16], extending some notions and results already known from Jacobowitz [Jac62] or O'Meara [O'M73], for instance.

Outline of the thesis

In Part I, we introduce and define the main tools used for this thesis. Most of the results are taken from outside literature. For the reader's convenience, we prove some of these results: for the other ones, we provide an appropriate reference where to find a proof. Note moreover that some further results which are not classically known, but certainly known to the experts, are also proved in that chapter. Such results may already appear in some papers (co)authored by the thesis' author [BMW25, MM25a, MM25b, Mul24, Mul25].

- In Section 1 we recall standard results about $\mathbb{Z}$-lattices and their isometries. We also develop on genera of integer $\mathbb{Z}$-lattices, show how their symbols are constructed, and we explain how to enumerate genera of even $\mathbb{Z}$-lattices.
- In Section 2 we discuss about the theory of embeddings for integral $\mathbb{Z}$-lattices. We review in particular the notions of overlattices and primitive extensions, before introducing Nikulin's procedure on the classification of primitive sublattices. We also describe algorithms, given in terms of pseudocode, for supporting the implementation of such methods.
- In Section 3 we study prime power cyclotomic fields and their completion at a ramified prime ideal. We use such fields several times in the thesis, especially while working with hermitian lattices. We prove some crucial results in that section.
- In Section 4 we define hermitian lattices, and describe their genera in a special case. We introduce the trace equivalence which plays an important role, theoretically and computationally, in the construction and classification of isometries of even indefinite $\mathbb{Z}$-lattices.
- In Section 5 we define irreducible holomorphic symplectic manifolds and we prove some first properties about them. We review the Torelli-type theorems and we also fix notation for the rest of the thesis.

In Sections 1, 2 and 4 we regularly comment about the computational aspects of what is introduced in each section. In such a way, we offer to the reader an overview of what is possible to do, computationally, and what are current limitations. This is relevant for the rest of thesis since many of our results rely on actual computations.

The main contribution of the thesis is concentrated in Parts II and III. In Part II, we translate the conditions of the Hodge-theoretic Torelli-type theorem, after Markman, into lattice-theoretic statements. In Part III, we give some geometric applications for the classification of finite group actions on IHS manifolds.

In Section 6 we start by reviewing some properties about birational automorphisms of IHS manifolds. We define, in particular, **symplectic** birational automorphisms which are the ones acting trivially on the $(2,0)$-part of the Hodge structures associated to the IHS manifolds they act on. For a known deformation type $\mathcal{T}$ of IHS manifolds, we fix a representative $\Lambda_{\mathcal{T}}$ for the isometry class of BBF forms associated to IHS manifolds of type $\mathcal{T}$. For any marked IHS manifold (X,η) with X of deformation type $\mathcal{T}$ and $\eta\colon H^2(X,\mathbb{Z})\to\Lambda_{\mathcal{T}}$ a marking, one can define an orthogonal representation

$$\mathrm{Bir}(X)\to O(\Lambda_{\mathcal{T}})$$

of the group of birational automorphisms of X. For any finite subgroup $G\leq\mathrm{Bir}(X)$, one can therefore associate its image $H\leq O(\Lambda_{\mathcal{T}})$ via this representation: any finite subgroup of $O(\Lambda_{\mathcal{T}})$ arising in that way is called **effective**. In this section, we give numerical criteria to decide whether a given finite subgroup $H\leq O(\Lambda_{\mathcal{T}})$ is effective (Theorem 6.9, Theorem 6.12). For a fixed marked pair $[(X,\eta)]$, seen as a point in the moduli space $\mathcal{M}_{\mathcal{T}}$ of marked pairs of deformation type $\mathcal{T}$, there exists a subgroup $\mathrm{Mon}^2(X)\leq O(H^2(X,\mathbb{Z}))$, called the **monodromy group** of X, which can be seen as the group of change of markings of X preserving the connected component of $\mathcal{M}_{\mathcal{T}}$ in which $[(X,\eta)]$ lies. This subgroup is known to be normal for the known examples of IHS manifolds (Table 4). There exists moreover a normal subgroup $\mathrm{Mon}^2(\Lambda_{\mathcal{T}})\leq O(\Lambda_{\mathcal{T}})$ corresponding to $\mathrm{Mon}^2(X)$ via the marking η, whose definition depends only on the deformation type $\mathcal{T}$ *in the known cases*. Changing the marking of X via an element on $\mathrm{Mon}^2(X)$ is the same as conjugating the image of the orthogonal representation

$$\mathrm{Bir}(X)\to O(\Lambda_{\mathcal{T}})$$

by an element of $\mathrm{Mon}^2(\Lambda_{\mathcal{T}})$. Thus, the determination of effective subgroups of $O(\Lambda_{\mathcal{T}})$ can be done up to certain conjugacy equivalence. From this observation, we describe five classification problems which structure a general strategy to classify finite effective subgroups of $O(\Lambda_{\mathcal{T}})$, up to conjugacy (Figure 1).

In Section 7, we cover two of the previous problems. They concern subgroups of $O(\Lambda_{\mathcal{T}})$ which arise from representing finite groups of *symplectic* birational automorphisms of IHS manifolds on $\Lambda_{\mathcal{T}}$. Any such finite subgroup $H\leq O(\Lambda_{\mathcal{T}})$ is called **symplectic**. Given an isometry $f\in O(\Lambda_{\mathcal{T}})$, or a subgroup $H\leq O(\Lambda_{\mathcal{T}})$, we say f and H are **stable** if they act trivially on the **discriminant group** $D_{\Lambda_{\mathcal{T}}}$ of $\Lambda_{\mathcal{T}}$. A major part of this section is dedicated to the construction and classification of finite *stable symplectic* subgroups $H\leq O(\Lambda_{\mathcal{T}})$ which satisfies certain maximality conditions: we call them **stably saturated**. We develop a procedure to tackle this problem computationally, and we apply it to two of the known deformation families, namely $\mathrm{K3}^{[3]}$ and OG10.

Theorem (see Theorem 7.48)**.** *For $\mathcal{T}=\mathrm{K3}^{[3]}$, there are exactly 219 $\mathrm{Mon}^2(\Lambda_{\mathrm{K3}^{[3]}})$-conjugacy classes of finite stable symplectic subgroups $H\leq O(\Lambda_{\mathrm{K3}^{[3]}})$ which are stably saturated.*

Theorem (see Theorem 7.54)**.** *For $\mathcal{T}=\mathrm{OG10}$, there are exactly 192 $\mathrm{Mon}^2(\Lambda_{\mathrm{OG10}})$-conjugacy classes of finite stable symplectic subgroups $H\leq O(\Lambda_{\mathrm{OG10}})$ which are stably saturated.*

Representatives for each of the conjugacy classes determined in the previous theorems, given in terms of matrices, are available in the respective databases [BMW24] and [MM25c]. These databases have been built for the works [BMW25] and [MM25b], and they both contain notebooks with snippets of codes which have been used for the purpose of the aforementioned articles. We end this section with a discussion about **nonstable symplectic involutions**, based on Nikulin's approach to classifying 2-elementary sublattices of even unimodular $\mathbb{Z}$-lattices [Nik83]. As an application, for IHS manifolds of deformation type OG10, we prove the following.

Theorem (see Theorem 7.57)**.** *For* $\mathcal{T} = \mathrm{OG10}$*, there are exactly 4* $\mathrm{Mon}^2(\Lambda_{\mathrm{OG10}})$*-conjugacy classes of nonstable symplectic involutions in* $O(\Lambda_{\mathrm{OG10}})$*.*

In Section 8 we study nonsymplectic effective isometries which do not admit any nontrivial symplectic iterates: we refer to them as **purely nonsymplectic**. After reviewing the state of the art on the classification of prime order nonsymplectic isometries, we explain how to construct isometries of even unimodular $\mathbb{Z}$-lattices following the assembly procedure of McMullen. We also show how to classify such isometries up to conjugacy. This is a very hard problem which can hardly be solved, theoretically, in full generality. We therefore focus our attention on isometries whose characteristic polynomial is of the form $\Phi_1^a\Phi_2^\epsilon\Phi_m^b$ where

(1) the Φ_i's are the ith cyclotomic polynomials;

(2) $a, b \in \mathbb{Z}$ are positive integers;

(3) $\epsilon = 0, 1$ and $m \geq 3$ is an integer which is even if $\epsilon = 1$.

The motivation to study such kind of isometries follows from a work of Brandhorst and Cattaneo [BC23], where the authors unified the classification of prime order nonsymplectic isometries of the known BBF forms into classifying isometries of 4 given even unimodular $\mathbb{Z}$-lattices (Table 9). In the second part of this section, we generalize the approach of Brandhorst and Cattaneo to purely nonsymplectic isometries of the known BBF forms, arising from automorphisms acting trivially on the Picard group of the IHS manifolds they act on. We call such purely nonsymplectic automorphisms **algebraically trivial**. The isometries associated to such automorphisms have minimal polynomial $\Phi_1\Phi_m$. Depending on whether these isometries are stable or not, the minimal polynomials of the isometries obtained on the corresponding even unimodular $\mathbb{Z}$-lattices, through the approach of Brandhorst and Cattaneo, satisfy conditions (1)–(3) above. A particular result of this section which we highlight is a finiteness result related to nonstable purely nonsymplectic isometries.

Theorem (see Theorem 8.40)**.** *There are only finitely many pairs* $(\mathcal{T}, m)$ *where* $\mathcal{T}$ *is a known deformation type of IHS manifolds and* $m \geq 3$ *is an even integer so that there exists an IHS manifold* X *of deformation type* $\mathcal{T}$ *admitting an algebraically trivial purely nonsymplectic automorphism* f *of order* m *with nonstable action on* $H^2(X, \mathbb{Z})$*.*

The actual result is stronger than this. We actually show that for each such pair $(\mathcal{T}, m)$, there are only finitely many conjugacy classes of nonstable isometries in $O(\Lambda_{\mathcal{T}})$ of order m, which can arise from purely nonsymplectic automorphisms (Table 18). This result also brings to light the existence of families of symmetric IHS manifolds whose general member has Picard rank 1, and some of these families are actually zero-dimensional (Remark 8.80).

In Section 9, we review an **extension approach** as introduced and developed by Brandhorst, Hashimoto and Hofmann [BH21, BH23]. The idea of this approach is to determine and classify finite effective subgroups of $O(\Lambda_{\mathcal{T}})$, for a fixed deformation type $\mathcal{T}$ of IHS manifolds, starting from the classification of *symplectic* subgroups of $O(\Lambda_{\mathcal{T}})$. The procedure they describe was originally stated in the case of K3 surfaces. We prove it to apply in greater generality, for the known examples of IHS manifolds: similar techniques would actually apply for any deformation family of IHS manifolds and their singular analogs. We also present an adaptation of this extension approach for constructing and classifying finite subgroups of symplectic isometries in $O(\Lambda_{\mathcal{T}})$ from the classification of finite *stable symplectic* subgroups of $O(\Lambda_{\mathcal{T}})$. Note that currently, this last approach only applies for the known deformation types of IHS manifolds, since it relies on the image of $\mathrm{Mon}^2(\Lambda_{\mathcal{T}}) \to O(D_{\Lambda_{\mathcal{T}}})$ being cyclic (Lemma 6.15). Each of the extension approaches can be turned into an algorithm, and be implemeted. We apply the last approach to the previous classification of finite groups of stable symplectic isometries for the deformation type OG10, giving rise to the following.

Theorem (see Theorem 9.33)**.** *For $\mathcal{T} = \mathrm{OG10}$, any finite symplectic subgroup of $O(\Lambda_{\mathrm{OG10}})$ is $\mathrm{Mon}^2(\Lambda_{\mathrm{OG10}})$-conjugate to a subgroup of one among 375 (maximal) groups.*

As before, representatives for each of the 375 (maximal) conjugacy classes determined in the previous theorem are available, in terms of matrices, in the database [MM25c].

In Section 10, we apply the theory of **projective representations** of finite groups to study how to compute the homogeneous ideal defining a projective K3 surface of genus 5, starting from transcendental data known from [BH21, BH23]. This is one direction towards an effective Torelli-type theorem. The work in this section brings further evidence that by working with symmetries, one can hope to recover a geometric description of some IHS manifolds from their periods. We apply the procedure described in this section to find equations for the K3 surface 77b from [BH21].

Theorem (see Theorem 10.1)**.** *The polarized K3 surface (S, L) corresponding to the case 77b in [BH21] admits a projective model in $\mathbb{P}^5_{\mathbb{C}}$ given by*

$$S\colon \left\{ \begin{array}{rcrcrcrcrl} ix_0x_1 & + & x_0x_2 & + & x_1x_3 & + & ix_2x_3 & + & x_5^2 & = 0 \\ ix_0x_1 & - & x_0x_2 & - & x_1x_3 & + & ix_2x_3 & + & x_4^2 & = 0 \\ -x_0x_3 & & & - & x_1x_2 & & & - & x_4x_5 & = 0 \end{array} \right. .$$

We conclude the section by commenting on how the action prescribed on such a K3 surface extends to other IHS manifolds via known geometric constructions. This section is the content of the published work [Mul24].

In Section 11, we apply the classification of finite stable symplectic subgroups of $O(\Lambda_{\mathrm{K3}^{[3]}})$ (Theorem 7.48) to geometrically describe examples of **double EPW-cubes** of Picard rank 21. Such double EPW-cubes are described as follows. Let W be a complex vector space of dimension 6, and fix a volume form on $\bigwedge^3 W$. Such a form determines a symplectic form η on $\bigwedge^3 W$. For any **η-Lagrangian subspace** $A \leq \bigwedge^3 W$, one can associate a sixfold $Z_A \subseteq \mathrm{Gr}(3, W)$ which is defined as the corank 2 degeneracy locus of a certain map of vectors bundles over the Grassmannian $\mathrm{Gr}(3, W)$. The associated double EPW-cube is a natural double cover

$$\pi\colon \widetilde{Z_A} \to Z_A.$$

Let $\mathrm{LG}_\eta(10, \bigwedge^3 W)$ be the 55-dimensional projective variety parametrizing η-Lagrangian spaces in $\bigwedge^3 W$. If A is general enough in $\mathrm{LG}_\eta(10, \bigwedge^3 W)$, i.e. A lies outside the union of two distinguished divisors $\Sigma, \Gamma \subseteq \mathrm{LG}_\eta(10, \bigwedge^3 W)$, it follows that $\widetilde{Z_A}$ is an IHS manifold of deformation type $\mathrm{K3}^{[3]}$. In this section, we determine the group of automorphisms of $\widetilde{Z_A}$ which respect the double cover π, for a given A satisfying some generality assumptions (Proposition 11.11). We show in particular that for a very general choice of A, the IHS manifold $\widetilde{Z_A}$ has no nontrivial symplectic birational automorphisms (Proposition 11.18). The 20-dimensional family of double EPW-cubes admit a codimension 1 subfamily, parametrized by $\Gamma \setminus \Sigma$, consisting of double EPW-cubes which are singular in a finite number of isolated points. The very general element of this family has a unique singular point, and it admits two projective resolutions which are again IHS manifolds. We prove that such resolutions are actually isomorphic, as projective manifolds. An idea to prove this is to compute the **movable cone** of the resolutions of a very general singular double EPW-cube, and deduce from it the existence, via the Torelli-type theorems, of an isomorphism between both resolutions. Similarly, from this description, we can as well determine the birational automorphism group of such a resolution.

Theorem (see Lemma 11.20 and Proposition 11.23)**.** *Let $A \in \Gamma \setminus \Sigma$ be a very general Lagrangian subspace. Then the small resolution $\widetilde{Z_A}^{\epsilon} \to \widetilde{Z_A}$ has Picard rank 2 and moreover,*

$$\mathrm{Bir}(\widetilde{Z_A}^{\epsilon}) = \langle \hat{\iota} \rangle$$

where $\hat{\iota}$ is a nonsymplectic birational involution induced by the covering involution associated to π.

Finally, in Section 12, we discuss the geometric realizations for the classification of finite symplectic subgroups of $O(\Lambda_{\mathrm{OG10}})$, obtained in Theorem 9.33.

Programming work

This thesis comes along with an active contribution of the author to the development of the computer algebra systems Hecke [FHHJ17] and OSCAR [DEF+25], written on Julia [BEKS17]. The author is co-maintaining the codes on *quadratic and hermitian forms* on both systems. For the purpose of the computations presented in this thesis, the author contributed to the implementation of the following:

(1) a package SymmetricIntersections for working with linear and projective representations of finite groups, given in terms of matrices. This is the main algorithmic support for the proof of Theorem 10.1, regarding the computations of equations of symmetric K3 surfaces.

(2) a package QuadFormAndIsom which provides an infrastructure to work with $\mathbb{Z}$-lattices equiped with one isometry, or a group of such. Besides standard methods, such a package also features:

   (a) the implementation of an algorithm of Brandhorst and Hofmann [BH23, Algorithms 1–8] for computing a complete set of representatives for the isomorphism classes of pairs (L, f) consisting of an even $\mathbb{Z}$-lattice L in a fixed genus and a finite order isometry $f \in O(L)$;

   (b) a generic implementation of Nikulin's procedure for classifying isomorphism classes of (equivariant) primitive extensions;

   (c) a generic implementation of Nikulin's procedure for classifying isomorphism classes of primitive embeddings and primitive sublattices.

   The package offers important computational tools which we use several time in this thesis, e.g. for the classification results from Section 7 and Section 9.

(3) algorithms supporting the implementation of the package QuadFormAndIsom, such as codes related to the trace equivalence, related to the comparison and classification of finite bilinear/quadratic modules, or related to genera of hermitian lattices.

(4) an infrastructure for enumerating genera of definite $\mathbb{Z}$-lattices of large rank, following Kneser's neighbor method and further improvements. This supports the implementation of Brandhorst–Hofmann algorithms, and the classification result from Proposition 7.65.

Part I.
Preliminaries

1. Integer lattices

Let us start by introducing some basic notion about lattices and their computational aspect, since they play a crucial role in this thesis.
References: [O'M73], [Nik80], [CS99], [Ebe02], [Kne02].

1.1. Definitions and notations

Let K be a field of characteristic zero. A *quadratic space* (V, b) over K consists of a finite-dimensional K-vector space V equipped with a nondegenerate symmetric bilinear form

$$b \colon V \times V \to K.$$

An *isometry* between two quadratic spaces (V_1, b_1) and (V_2, b_2) over K is a K-linear map $f \colon V_1 \to V_2$ such that $b_2(f(x), f(y)) = b_1(x, y)$ for all $x, y \in V_1$. In the case where $K = \mathbb{R}$, a real quadratic space (V, b) of dimension $n := \dim_{\mathbb{R}}(V) \geq 1$ is determined, up to isometry, by its *signatures* (l_+, l_-) (by Sylvester's law of inertia, see for instance [CS99, Chapter 15, §6.2]). These signatures correspond to the numbers l_+ and l_- of positive and negative diagonal entries, respectively, of the *Gram matrix* $(b(e_i, e_j))_{1 \leq i,j \leq n}$ where $\{e_i\}_{1 \leq i \leq n}$ is any basis of V which is orthogonal with respect to b. We define the *dimension* of (V, b) to be the K-dimension of V.

Definition 1.1. Let (V, b) be a real quadratic space of dimension $n \geq 1$ and signatures (l_+, l_-).

(1) We say (V, b) is *definite* if $l_+ l_- = 0$, and *indefinite* otherwise.

(2) We say (V, b) is *positive definite* (resp. *negative definite*) if $l_- = 0$ (resp. $l_+ = 0$).

Let $R := \mathbb{Z}$ or $R := \mathbb{Z}_p$ for some prime number p, and let $K := \mathrm{Frac}(R)$ be the field of fractions of R. We call *R-lattice* any pair (L, b) where L is a finitely generated free R-module and

$$b \colon (L \otimes_R K) \times (L \otimes_R K) \to K$$

is a nondegenerate symmetric bilinear form. We call the quadratic space $(L \otimes_R K, b)$ the *rational span* of (L, b). Two R-lattices (L_1, b_1) and (L_2, b_2) are *isometric* if there exists an R-module isomorphism $f \colon L_1 \to L_2$ such that $b_2(f(x), f(y)) = b_1(x, y)$ for all $x, y \in L_1$.

Definition 1.2. Let (L, b) be a $\mathbb{Z}$-lattice, and let $(L_{\mathbb{R}}, b_{\mathbb{R}}) := (L, b) \otimes_{\mathbb{Z}} \mathbb{R}$ be the associated real quadratic space. We call (L, b) *positive definite* (resp. *negative definite*, *indefinite*) if so is $(L_{\mathbb{R}}, b_{\mathbb{R}})$. Similarly, we define the *(real) signatures* of (L, b) to be the signatures of $(L_{\mathbb{R}}, b_{\mathbb{R}})$.

If there is no confusion possible, we sometimes drop b from the notation. In particular, for all $x, y \in L \otimes_R K$, we denote $x^2 := b(x, x)$ and $x.y := b(x, y)$. We call the former the *norm* of x, and the latter is referred to as the *product* of x and y.

Notation. If $a_1, \ldots, a_k \in K$, we define $\langle a_1, \ldots, a_k \rangle$ to be the R-lattice whose Gram matrix in a given basis is diagonal with entries $a_1, \ldots, a_k$.

We often denote an R-lattice by the Gram matrix associated to one of its bases.

Example 1.3. We denote by $U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the *hyperbolic plane lattice*.

Example 1.4. We denote by A_n $(n \geq 1)$, D_n $(n \geq 4)$ and E_n $(n \in \{6, 7, 8\})$ the negative definite $\mathbb{Z}$-lattices associated to the corresponding Dynkin diagrams [Ebe02, §1.4]. These are constructed in the following way: each vertex of the corresponding Dynkin diagram defines a basis vector of norm -2, two vertices connected by an edge have product equal to 1, and 0 otherwise. For instance, we define $A_2 := \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$. We refer to them as *(even) root lattices.*

We define a series of invariants, by isometry, for R-lattices.

Definition 1.5. For an R-lattice (L, b), we define:

(1) the *scale* $s(L, b)$ of (L, b) to be the R-ideal $b(L, L)$;

(2) the *norm* $n(L, b)$ of (L, b) to be the R-ideal $\sum_{x\in L} b(x, x)R$;

(3) the *determinant* $d(L, b)$ of (L, b) to be the class in $K^\times/(R^\times)^2$ of the determinant $d_B(L, b)$ of the Gram matrix of b associated to any R-basis B of L.

Remark 1.6. If (L, b) is an R-lattice, then any change of basis for L, as free R-module, is given by an R-module isomorphism $f \colon L \to L$. Since such an f is invertible, we have that $\det(f) \in R^\times$. If $\{x_1, \dots, x_r\}$ is an R-basis of L, then the determinants of the Gram matrices respectively associated to the bases $\{x_1, \dots, x_r\}$ and $\{f(x_1), \dots, f(x_r)\}$, with respect to b, differ by $\det(f)^2$. If $R = \mathbb{Z}$, we have that $(R^\times)^2 = \{1\}$ so the determinant $d_B(L, b)$ does not depend on the choice of B. If $R = \mathbb{Z}_p$ for a prime number p, we have that $(R^\times)^2$ is nontrivial. Hence, in this last case, the determinant of (L, b) is well-defined only up to squares of units.

Notation.

(1) If (L, b) is a $\mathbb{Z}$-lattice, then we see $\det(L, b) \in \mathbb{Q}^\times$ as a rational number, since it does not depend on any choice of a R-basis for L. We moreover call $|\det(L, b)|$ the *absolute determinant* of (L, b). Note that if (L, b) has signatures (l_+, l_-), then $\det(L, b) = (-1)^{l_-}|\det(L, b)|$.

(2) If (L, b) is a $\mathbb{Z}_p$-lattice where p is a prime number, then $|\det(L, b)|_p := p^{-\mathrm{val}_p(\det(L,b))}$ does not depend on a choice of a representative of $\det(L, b)$, where val_p denotes the p-adic valuation. We call $|\det(L, b)|_p^{-1}$ the *absolute determinant* of (L, b), and we will call the class $\det(L, b) \cdot |\det(L, b)|_p \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$ the *unit determinant* of (L, b).

Let (L, b) be an R-lattice. Then one has that $2s(L, b) \subseteq n(L, b) \subseteq s(L, b)$. We call (L, b) *integral* if $s(L, b) \subseteq R$, and if it is integral, we say (L, b) is *unimodular* if it has absolute determinant 1. Finally, we say (L, b) is *even* if $n(L, b) \subseteq 2R$.

Remark 1.7. If p is an odd prime number, then 2 is invertible in $\mathbb{Z}_p$ and in particular, any integral $\mathbb{Z}_p$-lattice (L, b) satisfies $n(L, b) \subseteq \mathbb{Z}_p = 2\mathbb{Z}_p$, meaning that (L, b) is even.

Let $L^\vee := \mathrm{Hom}_R(L, R)$ be the dual module of L. Since the bilinear form b is nondegenerate, we have a K-linear isomorphism

$$L \otimes_R K \to (L \otimes_R K)^\vee,\ x \mapsto (y \mapsto b(x, y)).$$

It induces an isomorphism of R-modules

$$\{x \in L \otimes_R K \ : \ b(x, L) \subseteq R\} \to L^\vee,\ x \mapsto (y \mapsto b(x, y)).$$

With this description, we can endow $L^\vee$ with the form b turning $(L^\vee, b)$ into an R-lattice. When (L, b) is integral, we have that $L \leq L^\vee$ as free R-modules, and we denote by $D_L := L^\vee/L$ the so-called *discriminant group* of L.

Definition 1.8.

(1) An integral $\mathbb{Z}$-lattice L is said to be *n-elementary* for some $n \geq 1$ if $nL^\vee \subseteq L$.

(2) An integral $\mathbb{Z}_p$-lattice L is said to be *p^i-modular* for some integer $i \in \mathbb{Z}$ if $p^i L^\vee = L$.

The group D_L is a finite abelian group and its order satisfies $\#D_L = |\det(L, b)|$ (see for instance [Ebe02, §1.1]). It is moreover equipped with a *torsion bilinear form*

$$b_L : D_L \times D_L \to K/R,\ (x + L, y + L) \mapsto b(x, y) + R.$$

When (L, b) is even, one also defines a *torsion quadratic form* q_L on D_L given by

$$q_L : D_L \to K/2R,\ x + L \mapsto b(x, x) + 2R.$$

The pair (D_L, q_L) is called the *discriminant form* of (L, b): for simplicity, we omit q_L from the notation whenever possible. In general, any finite abelian group A equipped with a torsion quadratic form as before will be referred to as *torsion quadratic module.*

Notation. Given a finitely generated abelian group A, we denote by $l(A)$ the minimum number of elements generating A.

Remark 1.9. As for R-lattices, we can represent (D_L, q_L) by a matrix M. For a given set of generators $\{x_1 + L, \ldots, x_r + L\}$ for D_L, the diagonal entries of M refer to representatives of $q_L(x_i + L) \in K/2R$, and the off-diagonal entries of M give representatives of $b_L(x_i + L, x_j + L) \in K/R$ for all $i \neq j$.

Example 1.10. Let $A_1 := (-2)$ be the root lattice. We observe that $|\det(A_1)| = 2$ and therefore, as abelian groups, $D_{A_1} \cong \mathbb{Z}/2\mathbb{Z}$. The $\mathbb{Z}$-lattice A_1 is even, and we denote $v \in A_1$ such that $v^2 = -2$. Note that $\frac{v}{2} \in A_1^\vee \setminus A_1$ and the group D_{A_1} has order 2; it is therefore generated by $\frac{v}{2} + A_1$. Moreover D_{A_1} is equipped with the torsion quadratic form

$$q_{A_1} \colon D_{A_1} \to \mathbb{Q}/2\mathbb{Z},\ \frac{v}{2} + A_1 \mapsto -\frac{1}{2} + 2\mathbb{Z}.$$

In that case, we write $D_{A_1} = \big(-1/2\big)$.

Given an even R-lattice (L, b) and a nonzero scalar $a \in K$, we denote by $(L, b)(a) := (L, a \cdot b)$ the rescaled lattice. Similarly, for any submodule $H \leq (D_L, q_L)$, we let $H(-1)$ to be the same as H as finite abelian group but equipped with the opposite form $(-q_L)_{|H}$.

Definition 1.11. An integral $\mathbb{Z}$-lattice L is said to be *n-divisible* for some $n \geq 2$ if there exists an integral $\mathbb{Z}$-lattice M such that $L = M(n)$. If no such n exists, we say that L is *indivisible.*

Remark 1.12. The $\mathbb{Z}$-lattice L is n-divisible if and only if $s(L) \subseteq n\mathbb{Z}$.

To conclude, we make the following definitions related to vectors in R-lattices.

Definition 1.13. Let (L, b) be a $\mathbb{Z}$-lattice and let $x \in L$ be a vector in L. We call the positive generator $\operatorname{div}(x, L)$ of the fractional ideal $b(x, L)$ of $\mathbb{Z}$ the *divisibility* of x in L. We moreover call the pair $(b(x, x),\ \operatorname{div}(x, L))$ the *type* of x (in L).

When L is integral, the divisibility $\operatorname{div}(x, L)$ of any vector $x \in L$ is the largest positive integer γ such that $x/\gamma \in L^\vee$.

Definition 1.14. Let (L, b) be an integral R-lattice. For an anisotropic vector $v \in L \otimes_R K$, we define the *reflection in v* to be the K-linear map

$$\tau_v \colon L \otimes_R K \to L \otimes_R K,\ x \mapsto x - 2\frac{x.v}{v^2}v.$$

Note that τ_v is actually an isometry for the form b. Any vector $v \in L$ is called a *root* of L if τ_v actually defines a R-module automorphism of L.

Remark 1.15. Let L be an integral $\mathbb{Z}$-lattice and let $v \in L$. If $2\mathrm{div}(v, L)$ is divisible by v^2, then v is a root of L.

Example 1.16. Let L be a definite integral $\mathbb{Z}$-lattice. Then any vector v with $|v^2| \leq 2$ is a root of L. We define the *root sublattice* of L to be the sublattice $L_0 \leq L$ generated by such vectors. In the case $L = L_0$, we call L a *root lattice*. Negative definite even root lattices are exactly direct sums of the ADE $\mathbb{Z}$-lattices mentioned in Example 1.4.

For an integral $\mathbb{Z}$-lattice L and a nonzero integer n, we call *n-vector* any vector $v \in L$ so that $v^2 = n$. If moreover v is a root of L, we call it an *n-root* of L.

Computational comments. Any $\mathbb{Z}$-lattice (L, b) can be described as the $\mathbb{Z}$-span of a set of $\mathbb{Q}$-linearly independent vectors in a given quadratic space (V, b), which we refer to as the *ambient space* of (L, b). It is therefore possible to represent any lattice (L, b) by a pair of matrices (B, G) where G is the Gram matrix of the ambient space (V, b) of (L, b) in a fixed basis, and B is a *basis matrix* of $L \otimes_{\mathbb{Z}} \mathbb{Q}$ with respect to the fixed basis of V. Using standard linear and commutative algebra, it is thus effectively possible to compute:

(1) the Gram matrix of (L, b) associated to any basis of L;

(2) the signatures (l_+, l_-) of (L, b), via the Gram–Schmidt orthogonalization procedure;

(3) the scale $s(L, b)$, the norm $n(L, b)$ and the determinant $d(L, b)$ of (L, b);

(4) a basis for the dual $\mathbb{Z}$-lattice $L^\vee$;

(5) a presentation of D_L and a Gram matrix of q_L associated to a set of generators of D_L;

(6) the type of any vector $x \in L$.

In particular, one can effectively determine whether L is definite, integral, even or unimodular. For the rest of the thesis, we assume that items (1)–(6) are computationally accessible.

1.2. Genera of integer lattices

Let L be an integral $\mathbb{Z}$-lattice. What we describe in this section extends to nonintegral $\mathbb{Z}$-lattices: since we mostly work with even $\mathbb{Z}$-lattices in this thesis, we often focus the discussion on this specific case. For each prime number p, we denote by $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ the associated $\mathbb{Z}_p$-lattice, where we often denote by b_p the associated form.

Definition 1.17. Let (V, b) be a quadratic space over $\mathbb{Q}$, and let S be a free $\mathbb{Z}$-submodule of V of maximal rank. We define

(1) the *isometry class* of the $\mathbb{Z}$-lattice (S, b) to be the set of free $\mathbb{Z}$-submodules $T \leq V$ of maximal rank such that $(S, b) \simeq (T, b)$;

(2) the *genus* (plural *genera*) of the $\mathbb{Z}$-lattice (S, b) to be the set of free $\mathbb{Z}$-submodules $T \leq V$ of maximal rank such that $(S_p, b_p) \simeq (T_p, b_p)$ for each prime number p.

Two $\mathbb{Z}$-lattices S and T are said to be *in the same genus* if there exists a $\mathbb{Z}$-embedding $f\colon S \to T \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $f(S)$ lies in the genus of T.

Two $\mathbb{Z}$-lattices which are isometric are necessarily in the same genus. The converse of the previous statement does not always hold in general. However, the genus of any given $\mathbb{Z}$-lattice consists of finitely many isometry classes [Kne02, Satz (21.3)].

Remark 1.18. Let us make an important remark on the problem of testing whether two given $\mathbb{Z}$-lattices are isometric.

(1) If a $\mathbb{Z}$-lattice L is of rank 1, it is clear that $L = \langle \det(L) \rangle$ so this case is trivial as L is uniquely determined by its determinant.

(2) The binary case, i.e. for $\mathbb{Z}$-lattices of rank 2, is a topic on its own and has a long history. In this case, the theory is intrinsically related to the theory of quadratic field extensions, and the comparison problem for $\mathbb{Z}$-lattices of rank 2 has been solved (see for instance [BV07, Chapter 2] for an algorithmic approach).

(3) For rank larger than or equal to 3 the problem of comparing two given $\mathbb{Z}$-lattices is handled in a different way. In the definite case, we are given two definite $\mathbb{Z}$-lattices L and M of the same rank $r \geq 3$: in that case, Plesken and Souvignier describe an algorithm in [PS97] which decides whether L and M are isometric. Moreover, such an algorithm can be used to produce an explicit isometry between L and M.

    In the indefinite case, the result is less obvious. In Section 1.5 we discuss the problem of *enumerating* genera of even $\mathbb{Z}$-lattices of rank bigger than 3. In particular, we explain how one can determine a complete set of representatives for the isometry classes of $\mathbb{Z}$-lattices in a given genus. Among the notions introduced in that section, we define the so-called *spinor genera* of a genus of $\mathbb{Z}$-lattices. An important property of these spinor genera is that in the indefinite case, all $\mathbb{Z}$-lattices in a given spinor genus are pairwise isometric. Moreover, starting from a given even $\mathbb{Z}$-lattice L, it is possible to determine a complete list of pairwise nonisometric $\mathbb{Z}$-lattices in the genus of L representing all spinor genera in that genus. In some cases, this approach can actually be used to show that a given indefinite $\mathbb{Z}$-lattice is unique in its genus, up to isometry.

A reason for the introduction of the notion of genus is to measure the failure of the *local-global principle* for integral $\mathbb{Z}$-lattices. In fact, by a classical theorem of Hasse and Minkowski, two quadratic spaces (V, b) and (V', b') over $\mathbb{Q}$ are isometric if and only if they have the same signatures and $(V \otimes_{\mathbb{Q}} \mathbb{Q}_p, b_p) \simeq (V' \otimes_{\mathbb{Q}} \mathbb{Q}_p, b'_p)$ for all prime numbers p. This fails to hold in general for $\mathbb{Z}$-lattices and the notion of genus compensates this. Moreover, we show now that comparing integral $\mathbb{Z}_p$-lattices, for p a prime number, is much easier than comparing integral $\mathbb{Z}$-lattices. In particular, deciding whether two integral $\mathbb{Z}$-lattices are in the same genus is a rather simple routine.

Definition 1.19 (Jordan decompositions)**.** Let p be a prime number, and let $n \geq 1$ be a positive integer. For $1 \leq i \leq n$, let L_i be a p^{s_i}-modular $\mathbb{Z}_p$-lattice, where $s_i \in \mathbb{Z}$. We say $\bigoplus_{i=1}^n L_i$ is a *Jordan decomposition* if $s_1 < s_2 < \cdots < s_n$. Two such Jordan decompositions $\bigoplus_{i=1}^n L_i$ and $\bigoplus_{i=1}^m L'_i$ are said to be of the *same type* if $n = m$ and if for all $1 \leq i \leq n$ the following hold:

(1) L_i and L'_i have the same rank, as $\mathbb{Z}_p$-modules;

(2) $s(L_i) = s(L'_i)$;

(3) $n(L_i) = s(L_i)$ if and only if $n(L'_i) = s(L'_i)$.

Given a Jordan decomposition $L = \bigoplus_{i=1}^n L_i$ we call the L_i's *Jordan constituents of L*.

Theorem 1.20 ([O'M73, §91C], [CS99, Chapter 15, §4.4, Theorem 2])**.** *Let p be a prime number, and let L be a $\mathbb{Z}_p$-lattice. Then L admits at least one Jordan decomposition, and any two Jordan decompositions of L are of the same type. Moreover:*

(1) *if p is odd, then each Jordan constituent of L admits an orthogonal basis;*

(2) *if $p = 2$, then each Jordan constituent of L is the orthogonal direct sum of $\mathbb{Z}_2$-lattices of rank 1 or 2.*

Proof. For the reader's convenience, we give a proof of existence. Let L be an integral $\mathbb{Z}_p$-lattice, let $\{x_1, \dots, x_n\}$ be a basis of L, and let G be the Gram matrix of L associated to such a basis. The proof follows by induction.

(1) If p is odd, we let k be the minimum among the p-adic valuations of the entries of G.

   (a) Suppose that there exists $1 \leq i \leq n$ such that $\mathrm{val}_p(x_i^2) = k$. Without loss of generality, we may assume that $i = 1$, and $x_1^2 = p^k\alpha$ where $\alpha \in \mathbb{Z}_p^\times$. By the definition of k, we know that for all $1 < j \leq n$, the equality $x_1.x_j = p^k\alpha_j$ holds for some $\alpha_j \in \mathbb{Z}_p$. Hence, for all $1 < j \leq n$, we can define $x_j' := x_j - \frac{\alpha_j}{\alpha}x_1$ to obtain a new basis $\{x_1, x_2', \dots, x_n'\}$ of L such that the sublattices respectively spanned by x_1 and $\{x_2', \dots, x_n'\}$ are in orthogonal direct sum in L, and span L.

   (b) If no such $1 \leq i \leq n$ exists, then there exists a pair (i, j) of distinct integers $1 \leq i, j \leq n$ such that $x_i.x_j = p^k\alpha$ with $\alpha \in \mathbb{Z}_p^\times$. Again, without loss of generality, we may assume $(i, j) = (1, 2)$. From that point, we can reduce to case (a) by replacing x_1 with $x_1 + x_2$ whose square $x_1^2 + 2p^k\alpha + x_2^2$ has p-adic valuation k since $p \neq 2$ and $\mathrm{val}_p(x_l^2) > k$ holds for every $1 \leq l \leq n$, by assumption.

(2) If $p = 2$, we let k again be the minimum among the 2-adic valuations of the entries of G.

   (a) If there exists $1 \leq i \leq n$ such that $\mathrm{val}_2(x_i^2) = k$, then the proof is the same as in (1)(a).

   (b) If no such $1 \leq i \leq n$ exists, we suppose again without loss of generality that $x_1.x_2 = 2^k\alpha$ with α invertible in $\mathbb{Z}_2$. Then the Gram matrix of the $\mathbb{Z}_2$-lattice $\mathbb{Z}_2x_1 + \mathbb{Z}_2x_2 \leq L$ is of the form
$$\begin{pmatrix} 2^k a & 2^k\alpha \\ 2^k\alpha & 2^k c \end{pmatrix} \tag{1}$$
   where $a, c \in 2\mathbb{Z}_2$. Since α is a unit in $\mathbb{Z}_2$ and a, c are not, we have that $d := ac - \alpha^2 \in \mathbb{Z}_2^\times$. Therefore if one defines, for all $2 < j < n$,
$$x_j' := x_j - \frac{cx_1.x_j - \alpha x_2.x_j}{d}x_1 - \frac{ax_2.x_j - \alpha x_1.x_j}{d}x_2$$
   we obtain a new basis $\{x_1, x_2, x_3', \dots, x_n'\}$ of L such that the sublattices respectively spanned by $\{x_1, x_2\}$ and $\{x_3', \dots, x_n'\}$ are in orthogonal direct sum in L, and span L.

This proves existence, and it also provides a proof of Items (1) and (2) of the statement. For the rest of the proof, we refer to [O'M73, Theorem (91.9)]. □

Remark 1.21. Let $p = 2$, let L be as in (2)(b) of the proof of Theorem 1.20, let again $x_1, x_2 \in L$, and let $a, c \in 2\mathbb{Z}_2$ and $\alpha \in \mathbb{Z}_2^\times$ be as in Equation (1). Then we have that $d := ac - \alpha^2$ is a unit in $\mathbb{Z}_2$. Suppose that there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{Z}_2$ such that
$$(\lambda_1x_1 + \lambda_2x_2).(\lambda_3x_1 + \lambda_4x_2) = 0.$$

Then we have that

$$\lambda_1\lambda_3 a + \lambda_2\lambda_4 c + (\lambda_1\lambda_4 + \lambda_2\lambda_3)\alpha = 0.$$

Since α is a unit, we must have that $\lambda_1\lambda_4 + \lambda_2\lambda_3 \in 2\mathbb{Z}_2$. Therefore, $\lambda_1\lambda_4 - \lambda_2\lambda_3 = 2\lambda_1\lambda_4 - (\lambda_1\lambda_4 + \lambda_2\lambda_3) \in 2\mathbb{Z}_2$. This means in particular that the matrix

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \in \mathbb{Z}_2^{2\times 2}$$

is not invertible. Hence, the sublattice $\mathbb{Z}_2 x_1 + \mathbb{Z}_2 x_2$ does not admit any orthogonal basis. The existence of such *nondiagonal* sublattices of L is a difference between the cases p odd and $p = 2$.

According to Theorem 1.20 and Remark 1.21, every integral $\mathbb{Z}_p$-lattice consists of irreducible modular $\mathbb{Z}_p$-lattices of rank 1 or 2. We refer to them as *elementary Jordan constituents*. Following the arguments of the proof of Theorem 1.20, it is possible to determine all isometry classes of such elementary Jordan constituents for each prime number p and each scale p^k, where $k \geq 0$.

Corollary 1.22 ([Nik80, Proposition 1.8.1])**.** *Let p be a prime number. Then any integral $\mathbb{Z}_p$-lattice is isometric to the orthogonal direct sum of elementary Jordan constituents. Moreover, up to isometry, there is a finite list of such consistuents for each scale p^k where $k \geq 0$. We give in the following table the scale p^k of such constituents ($k \geq 0$), their Gram matrix, their unit determinant and a Gram matrix of the associated torsion quadratic form for when $k \geq 1$. For all odd prime number p, we let $\theta_p \in \mathbb{Z}_p^\times \setminus (\mathbb{Z}_p^\times)^2$ be a nonsquare p-adic unit.*

p	scale	Gram matrix	unit determinant	discriminant form when $k \geq 1$
p odd	$p^k, \quad k \geq 0$	$\begin{pmatrix} p^k \end{pmatrix}$	$(\mathbb{Z}_p^\times)^2$	$\begin{pmatrix} 1/p^k \end{pmatrix}$
		$\begin{pmatrix} \theta_p p^k \end{pmatrix}$	$\theta_p(\mathbb{Z}_p^\times)^2$	$\begin{pmatrix} \theta_p/p^k \end{pmatrix}$
$p = 2$	$2^k, \quad k \geq 0$	$\begin{pmatrix} 2^k \end{pmatrix}$	$(\mathbb{Z}_2^\times)^2$	$\begin{pmatrix} 1/2^k \end{pmatrix}$
		$\begin{pmatrix} 3\cdot 2^k \end{pmatrix}$	$3(\mathbb{Z}_2^\times)^2$	$\begin{pmatrix} 3/2^k \end{pmatrix}$
		$\begin{pmatrix} 5\cdot 2^k \end{pmatrix}$	$5(\mathbb{Z}_2^\times)^2$	$\begin{pmatrix} 5/2^k \end{pmatrix}$
		$\begin{pmatrix} 7\cdot 2^k \end{pmatrix}$	$7(\mathbb{Z}_2^\times)^2$	$\begin{pmatrix} 7/p^k \end{pmatrix}$
		$\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}$	$7(\mathbb{Z}_2^\times)^2$	$\begin{pmatrix} 0 & 1/2^k \\ 1/2^k & 0 \end{pmatrix}$
		$\begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}$	$3(\mathbb{Z}_2^\times)^2$	$\begin{pmatrix} 1/2^{k-1} & 1/2^k \\ 1/2^k & 1/2^{k-1} \end{pmatrix}$

For a given prime number p and a given $k \geq 0$, the p^k-modular elementary Jordan constituents described in Corollary 1.22 are pairwise nonisometric. However, some given combinations of them could give rise to isometric $\mathbb{Z}_p$-lattices (see for instance [Nik80, Proposition 1.8.2]).

Example 1.23.

(1) The unit $2 \in \mathbb{Z}_3^\times$ is not a square, so the two $\mathbb{Z}_3$-lattices $\begin{pmatrix} 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \end{pmatrix}$ are not isometric. However,

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

are indeed isometric, via the $\mathbb{Z}_3$-linear mapping $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(2) Another well-known relation is that for all $k \geq 0$, we have that

$$\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix} \oplus \begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}$$

are isometric as $\mathbb{Z}_2$-lattices.

Remark 1.24. Let L be a modular $\mathbb{Z}_p$-lattice where p is a prime number, and let $k \in \mathbb{Z}$. Then by Corollary 1.22 we see that L is p^k-modular if and only if $s(L) = p^k\mathbb{Z}_p$. In particular, if L is a p^k-modular $\mathbb{Z}_p$-lattice for some prime number p and positive integer $k \geq 0$, then $L = L_k(p^k)$ where L_k is unimodular.

By similar techniques as in the proof of Theorem 1.20 it follows that for each prime number p, any torsion quadratic form $q_p \colon A_p \to \mathbb{Q}_p/2\mathbb{Z}_p$, where A_p is a finite abelian p-group, can be written as the orthogonal direct sum of elementary blocks as described in the last column of the table of Corollary 1.22 [Nik80, Proposition 1.8.1]. The proof of the next result is omitted.

Lemma 1.25 ([Nik80, Corollary 1.9.3])**.** *Let p be a prime number and let L be an even $\mathbb{Z}_p$-lattice. Then the isometry class of L is uniquely determined by its rank, its quadratic form q_L and its unit determinant.*

Remark 1.26. The previous result of Nikulin is a bit stronger than how stated above since it also gives a representative for the isometry class of L. For our purpose we would only need to know about the invariants associated to any even $\mathbb{Z}_p$-lattice. Let us comment on why these invariants are actually necessary.

(1) Let L be as in the statement of Lemma 1.25. By definition of the quadratic form D_L, if L admits a Jordan decomposition of the form

$$L_0 \oplus L_{i_1} \oplus \cdots \oplus L_{i_k}$$

where $0 < i_1 < \cdots < i_k$ and L_0 unimodular, then we would have that, as torsion quadratic forms,

$$D_L \simeq D_{L_{i_1}} \oplus \cdots \oplus D_{L_{i_k}}$$

meaning that the discriminant form does not see the unimodular constituent of any Jordan decomposition of L. In particular, the invariants of L and $\langle 1 \rangle \oplus L$ as in Lemma 1.25 are the same at the exception of the ranks.

(2) From the description given in Corollary 1.22, we see that the $\mathbb{Z}_2$-lattices

$$\begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

have the same rank, and their unit determinants $7(\mathbb{Z}_2^\times)^2$ are the same. However, those are not isometric and this can be depicted by comparing their respective discriminant forms.

(3) As described in [Nik80, Theorem 1.9.1], we observe that the torsion quadratic forms on $\mathbb{Z}/2\mathbb{Z}$, with values in $\mathbb{Q}_2/2\mathbb{Z}_2$, given by

$$(1/2) \quad \text{and} \quad (5/2)$$

are actually equal. This means that the two nonisometric $\mathbb{Z}_2$-lattices (2) and (10) have the same quadratic forms, and same ranks. However, their respective unit determinants are distinct.

Let now L be an even $\mathbb{Z}$-lattice, and let $q_L\colon D_L \to \mathbb{Q}/2\mathbb{Z}$ be the discriminant form of L. We can decompose, as finite abelian groups,

$$D_L \cong \bigoplus_{p \mid \# D_L} (D_L)_p$$

where for all prime numbers p dividing $\# D_L$, we denote by $(D_L)_p$ the p-Sylow subgroup of D_L. Note that for all prime numbers p dividing $\# D_L$, we have that $(D_L)_p \cong D_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and one can therefore identify $(D_L)_p \cong D_{L_p}$, as finite abelian groups. For every prime number p we have that $(D_L)_p$ is a p-group, by definition. Thus, the restriction of q_L to $(D_L)_p$ takes values in $2\mathbb{Z}[p^{-1}]/2\mathbb{Z} \cong \mathbb{Q}_p/2\mathbb{Z}_p$, where we denote by $\mathbb{Z}[p^{-1}]$ the ring of rational numbers with denominator a power of p. This induces a decomposition of torsion quadratic modules

$$D_L \simeq \prod_{p \mid \# D_L} D_{L_p} \tag{2}$$

[Nik80, Proposition 1.2.3]. In particular, the isometry class of the discriminant form D_L of L and the collection of isometry classes of the D_{L_p} for all $p \mid \# D_L$ determine each other.

Remark 1.27. For each prime number $p \mid \# D_L$, the discriminant form D_{L_p} is isometric to a canonical torsion quadratic form [MM09, Chapter IV]. Using Equation (2) and the previous canonical representatives, one can define a so-called *normal form* of D_L, which is a canonical representative of its isometry class, as torsion quadratic module.

Proposition 1.28 ([Nik80, Corollary 1.9.4])**.** *Let L be an even $\mathbb{Z}$-lattice. The genus of L is uniquely determined by the signatures (l_+, l_-) of L and the discriminant form D_L.*

Proof. We already know that the signatures of L determine uniquely the isometry class of $L \otimes_{\mathbb{Z}} \mathbb{R}$. Now, we have just seen that knowing the isometry class of the discriminant form D_L of L is the same as knowing the isometry class of D_{L_p} for all prime numbers $p \mid \det(L)$. Moreover, note that

$$\det(L) = (-1)^{l_-} \# D_L = (-1)^{l_-} \prod_{p \mid \det(L)} \# D_{L_p}.$$

From that point, using Lemma 1.25 and the fact that $l_+ + l_- = \operatorname{rank}_{\mathbb{Z}}(L) = \operatorname{rank}_{\mathbb{Z}_p}(L_p)$ for all prime numbers p, we obtain that (l_+, l_-) and D_L completely determine the isometry class of L_p for all prime numbers p. This proves the proposition. □

As a direct consequence, we obtain the following.

Corollary 1.29. *Let S and T be two even $\mathbb{Z}$-lattices. They are in the same genus if and only if they have the same signatures and their respective discriminant quadratic forms are isometric as torsion quadratic modules.*

1.3. Genus symbols

In the previous section, we have seen that determining whether two even $\mathbb{Z}$-lattices are in the same genus can be done by comparing their respective real signatures and discriminant forms. In particular, by using normal forms [MM09, Chapter IV], these comparisons are computationally accessible. In this subsection, we present Conway–Sloane's convention on genus symbols [CS99, Chapter 15]. The idea is to represent genera of integral $\mathbb{Z}$-lattices with a small set of data which can be condensed into a human readable format. This turns out to be a convenient tool to describe genera, which we use throughout this thesis.

Let p be a prime number, and let L be an integral $\mathbb{Z}_p$-lattice. We let

$$L = L_0 \oplus L_1(p) \oplus L_2(p^2) \oplus \cdots \oplus L_k(p^k)$$

be a Jordan decomposition of L, where the L_i's are unimodular of rank $n_i \geq 0$ and $n_k \neq 0$. In particular, since for all $0 \leq i \leq k$ the $\mathbb{Z}_p$-lattice L_i is unimodular, we have that $|\det(L_i)|_p = 1$ and the rescaled lattice $L_i(p^i)$ is p^i-modular (or unimodular when $i = 0$). We define the *sign* of $L_i(p^i)$ to be $\epsilon_i := \left(\frac{\det(L_i)}{p}\right)$: note that this definition does not depend on a choice of a representative of $\det(L_i) \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$.

Theorem 1.30 ([CS99, Chapter 15, Theorem 9])**.** *Suppose that $p \neq 2$ is odd. Then the set $\{(i, n_i, \epsilon_i)\}_{0\leq i\leq k}$ forms a complete set of invariants for the isometry class of L.*

Proof. This is a direct consequence of Corollary 1.22: remark that $\det(L_i)$ is in fact the unit determinant of $L_i(p^i)$. □

In the case where $p = 2$, the set of triples (i, n_i, ϵ_i) is not sufficient to characterize a $\mathbb{Z}_2$-lattice up to isometry: even though these are isometry invariants of $L_i(p^i)$ for each $i \in \{1, \ldots, k\}$, together they do not characterize uniquely the isometry class of L.

Example 1.31. Consider the $\mathbb{Z}_2$-lattices with respective Jordan decomposition

$$(1) \oplus (2) \quad \text{and} \quad (3) \oplus (6).$$

Then, those are isometric, via the $\mathbb{Z}_2$-linear change of bases $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. However, in the first case, we have that $\epsilon_0 = 1$, while $\epsilon_0 = -1$ in the second case.

In order to remedy this, we introduce two other isometry invariants for modular $\mathbb{Z}_2$-lattices. Let

$$L = L_0 \oplus L_1(p) \oplus L_2(p^2) \oplus \cdots \oplus L_k(p^k)$$

be an integral $\mathbb{Z}_2$-lattice as before, and let $0 \leq i \leq k$. We define the *type* S_i of $L_i(p^i)$ to be II if L_i is even, and we define $S_i :=$ I otherwise. Moreover, if L_i is odd (i.e. of type I) we define the *oddity* t_i of $L_i(p^i)$ to be the trace of a Gram matrix of L_i modulo 8. If L_i is even, we define $t_i := 0$. Note that S_i and t_i are isometry invariants.

Theorem 1.32 ([CS99, Chapter 15, Theorem 10], [Xu03, Theorem 3.2])**.** *Let L, L' be two integral $\mathbb{Z}_2$-lattices and let $\{(i, n_i, \epsilon_i, S_i, t_i)\}_{0\leq i\leq k}$ and $\{(j, n'_j, \epsilon'_j, S'_j, t'_j)\}_{0\leq j\leq l}$ be the sets of invariants associated to respective fixed Jordan decompositions of L and L'. Then L and L' are isometric if and only if*

(1) *$k = l$;*

(2) *$(n_i, S_i) = (n'_i, S'_i)$ for all $0 \leq i \leq k$;*

(3) *for each $0 \leq j \leq k$ such that L_j is even, we have*

$$\sum_{0\leq i<j} (t_i - t'_i) \equiv 4 \sum_{\substack{0\leq i<j \\ \epsilon_i \neq \epsilon'_i}} \min(i, j) \mod 8.$$

We therefore see that an isometry class of $\mathbb{Z}_p$-lattices can be characterized by a finite list of isometry invariants. We describe now a way to condense this information into a *symbol*.

Definition 1.33. Let $p \neq 2$ be an odd prime number, and let

$$L = L_0 \oplus L_1(p) \oplus L_2(p^2) \oplus \cdots \oplus L_k(p^k)$$

be an integral $\mathbb{Z}_p$-lattice with associated invariants $\{(i, n_i, \epsilon_i)\}_{0 \leq i \leq k}$. We define the *p-adic symbol* of L to be

$$1^{\epsilon_0 n_0} p^{\epsilon_1 n_1} (p^2)^{\epsilon_2 n_2} \cdots (p^k)^{\epsilon_k n_k}.$$

For $0 \leq i \leq n_i$, we call $(p^i)^{\epsilon_i n_i}$ a *factor* and we call n_i the rank of such a factor. Moreover, the factor $1^{\epsilon_0 n_0}$ is referred to as *unimodular factor.*

Example 1.34. Let $p = 3$ and suppose that L is a $\mathbb{Z}_3$-lattice with Gram matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 0 \\ 0 & 0 & 9u & 0 & 0 \\ 0 & 0 & 0 & 9u & 0 \\ 0 & 0 & 0 & 0 & 27 \end{pmatrix}$$

where $u \in \mathbb{Z}_3^\times$ is not a square modulo 3. Then the 3-adic symbol of L is

$$1^{-2}3^0 9^{+2} 27^{+1} \text{ or, simply, } 1^{-2}9^2 27^1.$$

As for the case of odd prime numbers, one can construct a symbol for integral $\mathbb{Z}_2$-lattices. This time, such a symbol actually depends on a choice of a Jordan decomposition. One can use Theorem 1.32 to decide whether two 2-adic symbols represent isometric integral $\mathbb{Z}_2$-lattices.

Definition 1.35. Let L be an integral $\mathbb{Z}_2$-lattice, and let

$$L = L_0 \oplus L_1(2) \oplus L_2(2^2) \oplus \cdots \oplus L_k(2^k)$$

be a Jordan decomposition of L, with associated invariants $\{(i, n_i, \epsilon_i, S_i, t_i)\}_{0 \leq i \leq k}$. We define the 2-*adic symbol* of L associated to such a Jordan decomposition to be

$$1^{\epsilon_0 n_0}_{\alpha_0} 2^{\epsilon_1 n_1}_{\alpha_1} 4^{\epsilon_2 n_2}_{\alpha_2} \cdots (2^k)^{\epsilon_k n_k}_{\alpha_k}$$

where for all $0 \leq i \leq k$, $\alpha_i = t_i$ if $S_i = \mathrm{I}$, and $\alpha_i = \mathrm{II}$ otherwise. Again, for $0 \leq i \leq n_i$, we call $(2^i)^{\epsilon_i n_i}_{\alpha_i}$ a *factor* and we call n_i the rank of such a factor. Moreover, the factor $1^{\epsilon_0 n_0}_{\alpha_0}$ is referred to as *unimodular factor.*

Example 1.36. Let L be the $\mathbb{Z}_2$-lattice defined by the following Gram matrix:

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}.$$

Then, a 2-adic symbol of L is given by

$$1^{-1}_5 2^{+2}_{\mathrm{II}} 4^0_{\mathrm{II}} 8^0_{\mathrm{II}} 16^{+1}_1 \text{ or, simply, } 1^{-1}_5 2^2_{\mathrm{II}} 16^1_1.$$

Table 1: Elementary Jordan constituents and their invariants

p	scale	Gram matrix	Invariants	p-adic symbol
p odd	$p^k, \quad k \geq 0$	$\begin{pmatrix} p^k \end{pmatrix}$	$(k, 1, 1)$	p^k
		$\begin{pmatrix} \theta_p p^k \end{pmatrix}$	$(k, 1, -1)$	$(p^k)^{-1}$
$p = 2$	$2^k, \quad k \geq 0$	$\begin{pmatrix} 2^k \end{pmatrix}$	$(k, 1, 1, \mathrm{I}, 1)$	$(2^k)^1_1$
		$\begin{pmatrix} 3 \cdot 2^k \end{pmatrix}$	$(k, 1, -1, \mathrm{I}, 3)$	$(2^k)^{-1}_3$
		$\begin{pmatrix} 5 \cdot 2^k \end{pmatrix}$	$(k, 1, -1, \mathrm{I}, 5)$	$(2^k)^{-1}_5$
		$\begin{pmatrix} 7 \cdot 2^k \end{pmatrix}$	$(k, 1, 1, \mathrm{I}, 7)$	$(2^k)^1_7$
		$\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}$	$(k, 2, 1, \mathrm{II}, 0)$	$(2^k)^2$
		$\begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}$	$(k, 2, -1, \mathrm{II}, 0)$	$(2^k)^{-2}$

In the case of factors representing even Jordan constituents, we sometimes drop the type II from the notation.

Similarly to Corollary 1.22, we give in Table 1 the invariants associated to elementary Jordan constituents for all prime numbers p, as well as their p-adic symbols.

Remark 1.37. By the knowledge of a p-adic symbol for any even $\mathbb{Z}_p$-lattice L, where p is a prime number, we are able to determine the isometry class of L. In particular, according to Lemma 1.25, one can recover what is the isometry class of D_L from the p-adic symbol of L.

Example 1.38. Let L be a $\mathbb{Z}_5$-lattice whose 5-adic symbol is $1^1 5^{-2} 25^{-1}$. Using the fact that $2 + 5\mathbb{Z}$ is not a square in $\mathbb{Z}/5\mathbb{Z}$, we deduce that L is isometric to the $\mathbb{Z}_5$-lattice

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 50 \end{pmatrix}.$$

In particular, we have that

$$D_L \simeq \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 2/25 \end{pmatrix}.$$

Following Theorem 1.32, we have that two $\mathbb{Z}_2$-lattices are isometric if their respective 2-adic symbols satisfy the second part of the statement of the theorem: we therefore call such symbols *equivalent*.

Let L now be an integral $\mathbb{Z}$-lattice. If p is an odd prime number not dividing $\det(L)$, then the $\mathbb{Z}_p$-lattice L_p is unimodular and according to Theorem 1.30, its isometry class is uniquely determined by the rank of L and the class $\det(L)(\mathbb{Z}_p^\times)^2$. In the case $\det(L)$ is odd, we show in the following lemma that knowing invariants about the unimodular $\mathbb{Z}_2$-lattice L_2 is necessary to fully understand the genus of L.

Lemma 1.39. *Let L be an integral $\mathbb{Z}$-lattice, and let L_0 be the unimodular constituent of any Jordan decomposition of the $\mathbb{Z}_2$-lattice L_2. Then L is even if and only if L_0 is either trivial, or even.*

Proof. Since L is integral, we know it is even if and only if $n(L) \subseteq 2\mathbb{Z}$. However, the latter holds if and only if $n(L_p) \subseteq 2\mathbb{Z}_p$ for all prime numbers p. Since for all prime numbers p, we have that $n(L_p) \subseteq s(L_p) \subseteq \mathbb{Z}_p$ (because L is integral), we already know that L_p is even for all odd prime numbers p. Now, let

$$L_2 = L_0 \oplus L_1(2) \oplus L_2(2^2) \oplus \cdots \oplus L_k(2^k)$$

be any Jordan decomposition of L_2. Then for all $1 \leq i \leq k$, we have that $n(L_i(2^i)) \subseteq 2\mathbb{Z}_2$, and the result follows. □

Proposition 1.40. *Let L be an integral $\mathbb{Z}$-lattice of determinant d and signatures (l_+, l_-). Let P be the finite list of odd prime numbers dividing d. Then,*

(1) *the pair (l_+, l_-),*

(2) *the p-adic symbol of L_p for each $p \in P$, and*

(3) *the equivalence class of the 2-adic symbol associated to a Jordan decomposition of L_2*

form a complete set of invariants for the genus of L.

Proof. Follows from Theorem 1.30, Theorem 1.32 and Lemma 1.39. □

Remark 1.41. Given an integral $\mathbb{Z}$-lattice L, knowing the signatures of L and the invariants associated to respective Jordan decompositions of the L_p's for all $p \mid 2\det(L)$ prime allows us to recover several invariants of L such as its scale, its norm and its determinant.

We have seen that if p is an odd prime number not dividing $\det(L)$, then the p-adic symbol of L_p does not provide information which cannot be recovered from the q-adic symbols for all prime numbers $q \mid 2\det(L)$, and the signatures of L. However, we have shown that the 2-adic symbol of L_2 carries the information on whether L is even, also when L_2 is unimodular. In order to get rid of any p-adic symbols for the prime number p such that L_p is unimodular, we introduce a last notation. We say that a genus of integral $\mathbb{Z}$-lattices if *of type* I if it represents odd $\mathbb{Z}$-lattices, and of type II otherwise.

Definition 1.42. Let L be an integral $\mathbb{Z}$-lattice of signatures (l_+, l_-), and let P be the sorted list of prime numbers dividing $\det(L)$. We define a *genus symbol* $g(L)$ of L to be a symbol made of $\mathrm{I}_{(l_+,l_-)}$ if L is odd and $\mathrm{II}_{(l_+,l_-)}$ if L is even, followed in order by the p-adic symbols of L_p for all $p \in P$. For simplicity, we remove the unimodular factors and the factors of rank 0 of each such p-adic symbol.

Example 1.43. Let $A_2 := \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ be the root lattice, and let $e_1, e_2 \in A_2$ be such that $e_1^2 = e_2^2 = -2$ and $e_1.e_2 = 1$. The lattice A_2 is even of determinant 3 and has signatures $(0, 2)$. Moreover, over the 3-adic integers, the change of basis $e_1 \mapsto e_1 + e_2$, $e_2 \mapsto e_1 - e_2$ is invertible, and so the Gram matrix of $(A_2)_3$ in the new basis obtained in that way is

$$\begin{pmatrix} -2 & 0 \\ 0 & -6 \end{pmatrix}.$$

From this, we see that the 3-adic symbol of $(A_2)_3$ is $1^1 3^1$. Hence, we obtain that a genus symbol for A_2 is $\mathrm{II}_{(0,2)}3^1$. We write $g(A_2) = \mathrm{II}_{(0,2)}3^1$, or $A_2 \in \mathrm{II}_{(0,2)}3^1$.

Remark 1.44. Similarly to what was remarked in Remark 1.37, we obtain that knowing a genus symbol of an even $\mathbb{Z}$-lattice L allows one to determine explicitly D_L up to isometry (see Proposition 1.28 and Proposition 1.40).

Because of the 2-adic symbols not being canonical, we cannot conclude that a genus of $\mathbb{Z}$-lattices admits a unique symbol representing it. However, using Theorem 1.32 we can compare two genus symbols: two genus symbols g and g' with same signatures, same invariants at all odd prime numbers and equivalent 2-adic symbols are said to be *equivalent*, and we write $g = g'$. Note that two genus symbols of a given integral $\mathbb{Z}$-lattice are equivalent.

Corollary 1.45. *Let L, L' be two integral $\mathbb{Z}$-lattices. They are in the same genus if and only if their genus symbols are equivalent.*

Proof. This is a translation of the content of Proposition 1.40. □

From now on, we will often refer to a genus of integral $\mathbb{Z}$-lattices by its symbol.

Notation. Let L be an integral $\mathbb{Z}$-lattice. If the genus of L can be represented by a given symbol g, then we write $L \in g$, or $g(L) = g$. Similarly, we write $L \in g_p$, or $g_p(L) = g_p$ for some prime number p if the p-adic symbol of L_p can be represented by g_p.

Note that for all prime numbers p, every p-adic symbol defined as in this section describes a (unique) isometry class of $\mathbb{Z}_p$-lattices. This is no longer the case for genus symbols of $\mathbb{Z}$-lattices. If for a given genus symbol g of $\mathbb{Z}$-lattices, there is no $\mathbb{Z}$-lattice whose genus symbol is equivalent to g, we therefore say that g is *empty*. For instance, even unimodular $\mathbb{Z}$-lattices only exist for certain values of their signatures.

Theorem 1.46 ([Ser70, Chapitre V, Section 1.5, §2, Théorème 2, Corollaire 1])**.** *Let $l_+, l_- \geq 0$ with $l_+ + l_- \geq 1$. The genus $\mathrm{II}_{(l_+,l_-)}$ of even unimodular $\mathbb{Z}$-lattices is nonempty if and only if $l_+ \equiv l_- \mod 8$.*

In general, there are certain compatibility statements which are required for a pair of signatures and a finite set of p-adic symbols to define a nonempty genus of integral $\mathbb{Z}$-lattices.

Definition 1.47. We make the following definitions:

(1) Let p be an odd prime number, and let g_p be a p-adic symbol with invariants $\{(i, n_i, \epsilon_i)\}_{0\leq i\leq k}$ where $k \geq 0$. We define the *p-excess* of g_p, denoted by $p - \mathrm{excess}(g_p)$, to be the congruence class modulo 8

$$\left(\sum_{0\leq i\leq k} n_i(p^i - 1) + 4\#\{0 \leq i \leq k \text{ odd } : n_i \neq 0 \text{ and } \epsilon_i = -1\} \mod 8\right).$$

(2) Let g_2 be a 2-adic symbol with invariants $\{(i, n_i, \epsilon_i, S_i, t_i)\}_{0\leq i\leq k}$ where $k \geq 0$ and $t_i = 0$ if $S_i = \mathrm{II}$. We define the *oddity* of g_2, denoted by $\mathrm{oddity}(g_2)$, to be the congruence class modulo 8

$$\left(\sum_{0\leq i\leq k} t_i + 4\#\{0 \leq i \leq k \text{ odd } : n_i \neq 0 \text{ and } \epsilon_i = -1\} \mod 8\right).$$

Theorem 1.48 ([CS99, Chapter 15, Theorem 11]). *Let (l_+, l_-) be a pair of nonnegative integers such that $l_+ + l_- > 0$, and let P be a finite list of pairwise distinct prime numbers. For all odd prime numbers $p \in P$, we let g_p be a p-adic symbol with invariants $\{(i, n_{p,i}, \epsilon_{p,i})\}_{0\leq i\leq k_p}$ such that $k_p \geq 0$ and $\sum_{0\leq i\leq k_p} n_{p,i} = l_+ + l_-$. Moreover, we let g_2 be a 2-adic symbol with invariants $\{(i, n_{2,i}, \epsilon_{2,i}, S_i, t_i)\}_{0\leq i\leq k_2}$ such that $k_2 \geq 0$ and $\sum_{0\leq i\leq k_2} n_{2,i} = l_+ + l_-$. Then, there exists an even $\mathbb{Z}$-lattice L of signatures (l_+, l_-), of determinant only divisible by primes in $P \cup \{2\}$, such that the p-adic symbol of L_p is g_p for all $p \in P$ odd and such that there exists a Jordan decomposition of L_2 whose associated 2-adic symbol is equivalent to g_2 if and only if the following conditions hold (in the numbering of [CS99, Chapter 15, §7.7])*

(29) *for all $p \in P \cup \{2\}$:*

$$\prod_{0\leq i\leq k_p} \epsilon_{p,i} = \left(\frac{-1}{p}\right)^{l_-}\left(\frac{a}{p}\right) \quad \text{where} \quad a := \prod_{q\in(P\cup\{2\})\setminus\{p\}} \left(\prod_{0\leq i\leq k_q} q^{in_{q,i}}\right),$$

(30) *the* oddity formula *holds:*

$$l_+ - l_- + \sum_{p\in P \text{ odd}} p - \mathrm{excess}(g_p) \equiv \mathrm{oddity}(g_2) \mod 8,$$

(31) *for every $p \in P$ odd and all $0 \leq i \leq k_p$:*

$$(n_{p,i} = 0) \implies (\epsilon_{p,i} = +1),$$

(32) *for all $0 \leq i \leq k_2$:*

$$(n_{2,i} = 0) \implies (S_i = \mathrm{II} \text{ and } \epsilon_{2,i} = +1),$$

(33) *for all $0 \leq i \leq k_2$:*

$$(n_{2,i} = 1) \implies \left(\epsilon_{2,i} = \left(\frac{t_i}{2}\right)\right),$$

(34) *for all $0 \leq i \leq k_2$ such that $S_i = \mathrm{I}$:*

$$(n_{2,i} = 2) \implies (t_i = 2, 6 \text{ or } (\epsilon_{2,i}, t_i) \in \{(+1, 0), (-1, 4)\}),$$

(35) *for all $0 \leq i \leq k_2$, $t_i \equiv n_{2,i} \mod 2$ and*

$$(n_{2,i} \text{ odd}) \implies (S_i = \mathrm{I}).$$

As an application, Conway and Sloane provide a complete characterization of genus symbols for even p-elementary $\mathbb{Z}$-lattices for all prime numbers p.

Theorem 1.49 ([Nik83, Theorem 4.3.1], [CS99, Chapter 15, Theorem 13]). *Let $l_+, l_- \geq 0$ and $n \in \mathbb{N}$ be such that $l_+ + l_- \geq n$, and let $\epsilon \in \{\pm 1\}$.*

(a) For an odd prime number $p \geq 3$, the genus $\mathrm{II}_{(l_+,l_-)}p^{\epsilon n}$ is nonempty if and only if the following hold:

(i) $l_+ - l_- \equiv 2\epsilon - 2 - (p-1)n \mod 8$;

(ii) if $l_+ + l_- = n$, then $\epsilon = \left(\frac{-1}{p}\right)^{l_-}$.

(b) *The genus* $\mathrm{II}_{(l_+,l_-)}2_{\mathrm{II}}^{\epsilon n}$ *is nonempty if and only if the following hold:*

   (i) $l_+ + l_- \equiv n \equiv 0 \mod 2$;

   (ii) $l_+ - l_- \equiv 2 - 2\epsilon \mod 8$;

   (iii) *if* $n = 0$ *or* $n = l_+ + l_-$, *then* $\epsilon = +1$.

(c) *For* $0 \le \delta \le 7$, *the genus* $\mathrm{II}_{(l_+,l_-)}2_{\delta}^{\epsilon n}$ *is nonempty if and only if the following hold:*

   (i) $n > 0$ *and* $n \equiv \delta \equiv l_+ + l_- \mod 2$;

   (ii) $l_+ - l_- \equiv \delta + 2 - 2\epsilon \mod 8$;

   (iii) *if* $n = l_+ + l_-$, *then* $\epsilon = +1$;

   (iv) *if* $n = 1$, *then* δ *is odd and* $\epsilon = \left(\frac{\delta}{2}\right)$;

   (v) *if* $n = 2$, *then* $l_+ - l_- \not\equiv 4 \mod 8$.

In order to see the previous characterization of Conway–Sloane (Theorem 1.48) in application, let us prove the following, which we will need later in this thesis.

Proposition 1.50. *Let* $l_+, l_- \ge 0$, *let* $n \in \mathbb{N}_0$, *let* $\delta \in \{1,3,5,7\}$ *and let* $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. *The genus* $\mathrm{II}_{(l_+,l_-)}2_{\mathrm{II}}^{\epsilon_1 n}4_{\delta}^{\epsilon_2}$ *is nonempty if and only if the following hold:*

(1) n *is even and* $\epsilon_2 = \left(\frac{\delta}{2}\right)$;

(2) $l_+ + l_- \ge n + 1$ *with equality only if* $\epsilon_1 = \epsilon_2$;

(3) $l_+ - l_- \equiv \delta + 2 - 2\epsilon_1 \mod 8$;

(4) *if* $n = 0$, *then* $\epsilon_1 = +1$.

Proof. Let $\epsilon_0 \in \{\pm 1\}$ be such that $\epsilon_0 = 1$ if $l_+ + l_- = n + 1$: it is the sign of the unimodular factor in the 2-adic symbol. According to Theorem 1.48, the genus $\mathrm{II}_{(l_+,l_-)}2_{\mathrm{II}}^{\epsilon_1 n}4_{\delta}^{\epsilon_2}$ is nonempty if and only if the following conditions are satisfied:

(29) $\epsilon_0\epsilon_1\epsilon_2 = +1$;

(30) $l_+ - l_- \equiv \mathrm{oddity}(2_{\mathrm{II}}^{\epsilon_1 n}4_{\delta}^{\epsilon_2}) \mod 8$;

(32) if $n = 0$, then $\epsilon_1 = +1$;

(33) $n \neq 1$ and $\epsilon_2 = \left(\frac{\delta}{2}\right)$;

(35) n is even.

Now, by definition

$$\mathrm{oddity}(2_{\mathrm{II}}^{\epsilon_1 n}4_{\delta}^{\epsilon_2}) \equiv 0 + \delta + 2 - 2\epsilon_1 \mod 8$$

and moreover, if $l_+ + l_- = n+1$, then $\epsilon_0 = +1$ and so $\epsilon_1\epsilon_2 = +1$. Otherwise, when $l_+ + l_- > n+1$, then the unimodular factor in the 2-adic symbol has positive rank, and one can always choose $\epsilon_0 \in \{\pm 1\}$ such that condition (29) holds. □

Computational comments. Let L be an integral $\mathbb{Z}$-lattice, and let p be a prime number dividing $2\det(L)$. By choosing an appropriate power of p, it is possible to determine a Jordan decomposition of L_p up to a given precision. In particular, the set of invariants associated to such a Jordan decomposition are effectively computable. Together with the fact that the conditions from Definition 1.19 and Theorem 1.48 are purely arithmetical, it is computationally possible:

(1) to compute a symbol for the genus of L;

(2) to compare two given genus symbols;

(3) to determine whether a symbol defines a nonempty genus of integral $\mathbb{Z}$-lattices;

(4) to determine whether there exists an even $\mathbb{Z}$-lattice with given signatures and discriminant form;

(5) to determine a representative for the isometry class of discriminant forms for even $\mathbb{Z}$-lattices with given genus symbol.

Note that given a nonempty genus symbol g, finding an explicit integral $\mathbb{Z}$-lattice L such that $L \in g$ is feasible but nontrivial: we refer to [Kir16, §3.4, §3.5] for more details. For the rest of thesis, we assume that items (1)–(5) above, as well as finding an explicit representative for every nonempty genus symbol, are computationally feasible.

1.4. Isometries

Let again $R = \mathbb{Z}, \mathbb{Z}_p$ for some p prime, let K be the field of fractions of R, and let (L, b) be an R-lattice. We denote by $O(L, b)$ the *group of isometries* of (L, b).

Remark 1.51. Computing generators of $O(L, b)$ for (L, b) a $\mathbb{Z}$-lattice is a very difficult task, in general. If $\operatorname{rank}_{\mathbb{Z}}(L) = 1$, then $O(L, b) = \{\pm \operatorname{id}_L\}$ so this case remains again trivial. In the case (L, b) is definite, one can invoke again the algorithm of Plesken–Souvignier [PS97] which allows to effectively (computationally) determine generators for $O(L, b)$ in terms of matrices. In the indefinite case it is, to the author's knowledge, almost impossible except in a few particular cases. For instance, Brandhorst and Hofmann explain in [BH23, Remark 4.27] how one can solve this problem for rank 2 indefinite $\mathbb{Z}$-lattices.

For any anisotropic vector $v \in L \otimes_R K$, we define the *spinor norm* of the associated reflection τ_v to be the coset

$$-b(v, v)(K^\times)^2. \tag{3}$$

By the Cartan–Dieudonné theorem, any isometry in $O((L, b) \otimes_R K)$ can be decomposed as the product of at most $\operatorname{rank}_R(L)$ reflections in anisotropic vectors of $L \otimes_R K$. There is a unique group homomorphism [O'M73, 54:6], referred to as *spinor norm morphism*,

$$\sigma_K \colon O(L, b) \to K^\times/(K^\times)^2$$

such that $\sigma_K(\tau_v) = -b(v, v)(K^\times)^2$ for all $v \in L \otimes_R K$ anisotropic. We denote $O^+(L, b) := \ker \sigma_K$.

Remark 1.52. The definition of spinor norm for a reflection τ_v as in Equation (3) is not fixed, and different people in the literature might as well choose $b(v, v)(K^\times)^2$ instead. By the convention we have chosen above, we observe that if (L, b) is a $\mathbb{Z}$-lattice of real signatures (l_+, l_-), then $\sigma_{\mathbb{Q}}(-\operatorname{id}_L) = (-1)^{l_+}(\mathbb{Q}^\times)^2$.

From now on, let (L, b) be a $\mathbb{Z}$-lattice. If (L, b) is even, any isometry $f \in O(L, b)$ induces a group isomorphism $D_f \colon D_L \to D_L$ which is an isometry with respect to the form q_L. This defines a group homomorphism

$$\pi_L \colon O(L, b) \to O(D_L, q_L),\ f \mapsto D_f.$$

We denote its kernel by $O^\#(L, b) := \ker \pi_L$, and we call it the *stable subgroup* of $O(L, b)$ (see Remark 2.22 for an explanation on this terminology). Any isometry $f \in O^\#(L, b)$, or subgroup of isometries $G \leq O^\#(L, b)$, will be called *stable*.

Remark 1.53. Even though computing generators for $O(L, b)$ is not always possible (Remark 1.51), we remark the following:

(1) computing generators for the finite group $O(D_L, q_L)$, when (L, b) is even, is possible thanks to an algorithm of Brandhorst and Veniani [BV24];

(2) the image of π_L in $O(D_L, q_L)$, in the case where (L, b) is indefinite and $\mathrm{rank}_{\mathbb{Z}}(L) \geq 3$, has been described through the so-called *Miranda–Morrison theory*. See [MM09, Chapter VIII, Theorem 5.1] for a description of the image, and [MM85, MM86] for the case π_L surjective. Then using the generation algorithm of [BV24] and an algorithm of Shimada [Shi18, §5], computing generators for the image of π_L is computationally feasible.

For any $f \in O(L, b)$, we let $O(L, b, f)$ be the centralizer of f in $O(L, b)$, and similarly we let $O(D_L, q_L, D_f)$ be the centralizer of D_f in the group of isometries of (D_L, q_L).

Notation. Any isometry $f \in O(L, b)$ is in particular a $\mathbb{Z}$-module isomorphism, and its determinant is thus either 1 or -1. We denote by $SO(L, b)$ the subgroup of isometries of positive determinant.

Let us give some extra definitions for general subgroups of isometries.

Definition 1.54. Let $G \leq O(L, b)$ be a subgroup.

(1) We call the $\mathbb{Z}$-lattice $L^G := \{x \in L \ : \ g(x) = x, \forall g \in G\} \leq L$ the *invariant sublattice* of G;

(2) We call the orthogonal complement $L_G := (L^G)^{\perp}_L$ the *coinvariant sublattice* of G;

(3) We call the restriction $(\pi_L)_{|G} \colon G \to O(D_L, q_L)$ the *discriminant representation* of G, and we denote its image by $\overline{G}$.

Similar definitions as in (1) and (2) apply if we replace G by any isometry $f \in O(L, b)$.

Notation. For all $G \leq O(L, b)$, we denote by $G^+ := G \cap O^+(L, b)$ the normal subgroup of isometries with trivial spinor norm, and by $G^\# := G \cap O^\#(L, b)$ the stable subgroup of G. We moreover denote $G^{+,\#} := G^+ \cap G^\#$.

Definition 1.55. For a chain of subgroups $H \leq G \leq O(L, b)$, we call the *saturation* of H in G, denoted by $\mathrm{Sat}_G(H)$, the largest subgroup $H \leq \mathrm{Sat}_G(H) \leq G$ such that $L^H = L^{\mathrm{Sat}_G(H)}$. In the literature, the saturation $\mathrm{Sat}_G(H)$ is also called the *pointwise stabilizer* of L^H in G.

The notion of saturation is useful when one wants to compare and classify groups with similar invariant, or coinvariant, sublattice. We see by definition that the saturation is the maximal element in the poset of subgroups of $O(L, b)$ with given invariant sublattice. Sometimes, in this thesis, the question of knowing the saturation in the stable subgroup of $O(L, b)$ arises. We thus make the following definitions.

Definition 1.56. Let $H \leq G \leq O(L, b)$ be a chain of subgroups. We say H is *saturated* in $G \leq O(L, b)$ if $H = \mathrm{Sat}_G(H)$. Moreover, H is said to be *stably saturated* in $G \leq O(L, b)$ if $H^\#$ is saturated in $G^\#$.

We prove the following lemma, which we use several times throughout the thesis.

Lemma 1.57. *Let $H \leq G \leq O(L)$ be a chain of subgroups. If H is saturated in G, then it is stably saturated.*

Proof. Let $g \in G^\#$ such that g acts trivially on $L^{H^\#}$. Since $L^H \subseteq L^{H^\#}$, we have that g is the identity on L^H and thus $g \in \mathrm{Sat}_G(H) = H$. Hence $g \in G^\# \cap H = H^\#$, and $\mathrm{Sat}_{G^\#}(H^\#) = H^\#$. □

A core part of this thesis relies on studying, constructing and classifying finite groups of isometries of even indefinite $\mathbb{Z}$-lattices. We thus generalize the category of objects we work with to ease our notation, and to make isometries part of the objects we study and not only tools.

Definition 1.58. Given a $\mathbb{Z}$-lattice L and an isometry $f \in O(L)$, we call the pair (L, f) a *lattice with isometry*. Two such lattices with isometry (L_1, f_1) and (L_2, f_2) are *isomorphic* if there exists an isometry $\psi\colon L_1 \to L_2$ such that $f_2 \circ \psi = \psi \circ f_1$. If one replaces f_i by a subgroup $G_i \leq O(L_i)$, we call the pairs (L_1, G_1) and (L_2, G_2) *conjugate* if there exists an isometry $\psi\colon L_1 \to L_2$ such that $G_1 = \psi^{-1} G_2 \psi$.

Let $\mu(X) \in \mathbb{Q}[X]$ be a monic polynomial. We call *μ-lattice* any lattice with isometry (L, f) such that $\mu(f) = 0$, as $\mathbb{Z}$-module endomorphisms. We moreover say (L, f) is a *μ^*-lattice* if μ is the minimal polynomial of f. Given a lattice with isometry (L, f), and given a polynomial $\mu(X) \in \mathbb{Q}[X]$, we call the $\mathbb{Z}$-lattice $L^{\mu(f)} := \ker(\mu(f))$ the *μ-kernel sublattice* of (L, f) .

Remark 1.59. Let L be a $\mathbb{Z}$-lattice, let $f \in O(L)$ be an isometry and let χ be the characteristic polynomial of f. Recall that $\det(f) = \pm 1$ and therefore $\chi(0) = \pm 1$. Then it is known that

$$t^{\deg(\chi)} \chi(1/t) = \chi(0)\chi(t)$$

as polynomials in $\mathbb{Z}[t]$: we say χ is *$\chi(0)$-symmetric* ([Mil69], [BF15, Proposition 1.1]). If moreover f is of finite order, then χ is a product of cyclotomic polynomials.

Let L be a $\mathbb{Z}$-lattice and let $f \in O(L)$ be of finite order $m \geq 1$. We denote by

$$\chi_f(X) = \prod_{n|m} \Phi_n^{d_n(f)}(X) \in \mathbb{Z}[X] \quad \text{and} \quad m_f(X) = \prod_{n|m,\ d_n(f)\neq 0} \Phi_n(X) \in \mathbb{Z}[X]$$

the characteristic and minimal polynomials of f respectively, where $d_n(f) \in \mathbb{N}_0$. For all $n \mid m$, we let $L^{\Phi_n(f)} = \ker(\Phi_n(f))$ be the corresponding kernel sublattice: it satisfies $\operatorname{rank}_{\mathbb{Z}}(L^{\Phi_n(f)}) = d_n(f)\varphi(n)$.

Remark 1.60. For all $n \geq 1$ dividing m and such that $d_n(f) \neq 0$, the lattice $L^{\Phi_n(f)}$ equipped with the isometry $f_n := f_{|L^{\Phi_n(f)}} \neq 0_{L^{\Phi_n(f)}}$ is a Φ_n^*-lattice.

Notation. If $n = 1$, we obtain $L^{\Phi_1(f)} = L^f$, which we therefore call the *invariant sublattice* of (L, f). In a similar vain, if m is even and $n = 2$, then $L^{\Phi_2(f)} = L^{-f}$ is referred to as the *(-1)-sublattice of* (L, f).

The characteristic and minimal polynomials of f therefore encode information on particular sublattices of L which are preserved by f. Knowing the minimal polynomial of isometries in $O(L)$, and the kernel sublattices associated to irreducible divisors of their minimal polynomials, can help us distinguish between different conjugacy classes in $O(L)$.

For any $n \geq 3$ dividing the order m of $f \in O(L)$, the minimal polynomial of f_n is Φ_n so we can see $L^{\Phi_n(f)}$ as a $\mathbb{Z}[\zeta_n]$-module where ζ_n is a primitive nth root of unity: multiplication by ζ_n is given by the action of f_n. We can thus define a structure of *hermitian $\mathbb{Z}[\zeta_n]$-lattice* on $L^{\Phi_n(f)}$ (see Section 4). These kernel sublattices for finite order isometries allow one to define an invariant of lattices with isometry of finite order called the *type* [BH23, Definition 4.18].

Definition 1.61. Let (L, f) be a lattice with isometry of finite order $m \geq 1$. For a positive divisor n of m, let H_n be the hermitian structure of (L_n, f_n) (see Definition 4.20) and let A_n be the $(X^n - 1)$-kernel sublattice of $(L^{\Phi_n(f)}, f)$. The finite collection of pairs of genera $\{(g(H_n), g(A_n))\}_{n|m}$ is called the *type* of (L, f).

Two isomorphic lattices with isometry share the same type, but a type might be represented by several distinct isomorphism classes of lattices with isometry. Note moreover that the notion of type is not a generalization of the notion of genus of $\mathbb{Z}$-lattices (Definition 1.17). In particular, having (L, f) and (M, g) two lattices with isometry of the same type does not imply that there exists an isometry $\psi\colon L_p \to M_p$ such that $\psi \circ f_p = g_p \circ \psi$ for all prime numbers p.

Computational comments. Let L be an even $\mathbb{Z}$-lattice. We have seen in Remark 1.51 that if L is definite or of rank at most 2, then it is possible to compute generators for $O(L)$, given in terms of matrices. In particular, in these cases, it is possible to also determine what are $O^+(L)$ and $SO(L)$. We moreover know that one can compute generators for $O(D_L)$ and the image of $O(L) \to O(D_L)$ (Remark 1.53). Throughout the thesis, we will therefore assume that the previous are computable. Now let $G \leq O(L)$ be a finite subgroup, given in terms of matrices for a fixed basis of L. Then one can effectively:

(1) compute the spinor norm of any $f \in G$;

(2) construct the image $\overline{G} := \pi_L(G) \leq O(D_L)$;

(3) determine generators for the stable subgroup $G^\# \leq G$;

(4) compute the invariant and coinvariant sublattices of G;

(5) compute generators for the centralizer in G of any $f \in O(L)$;

(6) compute generators for the centralizer of D_f in $O(D_L)$ for any $f \in O(L)$;

(7) determine the characteristic and minimal polynomial of any $f \in O(L)$;

(8) compute bases for the kernel sublattices associated to any isometry $f \in O(L)$.

As before, we assume from now on that items (1)–(8) are computationally accessible.

1.5. Genus enumeration

We conclude this first part of preliminaries with a discussion about enumerating genera of even $\mathbb{Z}$-lattices. We explain how one can, in practice, determine a complete set of representatives for the isometry classes in a given genus g of even $\mathbb{Z}$-lattices. Note that this approach is generalizable to any integral $\mathbb{Z}$-lattice.

1.5.1. Spinor genera

Let us start by introducing the notion of spinor genus.

Notation. For this section, and this section only, we change our conventions and for any R-lattice (L, b), where $R = \mathbb{Z}, \mathbb{Z}_p$ (p prime) and $K = \mathrm{Frac}(R)$, we define the spinor norm of any reflection in an anisotropic vector $v \in L \otimes_R K$ to be $b(v, v)(K^\times)^2$.

Let g be a genus of even $\mathbb{Z}$-lattices inside a fixed quadratic space V over $\mathbb{Q}$. In what follows, we denote by $O(V)$ the group of isometries of V, and we define $O^+(V)$ the group of isometries with positive spinor norm. Similarly, for every prime number p, we let $O^+(V_p)$ the group of isometries of V_p with trivial spinor norm, and we let $SO^+(V_p)$ the subgroup of such isometries with determinant 1.

Definition 1.62. Let $L, M \leq V$ be two $\mathbb{Z}$-lattices in g. We say L and M are in the same *spinor genus* if there exists an isometry $\sigma \in O(V)$ such that for all p a prime number there exists an isometry $\varphi_p \in SO^+(V_p)$ so that

$$L_p = \sigma_p(\varphi_p(M)).$$

We denote by $\operatorname{spin}(L)$ the set of $\mathbb{Z}$-lattices $M \leq V$ which are in the same spinor genus as L. Note that this definition defines an equivalence relation on g: we define a *spinor genus* in g to be a subset consisting of all $\mathbb{Z}$-lattices in g which are in the same spinor genus.

According to [O'M73, Theorem 102:8], a genus g of even $\mathbb{Z}$-lattices consists of finitely many spinor genera. Note the following:

Theorem 1.63 ([O'M73, Theorem 104:5], [Kne02, Satz (25.2)])**.** *Let L be an even indefinite $\mathbb{Z}$-lattice of rank greater than or equal to 3. Then* $\operatorname{spin}(L)$ *consists of a unique isometry class. In particular, the number of isometry classes in $g(L)$ is equal to its number of spinor genera.*

The theorem is a consequence of a more general result called Strong Approximation Theorem (see for instance [O'M73, 104:4]).

Remark 1.64. Given a genus g of even $\mathbb{Z}$-lattices, and a $\mathbb{Z}$-lattice $L \in g$, it is computationally possible to recover one representative for each spinor genus in g [BH83].

In what follows we give a description of genus enumeration for genera consisting of a single spinor genus. Moreover, we focus on the definite case.

1.5.2. Kneser's neighbor method

Let g be a genus of definite even $\mathbb{Z}$-lattices and suppose that g consists of exactly one spinor genus. Let us fix a representative $L \in g$, which is computable (see for instance [Kir16, §3.4, §3.5]), and let us denote by b the bilinear form on L.

Definition 1.65. Let p be a prime number. We say that the $\mathbb{Z}_p$-lattice (L_p, b_p) is *maximal integral* if (L_p, b_p) is integral and for any $\mathbb{Z}_p$-module $L_p \leq M \leq V_p$, the $\mathbb{Z}_p$-lattice (M, b_p) is integral if and only if $M = L_p$.

Definition 1.66. Let p be a prime number. We say p is a *Kneser prime* for g if the following hold:

(1) the $\mathbb{Z}_p$-lattice (L_p, b_p) is isotropic;

(2) the $\mathbb{Z}_p$-lattice (L_p, b_p) is maximal integral.

Note that this definition does not depend on the choice of L as a representative of g. In fact, by definition, for every $L' \in g$ and for all prime numbers p, we have that $L_p \simeq L'_p$: in particular, one is isotropic if so is the other. The second condition can be checked directly on the torsion bilinear form b_{L_p} of L_p. Indeed, if there exists a $\mathbb{Z}_p$-module $L_p \leq M \leq V_p$ such that b_p is integral over M, then we have that $L_p \leq M \leq L_p^\vee$ and M/L_p defines an isotropic submodule of $L_p^\vee/L_p$ (for the torsion bilinear form b_p). Therefore, such an M exists if and only if L_p is not unimodular and (D_{L_p}, b_{L_p}) has a totally isotropic submodule.

Remark 1.67. As a consequence, we see that both conditions from Definition 1.66 can be checked directly from any symbol representing g.

We review now results of Kneser on *neighbors*. The goal is to represent g as a finite graph whose nodes represent the isometry classes in g. Every two nodes will be connected by an edge if they satisfy the following.

Definition 1.68. ([Kne02, §28]) Let p be a prime number and let L' be a full rank $\mathbb{Z}$-lattice in V. Then L' is called a *p-neighbor* of L if, as $\mathbb{F}_p$-vector spaces,

$$L/(L \cap L') \simeq L'/(L \cap L') \simeq \mathbb{F}_p.$$

What remains to be understood is that such a graph is actually finite, and connected. Moreover, we aim to describe a procedure to reconstruct (partially) such a graph from $L \in g$. For that purpose, let us define an element $y \in L$ to be *p-admissible* if $y \in L \setminus pL$ and $b(y, y) \in 2p^2\mathbb{Z}$, for p a Kneser prime of g.

Theorem 1.69 ([SP91, §1], [Kne02, §28])**.** *Let p be a Kneser prime for g, let $y \in L$ be p-admissible and let*

$$L^p(y) := L_{y,p} + \mathbb{Z}\frac{1}{p}y \quad \textit{where} \quad L_{y,p} = \{x \in L \ : \ b(x, y) \in p\mathbb{Z}\}. \tag{4}$$

Then $L^p(y)$ is a p-neighbor of L. Moreover, $L^p(y)$ lies in g and any p-neighbor of L lying in g is of the form $L^p(y)$ for a p-admissible vector $y \in L$.

A priori the previous definition only gives us that there are infinitely many p-neighbors of L. The following shows that actually there are finitely many of them, and that $O(L)$ acts on the set of p-neighbors of L.

Proposition 1.70 ([Kne02, Hilfssatz (28.7)])**.** *Let p be a Kneser prime and let $y, y' \in L$ be p-admissible.*

(1) *If $y + pL = y' + pL$, then $L^p(y) = L^p(y')$.*

(2) *If there exists $f \in O(L)$ such that $f(y) - y' \in pL$, then f induces an isometry between $L^p(y)$ and $L^p(y')$.*

So in particular, in order to determine p-neighbors of L up to isometry, it is enough to determine $O(L)$-orbits of isotropic lines in $L/pL \simeq \mathbb{F}_p^{\mathrm{rank}_{\mathbb{Z}}(L)}$, and consider p-admissible representatives. Note that not all such isotropic lines admit a p-admissible representative:

Lemma 1.71. *Let p be a Kneser prime for g, let $l_p \in \mathbb{P}(L/pL)$ be a line, and let $w \in L \setminus pL$ be a representative of l_p, i.e. $l_p = \mathbb{F}_p \cdot (w + pL)$. Then, l_p admits a p-admissible representative if and only if $b(w, w) \in 2p\mathbb{Z}$ and*

(1) *either w is p-admissible;*

(2) *or $w \notin pL^\vee$.*

In the second case, there exists a p-admissible vector $y \in w + pL$.

Proof. Suppose that w is not p-admissible. Already note that if $w \in pL^\vee$, then for all $y = w + px$ where $x \in L$, we have that $b(w, x) \in p\mathbb{Z}$ and thus

$$b(y, y) = b(w, w) + 2pb(w, x) + p^2 b(x, x) \equiv b(w, w) \not\equiv 0 \mod 2p^2.$$

Now, assume that $w \notin pL^\vee$.

(i) If p is odd, since $w \notin pL^\vee$, there exists $x \in L$ such that $b(x, w) \equiv a \mod p$ for some $1 \leq a \leq p - 1$. Then, for any $\alpha \in \mathbb{Z}$ satisfying

$$2ap\alpha \equiv -b(w, w) \mod p^2$$

we have that $y := w + \alpha px \notin pL$ and moreover

$$b(y, y) = b(w, w) + 2ap\alpha + \alpha^2 p^2 b(x, x) \equiv 0 \mod p^2.$$

(ii) Now if $p = 2$, we proceed similarly, but this time we choose $4\alpha \equiv -b(w,w) \mod 8$. $\square$

Now, let us fix p a Kneser prime for g. We define $C(g)$ to be the set of all the (finitely many) isometry classes of lattices in g, and $E_p(g)$ to be the set

$$E_p(g) := \left\{([L],[L']) \in C(g)^2 \;:\; L, L' \text{ are } p\text{-neighbors}\right\}.$$

We define $\mathrm{Kne}_p(g) := (C(g), E_p(g))$ to be the *p-neighbor graph* of g (see [SH98, page 742]), where we see $C(g)$ as the vertices of $\mathrm{Kne}_p(g)$ and $E_p(g)$ as its edges. By definition, $\mathrm{Kne}_p(g)$ consists of finitely many nodes, and two nodes $[L], [L']$ are connected by an edge if and only if $([L],[L']) \in E_p(g)$. Note that the number of connected components of $\mathrm{Kne}_p(g)$ does not depend on the choice of a Kneser prime p.

The neighbor construction Equation (4) offers a practical way of constructing representatives of isometry classes in g and it is the starting point for Kneser's neighbor procedure [Kne02, Satz 28.4]. In particular, it lets us reconstruct $C(g)$ starting from a single isometry class $[L]$ and any Kneser prime number p for g.

The general idea is the following: we start by constructing all the p-neighboring isometry classes $[L']$ of $[L]$ obtained from the neighbor construction. Then, we iterate this process to all the new isometry classes obtained until we have exhausted the p-neighbor graph, i.e. we cannot construct any new isometry class from the ones already obtained. Note that an isometry class can be reached by several different other ones in $\mathrm{Kne}_p(g)$: one has to compare any new neighboring $\mathbb{Z}$-lattice to representatives of the isometry classes already computed. The following result ensures that this process allows us to recover $C(g)$.

Theorem 1.72 ([SP91, §1 (ix)]). *If the genus g consists only of one spinor genus, $\mathrm{rank}_{\mathbb{Z}}(L) \geq 3$ and p is a Kneser prime for g, then the previous algorithm returns representatives of all isometry classes in g.*

The proof of the previous theorem relies on Strong Approximation Theorem, which requires to work with isotropic quadratic spaces of rank at least 3 over $\mathbb{Q}_p$. Moreover, in order to ensure that p-neighbors of L are in the same genus as L, we need that L_p is maximal for the bilinear form b_p: hence the conditions for the definition of Kneser prime.

Remark 1.73. In the case where g has several spinor genera, one would need to make an extra assumption on p, namely that the p-neighbor construction respects spinor genera. Note that such a condition is also only dependent on g.

Common implementations of Kneser's procedure do not exhaust all the edges in the p-neighbor graph $\mathrm{Kne}_p(g)$. In fact, for genera of definite $\mathbb{Z}$-lattices, there exists an invariant, called the *mass*, which allows us to determine whether or not we have enumerated $C(g)$.

Definition 1.74. Let g be a genus of definite $\mathbb{Z}$-lattices. We call the *mass* of g, which we denote by $m(g)$, the following sum

$$m(g) := \sum_{[L]\in C(g)} \frac{1}{\#O(L)}. \tag{5}$$

For any $[L] \in C(g)$, we call the term $w([L]) := \frac{1}{\#O(L)}$ the *weight* of $[L]$ in g.

Thanks to the Smith–Minkowski–Siegel formula (see for instance [CS99, §2, Eq. (2)]), one can actually compute the mass of a genus of definite $\mathbb{Z}$-lattices without enumerating its isometry classes. Therefore, while walking through $\mathrm{Kne}_p(g)$, one can associate to each visited node the weight of the corresponding isometry class. Adding the weights of the isometry classes already found gives us information on which proportion of $C(g)$ we have enumerated. In particular, if the previous sum agrees with the mass of g, then all the isometry classes in g have been enumerated.

1.5.3. Practical implementation and possible improvements

We have explained the needed theory behind the enumeration of the genus g (we refer to [SH98] for more details). Let us now make some comments on practical implementation and possible improvements.

Let again g be a genus of even definite $\mathbb{Z}$-lattices such that g consists of exactly one spinor genus, and let p be a Kneser prime. Let moreover $L \in g$ be a representative, and let $n \geq 3$ be its rank. We have seen in the previous section that in order to find p-admissible vectors in L, it suffices to enumerate lines l_p in $L/pL \cong \mathbb{F}_p^n$ which are isotropic with respect to the form induced by the one on L, and choose any p-admissible lift in L of a representative of l_p. The set $\mathcal{L}_{p,n}$ of lines in $\mathbb{F}_p^n$ is of cardinality $(p^n-1)/(p-1)$. When p and n are large (heuristically, whenever $p^n - 1 \geq 10^7(p-1)$), enumerating isotropic elements in $\mathcal{L}_{p,n}$ at each iteration of the neighbors construction is infeasible: for instance, in the case $p = 5$ and $n = 15$, the set $\mathcal{L}_{5,15}$ is of size $7629394531 >> 10^7$. The whole process of enumerating all the isotropic lines, looking for p-admissible lifts, constructing the corresponding neighbors, comparing each new neighbor to the already visited nodes in $C(g)$ and computing the weight of every new visited node can take, in such situation, more than 5 hours. Sometimes such a process can actually be much longer than that. If we do apply this procedure multiple times for different isometry classes, depending on the size of $C(g)$, the enumeration can take several weeks, or even months.

Remark 1.75. As we have seen in Proposition 1.70 we only have to look for representatives of $O(L)$-orbits of isotropic lines in L/pL. In other words, walking through $\mathrm{Kne}_p(g)$ only requires to lift representatives of $O(L)$-orbits of isotropic elements in $\mathcal{L}_{p,n}$ since two isotropic lines in the same orbit give rise to isometric neighbors, which correspond to the same class in $C(g)$. However, even though $\#O(L)$ is finite for a definite $\mathbb{Z}$-lattice L, it turns out that the available algorithms, to the author's knowledge, to compute representatives for the $O(L)$-orbits in $\mathcal{L}_{p,n}$ require either too much memory space or too much computation time, for large p and n (see for instance [O'B90] for a more general approach to orbits of vector spaces over finite fields).

A way to bypass these complexity matters is to add randomization. We iteratively proceed as follows: we fix p to be the smallest Kneser prime for g, we choose a random isometry class among those already visited in $C(g)$, and we collect a sample of 10 to 50 random elements in $\mathcal{L}_{p,n}$ to construct neighbors in $\mathrm{Kne}_p(g)$ from the isotropic lines among them.

Computational comments. From the infrastructure we have already defined in the previous sections, one can enumerate any given genus of even $\mathbb{Z}$-lattices. A generic implementation of genus enumeration for integral $\mathbb{Z}$-lattices, taking into account spinor genera and using Kneser's neighbor method in the definite cases, has been implemented on the computer algebra system Hecke [FHHJ17]. Hence, for the rest of thesis, we assume that we are in measure of systematically enumerating any genus of even $\mathbb{Z}$-lattices, at least up to rank 13. For higher ranks, it may actually happen that even randomization is not enough; see the following discussion.

A final issue one can face with the randomized implementation previously described is that random walks in $\mathrm{Kne}_p(g)$ can eventually lead to vain iterations. In this case, there might be a little number of nodes which have not been visited yet and the probability to find them via random construction of neighbors is low. This is the case, for instance, for isometry classes of $\mathbb{Z}$-lattices with relatively small weight: their isometry group has big cardinality and therefore, there are few edges connecting to the corresponding node in $\mathrm{Kne}_p(g)$ for all small primes p. It is possible, at this stage, to complete the enumeration differently.

Let $C_{al}(g)$ be the list of isometry classes in g which have been already computed, and let

$$m := m(g) - \sum_{[L] \in C_{al}(g)} w([L]) \in \mathbb{Q}.$$

Let us assume that $m \notin \mathbb{Z}$, and let q be the largest prime number dividing the denominator of m. This prime q must divide the order of the isometry group of a class in $C(g) \setminus C_{al}(g)$, and thus such an isometry class contains a $\mathbb{Z}$-lattice admitting at least one isometry of order q. In the paper [BH23], Brandhorst and Hofmann have developed methods to compute, given a genus g of even $\mathbb{Z}$-lattices and a prime number q, a complete set of representatives for the isometry classes of lattices with isometry (L, f) such that $L \in g$ and $f \in O(L)$ has order q. Their algorithms have been implemented in the Oscar package [OSC25, QuadFormAndIsom] by the author of the thesis. By enumerating all such pairs (L, f), we can find new $\mathbb{Z}$-lattices in g not belonging to any class in $C_{al}(g)$, and having an isometry of order q. Subtracting the weights of their respective isometry classes to m should eventually clear out q from the divisors of the denominator of m. If m is still nonzero, we keep going with the largest prime $q' < q$ dividing the denominator of m, and so on until $m = 0$.

Remark 1.76. If m was initially an integer, then one can choose q to be the largest prime number such that $q - 1 \leq n$. This is the largest prime order possible for any isometry of a rank n $\mathbb{Z}$-lattice. Then one iterates as before by choosing any new prime being smaller than q. This value of q is not optimal, but the enumeration algorithm of Brandhorst and Hofmann is fast for large prime numbers.

2. Embeddings of $\mathbb{Z}$-lattices

In this section, we review some known facts about embeddings of lattices, and their equivariant analogs, which play an important role for the computations in this thesis. The goal of this section is to describe algorithms which we use several times. This will impact, in particular, the formulation of the results presented in what follows. The results presented in this section are mostly due to Nikulin [Nik80].

2.1. Overlattices

In the category of $\mathbb{Z}$-lattices, any morphism is injective: we therefore talk about *embeddings*, and given an embedding $S \hookrightarrow L$ of $\mathbb{Z}$-lattices, we shall therefore see S as a sublattice of L.

Definition 2.1. Let $S \leq L$ be a sublattice of a given $\mathbb{Z}$-lattice L. We say L is an *overlattice* of S if $\mathrm{rank}_{\mathbb{Z}}(S) = \mathrm{rank}_{\mathbb{Z}}(L)$.

Suppose that S and L are even $\mathbb{Z}$-lattices, and L is a proper overlattice of S (i.e. $L \neq S$). There is a succession of embeddings

$$S \leq L \leq L^\vee \leq S^\vee$$

where we see all the four above as $\mathbb{Z}$-submodules of $L \otimes_{\mathbb{Z}} \mathbb{Q} = S \otimes_{\mathbb{Z}} \mathbb{Q}$. We thus obtain a chain of embeddings of torsion quadratic modules

$$L/S \leq L^\vee/S \leq D_S.$$

Since L and S are even, we have that the submodule $L/S \leq D_S$ is isotropic for the torsion quadratic form q_S on D_S. Moreover, by the characterization

$$L^\vee = \{v \in L \otimes_{\mathbb{Z}} \mathbb{Q} \ : \ v.L \subseteq \mathbb{Z}\}$$

and the fact that L/S is isotropic, we have that $L/S \leq (L/S)^\perp = L^\vee/S \leq D_S$ for the torsion bilinear form b_S on D_S. In particular $D_L \simeq (L^\vee/S)/(L/S)$ as torsion quadratic modules, where the quadratic form is induced by the one on D_S.

Lemma 2.2. *Any submodule $H \leq D_S$ is of the form L/S for $S \leq L \leq S^\vee$ an overlattice. In particular, the set of overlattices $S \leq L \leq S^\vee$ is in bijection with the set of submodules $H \leq D_S$.*

Proof. First of all, remark from the discussion above that for any $\mathbb{Z}$-lattice T such that $S \leq T \leq S^\vee$, then T is an overlattice of S and $T/S \leq D_S$. Conversely, given $H \leq D_S$, we have that for any $h \in H \leq S^\vee/S$ there exists $v \in S^\vee$ such that $h = v + S$. We therefore define

$$T := \{v \in S^\vee \ : \ v + S \in H\}$$

to conclude. □

Example 2.3. Let again $L := A_2$ be the root lattice. Then $D_{A_2} \cong \mathbb{Z}/3\mathbb{Z}$ and we see that there are exactly two overlattices $A_2 \leq T \leq A_2^\vee$: namely A_2 and $A_2^\vee$.

Note that by definition of the dual, if S is an integral $\mathbb{Z}$-lattice, then any integral overlattice $S \leq L$ is a sublattice of $S^\vee$.

Proposition 2.4 ([Nik80, Proposition 1.4.1])**.** *Let S be an even lattice. Then there is a one-to-one correspondence*

$$\{q_S\text{-isotropic submodules } H \leq D_S\} \leftrightarrow \{\text{even overlattices } S \leq L\}.$$

Proof. It is an application of Lemma 2.2, and by using previous discussion. In fact, if $L/S \leq D_S$ is an isotropic submodule, then for any $v \in L \leq S^\vee$ we have that $q_S(v+S) = v^2 + 2\mathbb{Z} \in 2\mathbb{Z}$, meaning that v^2 is even. Hence the bijection. □

For an even $\mathbb{Z}$-lattice S, we say two integral overlattices $S \leq L_1, L_2 \leq S^\vee$ are *isomorphic over* S if there exists an isometry $f \in O(S)$ which extends to an isometry between L_1 and L_2. The previous should be seen as follows. Any isometry $f \in O(S)$ induces an isometry $f^\vee \in O(S^\vee)$: hence L_1 and L_2 are isomorphic over S if there exists $f \in O(S)$ such that $f^\vee(L_1) = L_2$.

Proposition 2.5 ([Nik80, Proposition 1.4.2])**.** *Two overlattices $S \leq L_1, L_2 \leq S^\vee$ are isomorphic over S if and only if the two submodules L_1/S, $L_2/S \leq D_S$ are in the same orbit for the action of $O(S)$ on D_S.*

Proof. One direction follows by definition: if L_1 and L_2 are isomorphic over S, then there exists $f \in O(S)$ such that $D_f(L_1/S) = L_2/S$. Conversely, suppose that there exists an isometry $f \in O(S)$ such that $D_f(L_1/S) = L_2/S$. Then, we have by definition that for all $v \in L_1 \leq S^\vee$

$$f^\vee(v) + S \in L_2/S$$

and thus $f^\vee(L_1) \leq L_2$ by Lemma 2.2. By applying similar arguments to f^{-1}, we obtain that $(f^\vee)^{-1}(L_2) \leq L_1$ so we can conclude. □

Putting together Propositions 2.4 and 2.5 we get a way to classify even overlattices of a given even $\mathbb{Z}$-lattice S. Before applying the previous to the theory of primitive extensions, let us recall the following well-known fact.

Lemma 2.6. *Let S be an even $\mathbb{Z}$-lattice, let $S \leq L$ be an even overlattice and let us denote by $[L : S] = \#(L/S)$ the* index *of S in L. Then*

$$\det(S) = [L : S]^2 \det(L).$$

In particular, $\det(S)$ *is divisible by* $\det(L)$*, and S has an even unimodular overlattice if and only if there exists $H \leq D_S$ a q_S-isotropic submodule such that $(\#H)^2 = \#D_S$.*

Proof. Note that since $S \otimes_{\mathbb{Z}} \mathbb{R} = L \otimes_{\mathbb{Z}} \mathbb{R}$, we have that $\det(S)$ and $\det(L)$ have the same sign. Now since b_S is a nondegenerate bilinear form on the finite abelian group D_S, for any submodule $H \leq D_S$, the following equality holds:

$$\#D_S = (\#H) \cdot (\#H^\perp).$$

The proof follows by recalling that $D_L \simeq (L^\vee/S)/(L/S)$ where $L^\vee/S = (L/S)^\perp$ for the form b_S. □

Example 2.7.

(1) Let $S := \langle -8 \rangle$. Then $D_S \simeq \left(-1/8\right)$ as torsion quadratic module, and the order 2 subgroup of $D_S \cong \mathbb{Z}/8\mathbb{Z}$ is isotropic for q_S. The associated overlattice is isometric to $\langle -2 \rangle$.

(2) Let L be an even $\mathbb{Z}$-lattice of signatures (l_+, l_-), and let $S := L \oplus L(-1)$. Then the graph $\Gamma \leq D_S \cong D_L \oplus D_{L(-1)}$ of any isometry $\gamma \colon D_L \to D_{L(-1)} \simeq D_L(-1)$ is q_S-isotropic. Moreover, $(\#\Gamma)^2 = (\#D_L)^2 = \#D_S$ and therefore S has an even unimodular overlattice of signatures $(l_+ + l_-, l_+ + l_-)$. Such an overlattice is isometric to $U^{\oplus \operatorname{rank}_{\mathbb{Z}}(L)}$ [Nik80, Theorem 1.1.1].

2.2. Primitive sublattices

The results about overlattices will be useful for us when classifying orbits of *primitive sublattices* of a given even $\mathbb{Z}$-lattice L, by the mean of *primitive extensions*. Let us describe this as follows.

Definition 2.8. Let L be an even $\mathbb{Z}$-lattice.

(1) We say S is *primitive*, as sublattice of L, if the quotient module L/S is torsionfree.

(2) An embedding $i\colon S \hookrightarrow L$ is said to be *primitive* if $i(S)$ is primitive in L.

Remark 2.9. Given a sublattice S of an integral $\mathbb{Z}$-lattice L, if L is an overlattice of S and S is primitive in L, then $S = L$.

Example 2.10. Let L be an even $\mathbb{Z}$-lattice and let $f \in O(L)$ be an isometry. Then for any monic polynomial $p(X) \in \mathbb{Z}[X]$, the sublattice $\ker(p(f)) \leq L$ is primitive.

Definition 2.11. Let L be an even $\mathbb{Z}$-lattice.

(1) Two primitive sublattices $S, T \leq L$ are called *isomorphic* if there exists an isometry $f \in O(L)$ such that $f(S) = T$.

(2) Given an even $\mathbb{Z}$-lattice S, two primitive embeddings $i_1, i_2\colon S \to L$ are said *isomorphic* if so are $i_1(S)$ and $i_2(S)$.

2.2.1. Primitive extensions

Let S and T be even $\mathbb{Z}$-lattices and let L be an overlattice of $S \oplus T$.

Definition 2.12. We call L a *primitive extension* of $S \oplus T$ if both composite embeddings $S \hookrightarrow S \oplus T \hookrightarrow L$ and $T \hookrightarrow S \oplus T \hookrightarrow L$ are both primitive.

Suppose that L is a primitive extension of $S \oplus T$. Note that as finite quadratic modules, we have a natural identification

$$D_{S\oplus T} \simeq D_S \oplus D_T.$$

Since L is in particular an overlattice of $S \oplus T$, we have a succession of inclusions

$$S \oplus T \leq L \leq L^\vee \leq S^\vee \oplus T^\vee$$

giving rise to the succession of inclusions of torsion quadratic modules

$$L/(S \oplus T) \leq L^\vee/(S \oplus T) \leq (S^\vee \oplus T^\vee)/(S \oplus T) \xrightarrow{\simeq} D_S \oplus D_T.$$

Let us denote by $H_S \leq D_S$ and $H_T \leq D_T$ the respective images of $L/(S \oplus T)$ along the two projections

$$\begin{tikzcd} & D_S \oplus D_T \arrow[dl, "p_s"'] \arrow[dr, "p_T"] & \\ D_S & & D_T \end{tikzcd}.$$

Lemma 2.13. *As finite abelian groups, we have that $H_S \cong L/(S \oplus T) \cong H_T$. Moreover, the associated isomorphism of abelian groups $\gamma := p_T \circ (p_S)^{-1}_{|H_s}\colon H_S \to H_T$ satisfies*

$$q_T(\gamma(x)) + q_S(x) = 0 + 2\mathbb{Z}$$

for all $x \in H_S$.

Proof. For the sake of the proof, let us prove that $p_S\colon L/(S\oplus T)\to D_S$ is injective. Let $x+(S\oplus T)$ be in the kernel of p_S, and write $x=x_S+x_T\in L$, where $x_S\in S^\vee$ and $x_T\in T^\vee$. Then, since $p_S(x+(S\oplus T))=x_S+S\in S$, we have that $x\in(S\oplus T^\vee)\cap L$. Hence, since $S\leq L$, we obtain that $x_T\in L$ and since $T^\vee/T$ has finite order, there exists $n\in\mathbb{Z}_{\geq 1}$ such that $nx_T\in T$. But T is primitive in L, so n must be 1 and $x_T\in T$. Thus $x\in S\oplus T$ and p_S is injective. This shows that $L/(S\oplus T)$ is the graph of an isomorphism of finite abelian groups $\gamma\colon H_S\to H_T$. In particular, for all $x\in H_S$, we have that $x+\gamma(x)\in L/(S\oplus T)$ is isotropic for the form $q_{S\oplus T}$ and thus

$$q_{S\oplus T}(x+\gamma(x))=q_S(x)+q_T(\gamma(x))=0+2\mathbb{Z}.\qquad\square$$

Definition 2.14. Let S and T be two even $\mathbb{Z}$-lattices.

(1) We call *glue map* any isometry $D_S\geq H_S\xrightarrow{\gamma}H_T\leq D_T$ between finite subgroups of D_S and D_T respectively, satisfying that

$$q_S(x)+q_T(\gamma(x))=0+2\mathbb{Z}$$

for all $x\in H_S$. Moreover, we call the subgroups H_S and H_T the *glue domains* of γ, and we say that γ is a glue map between H_S and H_T.

(2) We say that S and T *glue along* $H_S\leq D_S$ and $H_T\leq D_T$ if there exists a glue map between H_S and H_T.

(3) We call *gluing* of S and T any glue map between subgroups of their respective discriminant groups. We say that S and T glue if such a gluing exists.

Remark 2.15. Note that glue maps are a special case of the so-called *anti-isometries*: indeed, a glue map $\gamma\colon H_S\to H_T$ as before induces an isometry between the torsion quadratic modules H_S and $H_T(-1)$. In this case, we also say that H_S and H_T are *anti-isometric.*

Example 2.16. Given an even $\mathbb{Z}$-lattice L, we have that L and $L(-1)$ glue along their respective discriminant groups.

Proposition 2.17 ([Nik80, Proposition 1.5.1])**.** *Let S and T be two even $\mathbb{Z}$-lattices. Then gluings of S and T correspond bijectively to even primitive extensions of $S\oplus T$.*

Proof. We have already seen that if $S\oplus T\leq L$ is an even primitive extension, then L defines a gluing whose graph in $D_S\oplus D_T$ is exactly $L/(S\oplus T)$.

Let $D_S\geq H_S\xrightarrow{\gamma}H_T\leq D_T$ be a gluing of S and T. We denote by Γ its graph in $D_S\oplus D_T$. By definition of γ as a glue map, we know that Γ is an isotropic submodule and it is thus of the form $L_\gamma/(S\oplus T)$ where $S\oplus T\leq L_\gamma$ is an even overlattice (Proposition 2.4). Now let $x=x_S+x_T\in L_\gamma\leq S^\vee\oplus T^\vee$, and suppose there exists $n\in\mathbb{Z}_{\geq 1}$ such that $nx\in T\leq L_\gamma$. The spaces $S^\vee\otimes_{\mathbb{Z}}\mathbb{Q}$ and $T^\vee\otimes_{\mathbb{Z}}\mathbb{Q}$ are in direct sum in $L_\gamma\otimes_{\mathbb{Z}}\mathbb{Q}$, so having $nx=nx_S+nx_T\in T$ implies that $x_S=0$. Since $L_\gamma/(S\oplus T)$ is the graph of γ, we also have that

$$x_T+T=\gamma(x_S+S)=T$$

meaning that $x_T\in T$. Hence $T\leq L_\gamma$ is primitive, and similar arguments show that $S\leq L_\gamma$ is primitive. Hence $S\oplus T\leq L_\gamma$ is an even primitive extension. $\square$

Notation. We call L_γ the *overlattice relative to* γ, and we say that S and T *glue to* L_γ. We will often call H_S and H_T the *glue domains* of the respective primitive embeddings $S\hookrightarrow L_\gamma$ and $T\hookrightarrow L_\gamma$.

Remark 2.18. With the notation of Proposition 2.17, and using Lemma 2.6, we observe that

$$[L_\gamma : S \oplus T]^2 = (\#H_S) \cdot (\#H_T) = \frac{\det(S)\det(T)}{\det(L)}.$$

Given S and T two even $\mathbb{Z}$-lattices, we call two even primitive extensions $S \oplus T \leq L_1, L_2$ *isomorphic* if there exists an isometry $f\colon L_1 \to L_2$ such that $f(S) = S$ and $f(T) = T$. Similarly to Proposition 2.5, we have an explicit way to determine whether two primitive extensions are isomorphic.

Lemma 2.19 ([Nik80, Corollary 1.5.2])**.** *Let S and T be two even $\mathbb{Z}$-lattices and let $S \oplus T \leq L_{\gamma_1}, L_{\gamma_2}$ be two even primitive extensions obtained from respective gluings $\gamma_1\colon H_S^1 \to H_T^1$ and $\gamma_2\colon H_S^2 \to H_T^2$ of S and T. Then L_{γ_1} and L_{γ_2} are isomorphic as primitive extensions if and only if there exist two isometries $\psi \in O(S)$ and $\phi \in O(T)$ such that $D_\psi(H_S^1) = H_S^2$, $D_\phi(H_T^1) = H_T^2$, and such that the following diagram commutes*

$$\begin{tikzcd} H_S^1 \arrow[r, "\gamma_1"] \arrow[d, "D_\psi"'] & H_T^1 \arrow[d, "D_\phi"] \\ H_S^2 \arrow[r, "\gamma_2"'] & H_T^2 \end{tikzcd}.$$

Proof. Suppose that there exists $f\colon L_{\gamma_1} \to L_{\gamma_2}$ such that $f(S) = S$ and $f(T) = T$: let us denote by $\psi \in O(S)$ and $\phi \in O(T)$ the induced isometries. Then it follows that f induces an isometry $\overline{f}$ between $L_{\gamma_1}/(S \oplus T)$ and $L_{\gamma_2}/(S \oplus T)$, and by projecting into D_S and D_T, the isometry $\overline{f}$ gives that D_ψ maps $H_S^1 = p_S(L_{\gamma_1}/(S \oplus T))$ onto $H_S^2 = p_S(L_{\gamma_2}/(S \oplus T))$. Similarly, D_ϕ maps H_T^1 to H_T^2. Let now $x \in H_S^1$: we see that

$$\overline{f}(x + \gamma_1(x)) = D_\psi(x) + D_\phi(\gamma_1(x)) \in L_{\gamma_2}/(S \oplus T)$$

meaning that $D_\phi(\gamma_1(x)) = \gamma_2(D_\psi(x))$, and we can conclude.

By reversing the arguments, we see that if $\psi \in O(S)$ and $\phi \in O(T)$ satisfy the second part of the statements, then $\psi \oplus \phi$ defines an isometry between L_{γ_1} and L_{γ_2}. □

Example 2.20. Let S and T be two even $\mathbb{Z}$-lattices such that there exists a glue map between D_S and D_T. Then, there exists an even unimodular primitive extension $S \oplus T \leq L$. If we assume moreover that the morphism $O(T) \to O(D_T)$ is surjective, then by Lemma 2.19 we have that any isometry of S extends to an isometry of L.

Corollary 2.21. *Let L be an even $\mathbb{Z}$-lattice and let $S \leq L$ be a primitive sublattice. Then there exists a subgroup $G \leq O^\#(L)$ such that G fixes pointwise $S^\perp$ and whose restriction to S is $O^\#(S)$.*

Proof. Let us denote by $T := S_L^\perp$: it then follows that L is a primitive extension of $S \oplus T$, with associated glue map $D_S \geq H_S \xrightarrow{\gamma} H_T \leq D_T$. Since by definition $O^\#(S)$ acts trivially on H_S, we have that for all $\phi \in O^\#(S)$, the isometry $f := \phi \oplus \mathrm{id}_T \leq O(S^\vee \oplus T^\vee)$ preserves L. In fact, we also observe that $D_{\phi\oplus\mathrm{id}_T} = \mathrm{id}_{D_S} \oplus \mathrm{id}_{D_T} \in O(D_S \oplus D_T)$ acts trivially on $L^\vee/(S \oplus T)$. Therefore, the isometry f restricts to an isometry in $O^\#(L)$. We define

$$G := O^\#(S) \times \{\mathrm{id}_T\} = \{\phi \oplus \mathrm{id}_T \;:\; \phi \in O^\#(S)\} \leq O^\#(L),$$

and the proof follows. □

Remark 2.22. The naming *stable* for the normal subgroup $O^\#(L) \trianglelefteq O(L)$ comes from what is observed in Corollary 2.21: the stable subgroup of a primitive sublattice $S \leq L$ "extends" to a subgroup of stable isometries of L. In some sense, stability of an isometry is preserved under primitive embeddings. Note that G is the pointwise stabilizer of $T := S_L^\perp$ in $O^\#(L)$. In particular, if $O^\#(S)$ fixes no nontrivial vector in S, i.e. $S^{O^\#(S)} = \{0\}$, then the group G constructed in Corollary 2.21 satisfies that $L_G = S$ and G is saturated in $O^\#(L)$.

For computational reasons, we reformulate Lemma 2.19 as follows.

Proposition 2.23 ([Nik80, Corollary 1.5.2])**.** *Let S and T be two even $\mathbb{Z}$-lattices. Then the double cosets*

$$O(T)\backslash\{\gamma \text{ gluing of } S \text{ and } T\}/O(S)$$

are in bijection with the isomorphism classes of even primitive extensions of $S \oplus T$.

Proof. Let $(f, g) \in O(T) \times O(S)$, and let $D_S \geq H_S \xrightarrow{\gamma} H_T \leq D_T$ be a glue map. We define

$$f \cdot \gamma \cdot g := D_f \circ \gamma \circ D_g.$$

This determines well-defined left and right actions of $O(T)$ and $O(S)$ respectively on the set of glue maps between respective subgroups of D_S and D_T. From this, the proof of the proposition is a direct consequence of Lemma 2.19. ☐

Algorithm 1: Primitive extensions

Input: Two even $\mathbb{Z}$-lattices S and T.
Output: A complete set of representatives for the isomorphism classes of even primitive extensions $S \oplus T \leq L$.

Initialize the empty list $E = []$.
Let $\mathcal{H}_S$ be a complete set of representatives of $\overline{O(S)}$-orbits in $\{H_S \leq D_S\}$.
for $[H_S] \in \mathcal{H}_S$ **do**
    Let $\mathcal{H}_T$ be a complete of set of representatives of $\overline{O(T)}$-orbits in $\{H_T \leq D_T \ : \ H_T \simeq H_S(-1)\}$.
    for $[H_T] \in \mathcal{H}_T$ **do**
        Let γ be a glue map between H_S and H_T.
        $G_T \leftarrow \operatorname{im}(\operatorname{Stab}_{\overline{O(T)}}(H_T) \to O(H_T))$.
        $G_S^\gamma \leftarrow \gamma \operatorname{im}(\operatorname{Stab}_{\overline{O(S)}}(H_S) \to O(H_S))\gamma^{-1}$.
        for $G_T \cdot g \cdot G_S^\gamma \in G_T\backslash O(H_T)/G_S^\gamma$ **do**
            $\gamma_g \leftarrow g \circ \gamma$
            Let $S \oplus T \leq L_{\gamma_g}$ be the associated primitive extension.
            Append L_{γ_g} to E.
Return E.

Proposition 2.24. *Let S and T be two even $\mathbb{Z}$-lattices. Then Algorithm 1 returns a complete set of representatives for the isomorphism classes of even primitive extensions $S \oplus T \leq L$.*

Proof. Follows from Proposition 2.23. ☐

Computational comments. Let S, T, L be even $\mathbb{Z}$-lattices. Since it is possible to work with D_S explicitly, for instance, then we know that we can determine the overlattices of S explicitly by determining subgroups of D_S. Similarly, one can define glue maps between subgroups of D_S and D_T respectively. Thus, we can work with even $\mathbb{Z}$-lattices and primitive extensions of such in a concrete way. In particular, for the rest of this thesis we assume that the following are computationally feasible.

(1) Determining whether two torsion quadratic modules are isometric, and whether there exists a glue map between them.

(2) Constructing the primitive extension associated to a gluing of even $\mathbb{Z}$-lattices.

(3) Constructing the glue map associated to a given primitive extension $S \oplus T \leq L$.

Note that the computations in lines 2 and 4 in Algorithm 1 are hard in general, and they rely on some algorithms on p-groups generation [O'B90]. We use Miranda–Morrison theory (Remark 1.53) in order to obtain the actions induced by $O(S)$ and $O(T)$ respectively on the sets of submodules of D_S and D_T. Note however that depending on D_S and D_T, such computations can be expensive, and become serious bottlenecks in a program. For instance, if $D_S \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 12}$ as finite abelian groups then the set $\mathcal{H}_S$ cannot be computed in reasonable time (to the author's knowledge). For the computations of double cosets in line 9, one can find generators of $O(H_T)$ for a given submodule $H_T \leq D_T$ using an algorithm of [BV24].

For the rest of the thesis, we assume that Algorithm 1 is computationally accessible, and usable in the computational limits just mentioned. Note that it has actually been implemented in greater generality for integral $\mathbb{Z}$-lattices in Oscar [OSC25, QuadFormAndIsom]. Moreover, by requiring that the primitive extensions in output of Algorithm 1 lies in a given genus, one can improve the computations of $\mathcal{H}_S$ and $\mathcal{H}_T$, and also filter the outputs accordingly.

2.2.2. Embeddings into even unimodular $\mathbb{Z}$-lattices

Before describing an algorithm to classify orbits of primitive sublattices in even $\mathbb{Z}$-lattices of a given genus, let us review the easier case of embeddings into even unimodular $\mathbb{Z}$-lattices.

Lemma 2.25. *Let L be an even unimodular $\mathbb{Z}$-lattice and let $S \leq L$ be a primitive sublattice. Then the genus of $T := S^\perp$ is uniquely determined by the ones of S and L.*

Proof. Let $H_S \leq D_S$ and $H_T \leq D_T$ be the glue domains of $S \oplus T \leq L$. According to Remark 2.18, since L is unimodular, we have that

$$\frac{\det(S)}{\#H_S} \cdot \frac{\det(T)}{\#H_T} = 1.$$

Since both factors on the lefthand side are integers greater than, or equal to, 1, we obtain that $H_S = D_S$ and $H_T = D_T$. Thus, $D_T \simeq D_S(-1)$ is determined by the genus of S. Moreover, the signatures of T are determined by the ones of L and S. Hence, since the genus of T is characterized by its signatures and D_T (Proposition 1.28), we can conclude. □

Remark 2.26. Note that in particular, with the notation of Lemma 2.25, we have that $l(D_S) = l(D_T) \leq \mathrm{rank}_{\mathbb{Z}}(T)$. Therefore, if S embeds primitively into the even unimodular $\mathbb{Z}$-lattice L, we have that

$$\mathrm{rank}_{\mathbb{Z}}(S) + l(D_S) \leq \mathrm{rank}_{\mathbb{Z}}(L).$$

We therefore obtain a converse to what was observed in Example 2.20. We can now prove the following criterion for primitive embeddings into even unimodular $\mathbb{Z}$-lattices.

Proposition 2.27 ([Nik80, Theorem 1.12.2, Corollary 1.12.3]). *Let $l_+, l_- \geq 0$ be two nonnegative integers such that $l_+ + l_- \geq 1$ and $l_+ \equiv l_- \mod 8$. Let $0 \leq s_+ \leq l_+$ and $0 \leq s_- \leq l_-$ be two nonnegative integers such that $l_+ + l_- > s_+ + s_- \geq 1$ and let S be an even $\mathbb{Z}$-lattice of signatures (s_+, s_-). Then S embeds primitively into an even unimodular $\mathbb{Z}$-lattice of signatures (l_+, l_-) if* *and only if there exists an even $\mathbb{Z}$-lattice of signatures $(l_+ - s_+, l_- - s_-)$ and discriminant form $D_S(-1)$. In particular such an embedding exists whenever*

$$\mathrm{rank}_{\mathbb{Z}}(S) + l(D_S) \leq l_+ + l_- - 1.$$

Proof. The first part follows from Proposition 2.17: such an embedding exists if and only if there exists an even $\mathbb{Z}$-lattice T of signatures $(l_+ - s_+, l_- - s_-)$ such that $S \oplus T$ has an even unimodular primitive extension. By Lemma 2.25 we see that the very last condition is equivalent to $D_T \simeq D_S(-1)$.

For the last assertion, suppose that $\mathrm{rank}_{\mathbb{Z}}(S) + l(D_S) \leq l_+ + l_- - 1$, let us denote by (s_+, s_-) the signatures of S and let $t_\pm = l_\pm - s_\pm$. Let p be a prime number dividing $2\det(S)$. We have that the p-adic symbol of the non-unimodular part T_p' of any Jordan decomposition of $S_p(-1)$ satisfies conditions (31)–(35) from Theorem 1.48. Note that T_p' has rank at most $l(D_S)$. By assumption, we have that

$$t_+ + t_- \geq l(D_S) + 1:$$

we define $T_p := T_p' \oplus U_p$ where U_p is a unimodular $\mathbb{Z}_p$-lattice of rank $t_+ + t_- - \mathrm{rank}_{\mathbb{Z}_p}(T_p') \geq 1$. Note that since S is even and $t_+ + t_- \equiv s_+ + s_- \mod 2$, we can make sure that U_2 as before, for $p = 2$, is even too. Moreover, we choose the U_p's, for all prime numbers $p \mid 2\det(S)$, in such a way that the set of p-adic symbols $\{g_p(T_p)\}_{p \mid 2\det(S)}$ and the pair (t_+, t_-) satisfy condition (29) from Theorem 1.48 (which is always possible by changing U_p to another even unimodular $\mathbb{Z}_p$-lattice of the same rank but with different unit determinant, see Corollary 1.22). Finally, since condition (30) from Theorem 1.48 does not depend on the choice of the U_p's, and since

$$t_+ - t_- \equiv s_- - s_+ \mod 8$$

we have that $\{g_p(T_p)\}_{p \mid 2\det(S)}$ and the pair (t_+, t_-) satisfy condition (30). Hence, $\{g_p(T_p)\}_{p \mid 2\det(S)}$ and the pair (t_+, t_-) define a nonempty genus g_T of even $\mathbb{Z}$-lattices, and by construction any $T \in g_T$ satisfies that $D_T \simeq D_S(-1)$. Thus there exists an even $\mathbb{Z}$-lattice T of signatures (t_+, t_-) and discriminant group $D_S(-1)$ (see also [Nik80, Proposition 1.10.1, Corollary 1.10.2]). □

Remark 2.28. Similarly to what was done in the second part of the proof of Proposition 2.27, we can show that if there exists an even $\mathbb{Z}$-lattice T with signatures $(l_+ - s_+, l_- - s_-)$ and discriminant form $D_S(-1)$, then for any pair of nonnegative integers (t_+, t_-) such that $t_+ + t_- \geq l_+ + l_- - (s_+ - s_-)$ and such that

$$t_+ - t_- \equiv s_- - s_+ \mod 8,$$

there exists an even $\mathbb{Z}$-lattice of signatures (t_+, t_-) and discriminant form $D_S(-1)$. Thus, in this context, S embeds primitively into an even unimodular $\mathbb{Z}$-lattice of signatures (l_+, l_-) if and only if S embeds primitively into an even unimodular $\mathbb{Z}$-lattice of signatures $(s_+ + t_+, s_- + t_-)$ for all pairs of nonnegative integers (t_+, t_-) as before.

Let us conclude with the following theorem of Nikulin, and a consequence of it.

Theorem 2.29 ([Nik80, Theorem 1.14.2])**.** *Let T be an even indefinite $\mathbb{Z}$-lattice such that $\mathrm{rank}_{\mathbb{Z}}(T) \geq l(D_T) + 2$. Then T is unique in its genus, up to isometry, and the morphism $O(T) \to O(D_T)$ is surjective.*

Corollary 2.30. *Let L be an even unimodular $\mathbb{Z}$-lattice of signatures (l_+, l_-), and let $S \leq L$ be a primitive sublattice of with signatures $0 \leq s_+ < l_+$ and $0 \leq s_- < l_-$, and such that*

$$\mathrm{rank}_{\mathbb{Z}}(S) + l(D_S) \leq \mathrm{rank}_{\mathbb{Z}}(L) - 2.$$

Then there is a unique $O(L)$-orbit of primitive sublattices of L isometric to S.

Proof. From Lemma 2.25, we know that for any primitive sublattice $S' \leq L$ isometric to S, the genus of $(S')^{\perp}_{L}$ is uniquely determined by L and S. By assumption, we have that this genus is determined by $D_S(-1)$ and $t_{\pm} = l_{\pm} - s_{\pm} > 0$. In particular, any $\mathbb{Z}$-lattice T in this genus is even indefinite, and satisfies

$$\operatorname{rank}_{\mathbb{Z}}(T) = t_+ + t_- = \operatorname{rank}_{\mathbb{Z}}(L) - \operatorname{rank}_{\mathbb{Z}}(S) \geq l(D_S) + 2 = l(D_T) + 2.$$

In particular, T is unique in its genus, and $O(T) \to O(D_T)$ is surjective. Then the proof follows from Proposition 2.23 and Example 2.20. □

Remark 2.31. Another well-known consequence of Theorem 2.29 is the following. Let S and T be two even $\mathbb{Z}$-lattices: in particular, it is clear that $\operatorname{rank}_{\mathbb{Z}}(S) \geq l(D_S)$, and thus $U \oplus S$ satisfies the conditions of Theorem 2.29, where we recall that U is the hyperbolic plane lattice. Hence it is unique in its genus. Moreover, since U is unimodular, the genus of $U \oplus S$ uniquely determines the one of S. It then follows that S and T are in the same genus if and only if $U \oplus S \simeq U \oplus T$.

2.2.3. Embeddings into arbitrary even $\mathbb{Z}$-lattices

We are now ready to describe the following algorithm. It originates from the proof of [Nik80, Proposition 1.15.1], and can be generalized to integral $\mathbb{Z}$-lattices. We omit the technical details of the proof as it follows step-by-step the proof of Nikulin's proposition. In what follows, we define a primitive sublattice to consist of a pair of even $\mathbb{Z}$-lattices (L, S) such that $S \leq L$ is primitive. Two such primitive sublattices (L_1, S_1) and (L_2, S_2) will be called isomorphic if there exists an isometry $f\colon L_1 \to L_2$ such that $f(S_1) = S_2$,

Proposition 2.32. *Let g be a nonempty genus of even $\mathbb{Z}$-lattices and let S be an even $\mathbb{Z}$-lattice. Then Algorithm 2 returns a complete set of representatives for the isomorphism classes of primitive sublattices (L, S') where $S' \simeq S$ and $L \in g$.*

Proof. Let us first remark the following: if (L_1, S_1) and (L_2, S_2) are isomorphic primitive sublattices, then $(S_1)^{\perp}_{L_1}$ and $(S_2)^{\perp}_{L_2}$ are isometric as $\mathbb{Z}$-lattices.

(1) Assume first that D_g is trivial, i.e. the genus g is represented by even unimodular $\mathbb{Z}$-lattices. Then the proof follows by Proposition 2.27 and the comment above.

(2) If D_g is nontrivial, then it follows from the proof of [Nik80, Proposition 1.15.1]. Let us just note the following. Given an even primitive extension $S \oplus T \leq V$ as in line 17, we have that O_V consists of isometries of $D_S \oplus D_T$ preserving $V/(S \oplus T)$ and which are induced by an isometry of $O(S \oplus T)$. In particular, G_V is the image of the representation on D_V of the subgroup of isometries of $O(V)$ preserving the primitive extension $S \oplus T \leq V$. In particular since $O(T) \to O(D_T)$ is surjective and all the M_γ's are unique in their respective genus (Theorem 2.29), we have that the set of double cosets in line 28 parametrizes some isomorphism classes of primitive sublattices $S' \leq L$ such that $S' \simeq S$, $L \in g$ and $(S')^{\perp}_{L} \simeq K$, where K is defined in line 28. □

Computational comments. From what we have said earlier, all the steps in Algorithm 2 are computationally accessible, and therefore such an algorithm can also be implemented on any computer algebra system supporting the computational infrastructure described up to now.

Algorithm 2: Primitive embeddings

Input: A nonempty genus g of even $\mathbb{Z}$-lattices and an even $\mathbb{Z}$-lattice S.

Output: A complete set of representatives for the isomorphism classes of primitive sublattices (L, S') where $S' \simeq S$ and $L \in g$.

Initialize the empty list $E = []$.
Let (l_+, l_-) be the signatures associated to g.
Let (s_+, s_-) be the signatures of S.
Let D_g be a torsion quadratic module associated to g.
if *D_g is trivial* **then**
    if *there is no $\mathbb{Z}$-lattice of signatures $(l_+ - s_+, l_- - s_-)$ and discriminant form $D_S(-1)$* **then**
        Return E.
    Let g_K be the genus of even $\mathbb{Z}$-lattices determined by $(l_+ - s_+, l_- - s_-)$ and $D_S(-1)$ (Lemma 2.25).
    Let $\mathcal{K}$ be a complete set of representatives for the isometry classes in g_K (Section 1.5).
    for $K \in \mathcal{K}$ **do**
        Let $\mathcal{G}$ be a complete set of representatives for the isomorphism classes of even unimodular primitive extensions $S \oplus K \leq L$ (Algorithm 1).
        for $L \in \mathcal{G}$ **do**
            Append the pair (L, S) to E.
    Return E.
Let $L \in g$.
Let $T := U \oplus L(-1)$.
Let $\mathcal{V}$ be a complete set of representatives for the isomorphism classes of even primitive extensions $S \oplus T \leq V$ (Algorithm 1).
for $V \in \mathcal{V}$ **do**
    Let $D_S \geq H_S \xrightarrow{\gamma} H_T \leq D_T$ be the glue map associated to $S \oplus T \leq V$.
    $O_S \leftarrow \mathrm{Stab}_{\overline{O(S)}}(H_S)$.
    $O_T \leftarrow \mathrm{Stab}_{\overline{O(T)}}(H_T)$.
    $O_V \leftarrow \{(a,b) \in O_S \times O_T \ : \ b_{|H_T} \circ \gamma = \gamma \circ a_{|H_S}\}$.
    $G_V \leftarrow \mathrm{im}(O_V \to O(D_V))$.
    if *there is no $\mathbb{Z}$-lattice of signatures $(l_+ - s_+, l_- - s_-)$ and discriminant form $D_V(-1)$* **then**
        Continue the for loop with the next $V \in \mathcal{V}$.
    Let g_K be the genus of even $\mathbb{Z}$-lattices determined by $(l_+ - s_+, l_- - s_-)$ and $D_V(-1)$ (Lemma 2.25).
    Let $\mathcal{K}$ be a complete set of representatives for the isometry classes in g_K (Section 1.5).
    for $K \in \mathcal{K}$ **do**
        $\mathcal{G} \leftarrow \overline{O(K)} \backslash \{\gamma \colon D_V \to D_K \text{ glue map}\} / G_V$.
        for $\overline{O(K)} \cdot \gamma \cdot G_V \in \mathcal{G}$ **do**
            Let $V \oplus K \leq M_\gamma$ be the associated even unimodular primitive extension.
            $L' \leftarrow T^{\perp}_{M_\gamma}$.
            Append the pair (L', S) to E.
Return E.

2.3. Equivariant primitive extensions

Let us conclude the preliminaries on $\mathbb{Z}$-lattices with the analog of primitive extensions in the category of lattices with isometry.

Let (S, s) and (T, t) be two even lattices with isometry where $s \in O(S)$ and $t \in O(T)$. Let $D_S \geq H_S \xrightarrow{\gamma} H_T \leq D_T$ be a glue map. Such a glue map γ is called (s,t)-*equivariant* if H_S and H_T are respectively preserved by D_s and D_t, and if it satisfies the **equivariant gluing condition**

$$\gamma \circ (D_s)_{|H_S} = (D_t)_{|H_T} \circ \gamma. \tag{EGC}$$

Proposition 2.33. *The map γ is (s,t)-equivariant if and only if $s \oplus t$ extends along the primitive extension $S \oplus T \leq L_\gamma$ to an isometry f_γ of L_γ.*

Proof. It is a direct consequence of Lemma 2.19. □

We call (L_γ, f_γ) an *equivariant primitive extension* of (S, s) and (T, t).

Definition 2.34. Let $(S_1, s_1) \oplus (T_1, t_1) \subseteq (L_1, f_1)$ and $(S_2, s_2) \oplus (T_2, t_2) \subseteq (L_2, f_2)$ be two equivariant primitive extensions. They are said to be *isomorphic* if there exists an isomorphism $\psi\colon (L_1, f_1) \to (L_2, f_2)$ which restricts to isomorphisms $\psi_S\colon (S_1, s_1) \to (S_2, s_2)$ and $\psi_T\colon (T_1, t_1) \to (T_2, t_2)$.

The following is an analog of Proposition 2.23 in the category of lattices with isometry. The proof is omitted since it is a direct translation of the proof of Lemma 2.19 and Proposition 2.23 to the setting of lattices with isometry (see also [BH23, Proposition 2.2])

Proposition 2.35. *Let (S, s) and (T, t) be even lattices with isometry. Then the double cosets*

$$O(T,t)\backslash\{\gamma\ (s,t)\text{-equivariant gluing of } S \text{ and } T\}/O(S,s)$$

are in bijection with the isomorphism classes of equivariant primitive extensions $(S, s) \oplus (T, t) \leq (L_\gamma, f)$ such that $f_{|S} = s$ and $f_{|T} = t$.

Remark 2.36. In Proposition 2.35, if only one lattice has an isometry attached to it, it is effectively possible to decide whether the second lattice can be endowed with an isometry giving rise to an equivariant glue map. However, in practice, such an isometry is often only computable in the definite, or rank at most 2, cases.

Example 2.37. A simple example of an equivariant primitive extension is the following. Let L be an even $\mathbb{Z}$-lattice, and let $f \in O(L)$. Suppose that there exists a factorization $m_f(X) = p_1(X)p_2(X)$ of the minimal polynomial of f, with $p_1(X), p_2(X) \in \mathbb{Q}[X]$ coprime. The primitive sublattices $L^{p_1(f)}$ and $L^{p_2(f)}$ are in orthogonal direct sum in L. If we denote by f_1 and f_2 the respective restrictions of f to $L^{p_1(f)}$ and $L^{p_2(f)}$, then $(L^{p_1(f)}, f_1) \oplus (L^{p_2(f)}, f_2) \leq (L, f)$ is an equivariant primitive extension.

If we let γ be the associated (f_1, f_2)-equivariant glue map from Example 2.37, then from p_1 and p_2 one gets information on the primes dividing the order of the glue domains of γ. In fact, let us demonstrate this in what follows.

Definition 2.38. Let $p(X), q(X) \in \mathbb{Z}[X]$ be two polynomials which are coprime in $\mathbb{Q}[X]$. Then we define the *reduced resultant* of p and q, denoted $\mathrm{rres}(p, q)$, to be the positive generator d of the $\mathbb{Z}$-ideal

$$(p(X)\mathbb{Z}[X] + q(X)\mathbb{Z}[X]) \cap \mathbb{Z}.$$

From its definition, one sees that the reduced resultant of two polynomials $p(X), q(X) \in \mathbb{Z}[X]$ which are coprime in $\mathbb{Q}[X]$ divides their *resultant*

$$\operatorname{res}(p,q) := \prod_{\substack{(u,v)\in\mathbb{C}^2 \\ p(u)=q(v)=0}} (u-v) \in \mathbb{Z}.$$

Remark 2.39. If $p(X), q(X) \in \mathbb{Z}[X]$ have a common factor of positive degree, then we can define the resultant and reduced resultant similarly, and we obtain that $\operatorname{rres}(p,q) = \operatorname{res}(p,q) = 0$.

Proposition 2.40. *Let $(S,s) \oplus (T,t) \leq (L,f)$ be an equivariant primitive extension, and let $d := \operatorname{rres}(m_s(X), m_t(X))$ be the reduced resultant of the minimal polynomials of s and t respectively. Then*

$$dL \leq S \oplus T.$$

Proof. If m_s and m_t have a common factor of positive degree, then $d = 0$ and the result obviously follows. Let us assume now that m_s and m_t are coprime in $\mathbb{Q}[X]$. In particular, we have that $m_f = m_s m_t$ and we can see S and T are the respective kernel sublattices $\ker m_s(f)$ and $\ker m_t(f)$. Therefore, since S and T are both primitive in L, we observe that

$$\begin{aligned} m_s(f)(L) &\leq T \\ \text{and } m_t(f)(L) &\leq S. \end{aligned}$$

By definition of d, there exist $u(X), v(X) \in \mathbb{Z}[X]$ such that

$$d = u(X)m_s(X) + v(X)m_t(X).$$

Since d is constant, seen as an element of $\mathbb{Z}[X]$, we have that

$$d \cdot \mathrm{id}_L = u(f)m_s(f) + v(f)m_t(f)$$

meaning that

$$dL = (u(f)m_s(f) + v(f)m_t(f))(L) \leq u(f)m_s(f)(L) + v(f)m_t(f)(L) \leq S \oplus T.$$

Hence the result follows. □

Proposition 2.41 ([Fil02, Lemma 2])**.** *Let $1 < n < m$ be positive integers. Then*

$$\operatorname{rres}(\Phi_n, \Phi_m) = \operatorname{rres}(\Phi_m, \Phi_n) = \begin{cases} p & \text{if } \frac{m}{n} \text{ is a power of the prime number } p \\ 1 & \text{else} \end{cases}.$$

We conclude with the following.

Corollary 2.42. *Let p be a prime number and let n be a positive integer. Let (L,f) be a lattice with isometry such that f has order p and L is n-elementary. Let us denote by $F := L^f$ and $C := L_f$ the associated invariant and coinvariant sublattices. Then F and C are pn-elementary.*

Proof. According to Propositions 2.40 and 2.41, we have that $pL \leq F \oplus C$. In particular, since $nL^\vee \leq L$, we obtain

$$pnL^\vee \leq pL \leq F \oplus C.$$

But now, using the fact that $F \leq L$ is a primitive sublattice, we have that the morphism $\pi\colon L^\vee \to F^\vee$ is surjective. Therefore, by applying π to the inclusion $pnL^\vee \leq F \oplus C$, we obtain that

$$pnF^\vee \leq F$$

and F is pn-elementary (see [BH23, Proposition 4.10]). Similar arguments apply to C. $\square$

Computational comments. Similarly to what was done in Algorithm 1, one can turn Proposition 2.35 into an algorithm which returns, given two even lattices with isometry (S, s) and (T, t), a complete set of representatives for the isomorphism classes of even equivariant primitive extensions

$$(S, s) \oplus (T, t) \leq (L, f)$$

(see [BH23, Algorithm 2] for an example). The main difference between such an algorithm and Algorithm 1 is the computation of the double cosets as described in Proposition 2.35 and Proposition 2.23 respectively. Similarly to what was remarked in Remark 1.53, determining a set generators of $\overline{O(S, s)}$ and $\overline{O(T, t)}$, in terms of matrices, is computationally accessible using an implementation of the hermitian analog of Miranda–Morrison theory as described in [BH23, §6]. Note that such a procedure has been implemented in [OSC25, QuadFormAndIsom] by the author of this thesis (based on original scripts of Brandhorst and Hofmann). The infrastructure required for the implementation of such a program will be explained in Section 4 where we introduce *hermitian lattices*. From now on, we assume that one can effectively compute a complete of representatives for the isomorphism classes of such equivariant primitive extensions.

3. Prime power cyclotomic fields

The content of this section is collected from the author's work [Mul25, Appendix A]. In what follows, for $m \geq 1$, we denote by $\zeta_m \in \mathbb{C}$ a primitive mth root of unity.
References: [O'M73, Was97, Kir16].

3.1. General facts

Let $m \geq 3$ be an integer, and let $E := \mathbb{Q}(\zeta_m)$ be the mth cyclotomic field. It contains a totally real subfield K of index 2, which is generated by $\zeta_m + \zeta_m^{-1}$ over $\mathbb{Q}$, and the extension E/K is CM. The rings $\mathcal{O}_E := \mathbb{Z}[\zeta_m]$ and $\mathcal{O}_K := \mathbb{Z}[\zeta_m + \zeta_m^{-1}]$ are maximal orders in E and K respectively. The extension E/K is Galois, and $\mathrm{Gal}(E/K)$ is generated by the complex conjugation mapping $\iota : \ \zeta_m \mapsto \iota(\zeta_m) := \zeta_m^{-1}$.

Lemma 3.1 ([Was97, Propositions 2.3 and 2.15])**.**

(1) *A prime $\mathbb{Z}$-ideal $p\mathbb{Z}$ ramifies in $E/\mathbb{Q}$ if and only if p divides m;*

(2) *A prime $\mathcal{O}_K$-ideal $\mathfrak{p}$ ramifies in E/K if and only if m is a power (or twice a power) of a prime number p and $\mathfrak{p}$ divides $p\mathcal{O}_K$.*

In this thesis, we mostly work with cyclotomic fields for $m = p^k$ a prime power. In the following we prove some standard results we use in Section 8.1.

Proposition 3.2. *Suppose $m = p^k \geq 3$ for some prime number p, and some positive integer k. Let us denote $\zeta := \zeta_{p^k}$ and $\pi := 1 - \zeta$. The following hold:*

(1) *$\mathfrak{P} := \pi\mathcal{O}_E$ is a maximal $\mathcal{O}_E$-ideal and $p\mathcal{O}_E = \mathfrak{P}^{\varphi(m)}$;*

(2) *$\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_K$ is generated by $\pi\iota(\pi)$. It is the only prime $\mathcal{O}_K$-ideal which ramifies in E/K;*

(3) *$E_\mathfrak{p}/K_\mathfrak{p}/\mathbb{Q}_p$ is a tower of totally ramified extensions. In particular, $\mathcal{O}_E/\mathfrak{P} \simeq \mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_p$;*

(4) *the different ideals $\mathfrak{D}_{E/\mathbb{Q}}$ and $\mathfrak{D}_{E/K}$ are principal generated respectively by $\pi^{p^{k-1}(pk-k-1)}$ and π^e where $e := \gcd(2, p)$.*

Proof.

(1) This is a generalization of [Was97, Lemma 1.4] from prime order to prime power order.

(2) We have that $\mathfrak{p} = N_K^E(\mathfrak{P})$ and in particular, $\mathfrak{p}$ is generated by $N_K^E(\pi) = \pi\iota(\pi)$. The ramification statement follows from Lemma 3.1.

(3) We have that $[E : \mathbb{Q}] = \varphi(m)$, so $E_\mathfrak{p}/\mathbb{Q}_p$ is totally ramified according to (1). A fortiori, so is the tower $E_\mathfrak{p}/K_\mathfrak{p}/\mathbb{Q}_p$. In particular, we have that $\mathcal{O}_E/\mathfrak{P} \simeq \mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_p$.

(4) Since $p\mathbb{Z}$ is the unique prime $\mathbb{Z}$-ideal which ramifies in $E/\mathbb{Q}$, we know that $\mathfrak{P}$ is the unique prime $\mathcal{O}_E$-ideal which divides $\mathfrak{D}_{E/\mathbb{Q}}$. Now, the norm of $\mathfrak{D}_{E/\mathbb{Q}}$ is equal to the absolute discriminant of E: by [Was97, Proposition 2.1], it is equal to $p^{p^{k-1}(pk-k-1)}$. Since the norm of $\mathfrak{P}$ over $\mathbb{Q}$ is exactly p, we have that $\mathrm{val}_\mathfrak{P}(\mathfrak{D}_{E/\mathbb{Q}}) = p^{k-1}(pk - k - 1)$ as expected. The minimal polynomial of ζ over K is $\mu(t) := t^2 - (\zeta + \zeta^{-1})t + 1 \in K[t]$, so the relative different $\mathfrak{D}_{E/K}$ is generated by $\mu'(\zeta) = \zeta - \zeta^{-1} = \zeta\iota(\pi)(1 + \zeta^{-1})$. We conclude by remarking that ζ is a unit in $\mathcal{O}_E$, $\iota(\pi) \in \iota(\mathfrak{P}) = \mathfrak{P}$ and moreover, $(1 + \zeta^{-1}) \in \mathfrak{P}$ if and only if $2 \in \mathfrak{P}$. □

3.2. Local norms

Let $m = p^k \geq 3$ be a prime power, and let $\zeta := \zeta_m$. Since the prime $\mathcal{O}_K$-ideal $\mathfrak{p}$ above $p\mathbb{Z}$ ramifies in E/K, we have that $E_\mathfrak{p}/K_\mathfrak{p}$ is a totally ramified degree 2 extension of local fields over $\mathbb{Q}_p$. We define the local norm map

$$N_{K_\mathfrak{p}}^{E_\mathfrak{p}} \colon E_\mathfrak{p}^\times \to K_\mathfrak{p}^\times.$$

We call an element of $K \cap N_{K_\mathfrak{p}}^{E_\mathfrak{p}}(E_\mathfrak{p}^\times)$ a *local norm* at $\mathfrak{p}$. We aim to prove the following theorem.

Theorem 3.3. *Let $m = p^k \geq 3$ be a prime power. Then -1 is a local norm at $\mathfrak{p}$ if and only if $\varphi(m) \equiv 0 \mod 4$.*

Remark 3.4. For p odd, we have that $\varphi(p^k) \equiv 0 \mod 4$ if and only if $p \equiv 1 \mod 4$. For powers 2^k of 2 we have that $\varphi(2^k) \equiv 0 \mod 4$ if and only if $k \geq 3$.

In order to prove Theorem 3.3, we separate the odd case and the case of powers of 2. First, let us remark the following: E is generated over K by $\zeta - \zeta^{-1}$ whose square is

$$w := (\zeta - \zeta^{-1})^2 = \zeta^2 + \zeta^{-2} - 2 = (\zeta + \zeta^{-1})^2 - 4 \in \mathcal{O}_K^\times.$$

In particular, $E = K(\sqrt{w})$. In order to pursue, let us introduce the following.

Definition 3.5 ([O'M73, §63.B], [Kir16, Definition 3.1.6])**.** Let $\mathfrak{p}$ be a prime $\mathcal{O}_K$-ideal, and let $a, b \in K_\mathfrak{p}$ be nonzero. We define the *Hilbert symbol* by

$$(a,b)_\mathfrak{p} := \begin{cases} +1 & \text{if the equation } ax^2 + by^2 = z^2 \text{ admits a nonzero solution } (x,y,z) \in K_\mathfrak{p}^3 \\ -1 & \text{else} \end{cases}.$$

Let us recall some known properties of those Hilbert symbols (see [O'M73, §63.B] and [Kir16, Theorem 3.1.7] for further details).

Proposition 3.6. *Let $\mathfrak{p}$ be a prime $\mathcal{O}_K$-ideal, and let $a, b, c \in K_\mathfrak{p}^\times$ be nonzero.*

(1) *The Hilbert symbol is* symmetric, *i.e.* $(a,b)_\mathfrak{p} = (b,a)_\mathfrak{p}$*;*

(2) *If $a \in (K_\mathfrak{p}^\times)^2$ is a square, then $(a,b)_\mathfrak{p} = 1$;*

(3) *The Hilbert symbol is* multiplicative, *i.e.* $(ac,b)_\mathfrak{p} = (a,b)_\mathfrak{p}(c,b)_\mathfrak{p}$*;*

(4) *$(a, 1-a)_\mathfrak{p} = 1$;*

(5) *If $2 \notin \mathfrak{p}$, and $a, b \in \mathcal{O}_{K_\mathfrak{p}}$ with $a \in \mathcal{O}_{K_\mathfrak{p}}^\times$, then $(a,b)_\mathfrak{p} = \left(\frac{a}{\mathfrak{p}}\right)^{\mathrm{val}_\mathfrak{p}(b)}$.*

Proof.

(1) The first point is clear by definition.

(2) If $a = d^2$ is a square, then we have that $(1, 0, d)$ is a nontrivial solution of

$$ax^2 + by^2 = z^2$$

and so $(a,b)_\mathfrak{p} = 1$.

(3) For this third part, let us define $E_b := K_{\mathfrak{p}}(\sqrt{b})$. If b is a square in $K_{\mathfrak{p}}$, i.e. $E_b = K_{\mathfrak{p}}$, then the result follows from part (2). Now let us assume that b is not a square and let $N_{K_{\mathfrak{p}}}^{E_b} \colon u + v\sqrt{b} \mapsto u^2 - bv^2$ be the norm map of the Galois extension $E_b/K_{\mathfrak{p}}$. If $a \in K_{\mathfrak{p}}^{\times}$ is such that $(a,b)_{\mathfrak{p}} = 1$, then there exists $(x,y,z) \in K_{\mathfrak{p}}$ nonzero such that $ax^2 + by^2 = z^2$ with x nonzero since b is not a square. We thus observe that $a = N(z/x + \sqrt{b}y/x) \in N(E_b^{\times})$ is a norm in $K_{\mathfrak{p}}^{\times}$. Conversely, if there exists $u + \sqrt{b}v \in E_b^{\times}$ nonzero such that $a = N(u + \sqrt{b}v) = u^2 - bv^2$, then we have that $(1, v, u)$ is a nonzero solution to

$$ax^2 + by^2 = z^2$$

and thus $(a,b)_{\mathfrak{p}} = 1$. Hence, we obtain that $(a,b)_{\mathfrak{p}} = 1$ if and only if $a \in N(E_b^{\times})$. Since the same holds for $c \in K_{\mathfrak{p}}^{\times}$, and $[K_{\mathfrak{p}}^{\times} : N(E_{\mathfrak{p}}^{\times})] \leq 2$, the multiplicativity results follows.

(4) Observe that $(1,1,1)$ is a nonzero solution to

$$ax^2 + (1-a)y^2 = a(x^2 - y^2) + y^2 = z^2$$

so $(a, 1-a)_{\mathfrak{p}} = 1$.

(5) Follows from [Voi12, Proposition 5.5, Corollary 5.6]. □

Back to our problem, where we found that $E = K(\sqrt{w})$ for some w in $\mathcal{O}_K$, the proof of item (3) in Proposition 3.6 tells us that -1 is a local norm at $\mathfrak{p}$ if and only if $(-1, w)_{\mathfrak{p}} = 1$, seeing -1 and w as elements in $K_{\mathfrak{p}}$. So our problem reduces to a computation of Hilbert symbols in local fields. In the case where p is an odd prime number, we already have the following.

Lemma 3.7. *Suppose that p is odd. Then -1 is a local norm at $\mathfrak{p}$ if and only if $p \equiv 1 \mod 4$.*

Proof. If p is odd, then

$$w = (\zeta + \zeta^{-1})^2 - 4 = (\zeta + \zeta^{-1} - 2)(\zeta + \zeta^{-1} + 2) = -\pi\iota(\pi)(2 + \zeta + \zeta^{-1}) \in \mathfrak{p} \setminus \mathfrak{p}^2$$

and $\mathrm{val}_{\mathfrak{p}}(w) = 1$. Hence by Proposition 3.6 (5), we have that $(-1, w)_{\mathfrak{p}} = \left(\frac{-1}{\mathfrak{p}}\right)$ and therefore, -1 is a local norm at $\mathfrak{p}$ if and only if -1 is a square in $\mathcal{O}_{K_{\mathfrak{p}}}/\mathfrak{p} \simeq \mathbb{F}_p$. It is known that the latter holds if and only if $p \equiv 1 \mod 4$. □

Now suppose that $p = 2$: for convenience we will write $\beta := \pi\iota(\pi)$ for a generator of $\mathfrak{p}$. Since we chose $m = 2^k \geq 3$, we have that $k \geq 2$ and $\zeta^{2^{k-2}}$ is a square root of -1. But since E/K is CM, the number field K does not contain $\zeta^{2^{k-2}}$, and we can write $E = K(\zeta^{2^{k-2}})$. In particular, from now on we consider $w = -1$ and $E = K(\sqrt{-1})$. This time, to compute $(-1,-1)_{\mathfrak{p}}$, we follow an approach of Kirschmer using *quadratic defects* [Kir16, Algorithm 3.1.3].

Definition 3.8 ([O'M73, §63A],[Kir16, Definition 3.1.1])**.** Let $\mathbb{K}$ be a local field, and let $a \in \mathbb{K}$. The *quadratic defect* $\mathfrak{d}(a)$ of a is the fractional $\mathcal{O}_{\mathbb{K}}$-ideal defined by

$$\mathfrak{d}(a) := \bigcap_{b \in \mathbb{K}} (a - b^2)\mathcal{O}_{\mathbb{K}}.$$

We recall the setup in which we use these quadratic defects: $m = 2^k \geq 3$ is a power of two, ζ is a primitive mth root of unity, $K = \mathbb{Q}(\zeta + \zeta^{-1})$ is totally real with $[K : \mathbb{Q}] = 2^{k-2}$ and $\mathfrak{p}$ is the unique prime $\mathcal{O}_K$-ideal dividing $2\mathcal{O}_K = \mathfrak{p}^{[K:\mathbb{Q}]}$. We set $\mathbb{K} = K_{\mathfrak{p}}$, we consider $a = -1 \in K_{\mathfrak{p}}$, and we still denote by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}_{K_{\mathfrak{p}}}$.

Claim 3.9. $\mathrm{val}_{\mathfrak{p}}(\mathfrak{d}(-1)) = 2^{k-1} - 1$.

Proof of Claim 3.9. According to [Kir16, Lemma 3.1.2], we have that $\mathfrak{d}(-1)$ is the smallest among the following $\mathcal{O}_{K_\mathfrak{p}}$-ideals

$$(0) \subsetneq 4\mathcal{O}_{K_\mathfrak{p}} \subsetneq 4\mathfrak{p}^{-1} \subsetneq 4\mathfrak{p}^{-3} \subsetneq \ldots \subsetneq \mathfrak{p}$$

such that -1 is a square modulo $\mathfrak{d}(-1)$. We aim to prove that $\mathfrak{d}(-1) = 4\mathfrak{p}^{-1}$ — since $\mathrm{val}_\mathfrak{p}(4) = 2\mathrm{val}_\mathfrak{p}(2) = 2^{k-1}$, we can then conclude. First of all, we recall that $\beta = 2 - \zeta - \zeta^{-1}$ is a generator of $\mathfrak{p}$: in particular, $4\mathfrak{p}^{-1}$ is generated by $\alpha := \frac{4}{2-\zeta-\zeta^{-1}}$. Now, using the fact that $\zeta^{2^{k-1}} = -1$, one can show that

$$\alpha - 1 = -\frac{(1+\zeta)^2}{(1-\zeta)^2} = -(\zeta + \ldots + \zeta^{2^{k-1}-1})^2 = \left(-1 - \sum_{i=1}^{2^{k-2}-1} (\zeta^i + \zeta^{-i})\right)^2 \in (K_\mathfrak{p}^\times)^2.$$

Hence, $\alpha - 1$ is a square in $K_\mathfrak{p}$ meaning that $\mathfrak{d}(-1) \subseteq 4\mathfrak{p}^{-1}$. To conclude we remark the following:

(1) since $E_\mathfrak{p} = K_\mathfrak{p}(\sqrt{-1})$, we see that $X^2 + 1 \in K_\mathfrak{p}[X]$ is irreducible and so $\mathfrak{d}(-1) \neq (0)$; and

(2) if $\mathfrak{d}(-1) = 4\mathcal{O}_{K_\mathfrak{p}}$, by [Kir16, Theorem 3.1.7-5] we would have that $K_\mathfrak{p}(\sqrt{-1}) = E_\mathfrak{p}$ is unramified over $K_\mathfrak{p}$: this is absurd because $E_\mathfrak{p}/K_\mathfrak{p}$ is totally ramified. □

We are now ready to prove the equivalent of Lemma 3.7 in the case $p = 2$:

Lemma 3.10. *Suppose that $p = 2$. Then -1 is a local norm at $\mathfrak{p}$ if and only if $m = 2^k \geq 8$.*

Proof. Since $2^{k-1} - 1$ is odd for all $k \geq 2$, Claim 3.9 together with [Kir16, Lemma 3.1.2-4] tells us that there exists a unit $u \in -(\mathcal{O}_{K_\mathfrak{p}}^\times)^2$ such that $\mathfrak{d}(u) = \mathfrak{d}(-1) = 4\mathfrak{p}^{-1}$ and $\mathrm{val}_\mathfrak{p}(1-u) = 2^{k-1} - 1$. Since $-u \in (\mathcal{O}_{K_\mathfrak{p}}^\times)^2$ is a square, we get that $(-u, -1)_\mathfrak{p} = 1$ (Proposition 3.6 (2)) and by multiplicativity of Hilbert symbols (Proposition 3.6 (3)) we obtain

$$(u, -1)_\mathfrak{p} = (-u, -1)_\mathfrak{p}(-1, -1)_\mathfrak{p} = (-1, -1)_\mathfrak{p}. \tag{6}$$

Moreover, we know that $(u, 1-u)_\mathfrak{p} = 1$ according to Proposition 3.6 (4). Together with the fact that $2^{k-1} - 2$ is even for $k \geq 2$, we have that $\beta^{2^{k-1}-2} \in (\mathcal{O}_{K_\mathfrak{p}}^\times)^2$ is a square and we deduce that

$$\left(u, \frac{u-1}{\beta^{2^{k-1}-2}}\right)_\mathfrak{p} = (u, u-1)_\mathfrak{p} = (u, -1)_\mathfrak{p}\,(u, 1-u)_\mathfrak{p} = (u, -1)_\mathfrak{p} \tag{7}$$

with $\mathrm{val}_\mathfrak{p}\left(\frac{u-1}{\beta^{2^{k-1}-2}}\right) = \mathrm{val}_\mathfrak{p}(u-1) - 2^{k-1} + 2 = 1$. We denote $c := \frac{u-1}{\beta^{2^{k-1}-2}}$. By Equations (6) and (7), the scalar c satisfies $(u, c)_\mathfrak{p} = (u, -1)_\mathfrak{p} = (-1, -1)_\mathfrak{p}$. Moreover, one can compute

$$\begin{aligned}
\left(u(1 - \beta^{2^{k-1}-2}c),\, c\right)_\mathfrak{p} &= (u,\, c)_\mathfrak{p} \left(1 - \beta^{2^{k-1}-2}c,\, c\right)_\mathfrak{p} && (3.6\ (3)) \\
&= (u,\, c)_\mathfrak{p} \left(1 - \beta^{2^{k-1}-2}c,\, c\right)_\mathfrak{p} \left(1 - \beta^{2^{k-1}-2}c,\, \beta^{2^{k-1}-2}\right)_\mathfrak{p} && (3.6\ (2)) \\
&= (u, c)_\mathfrak{p} \left(1 - \beta^{2^{k-1}-2}c,\, \beta^{2^{k-1}-2}c\right)_\mathfrak{p} && (3.6\ (3)) \\
&= (u, c)_\mathfrak{p} && (3.6\ (4))
\end{aligned}$$

and $u(1 - \beta^{2^{k-1}-2}c) - 1 = u(2-u) - 1 = -(u-1)^2$ has $\mathfrak{p}$-adic valuation $2^k - 2$. Hence there are two cases.

(1) If $2^k = 4$, then $\mathrm{val}_\mathfrak{p}(u(2-u) - 1) = 2 = \mathrm{val}_\mathfrak{p}(4)$. So there exists a unit $\delta \in \mathcal{O}_{K_\mathfrak{p}}^\times$ such that $u(2-u) = 1 - 4\delta$. In particular $u(2-u)$ is a square modulo $4\mathcal{O}_{K_\mathfrak{p}}$, and either

$\mathfrak{d}(u(2-u)) = (0)$ or $\mathfrak{d}(u(2-u)) = 4\mathcal{O}_{K_\mathfrak{p}}$. The former is possible, by Hensel's Lemma, if and only if $1 - 4\delta \equiv (1+2\alpha)^2 \mod 4\mathfrak{p}$ for some $\alpha \in \mathcal{O}_{K_\mathfrak{p}}$, or equivalently $\delta \equiv \alpha^2 + \alpha \mod \mathfrak{p}$. However, the previous congruence system has a solution in $\mathcal{O}_{K_\mathfrak{p}}$ only if $X^2 + X + 1$ has a root in $\mathcal{O}_{K_\mathfrak{p}}/\mathfrak{p} \simeq \mathbb{F}_2$, which is not true (see also [Kir16, Algorithm 3.1.3] for a general argument). Hence $\mathfrak{d}(u(2-u)) = 4\mathcal{O}_{K_\mathfrak{p}}$, and since c has $\mathfrak{p}$-adic valuation 1, [O'M73, 63:11a] tells us that $(u(2-u), c)_\mathfrak{p} = -1$. We therefore conclude that $(-1,-1)_\mathfrak{p} = (u(2-u), c)_\mathfrak{p} = -1$ and that -1 is not a local norm at $\mathfrak{p}$.

(2) Otherwise, if $2^k \geq 8$, we have that $\mathrm{val}_\mathfrak{p}(u(2-u)-1) = 2^k - 2 > 2^{k-1} = \mathrm{val}_\mathfrak{p}(4)$. By [O'M73, 63:2], this implies that $\mathfrak{d}(u(2-u)) = (0)$ and $u(2-u)$ is therefore a square in $K_\mathfrak{p}$. In that case $(-1,-1)_\mathfrak{p} = (u(2-u), c)_\mathfrak{p} = 1$. □

Proof of Theorem 3.3. Follows from Lemmas 3.7 and 3.10. □

3.3. Congruence classes of units

Let again $m = p^k$ be a prime power, $\zeta := \zeta_m$ and $E := \mathbb{Q}(\zeta)$. For $\pi := 1 - \zeta$, we recall that $\mathfrak{P} = \pi\mathcal{O}$ is the unique prime $\mathcal{O}_E$-ideal dividing $p\mathcal{O}_E = \mathfrak{P}^{\varphi(p^k)}$. We study the congruence classes of some units in E modulo ideals of the form $1 + \mathfrak{P}^i$ for $i \geq 1$. Indeed, in order to prove Lemma 8.25, or more precisely Theorem 8.21, we need to count the number of classes of units of norm 1 in E modulo such ideals. Let $\iota \in \mathrm{Gal}(E/K)$ be the generator and let $\mathcal{F}(E) := \{e \in \mathcal{O}_E^\times : e\iota(e) = 1\}$ be the set of units of norm 1 in E. By [Was97, Theorem 4.12, Corollary 4.13], the set $\mathcal{F}(E)$ coincides with the set $\mu(E)$ of roots of unity in E. In particular $\mathcal{F}(E) = \{\pm\zeta^a : 0 \leq a \leq p^k - 1\}$ and it has order $\mathrm{lcm}(2, p^k)$. For all $1 \leq j \leq p^k$, we denote $\mathcal{F}_j(E) := \ker(\mathcal{F}(E) \to \mathcal{O}_E^\times/(1 + \mathfrak{P}^j))$.

Lemma 3.11. *Let $m = p^k \geq 3$ be a prime power. For all $1 \leq a \leq p^k - 1$, we have that*

$$\mathrm{val}_\mathfrak{P}(1 - \zeta^a) = p^{\mathrm{val}_p(a)}.$$

Proof. Let us write $a = p^l b$ where $\gcd(p, b) = 1$, and let $\xi := \zeta^a$. The algebraic number ξ is a primitive p^{k-l}th root of unity, and the extension $E/\mathbb{Q}(\xi)$ has degree

$$\frac{[E:\mathbb{Q}]}{[\mathbb{Q}(\xi):\mathbb{Q}]} = \frac{p^{k-1}(p-1)}{p^{k-l-1}(p-1)} = p^l.$$

Note that $(1-\xi)\mathcal{O}_{\mathbb{Q}(\xi)}$ is the unique prime $\mathcal{O}_{\mathbb{Q}(\xi)}$-ideal lying above $p\mathbb{Z}$ (Proposition 3.2). This implies that $(1-\xi)\mathcal{O}_{\mathbb{Q}(\xi)}$ totally ramifies in E and thus $(1-\xi)\mathcal{O}_E = \mathfrak{P}^{[E:\mathbb{Q}(\xi)]} = \mathfrak{P}^{p^l}$. We can conclude by remarking that $l = \mathrm{val}_p(a)$. □

Corollary 3.12. *Let $m = p^k \geq 3$ be a prime power. For all $1 \leq i \leq k$ and for all $p^{i-1} < j \leq p^i$, $\#\mathcal{F}_j(E) = p^{k-i}$ and $\#\mathcal{F}_1(E) = p^k$.*

Proof. First of all, we remark that $1 - \zeta^a \in \mathfrak{P}$ for all $1 \leq a \leq p^k - 1$ because $\mathfrak{P} = (1-\zeta)\mathcal{O}_E$. For a similar reason, we observe that $1 + \zeta \in \mathfrak{P}$ if and only if $p = 2$. This already tells us that $\#\mathcal{F}_1(E) = p^k$.

We conclude the rest of the proof by invoking Lemma 3.11 which tells us that for all $1 \leq i \leq k$, and for all $p^{i-1} < j \leq p^i$

$$\#\mathcal{F}_j(E) = \#\mathcal{F}_{p^i}(E) = \#\{1 \leq a \leq p^k - 1 : \mathrm{val}_p(a) \geq i\} + 1 = p^{k-i}.$$

□

4. Hermitian lattices

This section is adapted from [Mul25], and it is inspired by the work of Kirschmer [Kir16, Kir19]. See also [Jac62] and [O'M73] for further standard results about hermitian lattices, which we use throughout.

4.1. Definitions and notations

Let K be a number field and let E be a degree 2 extension of K. We have that $\mathrm{Gal}(E/K)$ has order 2 generated by an involution ι. For any place ν of K, we define K_ν and $E_\nu := E \otimes_K K_\nu$ to be the respective ν-adic completions. Let finally $\mathcal{O}_E$ and $\mathcal{O}_K$ be respective maximal orders of E and K.

A *hermitian space* (V, h) over E/K consists of a finite-dimensional E-vector space V equipped with a nondegenerate binary form

$$h\colon V \times V \to E$$

which is ι-sesquilinear i.e. h is E-linear on the first variable and $h(x, y) = \iota(h(y, x))$, for all $x, y \in V$. Note in particular that for all $x \in V$, one has that $h(x, x) = \iota(h(x, x)) \in K$. Given any place ν of K, we denote by $(V_\nu, h_\nu) := (V, h) \otimes_K K_\nu$ the corresponding ν-adic hermitian space. An *isometry* between two hermitian spaces (V_1, h_1) and (V_2, h_2) over E/K is an E-linear isomorphism $f\colon V_1 \to V_2$ such that $h_2(f(x), f(y)) = h_1(x, y)$ for all $x, y \in V_1$.

Remark 4.1. If $\mathfrak{q}$ is a finite place of K, there are three possibilities:

(1) $\mathfrak{q}$ is inert in E and $E_\mathfrak{q}/K_\mathfrak{q}$ is unramified of degree 2;

(2) $\mathfrak{q}$ splits in E and $E_\mathfrak{q} \simeq K_\mathfrak{q} \times K_\mathfrak{q}$;

(3) $\mathfrak{q}$ ramifies in E and $E_\mathfrak{q}/K_\mathfrak{q}$ is a ramified degree 2 extension of local fields.

Any finite place $\mathfrak{q}$ of K satisfying any of (1) or (2) is said to be *good*, otherwise we call it *bad*.

Definition 4.2. Let (V, h) be a hermitian space over E/K and let ν be a real place of K. We call the *signature* of (V, h) at ν the number $n(V_\nu, h_\nu)$ of negative entries in the diagonal of the Gram matrix of the real quadratic space (V_ν, h_ν). If (V, h) is understood from the context, we only write $n(\nu)$.

A hermitian space (V, h) over E/K is called *positive definite* (resp. *negative definite*) if K is totally real and if for all (real) infinite places $\nu \in \Omega_\infty(K)$, $n(\nu) = 0$ (resp. $n(\nu) = \dim_E(V)$). Otherwise, (V, h) is called *indefinite*.

Remark 4.3. For $m \geq 3$, if $E := \mathbb{Q}(\zeta_m)$ is the mth cyclotomic field then the subfield $K := \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ is totally real and it makes sense to talk about definite hermitian spaces over E/K.

A *hermitian $\mathcal{O}_E$-lattice* (L, h) consists of a finitely generated projective $\mathcal{O}_E$-module L which we equip with a nondegenerate ι-sesquilinear form

$$h\colon (L \otimes_{\mathcal{O}_E} E) \times (L \otimes_{\mathcal{O}_E} E) \to E.$$

Remark 4.4. Any hermitian $\mathcal{O}_E$-lattice can be defined as a finitely generated projective $\mathcal{O}_E$-module in a given hermitian space (V, h) over E/K.

An *isometry* between two hermitian $\mathcal{O}_E$-lattices (L_1, h_1) and (L_2, h_2) is an $\mathcal{O}_E$-module isomorphism $f\colon L_1 \to L_2$ such that $h_2(f(x), f(y)) = h_1(x, y)$ for all $x, y \in L_1$.

Definition 4.5. Let (L, h) be a hermitian $\mathcal{O}_E$-lattice. Let $x_1, \ldots, x_r$ be an E-basis of $L \otimes_{\mathcal{O}_E} E$ and suppose that there exist fractional $\mathcal{O}_E$-ideals $\mathfrak{A}_1, \ldots, \mathfrak{A}_r$ such that

$$L = \bigoplus_{i=1}^{r} \mathfrak{A}_i x_i.$$

Then the set $\{(\mathfrak{A}_i, x_i)\}_{1 \leq i \leq r}$ is called a *pseudobasis* of L.

Remark 4.6. Note that since $\mathcal{O}_E$ is in general not a factorial domain, we cannot properly define a Gram matrix for any given hermitian $\mathcal{O}_E$-lattice. Indeed, such lattices do not admit a basis as for $\mathbb{Z}$-lattices. However, they do always admit a pseudobasis [O'M73, Theorem 81:3].

Many of the notions defined for $\mathbb{Z}$-lattices admit an analog for hermitian lattices, with their equivalent in terms of fractional ideals of $\mathcal{O}_E$ and $\mathcal{O}_K$ [Kir16, §2]. For instance, we define:

(1) the *scale* of (L, h) to be the fractional $\mathcal{O}_E$-ideal $\mathfrak{s}(L, h) := h(L, L)$;

(2) the *norm* of (L, h) to be the fractional $\mathcal{O}_K$-ideal $\mathfrak{n}(L, h)$ generated by the elements of the form $h(x, x)$, for $x \in L$. It satisfies $\mathfrak{n}(L, h)\mathcal{O}_E \subseteq \mathfrak{s}(L, h)$;

(3) the *volume* of (L, h) to be the fractional $\mathcal{O}_E$-ideal

$$\mathfrak{v}(L, h) := \left(\prod_{1 \leq i \leq k} \mathfrak{A}_i \iota(\mathfrak{A}_i) \right) \det \left(h(x_i, x_j)_{1 \leq i,j \leq k} \right)$$

where $\{(\mathfrak{A}_i, x_i)\}_{1 \leq i \leq r}$ is a pseudobasis of L;

(4) the *dual* hermitian lattice of (L, h) to be the hermitian $\mathcal{O}_E$-lattice with underlying module

$$L^{\#} := \{x \in L \otimes_{\mathcal{O}_E} E : \ h(x, L) \subseteq \mathcal{O}_E\}$$

equipped with the form h;

(5) L to be $\mathfrak{A}$-*modular* if there exists a fractional $\mathcal{O}_E$-ideal $\mathfrak{A}$ such that $\mathfrak{A}L^{\#} = L$, and *unimodular* if $\mathfrak{A} = \mathcal{O}_E$;

(6) the free hermitian $\mathcal{O}_E$-lattice $\langle a_1, \ldots, a_k \rangle$, for $a_1, \ldots, a_k \in K^{\times}$, whose Gram matrix in a given basis is diagonal with entries $a_1, \ldots, a_k$;

(7) (L, h) to be *positive definite* (resp. *negative definite*, *indefinite*) if $(L, h) \otimes_{\mathcal{O}_E} E$ is positive definite (resp. negative definite, indefinite).

If there is no ambiguity, we often drop h from the notations.

The previous definitions of hermitian spaces, hermitian lattices and (1)–(6) above apply if we work over the completions $E_{\mathfrak{q}}/K_{\mathfrak{q}}$ for all finite places $\mathfrak{q}$ of K.

Remark 4.7. For any finite place $\mathfrak{q}$, the maximal order $\mathcal{O}_{E_{\mathfrak{q}}}$ is a principal ideal domain: hence, as in the case of $\mathbb{Z}$-lattices, we can define bases for hermitian $\mathcal{O}_{E_{\mathfrak{q}}}$-lattices and thus associate Gram matrices to them. In particular, it makes sense to define the *determinant* $\det(L) \in K_{\mathfrak{p}}^{\times}/N_{K_{\mathfrak{q}}}^{E_{\mathfrak{q}}}(\mathcal{O}_{E_{\mathfrak{q}}}^{\times})$ of a hermitian $\mathcal{O}_{E_{\mathfrak{q}}}$-lattice L.

Notation. For any element $e \in E_{\mathfrak{q}}^{\times}$ in a completion of E, we denote by $H(e)$ the hermitian $\mathcal{O}_{E_{\mathfrak{q}}}$-lattice with Gram matrix $\begin{pmatrix} 0 & e \\ \iota(e) & 0 \end{pmatrix}$.

Let $\mathfrak{q}$ be a finite place of K, and let us define $H_\mathfrak{q}$ to be the hermitian space over $E_\mathfrak{q}/K_\mathfrak{q}$ with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 4.8. A hermitian $\mathcal{O}_{E_\mathfrak{q}}$-lattice L is called *hyperbolic* if $L \otimes_{\mathcal{O}_{E_\mathfrak{q}}} E_\mathfrak{q}$ is isometric to $H_\mathfrak{q}^{\oplus r}$ for some $r \geq 1$.

Finally, we denote by $U(L, h)$ the *unitary group* of (L, h) which consists of $\mathcal{O}_E$-module isomorphisms $f \colon L \to L$ preserving the form h (and similar definition for the associated adic lattices).

Computational comments. Let E/K be a degree 2 extension of number fields, let $\mathfrak{q}$ be a finite place of K and let ν be a real place of K. Using standard algebra, one can construct and work computationally with the extension E/K. Most of the basics of algebraic number theory are computationally accessible, and in particular one can effectively describe $\mathfrak{q}$ and ν (up to certain precision). Computations with maximal orders and number rings, their modules and their quotients are also possible [Coh96, FH14, BFH17]. In particular, one can describe hermitian $\mathcal{O}_E$-lattices with a given pseudobasis, intersect them or add them. For the rest of this thesis, we therefore assume that the following are computationally feasible.

(1) Constructing hermitian spaces and lattices.

(2) Computing the signature of a hermitian space over E/K associated to a real place ν of K.

(3) Determining whether a hermitian space (resp. lattice) over E/K is definite.

(4) Comparing two hermitian $\mathcal{O}_E$-lattice, given in terms of pseudobases in a given hermitian space over E/K.

(5) Determining a basis for a hermitian $\mathcal{O}_{E_\mathfrak{p}}$-lattice.

(6) Computing the scale, the norm, the volume and the dual of a hermitian lattice.

(7) Determining whether a hermitian lattice is modular.

Moreover, as for $\mathbb{Z}$-lattices, determining generators for the unitary group of a hermitian $\mathcal{O}_E$-lattice is hard. The few cases where this has been implemented, to the author's knowledge, are for rank 1 hermitian lattices and definite ones [Kir16, Remark 2.4.4].

4.2. Genera of hermitian lattices

Let E/K be a degree 2 extension of number fields, and let $\mathcal{O}_E$ and $\mathcal{O}_K$ be maximal orders of E and K respectively. We denote by ι the generator of $\mathrm{Gal}(E/K)$. As for genera of $\mathbb{Z}$-lattices we can define a notion of genus for hermitian $\mathcal{O}_E$-lattices, and describe hermitian lattices over some completions of E at finite places of K.

For any hermitian $\mathcal{O}_E$-lattice (L, h) and for any place ν of K, we denote by $(L_\nu, h_\nu) := (L, h) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\nu}$ the associated hermitian $\mathcal{O}_{E_\nu}$-lattice.

Definition 4.9. Let (V, h) be a hermitian space over E/K, and let S of a projective $\mathcal{O}_E$-submodule of V of maximal rank. We define

(1) the *isometry class* of the hermitian $\mathcal{O}_E$-lattice (S, h) to be the set of projective $\mathcal{O}_E$-submodules $T \leq V$ of maximal rank such that $(S, h) \simeq (T, h)$ as hermitian $\mathcal{O}_E$-lattices;

(2) the *genus* of the hermitian $\mathcal{O}_E$-lattice (S, h) to be the set of projective $\mathcal{O}_E$-submodules $T \leq V$ of maximal rank such that $(S_\mathfrak{q}, h_\mathfrak{q}) \simeq (T_\mathfrak{q}, h_\mathfrak{q})$ for all finite places $\mathfrak{q}$ of K.

Two hermitian $\mathcal{O}_E$-lattices S and T are said to be *in the same genus* if there exists an $\mathcal{O}_E$-embedding $f\colon S \to T \otimes_{\mathcal{O}_E} E$ such that $f(S)$ lie in the genus of T.

As for $\mathbb{Z}$-lattices, two isometric hermitian $\mathcal{O}_E$-lattices are in the same genus, but the converse does not always hold. Moreover a genus of hermitian $\mathcal{O}_E$-lattices consists of finitely many isometry classes too [Kir16, Theorem 2.4.5].

Remark 4.10. Testing whether two hermitian $\mathcal{O}_E$-lattices are isometric is as hard as for the case of $\mathbb{Z}$-lattices. For definite lattices, one can apply again an adaptation of Plesken–Souvignier algorithm, and therefore also get an explicit isometry [Kir16, Remark 2.2.4]. Moreover, a rank 1 hermitian $\mathcal{O}_E$-lattice L is determined by a fractional ideal in E. Two rank 1 hermitian $\mathcal{O}_E$-lattices in the same genus can be compared by comparing their *Steinitz invariant* in the class group of E [BC23, Proposition 2.9].

In analogy to real quadratic spaces, for any real infinite place $\nu \in \Omega_\infty(K)$, the isometry class of the hermitian space $(L_\nu, h_\nu) := (L, h) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\nu}$ is uniquely determined by its rank and the number of negative entries $n(V_\nu, h_\nu)$ in a diagonal Gram matrix.

Definition 4.11. Let (L, h) be a hermitian $\mathcal{O}_E$-lattice, and let $(V, h) := (L, h) \otimes_{\mathcal{O}_E} E$. We call the collection $\{n(V_\nu, h_\nu)\}_{\nu \in \Omega_\infty(K)}$ the *signatures* of (L, h). Together with $\mathrm{rank}_{\mathcal{O}_E}(L)$ they uniquely determine the isometry class of L_ν at each real places ν of K.

Hence, the genus of a hermitian $\mathcal{O}_E$-lattice L is determined by its rank, its signatures and the isometry class of $L_\mathfrak{q}$ for all finite places $\mathfrak{q}$ of K. In what follows, we state without proof that for any finite place $\mathfrak{q}$ of K, hermitian $\mathcal{O}_{E_\mathfrak{q}}$-lattices also admits an orthogonal Jordan decomposition, whose direct summands are modular [Kir16, Theorem 3.3.3]. The proof actually follows similarly as the proof of Theorem 1.20. What will matter for us is the shape of the Jordan constituents when $\mathfrak{q}$ is ramified, and decide whenever two Jordan decompositions define isometric hermitian $\mathcal{O}_{E_\mathfrak{q}}$-lattices. In what follows, we fix $\mathfrak{q}$ a prime ideal (we identify prime ideals and the finite places they determine) of K, and we let $\mathfrak{Q}$ be the largest ι-invariant integral $\mathcal{O}_E$-ideal containing $\mathfrak{q}\mathcal{O}_E$, where ι generates $\mathrm{Gal}(E/K)$.

Definition 4.12 (Jordan decompositions)**.** Let $n \geq 1$ be a positive integer. For $1 \leq i \leq n$, let L_i be a $\mathfrak{Q}^{s_i}$-modular hermitian $\mathcal{O}_{E_\mathfrak{q}}$-lattices, where $s_i \in \mathbb{Z}$. We say $\bigoplus_{i=1}^n L_i$ is a *Jordan decomposition* if $s_1 < s_2 < \cdots < s_n$. Two such Jordan decomposition $\bigoplus_{i=1}^n L_i$ and $\bigoplus_{j=1}^m L'_j$ are said to be of the *same type* if $n = m$ and for all $1 \leq i \leq n$ the following hold

(1) L_i and L'_i have the same rank, as $\mathcal{O}_{E_\mathfrak{q}}$-modules;

(2) $\mathfrak{s}(L_i) = \mathfrak{s}(L'_i)$;

(3) $\mathfrak{n}(L_i)\mathcal{O}_{E_\mathfrak{q}} = \mathfrak{s}(L_i)$ if and only if $\mathfrak{n}(L'_i)\mathcal{O}_{E_\mathfrak{q}} = \mathfrak{s}(L'_i)$.

Given a Jordan decomposition $L = \bigoplus_{i=1}^n L_i$ we call the *L_i's Jordan constituents of L*.

Theorem 4.13 ([O'M73, Theorem 91.9], [Kir16, Theorem 3.3.3])**.** *Every hermitian $\mathcal{O}_{E_\mathfrak{q}}$-lattice L admits a Jordan decomposition. Moreover, two Jordan decompositions of L are of the same type.*

Similarly to the case of $\mathbb{Z}_p$-lattices, for p a prime number, we have that hermitian $\mathcal{O}_{E_\mathfrak{q}}$-lattices are made of elementary Jordan constituents of rank 1 or 2. The shape of these constituents actually depends on the ramification data of $\mathfrak{q}$ in E, and on whether $2 \in \mathfrak{q}$. For the purpose of this thesis, we only need to know those elementary Jordan constituents when $\mathfrak{q}$ ramifies in E: the hermitian $\mathcal{O}_E$-lattices L we work with will satisfy that $L_\mathfrak{q}$ is unimodular for all good finite places $\mathfrak{q}$ of K.

Theorem 4.14 ([Kir16, Theorem 3.3.6]). *Suppose that $\mathfrak{q}$ does not ramify in E or that $2 \notin \mathfrak{q}$. Let $L = \bigoplus_{i=1}^n L_i$ and $L' = \bigoplus_{j=1}^m L'_j$ be two Jordan decompositions of hermitian $\mathcal{O}_{E_\mathfrak{q}}$-lattices. Then $L \simeq L'$ if and only if $n = m$ and $L_i \simeq L'_i$ for all $1 \leq i \leq n$.*

Remark 4.15. If a finite place $\mathfrak{q}$ of K is good, then up to isomorphism there is a unique unimodular $\mathcal{O}_{E_\mathfrak{q}}$-lattice for any given rank.

Theorem 4.16 ([Jac62, Theorem 11.4]). *Suppose that $\mathfrak{q}$ ramifies in E and $2 \in \mathfrak{q}$ (we say $\mathfrak{q}$ is* dyadic*). Let $L = \bigoplus_{i=1}^n L_i$ and $L' = \bigoplus_{j=1}^m L'_j$ be two Jordan decompositions of hermitian $\mathcal{O}_{E_\mathfrak{q}}$-lattices. Then $L \simeq L'$ are isometric if and only if the following hold*

(1) *L and L' have the same type, as Jordan decompositions (in particular $n = m$);*

(2) $\det(L)/\det(L') \in N_{K_\mathfrak{q}}^{E_\mathfrak{q}}(\mathcal{O}_\mathfrak{q}^\times)$*;*

(3) $\mathfrak{n}(L_i) = \mathfrak{n}(L'_i)$ *for all $1 \leq i \leq n$;*

(4) *for all $1 \leq i < n$, we have that*

$$(\det(L_1 \oplus \cdots \oplus L_i)/\det(L'_1 \oplus \cdots \oplus L'_i))\mathcal{O}_{K_\mathfrak{q}} \subseteq 1 + \mathcal{O}_{K_\mathfrak{q}} \cap \mathfrak{n}(L_i)\mathfrak{n}(L_{i+1})\mathfrak{s}(L_i)^{-2}.$$

Let us now refocus the discussion to the case of hermitian lattices defined over cyclotomic fields. Let $m \geq 3$ be an integer. We let $E := \mathbb{Q}(\zeta_m)$ be the mth cyclotomic field with ζ_m a primitive mth root of unity. Similarly to Section 3.1, we let K be the fixed field for the $\mathbb{Q}$-linear involution $\iota\colon E \to E$, $\zeta_m \mapsto \zeta_m^{-1}$. We recall that $\mathcal{O}_E := \mathbb{Z}[\zeta_m]$ and $\mathcal{O}_K := \mathbb{Z}[\zeta_m + \zeta_m^{-1}]$ are respective maximal orders in E and K. Given a hermitian $\mathcal{O}_E$-lattice L, we want to know the shape of a Jordan decomposition of $L_\mathfrak{p}$ for a bad place $\mathfrak{p}$ of K: we have already seen when two such Jordan decompositions define isometric hermitian lattices. It follows from Lemma 3.1 that in the cyclotomic case, the real field K has either zero or one bad place. The latter happens only if m is (twice) a prime power p^k, in which case K has exactly one bad finite place $\mathfrak{p}$ which corresponds to the unique prime $\mathcal{O}_K$-ideal lying above $p\mathbb{Z}$. The associated ramified prime $\mathcal{O}_E$-ideal is denoted by $\mathfrak{P}$.

Proposition 4.17. *Let $m = p^k \geq 3$ be a prime power, let $E := \mathbb{Q}(\zeta_m)$ be the mth cyclotomic field and let $\mathfrak{P}$ the unique (ramified) prime $\mathcal{O}_E$-ideal lying above $p\mathbb{Z}$. Let us denote $\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_K$ where $K := \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ is the maximal real subfield of E. Let L be a $\mathfrak{P}^i$-modular hermitian $\mathcal{O}_{E_\mathfrak{p}}$-lattice of rank r, for some $i \in \mathbb{Z}$. Let us denote $\pi := 1 - \zeta_m$ and $\beta := \pi\iota(\pi)$ where $\iota \in \mathrm{Gal}(E_\mathfrak{p}/K_\mathfrak{p})$ is a generator.*

(1) *if p is odd, then:*

   (a) *either i is even and $L \simeq \langle \beta^{i/2}, \dots, u\beta^{i/2} \rangle$ where $uN_{K_\mathfrak{q}}^{E_\mathfrak{q}}(\mathcal{O}_\mathfrak{q}^\times) = \det(L)$;*

   (b) *or i is odd, r is even and $L \simeq H(\pi^i)^{\oplus r/2}$;*

(2) *if $p = 2$, then:*

   (a) *either r is odd, i is even, and*

$$L \simeq \langle a\beta^{i/2} \rangle \oplus H(\pi^i)^{\oplus (r-1)/2}$$

   with $aN_{K_\mathfrak{p}}^{E_\mathfrak{p}}(E_\mathfrak{p}^\times) = (-1)^{(r-1)/2}\det(L)$ and $\mathfrak{n}(L)\mathcal{O}_{E_\mathfrak{p}} = \mathfrak{s}(L)$;

(b) *or r is even, $\mathfrak{n}(L) = \mathfrak{p}^k$ with*

$$\mathfrak{P}^{i+2} \subseteq \mathfrak{p}^k \mathcal{O}_{E_\mathfrak{p}} \subseteq \mathfrak{P}^i$$

and $\mathfrak{p}^k \mathcal{O}_{E_\mathfrak{p}} = \mathfrak{P}^{i+2}$ if and only if $L \simeq H(\pi^i)^{\oplus r/2}$. Moreover,

$$u^\epsilon N_{K_\mathfrak{p}}^{E_\mathfrak{p}}(E_\mathfrak{p}^\times) = (-1)^{r/2}\det(L)$$

where $u \notin N_{K_\mathfrak{p}}^{E_\mathfrak{p}}(E_\mathfrak{p}^\times)$, and $\epsilon \in \{0, 1\}$ with $\epsilon = 0$ if and only if L is hyperbolic.

Proof. This a translation of [Kir16, Proposition 3.3.5 & Corollary 3.3.20] to the prime power cyclotomic case, together with Proposition 3.2 which tells us that the different ideal $\mathfrak{D}_{E_\mathfrak{p}/K_\mathfrak{p}}$ is generated by $\pi^{\gcd(2,p)}$. □

Remark 4.18. It is good to note that for the cases (2)(a) and (2)(b) in Proposition 4.17 we have a fine description of $\mathfrak{n}(L) = \mathfrak{p}^k$. In fact, suppose that $p = 2$ and let L be $\mathfrak{P}^i$-modular of rank $r \geq 1$, for some $i \geq 1$. If r is odd, then i is even, and since $\mathfrak{p}\mathcal{O}_{E_\mathfrak{p}} = \mathfrak{P}^2$, we have that $\mathfrak{n}(L) = \mathfrak{p}^{\frac{i}{2}}$. Now, if r is even, there are two cases. Either i is odd, in which case so is $i + 2$ and thus we must have $\mathfrak{n}(L) = \mathfrak{p}^{\frac{i+1}{2}}$. Otherwise, i is even and $\mathfrak{n}(L) \in \{\mathfrak{p}^{\frac{i+2}{2}}, \mathfrak{p}^{\frac{i}{2}}\}$ — the two cases are distinguished by L being isometric to $H(\pi^i)^{\oplus r/2}$ or not.

Let us conclude by remarking the following, about genera of indefinite or rank 1 hermitian $\mathcal{O}_E$-lattices, in the cyclotomic case. We denote by $C(E/K)$ the *relative class group* E, which is the kernel of the ideal norm map $C(E) \to C(K)$ induced by N_K^E.

Notation. The order of $C(E/K)$, called the *the relative class number* of E, is denoted by $h^-(E)$.

Let us denote by $\mathcal{I}(E)$ the set of fractional $\mathcal{O}_E$-ideals, and let $J \subset \mathcal{I}(E)$ be the subset of ideals $\mathfrak{A}$ such that

$$\mathfrak{A}\iota(\mathfrak{A}) = \mathcal{O}_E.$$

We denote $J_0 := J \cap \mathcal{P}(E)$ where $\mathcal{P}(E)$ is the set of principal fractional $\mathcal{O}_E$-ideals. Let C_0 be the set of classes $[\mathfrak{A}]$ in $C(E)$ such that $\mathfrak{A} = \iota(\mathfrak{A})$. The group C_0 is generated by the elements of $C(K)$ and the prime $\mathcal{O}_E$-ideals which ramify over K. One has that the morphism

$$C(E)/C_0 \to J/J_0, \;\; [\mathfrak{A}] \mapsto [\mathfrak{A}/\iota(\mathfrak{A})]$$

is an isomorphism. In the case where $h^-(E)$ is odd, there is actually an isomorphism

$$J/J_0 \to C(E/K), \;\; \mathfrak{A}J_0 \mapsto \mathfrak{A}C(K)$$

[BC23, Lemma 2.14]. The following is adapted from [BC23, Proposition 2.9].

Proposition 4.19. *Let ζ_m be a primitive mth root of unity for some $m \geq 3$. Let $E := \mathbb{Q}(\zeta_m)$, let $K := \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ and (L, h) be a hermitian $\mathcal{O}_E$-lattice. Suppose that (L, h) is indefinite or that $\operatorname{rank}_{\mathcal{O}_E}(L) = 1$. Then the number of isometry classes in the genus of (L, h) is the relative class number $h^-(E)$.*

Proof. The case where $\operatorname{rank}_{\mathcal{O}_E}(L) = 1$ has already been proven in greater generality in [BC23, Proposition 2.9]. The indefinite case has been proven for $m = p$ is an odd prime number in the same proposition. To generalize to any $m \geq 3$, one can follow the proof of the aforementioned proposition using also the results from [Kir19, §3].

Let P be the set of finite places of K which ramify in E. Note that for all $\mathfrak{q} \in P$, we have that $\operatorname{Gal}(E/K) \cong \operatorname{Gal}(E_\mathfrak{q}/K_\mathfrak{q})$: let ι be a generator.

It follows from [Was97, Theorem 4.14] that $[C : C_0] = h^-(E)$ is the relative class number of E/K. Let us further define $\mathcal{F}(E) := \{e \in \mathcal{O}_E :\ e\iota(e) = 1\}$ and for all $\mathfrak{q} \in P$,

$$\mathcal{F}(E_\mathfrak{q}) := \{e \in \mathcal{O}_{E_\mathfrak{q}}^\times :\ e\iota(e) = 1\} \quad \text{and} \quad \mathcal{F}_1(E_\mathfrak{q}) := \{e \in \mathcal{F}(E_\mathfrak{q}) :\ e \equiv 1 \mod \mathfrak{Q}\}$$

where $\mathfrak{Q}$ is the unique prime $\mathcal{O}_E$-ideal dividing $\mathfrak{q}\mathcal{O}_E$. For all $\mathfrak{q} \in P$, we let moreover $\mathcal{F}(L_\mathfrak{q}) := \det(U(L_\mathfrak{q})) \leq \mathcal{F}(E_\mathfrak{p})$. According to [Shi64, Theorem 5.24], and the fact that (L, h) is indefinite, the number of isometry classes in the genus of (L, h) is given by $[C : C_0][\mathcal{E}(L, h) : R(L, h)]$ where $\mathcal{E}(L, h) := \prod_{\mathfrak{q}\in P} \mathcal{F}(E_\mathfrak{q})/\mathcal{F}(L_\mathfrak{q})$ and $R(L, h) := \{(e\mathcal{F}(L_\mathfrak{q}))_{\mathfrak{q}\in P} :\ e \in \mathcal{F}(E)\}$.

Now, by Lemma 3.1, either m is composite and P is empty, or $m = p^k, 2p^k$ is (twice) a prime power and $P = \{\mathfrak{p}\}$ where $\mathfrak{p}$ is the unique prime $\mathcal{O}_K$-ideal dividing $p\mathcal{O}_K$. So the result holds for m composite. Otherwise, we have that $\mathcal{E}(L, h) = \mathcal{F}(E_\mathfrak{p})/\mathcal{F}(L_\mathfrak{p})$. By [Kir19, Theorem 3.7], we know that $\mathcal{F}_1(E_\mathfrak{p}) \leq \mathcal{F}(L_\mathfrak{p})$ and that the quotient group $\mathcal{F}(E_\mathfrak{p})/\mathcal{F}_1(E_\mathfrak{p})$ is cyclic of order 2, generated by some $\delta \in \mathcal{F}(E_\mathfrak{p})$ such that $(1 - \delta)\mathcal{O}_{E_\mathfrak{p}} = \mathfrak{P}^{-1}\mathfrak{D}_{E_\mathfrak{p}/K_\mathfrak{p}}$. Using Proposition 3.2, one can easily check then $\delta = -\zeta_m$ works and moreover, that $(-\zeta_m\mathcal{F}_1(E_\mathfrak{p})) \in R(L, h)$. This implies in particular that $[\mathcal{E}(L, h) : R(L, h)] = 1$. □

Computational comments. Similarly to what has been described for $\mathbb{Z}$-lattices, it is possible to describe a genus of a hermitian $\mathcal{O}_E$-lattice L with

(1) the rank of L,

(2) the signatures of L at the real places of K,

(3) the isometry class of $L_\mathfrak{q}$ for all prime ideals of $\mathfrak{q}$ of K for which $L_\mathfrak{p}$ is unimodular or $2 \in \mathfrak{q}$.

Note that for each prime ideal as in point (3), the isometry class of $L_\mathfrak{p}$ can be described by a finite set of invariants. Thus, technically, one can describe the genus of L is a human readable way: though this information can hardly be condensed into a proper symbol. Working with genera of hermitian $\mathcal{O}_E$-lattices is computationally accessible, as for genera of $\mathbb{Z}$-lattices, and one can for example

(1) determine the genus of a hermitian $\mathcal{O}_E$-lattice;

(2) compare two given genera;

(3) determine whether a set of invariants define a nonempty genus of hermitian $\mathcal{O}_E$-lattices [Kir16, §3];

(4) determine a hermitian $\mathcal{O}_E$-lattice lying in a given genus [Kir16, §3.4, 3.5];

(5) compute a complete set of representatives for the isometry classes in a given genus (see [Kir16, §5] and [BC23, Proposition 2.9]).

As always, we assume from now on that the previous items are accessible computationally.

4.3. The cyclotomic transfer construction

Let $m \geq 3$ be an integer and let again $E := \mathbb{Q}(\zeta_m)$ with maximal real subfield K. We follow the same notation as in the previous section.

Notation. Let us denote $S_m := \{1 \leq i \leq \lfloor m/2\rfloor :\ \gcd(m, i) = 1\}$. It has order $\#S_m = \varphi(m)/2 = s$ where s is the number of infinite places of K: in particular, there is a bijection between the set S_m and $\Omega_\infty(K)$, the set of (real) infinite places of K. This bijection is described as follows: to $i \in S_m$ corresponds the real infinite place of K whose associated $\mathbb{Q}$-embedding into $\mathbb{R}$ sends $\zeta_m + \zeta_m^{-1}$ to $\zeta_m^i + \zeta_m^{-i}$.

Let (L, b, f) be a Φ_m-lattice. The isometry f has minimal polynomial Φ_m so there is an action of $\mathcal{O}_E$ on (L, b) given by $\zeta_m \cdot x := f(x)$, for all $x \in L$. This defines a structure of projective $\mathcal{O}_E$-module on L. We can define the form

$$h\colon (L \otimes_{\mathcal{O}_E} E) \times (L \otimes_{\mathcal{O}_E} E) \to E, \quad (x, y) \mapsto \frac{1}{m} \sum_{0 \leq i \leq m-1} b(x, f^i(y))\zeta_m^i \tag{8}$$

which is ι-sesquilinear and nondegenerate, where $\mathrm{Gal}(E/K) = \langle \iota \rangle$.

Definition 4.20. The $\mathcal{O}_E$-module L equipped with h defines a hermitian $\mathcal{O}_E$-lattice, which we call the *hermitian structure* of (L, b, f).

Remark 4.21. When $f = \pm \mathrm{id}_L$, we let $E := \mathbb{Q}$ and $\iota = \mathrm{id}_E$. Together with Equation (8), we have that $h = b$ and the hermitian structure of (L, b, f) is (L, b) itself.

Conversely, given a hermitian $\mathcal{O}_E$-lattice (L, h), we define the form

$$b\colon (L \otimes_{\mathbb{Z}} \mathbb{Q}) \times (L \otimes_{\mathbb{Z}} \mathbb{Q}), \quad (x, y) \mapsto \mathrm{Tr}_{\mathbb{Q}}^E(h(x, y)) \tag{9}$$

which is symmetric, bilinear and nondegenerate. This turns (L, b) into a $\mathbb{Z}$-lattice. The multiplication by ζ_m given by the $\mathcal{O}_E$-module structure on L defines an isometry f of (L, b) so that $\Phi_m(f) = 0$. Therefore, (L, b, f) is a Φ_m-lattice which we call the *trace lattice* of (L, h). Note that in particular,

$$\mathrm{rank}_{\mathbb{Z}}(L) = \varphi(m)\mathrm{rank}_{\mathcal{O}_E}(L). \tag{10}$$

The following is well-known: we provide a proof for the reader's convenience.

Proposition 4.22. *The two constructions in Equations (8) and (9) are inverse of each other. Moreover, if (L, b, f) is a Φ_m-lattice with hermitian structure (L, h), then $O(L, b, f) = U(L, h)$.*

Proof. Note that the result is trivial for $m = 1, 2$. From now on, let us assume $m \geq 3$. We start by showing that both constructions are inverse to each other. In what follows, we denote, for all $1 \leq j \leq m$ coprime to m, the $\mathbb{Q}$-automorphism of E sending ζ_m to ζ_m^j by $\sigma_j \in \mathrm{Gal}(E/\mathbb{Q})$.

(1) Let (L, h) be a hermitian $\mathcal{O}_E$-lattice, and let $x, y \in L \otimes_{\mathcal{O}_E} E$. Then, we observe

$$\begin{aligned}
\sum_{i=0}^{m-1} \mathrm{Tr}_{\mathbb{Q}}^E(h(x, f^i(y)))\zeta_m^i &= \sum_{i=0}^{m-1} \mathrm{Tr}_{\mathbb{Q}}^E(h(x, \zeta_m^i y))\zeta_m^i \\
&= \sum_{i=0}^{m-1} \left(\sum_{\substack{1 \leq j \leq m \\ \gcd(j,m)=1}} \sigma_j(h(x, y)\zeta_m^{-i}) \right) \zeta_m^i \\
&= mh(x, y) + \sum_{i=0}^{m-1} \left(\sum_{\substack{1 < j \leq m \\ \gcd(j,m)=1}} \sigma_j(h(x, y))\zeta_m^{-ij} \right) \zeta_m^i \\
&= mh(x, y) + \sum_{\substack{1 < j \leq m \\ \gcd(j,m)=1}} \sigma_j(h(x, y)) \underbrace{\sum_{i=0}^{m-1} \zeta_m^{i(1-j)}}_{=0} \\
&= mh(x, y)
\end{aligned}$$

where the last equality follows from the fact that for $j \neq 1$, one has that ζ_m^{1-j} is a root of $\frac{X^m-1}{X-1} = \sum_{i=0}^{m-1} X^i$. Hence, we have that

$$h(x,y) = \frac{1}{m}\sum_{i=0}^{m-1} \mathrm{Tr}_{\mathbb{Q}}^{E}(h(x, f^i(y)))\zeta_m^i.$$

(2) Now let (L, b, f) be a Φ_m-lattice and let $x, y \in L \otimes_{\mathbb{Z}} \mathbb{Q}$. We define the *Möbius function*

$$\mu\colon \mathbb{Z}_{\geq 1} \to \{-1, 0, 1\},\ d \mapsto \begin{cases} 1 & \text{if } d = 1 \\ (-1)^k & \text{if } d \text{ is the product of } k \text{ distinct prime numbers} \\ 0 & \text{otherwise} \end{cases}.$$

By a result of Gauss, it is known that for all $d \in \mathbb{Z}_{\geq 1}$, the value $\mu(d)$ is equal to the sum of the primitive complex dth roots of unity. By remarking that $\{\zeta_m^i : i \in \{0, \ldots, m-1\}\}$ consists of all the roots of $X^m - 1$, we compute:

$$\begin{aligned}\mathrm{Tr}_{\mathbb{Q}}^{E}\left(\sum_{i=0}^{m-1} b(x, f^i(y))\zeta_m^i\right) &= \sum_{i=0}^{m-1} b(x, f^i(y))\mathrm{Tr}_{\mathbb{Q}}^{E}(\zeta_m^i) \\ &= \sum_{d\in\{0,\ldots,m-1\},\, d|m} \left(\sum_{i\in\{0,\ldots,m-1\},\, \Phi_d(\zeta_m^i)=0} b(x, f^i(y))\mathrm{Tr}_{\mathbb{Q}}^{E}(\zeta_m^i)\right).\end{aligned}$$

Now, for each divisor d of m, we observe that

(a) for every $i \in \{0, \ldots, m-1\}$ such that $\Phi_d(\zeta_m^i) = 0$, the following holds:

$$\mathrm{Tr}_{\mathbb{Q}}^{E}(\zeta_m^i) = \frac{\varphi(m)}{\varphi(d)}\mathrm{Tr}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_m^i)}(\zeta_m^i) = \frac{\varphi(m)}{\varphi(d)}\mu(d),$$

(b) since f has minimal polynomial Φ_m, the following holds:

$$\sum_{i\in\{0,\ldots,m-1\},\, \Phi_d(\zeta_m^i)=0} f^i = \mu(d)\mathrm{id}_L.$$

Put together, we obtain

$$\mathrm{Tr}_{\mathbb{Q}}^{E}\left(\sum_{i=0}^{m-1} b(x, f^i(y))\zeta_m^i\right) = b(x,y) \sum_{d\in\{0,\ldots,m-1\},\, d|m} \frac{\varphi(m)}{\varphi(d)}\mu(d)^2.$$

To conclude, we recall that by definition of the Euler totient φ, we have that

$$\frac{m}{\varphi(m)} = \prod_{p|m,\, p \text{ prime}} \left(1 + \frac{1}{p-1}\right) = \sum_{d\in\{0,\ldots,m-1\},\, d|m} \frac{\mu(d)^2}{\varphi(d)}$$

giving us the wanted equality:

$$\mathrm{Tr}_{\mathbb{Q}}^{E}\left(\sum_{i=0}^{m-1} b(x, f^i(y))\zeta_m^i\right) = mb(x,y).$$

Now let (L, b, f) be a Φ_m-lattice and let (L, h) be its hermitian structure where h is defined as in Equation (8). Let $g \in O(L, b, f)$: since the $\mathcal{O}_E$-module structure on L is given by f, we have that g defines an $\mathcal{O}_E$-module isomorphism of L. Moreover, for all $x, y \in L \otimes_{\mathcal{O}_E} E$, we have

$$h(g(x), g(y)) = \sum_{i=0}^{m-1} b(g(x), f^i(g(y)))\zeta_m^i = \sum_{i=0}^{m-1} b(g(x), g(f^i(y)))\zeta_m^i = \sum_{i=0}^{m-1} b(x, f^i(y))\zeta_m^i = h(x,y)$$

since $g \in O(L, b, f)$. In particular $g \in U(L, h)$. Conversely, if $g \in U(L, h)$, then it defines a $\mathbb{Z}$-module isomorphism of L which commutes with f, and for all $x, y \in L \otimes_{\mathbb{Z}} \mathbb{Q}$

$$b(g(x), g(y)) = \mathrm{Tr}_{\mathbb{Q}}^{E}(h(g(x), g(y)) = \mathrm{Tr}_{\mathbb{Q}}^{E}(h(x,y)) = b(x,y)$$

since $g \in U(L, h)$. Hence $O(L, b, f) = U(L, h)$. □

Remark 4.23. Actually, the last part of the proof of Proposition 4.22 can be extended to show that two hermitian $\mathcal{O}_E$-lattices (L_1, h_1) and (L_2, h_2) are isometric if and only if their respective trace lattices are isomorphic, as lattices with isometry. In particular, the constructions in Equations (8) and (9) define an equivalence between the category of Φ_m-lattices and the category of hermitian $\mathcal{O}_E$-lattices.

For a hermitian $\mathcal{O}_E$-lattice (L, h) with trace lattice (L, b, f), we have that

$$L^\vee = \mathfrak{D}_{E/\mathbb{Q}}^{-1} L^\# \tag{11}$$

and (L, b) is integral if and only if $\mathfrak{s}(L, h) \subseteq \mathfrak{D}_{E/\mathbb{Q}}^{-1}$. Moreover, if (L, b) is integral, then (L, b) is even if and only if $\mathfrak{n}(L, h) \subseteq \mathfrak{D}_{K/\mathbb{Q}}^{-1}$ [BH23, Lemma 6.6]. More generally, we prove the following lemma, which actually holds for any degree two extension of number fields E/K.

Lemma 4.24. *Let (L, b, f) be the trace lattice of a hermitian $\mathcal{O}_E$-lattice (L, h). Then,*

$$s(L,b) = \mathrm{Tr}_{\mathbb{Q}}^{E}(\mathfrak{s}(L,h)), \qquad n(L,b) = 2\mathrm{Tr}_{\mathbb{Q}}^{K}(\mathfrak{n}(L,h)) \quad \text{and} \quad |d(L,b)| = N_{\mathbb{Q}}^{E}\left(\mathfrak{D}_{E/\mathbb{Q}}^{n}\mathfrak{v}(L,h)\right),$$

where $n := \mathrm{rank}_{\mathcal{O}_E}(L)$.

Proof. The result for the scale is trivial since by definition $b = \mathrm{Tr}_{\mathbb{Q}}^{E} \circ h$, and the relation involving $d(L, b)$ has been proven in [Jur15, Proposition 3.1.4] and follows from

$$|d(L,b)| = \#D_L = \#(\mathfrak{D}_{E/\mathbb{Q}}^{-1} L^\#/L) = N_{\mathbb{Q}}^{E}\left(\mathfrak{D}_{E/\mathbb{Q}}^{n}\mathfrak{v}(L,h)\right).$$

Now, for the statement about the norm, we already remark that for all $x \in L$, we have $b(x,x) = \mathrm{Tr}_{\mathbb{Q}}^{E}(h(x,x)) = 2\mathrm{Tr}_{\mathbb{Q}}^{K}(h(x,x)) \in 2\mathrm{Tr}_{\mathbb{Q}}^{K}(\mathfrak{n}(L,h))$. For the other inclusion, we follow a similar strategy as in [BH23, Corollary 6.7]. Note that since

$$(L_p, b_p) \simeq \bigoplus_{\mathfrak{p}|p} (L_{\mathfrak{p}}, \mathrm{Tr}_{\mathbb{Q}_p}^{E_{\mathfrak{p}}} \circ h_{\mathfrak{p}}), \tag{12}$$

as $\mathbb{Z}_p$-lattices, it suffices to show that for all prime numbers p and every prime $\mathcal{O}_K$-ideal dividing $p\mathcal{O}_K$, we have the inclusion $2\mathrm{Tr}^{K_\mathfrak{p}}_{\mathbb{Q}_p}(\mathfrak{n}(L_\mathfrak{p}, h_\mathfrak{p})) \subseteq n(L_p, b_p)$. This would hold if we can prove that for all $a \in \mathcal{O}_{K_\mathfrak{p}}$ and for all $x \in L_\mathfrak{p}$, we have

$$2\mathrm{Tr}^{K_\mathfrak{p}}_{\mathbb{Q}_p}(ah_\mathfrak{p}(x,x)) = \mathrm{Tr}^{E_\mathfrak{p}}_{\mathbb{Q}_p}(ah_\mathfrak{p}(x,x)) \in n(L_p, b_p).$$

Let B be the $\mathbb{Z}$-module consisting of scalars $w \in K_\mathfrak{p}$ such that $\mathrm{Tr}^{E_\mathfrak{p}}_{\mathbb{Q}_p}(wh_\mathfrak{p}(x,x)) \in n(L_p, b_p)$ for all $x \in L_\mathfrak{p}$. We want to show that B contains $\mathcal{O}_{K_\mathfrak{p}}$. In order to do so we apply [BH23, Lemma 6.5] which tells us that if B contains 1, $\mathrm{Tr}^{E_\mathfrak{p}}_{K_\mathfrak{p}}(\mathcal{O}_{E_\mathfrak{p}})$ and $N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(\mathcal{O}_{E_\mathfrak{p}})B$, then B contains $\mathcal{O}_{K_\mathfrak{p}}$. By the decomposition in Equation (12), one can already show that B contains 1, and B is not empty. Now let $w \in B$ and let $\lambda \in \mathcal{O}_{E_\mathfrak{p}}$. One computes, for all $x \in L_\mathfrak{p}$,

$$\mathrm{Tr}^{E_\mathfrak{p}}_{\mathbb{Q}_p}(\lambda\iota(\lambda)wh_\mathfrak{p}(x,x)) = \mathrm{Tr}^{E_\mathfrak{p}}_{\mathbb{Q}_p}(wh_\mathfrak{p}(\lambda x, \lambda x)) \in n(L_p, b_p).$$

Hence, $N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(\mathcal{O}_{E_\mathfrak{p}})B \subseteq B$. Finally let again $\lambda \in \mathcal{O}_{E_\mathfrak{p}}$: for all $x \in L_\mathfrak{p}$ we have

$$\begin{aligned}\mathrm{Tr}^{E_\mathfrak{p}}_{\mathbb{Q}_p}((\lambda+\iota(\lambda))h_\mathfrak{p}(x,x)) =& \mathrm{Tr}^{E_\mathfrak{p}}_{\mathbb{Q}_p}(h_\mathfrak{p}((\lambda+1)x, (\lambda+1)x)) \\ &- \mathrm{Tr}^{E_\mathfrak{p}}_{\mathbb{Q}_p}(h_\mathfrak{p}(\lambda x, \lambda x)) - \mathrm{Tr}^{E_\mathfrak{p}}_{\mathbb{Q}_p}(h_\mathfrak{p}(x,x)) \in n(L_p, b_p).\end{aligned}$$

This implies that $\mathrm{Tr}^{E_\mathfrak{p}}_{K_\mathfrak{p}}(\mathcal{O}_{E_\mathfrak{p}}) \subseteq B$ and we can conclude. □

We conclude with the following. For all $i \in S_m$, we denote by $(k_i^+, k_i^-) \in \mathbb{N}^2$ the signatures of the real quadratic space $(K_i, b_\mathbb{R}) := \ker(f_\mathbb{R} + f_\mathbb{R}^{-1} - \zeta_m^i - \zeta_m^{-i})$. If (l_+, l_-) denotes the signatures of (L, b), one has that $l_\pm = \sum_{i \in S_m} k_i^\pm$. Note that for all $i \in S_m$,

$$K_i \otimes_\mathbb{R} \mathbb{C} = \ker(f_\mathbb{C} - \zeta_m^i) \oplus \overline{\ker(f_\mathbb{C} - \zeta_m^i)},$$

with $\dim_\mathbb{C}(\ker(f_\mathbb{C} - \zeta_m^i)) = \mathrm{rank}_{\mathcal{O}_E}(L)$. In particular,

$$k_i^+ + k_i^- = \dim_\mathbb{C}(K_i \otimes_\mathbb{R} \mathbb{C}) = 2\dim_\mathbb{C}(\ker(f_\mathbb{C} - \zeta_m^i)) = 2(l_+ + l_-)/\varphi(m)$$

does not depend on i. Moreover, for all $i \in S_m$, we have that $(K_i, b_\mathbb{R}) = \mathrm{Tr}^{E_{\mathfrak{q}_i}}_{K_{\mathfrak{q}_i}}(L_{\mathfrak{q}_i})$ and $k_i^- = 2n(\mathfrak{q}_i)$. Hence for all $i \in S_m$, both signatures of K_i are even and

$$l_- = 2\sum_{i \in S_m} n(\mathfrak{q}_i). \tag{13}$$

We summarize in Table 2 the correspondence between some (isometry) invariants of Φ_m-lattices and their hermitian structure, via the trace equivalence.

Table 2: Transfer construction and correspondence of invariants

(L,b,f)	$O(L,b,f)$	$L^\vee$	$s(L,b)$	$n(L,b)$	$\lvert d(L,b)\rvert$	l_-
(L,h)	$U(L,h)$	$\mathfrak{D}^{-1}_{E/K}L^\#$	$\mathrm{Tr}^E_\mathbb{Q}(\mathfrak{s}(L,h))$	$2\mathrm{Tr}^K_\mathbb{Q}(\mathfrak{n}(L,h))$	$N^E_\mathbb{Q}\left(\mathfrak{D}^n_{E/\mathbb{Q}}\mathfrak{v}(L,h)\right)$	$2\sum_{i\in S_m} n(\mathfrak{q}_i)$

Computational comments. The trace equivalence can be implemented using standard algebraic manipulations. In fact, since we can work explicitly with lattices with isometry, hermitian lattices, and number fields extension, both of the constructions Equations (8) and (9) are computationally accessible. From now on, we often work with Φ_m-lattices and their hermitian structures interchangeably, and we also assume that the trace equivalence to be effectively accessible.

4.4. A Galois action

Let $m \geq 3$ and let again $E := \mathbb{Q}(\zeta_m)$. The extension $E/\mathbb{Q}$ is Galois and we denote by G the group $\mathrm{Gal}(E/\mathbb{Q})$. Let moreover (L, b, f) be a Φ_m-lattice with hermitian structure (L, h).

Proposition 4.25. *Let $1 \leq k < m$ be coprime to m, and let $\sigma \in G$ be defined by $\sigma(\zeta_m) = \zeta_m^k$. Then, the hermitian structure of (L, b, f^k) is given by $(\sigma^* L, \sigma^{-1} \circ h)$.*

Proof. Since $\gcd(k, m) = 1$, we have that multiplication by k defines an automorphism of $\mathbb{Z}/m\mathbb{Z}$. Hence, we obtain that for all $x, y \in L \otimes_{\mathcal{O}_E} E$

$$\frac{1}{m} \sum_{0 \leq i \leq m-1} b(x, f^i(y))\zeta_m^i = \frac{1}{m} \sum_{0 \leq i \leq m-1} b(x, f^{ki}(y))\zeta_m^{ki} = \sigma\left(\frac{1}{m} \sum_{0 \leq i \leq m-1} b(x, f^{ki}(y))\zeta_m^i\right).$$

Finally, multiplication by ζ_m on $\sigma^* L$ coincides with multiplication by $\sigma(\zeta_m) = \zeta_m^k$ on L, hence the isometry of (L, b) induced by multiplication by ζ_m on $\sigma^* L$ is f^k. $\square$

In particular, there is a right action of G on the set of hermitian structures associated to generators of $\langle f \rangle \leq O(L, b)$.

Lemma 4.26. *The Galois group G acts on the set of genera of $\mathcal{O}_E$-lattices of a given rank, and for every such genus g we have that $g \cdot \iota = g$ where ι generates $\mathrm{Gal}(E/K) \trianglelefteq G$.*

Proof. The first part of the proof follows from prior discussion. Now remark that since $[E : K] = 2$, we have that G acts on the set of places of K, and it maps infinite (resp. finite) places to infinite (resp. finite) places. By the definition of a genus of hermitian $\mathcal{O}_E$-lattices (Section 4.2), we see that this action induces the action of G on the set of genera of hermitian $\mathcal{O}_E$-lattices of a given rank. However, note that since $K = E^\iota$ then ι acts trivially on the set of places of K, and therefore it preserves all genera of hermitian $\mathcal{O}_E$-lattices. $\square$

Therefore, the action of G on the set of genera of hermitian $\mathcal{O}_E$-lattices factors trough an action of $\mathrm{Gal}(K/\mathbb{Q})$ on such a set.

Definition 4.27. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two genera of hermitian $\mathcal{O}_E$-lattice of given rank $n \geq 1$. We call $\mathfrak{g}_1$ and $\mathfrak{g}_2$ *Galois equivalent* if there are in the same orbit under the action of $\mathrm{Gal}(K/\mathbb{Q})$.

Corollary 4.28. *For any $1 \leq k < m$ so that $\gcd(k, m) = 1$, we have that the genera of the hermitian structures of (L, b, f) and (L, b, f^k) are Galois equivalent.*

Proof. It is a direct consequence of Proposition 4.25. $\square$

Let us give an example, which we use later in this thesis to describe such an action. Let $m = p^k \geq 3$ be a prime power and let (L, b, f) be a p-elementary Φ_m-lattice of signatures $(2, *)$. Let (L, h) be the hermitian structure of (L, b, f), and let us denote its rank by $n \geq 0$. We denote again $E := \mathbb{Q}(\zeta_m)$ and $K := \mathbb{Q}(\zeta_m + \zeta_m^{-1})$, and we let $\mathfrak{P} := (1 - \zeta_m)\mathcal{O}_E$ the unique prime $\mathcal{O}_E$-ideal dividing $p\mathcal{O}_E$ (Proposition 3.2). Since (L, b) is p-elementary and $\mathfrak{D}_{E/\mathbb{Q}}$ is a power of $\mathfrak{P}$, it follows that $L_\mathfrak{q}$ is unimodular for all finite places of K different from $\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_K$. The $\mathbb{Z}$-lattice (L, b) has signatures $(2, *)$, so following Table 2 and the discussion prior that table, we have that there exists an infinite place ν_0 of K such that L_{ν_0} has signature $n(\nu_0) = n-1$ and L_ν has signature $n(\nu) = n$ for all other infinite places $\nu \neq \nu_0$ of K. Let now $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. Since $p\mathbb{Z}$ totally ramifies in K, we have that σ induces a $\mathbb{Q}_p$-automorphism of $K_\mathfrak{p}$ and we have that

$$(L \cdot \sigma)_\mathfrak{p} \simeq L_\mathfrak{p}.$$

Thus, the respective genera of (L, h) and $(L, h) \cdot \sigma$ are the same at all finite places of K. However, we have that $\mathrm{Gal}(K/\mathbb{Q})$ acts faihtfully transitively on the set $\Omega_\infty(K)$ of infinite places of K: therefore, we obtain that the genera of (L, h) and $(L, h) \cdot \sigma$ differ only by the associated signatures.

Lemma 4.29. *There is a 2-to-1 map*

$$\{\text{generators of } \langle f \rangle\} \xrightarrow{2:1} \Omega_\infty(K)$$

which sends any generator f^l of the cyclic group $\langle f \rangle$ to the real place $\nu_0 \in \Omega_\infty(K)$ for which the hermitian structure of (L, b, f^l) has signature $n-1$ at ν_0.

Proof. Let (L, h) be again the hermitian structure of (L, b, f). Then we have seen that there is an infinite place $\nu_0 \in \Omega_K$ such that $n((L, h), \nu_0) = n-1$ and $n((L, h), \nu) = n$ for every other $\nu_o \neq \nu \in \Omega_K$. But now, from Proposition 4.25 and Corollary 4.28, we have that the elements $\tau \in \mathrm{Gal}(E/\mathbb{Q})$ correspond bijectively to the the generators f^l of $\langle f \rangle$ and moreover

(1) if $\tau = \iota$, then $(L, h) \cdot \iota$ and (L, h) are in the same genus, and have the same signatures;

(2) if $\tau \neq \iota$, then τ restricts to a nontrivial element $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, and $n((L, h) \cdot \sigma, \sigma^*\nu_0) = n((L, h), \nu_0) = n-1$, but $n((L, g) \cdot \sigma, \nu_0) = n$.

Remark that the 2-to-1 mapping is induced by the action of $\mathrm{Gal}(E/\mathbb{Q})$ on $\Omega_\infty(K)$. □

A consequence of the previous statement is the following. In the situation where (L, b, f) is a p-elementary Φ_{p^k}-lattice of signature $(2, *)$, then there exists a primitive p^kth root of unity ζ such that

$$\ker(f + f^{-1} - \zeta - \zeta^{-1})$$

has real signatures $(2, *)$ and the real quadratic space

$$\ker(f + f^{-1} - \zeta' - (\zeta')^{-1})$$

is negative definite for any other primitive p^kth root of unity $\zeta' \neq \zeta, \zeta^{-1}$. Moreover, the Galois orbit of the genus of the hermitian structure (L, h) of (L, b, f) is uniquely determined by the isometry class of $L_\mathfrak{p}$, where $p \in \mathfrak{p}$.

5. Irreducible holomorphic symplectic manifolds

In this section we introduce irreducible holomorphic symplectic manifolds, give some first properties about them, and give some examples. We later discuss about Torelli-type theorems for such varieties, and we define many important tools to work with symmetries of such complex manifolds. Except otherwise stated, throughout the thesis we work over $\mathbb{C}$ and all analytic spaces are smooth. Moreover, we later specify that we mostly work in a projective setting — our lattice-theoretic approach will depend on this last assumption in some cases.

5.1. Definition and motivation

Definition 5.1. An *irreducible holomorphic symplectic manifold* X (also known as *hyperkähler manifold*) is a simply-connected compact Kähler manifold such that $H^0(X, \Omega_X^2)$ is 1-dimensional generated by a nowhere degenerate holomorphic 2-form σ_X.

For short, we usually refer to irreducible holomorphic symplectic manifolds as *IHS manifolds*. Let us review some first properties about such manifolds: these results are classical, see for instance [Huy99].

Proposition 5.2. *Let X be an irreducible holomorphic symplectic manifold. Then:*

(1) *X has even complex dimension;*

(2) *$H^2(X, \mathbb{Z})$ is torsionfree;*

(3) *the first Chern class $c_1(X)$ of X is trivial;*

(4) *we have an isomorphism $\mathrm{Pic}(X) \cong \mathrm{NS}(X)$;*

(5) *the canonical bundle ω_X of X is trivial;*

(6) *$\mathrm{Aut}(X)$ is discrete.*

Proof. In what follows, let us denote by $n \geq 1$ the complex dimension of X.

(1) The holomorphic 2-form σ_X on X being nowhere degenerate, if defines a nondegenerate symplectic form on T_xX for all $x \in X$. In particular, these tangent spaces are even dimensional, and since X is smooth, we have that $\dim_{\mathbb{C}}(X)$ is even too.

(2) Since X is simply-connected, we have that $H_1(X, \mathbb{Z})$ is trivial. Hence, by the Universal Coefficient Theorem, we obtain that $H^2(X, \mathbb{Z})$ is torsionfree.

(3) Let
$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 1$$
be the exponential sequence. Recall that it induces a long exact sequence in cohomology, from which we excerpt the following exact sequence:
$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^\times).$$
We have an identification $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$, the connecting morphism c_1 maps a line bundle on X to its first Chern class in $H^2(X, \mathbb{Z})$, and the image $\mathrm{im}(c_1) =: \mathrm{NS}(X)$ of the previous morphism is the so-called *Néron–Severi* group of X. For a rank r vector bundle E on X, we recall that $c_1(E) = c_1(\bigwedge^r E)$. By definition, the first Chern class of X is

$c_1(X) := c_1(T_X)$ where T_X is the tangent bundle. But now, since σ_X is nowhere degenerate, it induces an isomorphism of vector bundles $T_X \simeq \Omega_X := T_X^\vee$ on X. Therefore,

$$2c_1(X) := 2c_1(T_X) = c_1(T_X) + c_1(T_X) = c_1(T_X) + c_1(\Omega_X) = c_1(T_X \otimes \Omega_X) = c_1(\mathcal{O}_X) = 0.$$

In particular, since $H^2(X, \mathbb{Z})$ is torsionfree by (2), we get that $c_1(X)$ is trivial.

(4) By simply-connectedness of X, we know that all closed 1-forms are exact, and in particular $H^0(X, \Omega_X) \cong H^1(X, \mathcal{O}_X)$ is trivial. Hence, c_1 is injective and $\mathrm{Pic}(X) \cong \mathrm{im}(c_1) = \mathrm{NS}(X)$.

(5) The canonical bundle of X is by definition the line bundle $\omega_X := \bigwedge^n \Omega_X$. From the fact that $\Omega_X \simeq T_X$, the properties of c_1, and $c_1(X)$ trivial, we conclude

$$c_1(\omega_X) = c_1(\Omega_X) = c_1(T_X) = 0.$$

Therefore, since $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective, we have that $\omega_X \simeq \mathcal{O}_X$ is trivial.

(6) We have seen in the proof of (4) that $H^0(X, T_X) \cong H^0(X, \Omega_X)$ is trivial. This in particular implies that X has no infinitesimal automorphisms and $\mathrm{Aut}(X)$ is zero-dimensional, as a group scheme. Therefore it is discrete. □

From the proof of Proposition 5.2 (4), we obtain that 2-dimensional IHS manifolds are K3 surfaces. Observe, following (3), that IHS manifolds arise as one of the building blocks for compact Kähler manifolds with trivial (real) first Chern class [Bea83b].

Theorem 5.3 (Beauville–Bogomolov decomposition theorem)**.** *Let X be a compact Kähler manifold with trivial real first Chern class. There exists a finite étale cover $\widetilde{X} \to X$ which decomposes as*

$$\widetilde{X} \simeq T \times \prod_i V_i \times \prod_j Y_j$$

where

(1) *T is a complex torus,*

(2) *V_i is an IHS manifold for all i, and*

(3) *Y_j is a strict Calabi–Yau manifold for all j.*

Remark 5.4. A compact Kähler manifold Y is said to be *strict Calabi–Yau* if Y has complex dimension strictly larger than 2, the canonical bundle ω_Y is trivial and the vector spaces $H^0(Y, \Omega_Y^p)$ are trivial for all $0 < p < \dim_{\mathbb{C}}(Y)$. Note that if we allow Y to have dimension 1, then Y is an elliptic curve, and if Y had dimension 2, we would recover that Y is a K3 surface hence irreducible holomorphic symplectic. This is why we restrict Y to have dimension at least 3, and call it "strict".

The question of studying IHS manifolds and understanding their symmetries is therefore natural. We already see from Proposition 5.2 that they share common properties with K3 surfaces, and we show later some more interesting similarities. This is why IHS manifolds are often told to be the higher-dimensional analogs of K3 surfaces, in this regard.

IHS manifolds have been a central object of studies for the last 30+ years, and they turned out to be good candidates for testing conjectures. One could cite for instance the proof of Tate's conjecture for K3 surfaces (see for instance the survey [Tot17]), or the proof of Morrison–Kawamata cone conjecture for IHS manifolds by Amerik and Verbitsky [AV17a]

Another motivation for studying the symmetries of IHS manifolds, which goes along with the previous paragraph, is to construct examples with given finite groups of symmetries and specific geometric features. In fact, the definition of irreducible holomorphic symplectic manifolds being quite rigid, it is hard task to construct explicit examples in general (compared to complex tori and strict Calabi–Yau manifolds).

5.2. Known deformation types of IHS manifolds

Recall that IHS manifolds have even complex dimension. For an IHS manifold X, there exists a universal deformation family $\mathcal{X} \to \mathrm{Def}(X)$ such that $\mathrm{Def}(X)$ is smooth and connected, and $X = \mathcal{X}_0$. Note that any deformation of X in a small disk $0 \in \Delta \subseteq \mathrm{Def}(X)$ is again an IHS manifold (see for instance [Huy99, §1.12] and reference therein). For each even complex dimension $2n \geq 2$, it is therefore customary to study IHS manifolds up to deformation.

Remark 5.5. Some interesting invariants are preserved under deformation, which will be made precise later. Already though, let us point out that two deformation equivalent IHS manifolds X and Y have the same second Betti number $b_2(Y) = b_2(X) := \dim_{\mathbb{C}} H^2(X, \mathbb{C})$.

5.2.1. First examples

At the time this thesis is written, only a handful of *deformation types* of IHS manifolds are known for each dimension. We introduce them now.

5.2.1.1. K3 surfaces

As we have already seen, an IHS manifold X of dimension 2 is a K3 surface. Examples of (projective) K3 surfaces can be obtained in the following way:

(1) any double cover $\pi : S \to \mathbb{P}^2$ branched along a smooth sextic curve is a K3 surface;

(2) smooth complete intersections in projective spaces are K3 surfaces in the case of
   (a) a quartic hypersurface $Q_4 \subseteq \mathbb{P}^3$;
   (b) an intersection $Q_2 \cap Q_3$ of a quadric and a cubic hypersurfaces in $\mathbb{P}^4$;
   (c) a so-called *triquadric*, given as the intersection $Q_2^1 \cap Q_2^2 \cap Q_2^3$ of three quadric hypersurfaces in $\mathbb{P}^5$.

Together, projective K3 surfaces and *abelian surfaces* define the two classes of projective K-trivial surfaces i.e. with trivial canonical bundle. We show in the following series of constructions that such surfaces are fundamental for the theory, and the construction, of IHS manifolds.

5.2.1.2. Hilbert scheme of points on projective K-trivial surfaces

As in most of the part of the thesis, we suppose that we work with projective complex varieties. One can find the details of the proofs of the following statements in [Bea83a, §6], and the references therein.

Let S be a projective K-trivial surface and let $n \geq 2$ be a positive integer. We denote by $S^{[n]}$ the so-called *Hilbert–Douady scheme* of n points on S, which is the variety parametrizing 0-dimensional (analytic) subspaces Z of S of length $h^0(Z, \mathcal{O}_Z) = n$. By results of Varouchas and Fogarty, it is known that the space $S^{[n]}$ is smooth and Kähler, of dimension $2n$.

Another description of $S^{[n]}$ is the following. Let S^n be the n-fold product of S, and let $S^{(n)} := S^n/\mathcal{S}_n$ be the associated symmetric quotient, where $\mathcal{S}_n$ is the nth symmetric group. The space $S^{(n)}$ is singular and it admits a minimal resolution

$$S^{[n]} \xrightarrow{HC} S^{(n)}$$

where HC is called *Hilbert–Chow* map. For instance, for $n = 2$, this resolution is isomorphic to the blowup of $S^{(2)}$ along the diagonal $\Delta := \{(x, x) \in S^{(2)} \ : \ x \in S\}$.

Proposition 5.6 ([Bea83a, §6, Théorème 3]). *Let S be a projective K3 surface, and let $n \geq 2$. Then $S^{[n]}$ is a smooth projective IHS manifold of dimension $2n$ and $b_2(S^{[n]}) = 23$.*

In the case $n = 1$, one recovers that $S^{[1]} = S$ is a projective K3 surface.

Example 5.7 ([Bea83a, §6]). Let S be a projective K3 surface, and let us denote by $\Delta \subseteq S^2$ and $\Delta' \subseteq S^{(2)}$ the respective diagonals. Then, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Bl}_{\Delta} S^2 \arrow[r, "p"] \arrow[d, "q"'] & S^2 \arrow[d, "\pi"] \\
S^{[2]} = \mathrm{Bl}_{\Delta'} S^{(2)} \arrow[r, "b"] & S^{(2)}
\end{tikzcd} .$$

The surjective map π is the quotient by the involution $\iota \in \mathrm{Aut}(S^2)$ given by exchanging the two factors. The automorphism ι lifts along p to an involution $\tilde{\iota}$ on $\mathrm{Bl}_{\Delta} S^2$: the map q is the corresponding quotient map. Let (x_1, x_2) and (y_1, y_2) be local coordinates around two points of S, each of them living in one of the copies of S in S^2. Then $\omega := dx_1 \wedge dx_2 + dy_1 \wedge dy_2$ is a holomorphic 2-form on S^2, and the pullback $p^*\omega$ is preserved by $\tilde{\iota}$. Therefore, there exists a holomorphic 2-form σ on $S^{[2]}$ such that $q^*\sigma = p^*\omega$. It turns out that the resulting holomorphic 2-form σ on $S^{[2]}$ is nowhere degenerate and unique up to scaling. Since we chose S projective, we have that $S^{[2]}$ is projective hence Kähler. Moreover, simply-connectedness follows by remarking that since b is a blowup, we have that

$$\pi_1(S^{[2]}) \cong \pi_1(S^{(2)} \setminus \Delta') \cong \pi_1(S^{(2)}) = 1.$$

Now if A is an abelian surface, and $n \geq 2$, the space $A^{[n+1]}$ is not simply-connected because neither is A. To remedy this, we consider the following Albanese map

$$\mathrm{alb} \colon A^{[n+1]} \xrightarrow{HC} A^{(n+1)} \xrightarrow{\Sigma} A$$

where the second map associates to an $(n+1)$-tuple of points their sum in A. The fibers of alb are pairwise isomorphic, they are smooth and simply-connected. Moreover:

Proposition 5.8 ([Bea83a, §6, Théorème 4]). *Let A be an abelian surface, and let $n \geq 2$. Then $K_n(A) := \mathrm{alb}^{-1}(0_A)$ is a smooth projective IHS manifold of dimension $2n$ and $b_2(K_n(A)) = 7$.*

Similarly, if $n = 1$ one recovers that $K_1(A) = \mathrm{Kum}(A)$ is the *Kummer surface* of A, which is a K3 surface (see for instance [Nik75]). This is also why, in the literature, $K_n(A)$ is often called a *generalized Kummer variety* of A.

These two series of examples are crucial because they give, for every even dimension $2n \geq 4$, two examples of IHS manifolds which are not deformation equivalent: in fact, since the second Betti numbers of both examples do not agree, they cannot be equivalent under deformation.

5.2.1.3. O'Grady's examples

These are not the only known types of deformation. Thanks to O'Grady, we know two further deformation types in dimension 6 and 10, respectively denoted OG6 [O'G03] and OG10 [O'G99]. These were originally constructed by considering some moduli space of semistable sheaves on projective K-trivial surfaces. An important thing to note is that O'Grady sixfolds have second Betti number 8, and O'Grady's tenfolds have second Betti number 24. In particular, these are genuine new deformation types in dimension 6 and 10 respectively.

Remark 5.9. At the time this thesis is written, no other deformation types have been discovered apart from the one previously cited: we refer to them as *known deformation types*.

Definition 5.10. Let $n \geq 2$, and let X be a projective IHS manifold. We say X is of

(1) *$K3^{[n]}$-type*, or of deformation type $\mathrm{K3}^{[n]}$, if there exists a projective K3 surface S such that X and $S^{[n]}$ are deformation equivalent. In this case we write $X \sim \mathrm{K3}^{[n]}$.

(2) *Kum_n-type*, or of deformation type Kum_n, if there exists an abelian surface A such that X and $K_n(A)$ are deformation equivalent. In this case we write $X \sim \mathrm{Kum}_n$.

(3) *OG6-type*, or deformation type OG6, if there exists an O'Grady sixfold X_6 such that X and X_6 are deformation equivalent. In this case we write $X \sim$ OG6.

(4) *OG10-type*, or deformation type OG10, if there exists an O'Grady tenfold X_{10} such that X and X_{10} are deformation equivalent. In this case we write $X \sim$ OG10.

We will say that two IHS manifolds X and Y are *of the same deformation type* is they are deformation equivalent. For instance, if both $X, Y \sim OG10$, then they are of the same deformation type. We often work with IHS manifolds of abstract deformation type $\mathcal{T}$, and we therefore just write $X \sim \mathcal{T}$ to say that X is of deformation type $\mathcal{T}$.

5.2.2. The Fano variety of lines on a cubic fourfold

It is relevant to note that the projective IHS manifolds constructed in Section 5.2.1 all have Picard rank at least 2. In this section, we present one of the early nontrivial examples of projective IHS manifolds, which has generically Picard rank 1.

Let W be a 6-dimensional complex vector space, and let $V \subseteq \mathbb{P}(W)$ be a smooth cubic fourfold. In particular, there exists $f \in \mathrm{Sym}^3 W^\vee$ a homogeneous smooth cubic form such that $V = V(f)$. Let us consider the *Fano variety of lines* $F(V)$ of V. Set-theoretically, one describes $F(V)$ as the set of lines $l \subseteq \mathbb{P}^5$ which are contained in V. Seeing lines in $\mathbb{P}^5$ as corresponding to planes in the 6-dimensional vector space W, we can therefore identify $F(V)$ with a subset of $\mathrm{Gr}(2, W)$, the Grassmannian of planes in W. In order to describe a scheme structure on $F(V)$, we use the following practical result from representation theory. The proof is a direct consequence of the Borel–Weil theorem: we refer to [Wey03, Corollary 4.1.9] for a more general statement.

Theorem 5.11 (Borel–Weil theorem)**.** *Let $0 < k < n$ be integers and let W be an n-dimensional complex vector space. We denote by $X := \mathrm{Gr}(k, W)$ the Grassmannian of k-spaces in W, and we let $F := \mathcal{U}_X$ be the tautological bundle of X, i.e. the vector bundle $F \to X$ whose fiber over any $U \in X$ is U itself. Then for all $\alpha \geq 0$ and for all $0 \leq \beta \leq k$ one has that as $\mathbb{C}$-vector spaces*

$$H^0(X, \mathrm{Sym}^\alpha F^\vee) \simeq \mathrm{Sym}^\alpha W^\vee,$$

$$\textit{and } H^0\left(X, \bigwedge^\beta F^\vee\right) \simeq \bigwedge^\beta W.$$

Remark 5.12. The result of Borel and Weil is stated for bundles over a homogeneous variety G/P, associated to irreducible representations of G. In our case, we have that $\mathrm{Gr}(k,W)$ can be seen as a homogeneous variety with $G = \mathrm{GL}(W)$. There is a generalization of Borel–Weil theorem due to Bott, for higher degree cohomology groups (see for instance [FH91, §23.3]).

It follows from Borel–Weil theorem that we can see the cubic form f defining V as a global section in $H^0(\mathrm{Gr}(2,W), \mathrm{Sym}^3\mathcal{U}^\vee_{\mathrm{Gr}(2,W)}) \simeq \mathrm{Sym}^3 W$. Its zero locus is the set of planes in W on which f is identically 0: in particular $F(V) = V(f) \subseteq \mathrm{Gr}(2,W)$. Note that $F(V)$ is smooth because so is V: moreover we observe that since $\dim_{\mathbb{C}}(\mathrm{Gr}(2,W)) = 8$ and $\mathrm{Sym}^3\mathcal{U}^\vee_{\mathrm{Gr}(2,W)}$ is a bundle of rank $\binom{4}{3} = 4$, we have that $F(V)$ is a smooth fourfold.

Proposition 5.13 ([BD85, Proposition 2])**.** *The fourfold $F(V)$ is an IHS manifold of deformation type* $\mathrm{K3}^{[2]}$.

In order to show the previous theorem, Beauville and Donagi consider a special cubic fourfold, known as *Pfaffian cubic*, and show that its Fano variety of lines is isomorphic to $S^{[2]}$ where S is a projective K3 surface [BD85, Proposition 5]. The general result follows then using some deformation argument.

Remark 5.14. The authors in [IM08, §2.1] give an explicit description of the symplectic form $F(V)$ for a smooth cubic fourfold V.

Remark 5.15. Given a smooth cubic fourfold $V \subseteq \mathbb{P}(W)$, then its Fano variety of lines $F(V)$ is equipped with an ample line bundle, being the restriction of the hyperplane section from the Plücker space $\mathbb{P}(\bigwedge^2 W)$. In particular, $F(V)$ has Picard rank at least one. In [BD85, Proposition 6] the authors actually show that for V very general in the moduli space of cubic fourfolds, the variety $F(V)$ has Picard rank exactly 1.

5.3. Cohomology and projectivity criterion

We would like to state analogs of the Torelli-type theorems known for K3 surfaces, but for IHS manifolds in general. For this, we start by describing an integral $\mathbb{Z}$-lattice structure on the second integral cohomology of IHS manifolds. The content of the next sections can be recovered from standard references, such as [Huy99] and [Deb22].

Let X be an IHS manifold of complex dimension $2n \geq 2$. We have seen in Proposition 5.2 that $H^2(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module, and it is moreover of finite rank. Remark that $H^2(X,\mathbb{C})$ is endowed with a pure Hodge structure of weight 2 given by

$$H^{2,0}(X) \cong H^0(X,\Omega^2_X), \quad H^{0,2}(X) = \overline{H^{2,0}(X)} \cong H^2(X,\mathcal{O}_X), \text{ and } H^{1,1}(X) \cong H^1(X,\Omega_X).$$

Remark 5.16. By Lefschetz theorem on $(1,1)$-classes, we have moreover that $\mathrm{NS}(X) := \mathrm{im}(c_1) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$.

Let $\sigma \in H^{2,0}(X)$ be nonzero such that $\int_X (\sigma\overline{\sigma})^n = 1$ — in general this integral is always nonzero, so we choose σ, unique up to scaling, so that it is 1. For all $v \in H^2(X,\mathbb{C})$, we define

$$q(v) := \frac{n}{2}\int_X (\sigma\overline{\sigma})^{n-1}v^2 + (1-n)\int_X \sigma^{n-1}\overline{\sigma}^n v \cdot \int_X \sigma^n\overline{\sigma}^{n-1}v.$$

Note that q defines a quadratic form on $H^2(X,\mathbb{C})$. Moreover, the following holds.

Theorem 5.17 ([Bea83a, Proposition 5])**.** *The quadratic form q is nondegenerate, and there exists a real constant $c_X > 0$ such that $c_X q$ restricted to $H^2(X,\mathbb{Z})$ gives the latter a structure of indivisible and integral $\mathbb{Z}$-lattice.*

The form q above is often referred to as *Beauville–Bogomolov* quadratic form, and c_X is called *Fujiki constant*. Observe that by definition we have that

$$q(\sigma) = q(\overline{\sigma}) = 0.$$

From now on, we will always assume that $H^2(X, \mathbb{Z})$ comes equipped with the restriction $q_X := (c_X q)_{|H^2(X,\mathbb{Z})}$ which we call *Beauville–Bogomolov–Fujiki* quadratic form, or BBF form for short. We denote again by q_X the symmetric bilinear form associated to q_X on $H^2(X, \mathbb{Z})$. According to [Bea83a, Proposition 5], we have that the $\mathbb{Z}$-lattice $(H^2(X, \mathbb{Z}), q_X)$ has real signatures $(3, b_2(X)-3)$: in fact, the two real classes $\sigma + \overline{\sigma}$ and $i(\sigma - \overline{\sigma})$ have positive norm with respect to q. Moreover, for every class $\lambda \in H^2(X, \mathbb{R})$ of a Kählerian metric on X, we have that $q(\lambda) > 0$.

Remark 5.18. The form q_X and the constant c_X are invariant under deformation and birational equivalences. In particular, these are intrinsic invariants of any deformation type, and they have been determined for all the known deformation types. Moreover, $(H^2(X, \mathbb{Z}), q_X)$ is an even $\mathbb{Z}$-lattice in the known cases. See [Bea83a, Propositions 6 and 8] for the $\mathrm{K3}^{[n]}$ types and the Kum_n types respectively, [Rap07, Corollary 3.5.13] for the deformation type OG6, and [Rap08, Theorem 4.3] for the deformation type OG10. For a known deformation type $\mathcal{T}$, we collect in Table 3 the second Betti number of any $X \sim \mathcal{T}$, the signatures and the abstract isometry class $\Lambda_{\mathcal{T}}$ of $(H^2(X, \mathbb{Z}), q_X)$, and the associated Fujiki constant $c_{\mathcal{T}} := c_X$.

Table 3: Known BBF forms and Fujiki constants

$\mathcal{T}$	b_2	sign	$\Lambda_{\mathcal{T}}$	$c_{\mathcal{T}}$
Kum_n	7	$(3, 4)$	$U^{\oplus 3} \oplus A_1(n+1)$	$(n+1)\frac{(2n)!}{n!2^n}$
OG6	8	$(3, 5)$	$U^{\oplus 3} \oplus A_1^{\oplus 2}$	60
K3	22	$(3, 19)$	$U^{\oplus 3} \oplus E_8^{\oplus 2}$	1
$\mathrm{K3}^{[n]}$	23	$(3, 20)$	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_1(n-1)$	$\frac{(2n)!}{n!2^n}$
OG10	24	$(3, 21)$	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$	945

Example 5.19. Let S be a projective K3 surface and let $n \geq 2$. We denote again by $HC\colon S^{[n]} \to S^{(n)}$ the Hilbert–Chow morphism. According to [Bea83a, Proposition 6], there exists an injective homorphism

$$i\colon H^2(S, \mathbb{C}) \hookrightarrow H^2(S^{[n]}, \mathbb{C})$$

such that

$$H^2(S^{[n]}, \mathbb{C}) = i(H^2(S, \mathbb{C})) \oplus \mathbb{C}E$$

where E is the class of the exceptional divisor of HC. Note that $E \in H^{1,1}(S^{[n]})$ is of $(1,1)$-type and $q_{S^{[n]}}(E, i(H^2(S, \mathbb{C}))) = 0$. Moreover, the homomorphism i is compatible with the Hodge structures, i.e. $i(H^{2,0}(S)) \subseteq H^{2,0}(S^{[n]})$ and $i(H^{1,1}(S)) \subseteq H^{1,1}(S^{[n]})$, and it induces a primitive embedding of even $\mathbb{Z}$-lattices

$$\iota\colon H^2(S, \mathbb{Z}) \hookrightarrow H^2(S^{[n]}, \mathbb{Z})$$

whose orthogonal complement is spanned by $\delta := E/2$. In particular, via i, we obtain that $\mathrm{NS}(S^{[n]}) \simeq \mathrm{NS}(S) \oplus \mathbb{Z}\delta$, which explains why $S^{[n]}$ has Picard rank at least 2 whenever S is projective.

Via the Hodge decomposition of $H^2(X,\mathbb{C})$, we can write any $v \in H^2(X,\mathbb{C})$ as $\alpha\sigma + \lambda + \beta\overline{\sigma}$ where $\lambda \in H^{1,1}(X)$ and $\alpha, \beta \in \mathbb{C}$. With such a description, we obtain by direct calculations that

$$q(v) = \frac{n}{2}\int_X (\sigma\overline{\sigma})^{n-1}\lambda^2 + \alpha\beta.$$

From this, we recover that $H^{2,0}(X)$ and $H^{0,2}(X)$ are isotropic with respect to q, and $H^{1,1}(X)^{\perp} = H^{2,0}(X) \oplus H^{0,2}(X)$. In that situation, we say that the Hodge structure on $H^2(X,\mathbb{C})$ is *polarized* with respect to q (see [DK07] for more details on polarized Hodge structures).

Remark 5.20. Remark that polarized weight 2 Hodge structures are completely characterized by the direct summand of $(2,0)$-type. In that situation, by the existence of the quadratic form q on $H^2(X,\mathbb{C})$, we see that one can recover the full weight 2 Hodge structure on the vector space $H^2(X,\mathbb{C})$ from $H^{2,0}(X)$.

Finally, we define $\mathrm{T}(X)$ to be the smallest primitive sublattice of $(H^2(X,\mathbb{Z}), q_X)$ whose $\mathbb{C}$-span $\mathrm{T}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ contains the symplectic forms of X. It is called the *transcendental lattice* of X and by what we have previously obersevd, it has positive real signature at least equal to 2. The following projectivity criterion from Huybrechts tells us exactly when there is equality.

Theorem 5.21 ([Huy99, Thereom 3.11])**.** *Let X be an IHS manifold. Then X is projective if and only if there exists a line bundle L on X such that $q_X(c_1(L)) > 0$.*

Note that this is equivalent to the existence of a vector of positive norm in $\mathrm{NS}(X)$. Hence we deduce that X is projective if and only if $\mathrm{NS}(X)$ is hyperbolic (i.e. has signatures $(1,*)$), if and only if $\mathrm{T}(X)$ has positive signature 2.

Remark 5.22. According to [Huy99], for every ample line bundle L on a projective IHS manifold X, we have $q_X(c_1(L)) > 0$, and in particular $c_1(L)$ is Kähler. However the converse does not always hold: a line bundle L on X satisfying $q_X(c_1(L)) > 0$ is not necessarily ample.

Definition 5.23. Let X be an IHS manifold. We call *polarization* on X any ample line bundle L such that $c_1(L) \in \mathrm{NS}(X)$ is primitive. If L is not ample but $c_1(L) \in \mathrm{NS}(X)$ is primitive of positive norm, then we say L is a *quasipolarization*.

Example 5.24. Let $V \subseteq \mathbb{P}(W)$ be a smooth cubic fourfold, and let $F(V)$ be its Fano variety of lines (Section 5.2.2). Then the line bundle $\mathcal{O}_{F(V)}(1)$, which is the restriction of the hyperplane class of the Plücker space $\mathbb{P}(\bigwedge^2 W)$ to $F(V)$, is very ample. In particular, it is a polarization and $c_1(\mathcal{O}_{F(V)}(1))$ has type $(6,2)$ in $H^2(X,\mathbb{Z})$ (Definition 1.13).

Definition 5.25. Let X be an IHS manifold, and let L be a (quasi)polarization on X. Let (n,d) be the type of $c_1(L)$ as a primitive vector in $H^2(X,\mathbb{Z})$, i.e. $c_1(L)^2 = n$ and $\mathrm{div}(c_1(L), H^2(X,\mathbb{Z})) = d$. Then we say L is a *(n,d)-(quasi)polarization*. If X admits an (n,d)-(quasi)polarization for a pair $(n,d) \in \mathbb{Z}_{\neq 0} \times \mathbb{N}$, then we say that X is *(n,d)-(quasi)polarized*.

5.4. Periods and Torelli-type theorems

From now on, we fix $\mathcal{T}$ a deformation type of IHS manifold, and we let $\Lambda_{\mathcal{T}}$ be the abstract $\mathbb{Z}$-lattice associated to that deformation type.

Definition 5.26. Let $X \sim \mathcal{T}$ be an IHS manifold. We call *marking* any isometry $\eta : H^2(X,\mathbb{Z}) \to \Lambda_{\mathcal{T}}$. Together, we call (X,η) a *marked pair*. Two marked pairs (X,η) and (X',η') with $X, X' \sim \mathcal{T}$ are called *equivalent* if there exists an isomorphism $f : X \to X'$ such that $\eta' = \eta \circ f^*$.

For a fixed deformation type $\mathcal{T}$, there exists a coarse moduli space $\mathcal{M}_{\mathcal{T}}$ parametrizing marked pairs (X, η), with $X \sim \mathcal{T}$, up to equivalence. This moduli space has dimension $\mathrm{rank}_{\mathbb{Z}}(\Lambda_{\mathcal{T}}) - 2$ and it is in general neither connected nor Hausdorff. It is reasonable to question whether the choice of a marking η for a fixed IHS manifold $X \sim \mathcal{T}$ affects in which connected component of $\mathcal{M}_{\mathcal{T}}$ the class $[(X, \eta)]$ lies. Let us define first define the following (see [Mar11, Definition 1.1]).

Definition 5.27. Let X, X_1 and X_2 be IHS manifolds of deformation type $\mathcal{T}$.

(1) An isometry $f\colon H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is said to be a *parallel-transport operator* if there exist a smooth and proper family $\pi\colon \mathcal{X} \to B$ of IHS manifolds, over an analytic base B, two points $b_1, b_2 \in B$ so that $X_i \simeq \mathcal{X}_{b_i}$ for $i = 1, 2$, and a continuous path $\gamma\colon [0, 1] \to B$ satisfying that $\gamma(0) = b_1$, $\gamma(0) = b_2$, and such that f induces a parallel-transport in the local system $R^2\pi_*\underline{\mathbb{Z}}$ along γ.

(2) A *monodromy operator* of X is a parallel-transport operator $f\colon H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$. We denote by $\mathrm{Mon}^2(X) \leq O(H^2(X, \mathbb{Z}))$ the subgroup of parallel transport operators of X.

Note that for an IHS manifold X, the group $\mathrm{Mon}^2(X)$ is a subgroup of $O^+(H^2(X, \mathbb{Z}))$ of finite index [Mar11, Lemma 7.5]. It is not known whether $\mathrm{Mon}^2(X)$ is normal in $O(H^2(X, \mathbb{Z}))$ for X any IHS manifold: however we know it to be normal for the known examples of IHS manifolds (see Section 6.2).

Example 5.28. If $X \sim \mathrm{K3}^{[n]}$ for some $n \geq 2$, then

$$\mathrm{Mon}^2(X) = \{f \in O^+(H^2(X, \mathbb{Z})) \ : \ f \text{ or } -f \text{ is a stable isometry}\}$$

[Mar10, Lemma 4.2]. As claimed, in that case, the monodromy group of X is normal in $O(H^2(X, \mathbb{Z}))$ and it is maximal if and only if $n - 1$ is a prime power (possibly trivial). In fact, we have that $D_{H^2(X,\mathbb{Z})} \simeq \big(-1/(2n-2)\big)$ as torsion quadratic modules and $O(D_{H^2(X,\mathbb{Z})}) \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $2(n - 1)$ is prime or twice a prime power.

Proposition 5.29 ([Huy99, Theorem 4.6])**.** *Two classes $[(X, \eta)], [(X', \eta')] \in \mathcal{M}_{\mathcal{T}}$ lie in the same connected component if and only if $\eta^{-1} \circ \eta' : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is a parallel transport operator.*

Therefore, different choices of a marking η for $X \sim \mathcal{T}$ could make the pair $[(X, \eta)]$ move between different connected components of $\mathcal{M}_{\mathcal{T}}$. A particular consequence of the previous proposition is that the number of connected components of $\mathcal{M}_{\mathcal{T}}$ is given by the index of $\mathrm{Mon}^2(X)$ in $O(H^2(X, \mathbb{Z}))$. In order now to understand inseparable points in $\mathcal{M}_{\mathcal{T}}$, let us introduce the *period domain*

$$\Omega_{\mathcal{T}} := \{[\omega] \in \mathbb{P}(\Lambda_{\mathcal{T}} \otimes_{\mathbb{Z}} \mathbb{C}) \ : \ \omega^2 = 0, \ \omega.\overline{\omega} > 0\}$$

and the associated *period map*

$$\mathcal{P}_{\mathcal{T}} : \mathcal{M}_{\mathcal{T}} \to \Omega_{\mathcal{T}}, \ [(X, \eta)] \mapsto \eta(H^{2,0}(X)).$$

Note that the definition makes sense since we have already seen that the for an IHS manifold X, the space of 2-forms $H^{2,0}(X)$ on X is totally isotropic for the BBF form q_X. It is well-known that $\mathcal{P}_{\mathcal{T}}$ is a local isomorphism [Bea83a, Théorème 5]. Moreover:

Theorem 5.30 ([Huy99, Theorem 8.1])**.** *The period map $\mathcal{P}_{\mathcal{T}}$ is surjective when restricted to any connected component of $\mathcal{M}_{\mathcal{T}}$.*

Remark 5.31. Given a marked pair (X, η) of deformation type $\mathcal{T}$, one can identify the universal deformation space $\mathrm{Def}(X)$ of X with an open analytic neighborhood of $[(X, \eta)] \in \mathcal{M}_{\mathcal{T}}$. The period map $\mathcal{P}_{\mathcal{T}}$ can be obtained by gluing the local period maps of the form $\mathrm{Def}(X) \to \Omega_{\mathcal{T}}$ for (X, η) of deformation type $\mathcal{T}$, which are all local isomorphisms [Bea83a, Théorème 5].

In some sense, via the period map, one can characterize IHS manifolds of a fixed deformation by their period, which itself corresponds to a polarized weight 2 Hodge structure on $\Lambda_{\mathcal{T}} \otimes_{\mathbb{Z}} \mathbb{C}$. However, this period map loses some information along the way, and in particular periods do not distinguish birational models, except in the case of K3 surfaces.

Remark 5.32. A nontrivial result we should keep in mind is that two IHS manifolds which are birational onto each other are actually deformation equivalent [Huy03, Theorem 2.5].

Proposition 5.33 ([Huy99])**.** *Let $\mathcal{T}$ be a fixed deformation type of IHS manifolds.*

(1) *For a point $[\omega] \in \Omega_{\mathcal{T}}$, the fiber $\mathcal{P}_{\mathcal{T}}^{-1}([\omega])$ consists of pairwise inseparable points;*

(2) *If two points $[(X, \eta)], [(X', \eta')] \in \mathcal{M}_{\mathcal{T}}$ in the same connected component have the same period, then X, X' are birational.*

Proposition 5.33 is often referred to as *Global Torelli theorem*. We introduce another Torelli-type theorem, as stated by Markman and Verbitsky, which is sometimes referred to as *Hodge-theoretic Torelli theorem*, or *Strong Torelli theorem*.

Definition 5.34. Let X and Y be two IHS manifolds of the same deformation type.

(1) Any isometry $f\colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ which is compatible with the Hodge structures, i.e. such that $f_{\mathbb{C}}(H^{2,0}(X)) = H^{2,0}(Y)$ is said to be a *Hodge isometry*.

(2) Any parallel-transport operator $f\colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ which is a Hodge isometry is said to be *Hodge parallel-transport operator*.

(3) Similarly, any monodromy operator $f \in \mathrm{Mon}^2(X)$ which is a Hodge isometry is called a *Hodge monodromy*.

We denote by $\mathrm{Mon}^2_{Hdg}(X)$ the subgroup of $\mathrm{Mon}^2(X)$ consisting of Hodge monodromies.

Theorem 5.35 ([Mar11, Theorem 1.3], [Ver13, Theorem 1.17])**.** *Let X and Y be two IHS manifolds of the same deformation type $\mathcal{T}$. Then*

(1) *X and Y are birational if and only if there exists a Hodge parallel-transport operator $H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$.*

(2) *Let $g\colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ be a Hodge parallel-transport operator. There exists an isomorphism $f\colon Y \to X$ such that $g = f^*$ if and only if $g_{\mathbb{R}}$ maps a Kähler class of X to a Kähler class of Y.*

In Section 5.5, we rewrite this theorem using the so-called *Kähler cone* and *fundamental exceptional chamber* of IHS manifolds. Doing so, we translate a bit more the conditions from the Global Torelli theorem into lattice-theoretic criteria which could be applied outside of any geometric context.

5.5. Decompositions of the positive cone

From now on, suppose that X is an IHS manifold of complex dimension $2n$. For any subring $R \leq \mathbb{C}$, we denote by

$$H^{1,1}(X,R) := H^{1,1}(X) \cap H^2(X,R)$$

the associated R-modules, which we equip with the form q_X extended to R. The real quadratic form $(H^{1,1}(X,\mathbb{R}), q_X)$ is hyperbolic, and its positive cone

$$\mathcal{P}_X := \{v \in H^{1,1}(X,\mathbb{R}) \ : \ q_X(v) > 0\}$$

has two connected components: one of them contains a Kähler class of X, and we denote it by C_X. We call the latter the *positive cone* of X. The set of Kähler classes on X defines a real convex open subcone $\mathcal{K}_X \subseteq C_X$ which we call the Kähler cone of X.

Definition 5.36. We define the *birational Kähler cone* of X to be

$$\mathcal{BK}_X := \bigcup_{f\colon X \dashrightarrow X'} f^*\mathcal{K}_{X'}$$

where the union runs over all birational maps $f\colon X \dashrightarrow X'$.

Remark 5.37. The birational Kähler cone $\mathcal{BK}_X$ of an IHS manifold X is not a cone per se, however $\overline{\mathcal{BK}_X} \cap H^{1,1}(X,\mathbb{R})$ is. The latter coincides with the *movable cone* $\mathrm{Mov}(X) \subseteq \overline{(\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}) \cap C_X}$ which is the cone generated by classes of movable line bundles on X. We recall that a line bundle L on X is said to be *movable* if the base locus of the linear system $|L|$ has codimension at least 2 in X.

We would like to define two important walls and chambers decompositions of the positive cone C_X. For this, let us define the following; see [Mar11, §5 and 6] and [AV15, Definition 1.13] for more details.

Definition 5.38. Let X be an IHS manifold.

(1) A *prime divisor* D on X is a reduced and irreducible effective divisor $D \subseteq X$.

(2) A finite collection of prime divisors $\{E_1, \dots, E_n\}$ on X is said to be *exceptional* if the restriction of q_X to $\mathbb{R}E_1 + \cdots + \mathbb{R}E_n \leq H^{1,1}(X,\mathbb{R})$ is negative definite.

(3) An effective divisor $E \subseteq X$ on X is called *prime exceptional* if its support is exceptional, in the sense of (2).

Given a line bundle L on X, there exists a smooth hypersurface $\mathrm{Def}(X,L) \subseteq \mathrm{Def}(X)$ and a universal deformation family

$$(\mathcal{X},\mathcal{L}) \to \mathrm{Def}(X,L)$$

of the pair (X,L). This means that $\mathcal{X} \to \mathrm{Def}(X,L)$ is a deformation of $\mathcal{X}_0 = X$ and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ such that $\mathcal{L}_0 := \mathcal{L}_{|\mathcal{X}_0} \simeq L$. For every $t \in \mathrm{Def}(X,L)$ we let $(X_t, L_t) := (\mathcal{X},\mathcal{L})_t$ be the corresponding deformation [Huy99].

Definition 5.39. Let again X be an IHS manifold.

(1) A line bundle L on X is called *stably prime exceptional* if there exists a closed analytic subset $Z \subseteq \mathrm{Def}(X,L)$, of positive codimension, such that the linear system $|L_t|$ consists of a prime exceptional divisor E_t of X_t, for all $t \in \mathrm{Def}(X,L) \setminus Z$.

(2) A nonzero primitive class $v \in \mathrm{NS}(X)$ is called *stably prime exceptional* if $v = c_1(L)$ for L a stably prime exceptional line bundle on X.

(3) A nonzero class $v \in H^{1,1}(X, \mathbb{Q})$ of negative norm is called *monodromy birationally minimal* (MBM for short) if for all Hodge monodromies $\gamma \in \mathrm{Mon}^2(X)$ we have $\gamma(z)^\perp \cap \mathcal{BK}_X = \emptyset$.

In what follows, we denote by $\mathcal{W}(X)$ the set of primitive integral MBM classes of X and $\mathcal{W}^{pex}(X)$ the set of stably prime exceptional divisors. Remark that $\mathcal{W}^{pex}(X) \subseteq \mathcal{W}(X)$.

The idea behind the definition of stably prime exceptional line bundles on an IHS manifold X is to have a notion of prime exceptional line bundles L which stays prime exceptional on general enough deformations of the pair (X, L). Moreover, primitive integral MBM classes can be seen as classes whose orthogonal complement defines "walls" inside the movable cone of an IHS manifold X. Such walls bound the birational Kähler cone $\mathcal{BK}_X$ of X inside the positive cone C_X, and they separate the pullbacks of the Kähler cones for the different IHS manifolds birational to X.

Notation. For this reason, primitive integral MBM classes are also often called *wall divisors*; note however that such classes might not necessarily be the first Chern class of an effective divisor.

Note that stably prime exceptional and wall divisors of the known IHS manifolds are numerically characterized. By this we mean the following. Let $\mathcal{T}$ be a known deformation type of IHS manifolds, and let $\Lambda_\mathcal{T}$ be the associated even $\mathbb{Z}$-lattice (Table 3). Then, there exist two sets of vectors in $\Lambda_\mathcal{T}$

$$\mathcal{W}^{pex}(\Lambda_\mathcal{T}) \subseteq \mathcal{W}(\Lambda_\mathcal{T})$$

such that, for all $X \sim \mathcal{T}$ IHS manifold of deformation $\mathcal{T}$ and for any marking $\eta\colon H^2(X, \mathbb{Z}) \xrightarrow{\simeq} \Lambda_\mathcal{T}$ one has

$$\mathcal{W}^{pex}(X) = \eta^{-1}(\mathcal{W}^{pex}(\Lambda_\mathcal{T})) \cap H^{1,1}(X, \mathbb{R})$$

and

$$\mathcal{W}(X) = \eta^{-1}(\mathcal{W}(\Lambda_\mathcal{T})) \cap H^{1,1}(X, \mathbb{R}).$$

Remark 5.40. The proof of these statements actually follows by the characterization of wall divisors for the known IHS manifolds. See [Mar13, Theorem 1.11] and [Mon15, Corollary 2.9] for the $\mathrm{K3}^{[n]}$ types, see [Yos16, Proposition 5.4] for the Kum_n types, see also [HT10] for more on these two infinite families, see [MR21, Theorem 1.2] for OG6, and see finally [MO22, Theorems 3.2 and 5.5] for OG10.

Definition 5.41. The two sets $\mathcal{W}^{pex}(\Lambda_\mathcal{T}) \subseteq \mathcal{W}(\Lambda_\mathcal{T})$ will respectively be called *the set of numerical stably prime exceptional divisors* and *the set of numerical wall divisors* for the deformation type $\mathcal{T}$. For any $X \sim \mathcal{T}$, we define the *numerical type* of any $v \in \mathcal{W}(X)$ to be the type of the vector $\eta(v) \in \Lambda_\mathcal{T}$ (Definition 1.13) where η is any marking of X.

Example 5.42 (Numerical wall divisors for $\mathrm{K3}^{[2]}$)**.** We have

$$\mathcal{W}^{pex}(\Lambda_{\mathrm{K3}^{[2]}}) = \{v \in \Lambda_{\mathrm{K3}^{[2]}} \ : \ v \text{ has type } (-2, 1) \text{ or } (-2, 2)\}.$$

Moreover

$$\mathcal{W}(\Lambda_{\mathrm{K3}^{[2]}}) = \mathcal{W}^{pex}(\Lambda_{\mathrm{K3}^{[2]}}) \sqcup \{v \in \Lambda_{\mathrm{K3}^{[2]}} \ : \ v \text{ has type } (-10, 2)\}.$$

Example 5.43 (Numerical wall divisors for $\mathrm{K3}^{[3]}$)**.** We have

$$\mathcal{W}^{pex}(\Lambda_{\mathrm{K3}^{[3]}}) = \{v \in \Lambda_{\mathrm{K3}^{[3]}} \ : \ v \text{ has type } (-2, 1),\ (-4, 2) \text{ or } (-4, 4)\}.$$

Moreover

$$\mathcal{W}(\Lambda_{\mathrm{K3}^{[3]}}) = \mathcal{W}^{pex}(\Lambda_{\mathrm{K3}^{[3]}}) \sqcup \{v \in \Lambda_{\mathrm{K3}^{[3]}} \ : \ v \text{ has type } (-12, 2) \text{ or } (-36, 4)\}.$$

Remark 5.44. According to Markman [Mar13, Theorem 1.11], vectors of norm -2 are always stably prime exceptional for IHS manifolds of the deformation types $\mathrm{K3}^{[n]}$: we recover in particular the well-known decomposition of the positive cone for K3 surfaces.

Example 5.45 (Numerical walls divisors for OG6)**.** We have

$$\mathcal{W}^{pex}(\Lambda_{\mathrm{OG6}}) = \{v \in \Lambda_{\mathrm{OG6}} \ : \ v \text{ has type } (-2,2) \text{ or } (-4,2)\}.$$

Moreover

$$\mathcal{W}(\Lambda_{\mathrm{OG6}}) = \mathcal{W}^{pex}(\Lambda_{\mathrm{OG6}}) \sqcup \{v \in \Lambda_{\mathrm{OG6}} \ : \ v \text{ has type } (-2,1)\}.$$

Example 5.46 (Numerical wall divisors for OG10)**.** We have

$$\mathcal{W}^{pex}(\Lambda_{\mathrm{OG10}}) = \{v \in \Lambda_{\mathrm{OG10}} \ : \ v \text{ has type } (-2,1) \text{ or } (-6,3)\}.$$

Moreover

$$\mathcal{W}(\Lambda_{\mathrm{OG10}}) = \mathcal{W}^{pex}(\Lambda_{\mathrm{OG10}}) \sqcup \{v \in \Lambda_{\mathrm{OG10}} \ : \ v \text{ has type } (-4,1) \text{ or } (-24,3)\}.$$

The orthogonal complement of any wall divisor $v \in \mathcal{W}(X)$ defines a hyperplane in $H^{1,1}(X,\mathbb{R})$, also known as a *wall*. We call *exceptional* any chamber of

$$C_X \setminus \bigcup_{v \in \mathcal{W}^{pex}(X)} v^{\perp}.$$

We define the *fundamental exceptional chamber* of X to be the set

$$\mathcal{FE}_X := \{x \in C_X \ : \ x.v > 0, \ \forall v \in \mathcal{W}^{pex}(X)\}.$$

It is an exceptional chamber, as defined above, and it contains moreover $\mathcal{BK}_X$ by definition of $\mathcal{W}^{pex}(X)$. Note that in general, $\mathcal{BK}_X \subsetneq \mathcal{FE}_X$.

Proposition 5.47 ([Mar11, Theorem 6.18])**.** *Let X be an IHS manifold, and let $W_{pex}(X)$ be the subgroup of $O^+(H^2(X,\mathbb{Z}))$ generated by the reflections in elements in $\mathcal{W}^{pex}(X)$.*

(1) *The group $W_{pex}(X)$ is a normal subgroup of $\mathrm{Mon}^2_{Hdg}(X)$;*

(2) *The group $W_{pex}(X)$ acts transitively on the set of exceptional chambers in C_X, and $\mathcal{FE}_X$ is a fundamental domain for this action.*

If one denotes moreover $\mathrm{Mon}^2_{bir}(X)$ the stabilizer of $\mathcal{FE}_X$ under the action of $\mathrm{Mon}^2_{Hdg}(X)$ on the set of exceptional chambers, we have

$$\mathrm{Mon}^2_{Hdg}(X) = W_{pex}(X) \rtimes \mathrm{Mon}^2_{bir}(X).$$

We call *Kähler-type chamber* any connected component of

$$C_X \setminus \bigcup_{v \in \mathcal{W}(X)} v^{\perp}.$$

Note that Kähler-type chambers are open in C_X, and they coincide if and only if they have nonempty intersection. The Kähler cone $\mathcal{K}_X$ of X is the Kähler-type chamber determined as

$$\mathcal{K}_X := \{x \in C_X \ : \ x.v > 0, \ \forall v \in \mathcal{W}(X)\}.$$

It is contained in $\mathcal{FE}_X$. From these definitions, we reformulate Theorem 5.35 as follows.

Theorem 5.48 ([Mar11, Theorems 1.2, 1.6 and Corollary 5.7])**.** *Let X and Y be two IHS manifolds which are deformation equivalent, and let $\phi\colon H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ be a Hodge parallel transport operator.*

(1) *There exists an isomorphism $f\colon X \to Y$ such that $\phi = f^*$ if and only if $\phi(\mathcal{K}_Y) = \mathcal{K}_X$.*

(2) *There exists a birational map $f\colon X \dashrightarrow Y$ such that $\phi = f^*$ if and only if $\phi(\mathcal{FE}_X) = \mathcal{FE}_Y$.*

In the beginning of the second chapter of this thesis, we see how one can use the decompositions of the positive cones for IHS manifolds in order to translate the previous theorem into lattice-theoretic criteria. In particular, we will move the problem of studying birational automorphisms of IHS manifolds, into a classification of special isometries of the abstract $\mathbb{Z}$-lattices associated to each of the known deformation types. Note that it is a hard problem to construct explicit birational automorphisms of IHS manifolds, so such a procedure will make our task easier. However, a downside of such an approach is that we lose the geometry from the picture. In fact, even though Theorem 5.48 tells us which isometries of $\mathbb{Z}$-lattices are induced by birational isomorphisms of IHS manifolds, it does not provide a way to reconstruct such maps. In the third chapter of this thesis, we work on this problem, and try to realize some group actions on IHS manifolds whose existence is ensured by the Torelli-type theorems.

5.6. Moduli space of polarized manifolds

Let us fix $\mathcal{T}$ a deformation type of IHS manifolds, with associated $\mathbb{Z}$-lattice Λ (abstract BBF form). Let moreover (n,d) be a pair of positive integers so that Λ contains at least one vector of type (n,d), i.e. a vector $v \in \Lambda$ such that $v^2 = n$ and $\operatorname{div}(v,\Lambda) = d$. Given such a vector v of type (n,d), we denote

$$\Omega_{\mathcal{T},v} := \{w \in \mathbb{P}(v^\perp \otimes_{\mathbb{Z}} \mathbb{C}) \ : \ w^2 = 0, w.\overline{w} > 0\}.$$

This is a closed subset of the period domain $\Omega_{\mathcal{T}}$, and it is acted on by $O(\Lambda_{\mathcal{T}}, v)$, the group of isometries fixing v. Note that $\Omega_{\mathcal{T},v}$ has two connected components, whose stabilizer under the action of $O(\Lambda, v)$ is exactly $O^+(\Lambda, v) := O(\Lambda, v) \cap O^+(\Lambda)$ [GHS10, §1]. We let $\Omega^+_{\mathcal{T},v}$ be such a connected component

Theorem 5.49 ([GHS10, Theorem 1.5])**.** *There exists a coarse moduli space $\mathcal{M}_{\mathcal{T},v}$ parametrizing triples (X,η,L) where $X \sim \mathcal{T}$, η is a marking and L is an (n,d)-polarization on X such that $\eta(c_1(L))$ is in the $O^+(\Lambda)$-orbit of v. The space $\mathcal{M}_{\mathcal{T},v}$ is equipped with a period map*

$$\mathcal{P}_{\mathcal{T},v}\colon \mathcal{M}_{\mathcal{T},v} \to O^+(\Lambda, v)\backslash\Omega^+_{\mathcal{T},v}$$

which is a dominant morphism of quasiprojective varieties with finite fibers.

Note that $\mathcal{P}_{\mathcal{T},v}$ might not be surjective, contrary to $\mathcal{P}_{\mathcal{T}}$, since there are periods in $\Omega_{\mathcal{T},v}$ parametrizing triples (X,η,L) such that L is not ample, i.e. L is just a quasipolarization on X (Definition 5.25).

Now the $\mathbb{Z}$-lattice Λ may have several $O^+(\Lambda)$-orbits of vectors v of type (n,d), each of them giving rise to a moduli space $\mathcal{M}_{\mathcal{T},v}$. By taking the disjoint union over all such orbits, we obtain a moduli space $\mathcal{M}^{(n,d)}_{\mathcal{T}}$ of (n,d)-polarized IHS manifolds of deformation type $\mathcal{T}$. Note that according to Theorem 5.49, the irreducible components of $\mathcal{M}^{(n,d)}_{\mathcal{T}}$ are all equipped with a dominant period map. For each $O^+(\Lambda)$-orbit of vectors v of type (n,d), we call $O^+(\Lambda,v)\backslash\Omega^+_{\mathcal{T},v}$ a *period space*: it has complex dimension $\operatorname{rank}_{\mathbb{Z}}(\Lambda) - 3$.

Example 5.50. The Fano varieties of lines on smooth cubic fourfolds define a family of $(6,2)$-polarized IHS fourfolds of $\mathrm{K3}^{[2]}$-type. Since the moduli space of cubic fourfolds is a 20-dimensional complex variety, we have that the previous family has dimension 20. Note that the moduli space $\mathcal{M}^{(6,2)}_{\mathrm{K3}^{[2]}}$ is irreducible: in fact, the $\mathbb{Z}$-lattice $\Lambda := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_1$ has only one $O^+(\Lambda)$-orbit of vectors of type $(6,2)$, by Eichler's criterion [Eic52, Satz 10.4]. Let $v \in \Lambda$ be a representative. We can identify the family of Fano varieties of lines on smooth cubic fourfolds with a dense open subset of $\mathcal{M}_{\mathrm{K3}^{[2]},v}$: the image of this set under $\mathcal{P}_{\mathrm{K3}^{[2]},v}$ has maximal dimension $20 = \mathrm{rank}_{\mathbb{Z}}(\Lambda) - 3$. We say that the family of Fano varieties of lines of smooth cubic fourfolds define a *locally complete family* of $(6,2)$-polarized IHS manifolds of deformation type $\mathrm{K3}^{[2]}$.

Locally complete families of polarized IHS manifolds are interesting for several reasons. For instance, very general elements in such families have Picard rank 1, and in particular they are often rigid (i.e. have no birational automorphisms). Moreover, we know that if such a family exists, then the general element in the associated (irreducible component of the) moduli space of polarized IHS manifold is an element of this family. This can be useful to produce explicit projective models of some IHS manifolds with given symmetries, or geometrical properties (determined from their period). However, we know today only very few examples of locally complete families of IHS manifolds.

Example 5.51. We have already seen that Fano varieties of lines of smooth cubic fourfolds define a locally complete family of smooth polarized IHS fourfolds. Here are a few other examples:

(1) the IHS fourfolds known as *double EPW-sextics* [O'G06] (see also Section 11) define a locally complete family of $(2,1)$-polarized IHS fourfolds of $\mathrm{K3}^{[2]}$-type;

(2) the so-called *Debarre–Voisin* manifolds [DV10] (see also Section 10.4.2) define a locally complete family of $(22,2)$-polarized IHS fourfold of deformation type $\mathrm{K3}^{[2]}$;

(3) the varieties of sum of powers $\mathrm{VSP}(V,10)$ associated to a smooth cubic fourfold V [RS00, IR01, IR07] define a locally complete family of $(38,2)$-polarized IHS fourfold of deformation type $\mathrm{K3}^{[2]}$;

(4) similarly to double EPW-sextics, there exists a locally complete family of $(4,2)$-polarized IHS sixfolds of $\mathrm{K3}^{[3]}$-type known as *double EPW-cubes* [IKKR19] (see also Section 11).

Remark 5.52. More generally, for any primitive sublattice $N \leq \Lambda$ one can define a moduli space of *N-polarized manifolds* which are marked pairs (X,η) such that $\eta(\mathrm{NS}(X))$ contains N as a primitive sublattice (see [Cam16])

Part II. Transcendental classification

6. Classification of birational automorphisms of IHS manifolds

In this section, we present the problem of classifying birational automorphisms of IHS manifolds from a lattice-theoretic point of view, and we describe a strategy to tackle this problem computationally.

6.1. Birational automorphisms of IHS manifolds

As we will see in the next section, by fixing a connected component of $\mathcal{M}_{\mathcal{T}}$, one can translate the Torelli-type theorem Theorem 5.48 into a finite list of numerical conditions determining whether an isometry of $\Lambda_{\mathcal{T}}$ can be induced from a (birational) automorphism on an IHS manifold $X \sim \mathcal{T}$. Let us start by recalling some standard results about birational automorphisms of IHS manifolds.

Let X be an IHS manifold. We denote by $\mathrm{Bir}(X)$ its group of birational automorphisms, i.e. birational maps $f\colon X \dashrightarrow X$ which are well-defined in codimension 1. Let moreover $\mathrm{Aut}(X) \leq \mathrm{Bir}(X)$ be the subgroup of regular automorphisms of X. According to Theorems 5.35 and 5.48, the natural action of $\mathrm{Bir}(X)$ on $H^2(X,\mathbb{C})$ induces an integral orthogonal representation

$$\rho_X : \mathrm{Bir}(X) \to O(H^2(X,\mathbb{Z}), q_X),\ f \mapsto (f^{-1})^* \tag{14}$$

so that $\rho_X(\mathrm{Bir}(X)) = \mathrm{Mon}^2_{bir}(X)$ (see Proposition 5.47). The kernel of this map is known to be finite [Huy99, Proposition 9.1], it is invariant under deformation [HT13, Theorem 2.1], and it has been determined for the known deformation types. See [Bea83b, Proposition 10] for the $\mathrm{K3}^{[n]}$ cases, [BNWS11, Corollary 5] for Kum_n cases and [MW17, Theorems 3.1 and 5.2] for the deformation type OG10 and OG6 respectively. We refer to the last column of Table 4 for an explicit description of such kernels.

Table 4: Known deformation types and some of their invariants ($X \sim \mathcal{T}$)

$\mathcal{T}$	b_2	sign	Λ	D_Λ	$\mathrm{Mon}^2(\Lambda)$	$\ker \rho_X$
Kum_n	7	$(3,4)$	$U^{\oplus 3} \oplus A_1(n+1)$	$\mathbb{Z}/(2n+2)\mathbb{Z}$	$\{g \in O^+(\Lambda)\ :\ \det(g)D_g = \mathrm{id}\}$	$C_{n+1}^4 \rtimes C_2$
OG6	8	$(3,5)$	$U^{\oplus 3} \oplus A_1^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$O^+(\Lambda)$	C_2^8
K3	22	$(3,19)$	$U^{\oplus 3} \oplus E_8^{\oplus 2}$	$\{0\}$	$O^+(\Lambda)$	1
$\mathrm{K3}^{[n]}$	23	$(3,20)$	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_1(n-1)$	$\mathbb{Z}/(2n-2)\mathbb{Z}$	$\{g \in O^+(\Lambda)\ :\ D_g = \pm\mathrm{id}\}$	1
OG10	24	$(3,21)$	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$	$\mathbb{Z}/3\mathbb{Z}$	$O^+(\Lambda)$	1

Proposition 6.1 ([Bea83b, Proposition 7]). *Let X be a projective IHS manifold, and let $G \leq \mathrm{Bir}(X)$ be a subgroup. Then there exist a positive integer n and a surjective character $\chi\colon G \twoheadrightarrow \mu_n \leq \mathbb{C}^\times$ onto the group of primitive nth roots of unity such that*

$$\rho_X(g)(\sigma_X) = \chi(g)\sigma_X$$

where $\sigma_X \in H^{2,0}(X)$ is any nonzero symplectic 2-form. Moreover, the Euler totient $\varphi(n)$ divides the rank of $\mathrm{T}(X)$.

Beauville actually showed that if X is a projective IHS manifold, and $f \in \mathrm{Bir}(X)$, then the action of f on $H^{2,0}(X)$ is of finite order. For simplicity, in what follows, let us assume that f is of finite order and let us denote by $n \geq 1$ the order of the image of f in $\mathrm{GL}(H^{2,0}(X))$. We have that $\mathrm{T}(X) \leq H^2(X,\mathbb{Z})^{\Phi_n(\rho_X(f))}$ is a primitive sublattice. Indeed since $\mathrm{T}(X)$ is the smallest primitive sublattice of $H^2(X,\mathbb{Z})$ whose $\mathbb{C}$-span contains $H^{2,0}(X)$, it must be contained in the primitive sublattice $H^2(X,\mathbb{Z})^{\Phi_n(\rho_X(f))} \leq H^2(X,\mathbb{Z})$. Since $\rho_X(f)$ preserves $\mathrm{T}(X)$, because it is a Hodge monodromy, we have that $(\mathrm{T}(X), \rho_X(f)_{|\mathrm{T}(X)})$ is a Φ_n-lattice and the rank of $\mathrm{T}(X)$ is divisible by $\varphi(n)$. Hence either f acts trivially on $\mathrm{T}(X)$ or it defines a structure of hermitian $\mathbb{Z}[\zeta_n]$-lattice on $\mathrm{T}(X)$, for some $n \in \mathbb{Z}_{\geq 2}$ and ζ_n a primitive nth root of unity.

Definition 6.2. Let X be an IHS manifold, and let $G \leq \mathrm{Bir}(X)$ be a finite subgroup.

(1) We call the integer $n := \#\,\mathrm{im}(G \to \mathrm{GL}(H^{2,0}(X)))$ the *transcendental value* of G.

(2) We say G is *symplectic* if it has transcendental value 1.

(3) We say G is *antisymplectic* if it has transcendental value -1.

Similar notions apply for any $f \in \mathrm{Bir}(X)$. Moreover, we say $f \in \mathrm{Bir}(X)$ is *purely nonsymplectic* if no nontrivial iterate of f is symplectic.

According to Proposition 6.1, for any $G \leq \mathrm{Bir}(X)$ we have an exact sequence

$$1 \to G_s \to G \to \mathbb{C}^\times$$

where we denote by G_s the normal subgroup of G consisting of symplectic birational automorphisms: we call G_s the *symplectic subgroup* of G. Note that G is symplectic if $G = G_s$ and if G_s is trivial, then G is cyclic generated by a purely nonsymplectic birational automorphism. We say G is *mixed* if $1 < G_s < G$.

Notation. For an IHS manifold X, we denote by $\mathrm{Aut}_s(X) \leq \mathrm{Bir}_s(X)$ the respective subgroups of symplectic automorphisms and symplectic birational automorphisms.

Let us conclude this small introduction to birational automorphisms by mentioning some important results related to finite group actions on IHS manifolds.

Proposition 6.3 ([Bea83b, §4, Proposition 6])**.** *Let X be an IHS manifold. If there exists a finite order isometry $f \in \mathrm{Bir}(X) \setminus \mathrm{Bir}_s(X)$, then X is projective.*

Proposition 6.4 ([Huy99, Proposition 9.1])**.** *Let X be an IHS manifold. A subgroup $G \leq \mathrm{Bir}(X)$ is finite if and only if $\rho_X(G)$ fixes a vector in $\mathcal{FE}_X$.*

As a consequence, we prove the following well-known result.

Proposition 6.5. *Let X be an IHS manifold, let $G \leq \mathrm{Bir}(X)$ be a finite subgroup, and let $H := \rho_X(G)$. Then, one of the following holds:*

(1) *either $G = G_s$ is symplectic, there are two primitive embeddings $\mathrm{T}(X) \leq H^2(X,\mathbb{Z})^H$ and $H^2(X,\mathbb{Z})_H \leq \mathrm{NS}(X)$, and $H^2(X,\mathbb{Z})_H$ is negative definite;*

(2) *or $G \neq G_s$, there are two primitive embeddings $\mathrm{T}(X) \leq H^2(X,\mathbb{Z})_H$ and $H^2(X,\mathbb{Z})^H \leq \mathrm{NS}(X)$, and $H^2(X,\mathbb{Z})^H$ has signatures $(1,*)$.*

Similar results apply for any finite order birational automorphism $f \in \mathrm{Bir}(X)$.

Proof.

(1) Suppose that G is symplectic. In particular, we have that G acts trivially on $H^{2,0}(X)$. This implies that $H^2(X,\mathbb{Z})^H \otimes_{\mathbb{Z}} \mathbb{C}$ contains $H^{2,0}(X)$. Since by definition the transcendental lattice $\mathrm{T}(X)$ is the smallest primitive sublattice of $H^2(X,\mathbb{Z})$ whose $\mathbb{C}$-span contains $H^{2,0}(X)$, we deduce that $\mathrm{T}(X) \leq H^2(X,\mathbb{Z})^H$ is a primitive sublattice. Hence the latter has positive

signature at least 2. But from Proposition 6.4, we also know that H fixes a vector $h \in \mathcal{FE}_X \leq H^{1,1}(X,\mathbb{R})$. We know by definition that $\mathrm{T}(X) \otimes_{\mathbb{Z}} \mathbb{R} \leq h^\perp$, so the $\mathbb{Z}$-lattice $H^2(X,\mathbb{Z})^H$ has signatures $(3,*)$. In particular, $H^2(X,\mathbb{Z})_H = (H^2(X,\mathbb{Z})^H)^\perp$ is negative definite, and it must be primitively embedded into $\mathrm{NS}(X) = \mathrm{T}(X)^\perp$ by the fact that $\mathrm{T}(X) \leq H^2(X,\mathbb{Z})^H$ is a primitive sublattice.

(2) By similar arguments, since G is finite, we know that $H^2(X,\mathbb{Z})^H$ contains a vector of positive norm (see also the proof of [Bea83b, Proposition 6(i)]). Now we know that G has transcendental value $n \geq 2$. In particular, there exists an element $g \in G$ such that $\rho_X(g)(\sigma_X) = \zeta_n \sigma_X$ for some primitive nth root of unity ζ_n, and for all $\sigma_X \in H^{2,0}(X)$ nontrivial. Let us denote by G_s the symplectic subgroup of G, and let $H_s := \rho_X(G_s)$. Let us denote $h := \rho_X(g)$ and remark that h is nontrivial. Indeed, we know that $H^2(X,\mathbb{Z})^{\Phi_n(h)} \otimes_{\mathbb{Z}} \mathbb{C}$ contains $H^{2,0}(X)$. As in part (1), we deduce that we have a succession of primitive sublattices

$$\mathrm{T}(X) \leq H^2(X,\mathbb{Z})^{\Phi_n(h)} \leq H^2(X,\mathbb{Z})_h.$$

Now remark that $H = \langle H_s, h\rangle$ and since $H_s \leq H$, we have that $H^2(X,\mathbb{Z})^H \leq H^2(X,\mathbb{Z})^h$ is a primitive sublattice. In particular, by orthogonality, we obtain a succesion of primitive sublattices

$$\mathrm{T}(X) \leq H^2(X,\mathbb{Z})_h \leq H^2(X,\mathbb{Z})_H$$

and the latter has signatures $(2,*)$. We conclude by remarking that, by orthogonality again, we have that $H^2(X,\mathbb{Z})^H \leq \mathrm{NS}(X)$ is a primitive sublattice of signatures $(1,*)$.

Finally, if $f \in \mathrm{Bir}(X)$ has finite order, we observe that $\rho_X(f)$ and the cyclic group it generates have the same invariant and coinvariant sublattices, so the results follow similarly. □

6.2. Torelli setting

The content of this section is a generalization of results from [BC23, BH23].

Let us fix $\mathcal{T}$ to be a known deformation type of IHS manifolds, let $\Lambda := \Lambda_{\mathcal{T}}$ be the associated even $\mathbb{Z}$-lattice, and let $\mathcal{M}^\circ$ be a connected component of the moduli space $\mathcal{M} := \mathcal{M}_{\mathcal{T}}$ of marked pairs of deformation type $\mathcal{T}$. Recall that the markings associated to two points in $\mathcal{M}^\circ$ differ by a parallel transport operator (Proposition 5.29). Hence, for any class $[(X,\eta)] \in \mathcal{M}^\circ$, the group $\mathrm{Mon}^2_\circ(\Lambda) := \eta \mathrm{Mon}^2(X)\eta^{-1} \leq O^+(\Lambda)$ depends only on the connected component $\mathcal{M}^\circ$ of $\mathcal{M}$.

Notation. A priori, for a given deformation type $\mathcal{T}$ of IHS manifolds, the group $\mathrm{Mon}^2_\circ(\Lambda)$ might not be normal in $O(\Lambda)$. This is a reason why one usually fixes a connected component of $\mathcal{M}$. For the known deformation types though, the group $\mathrm{Mon}^2_\circ(\Lambda)$ has been determined, and it is known to be normal in $O(\Lambda)$. See [Mar10, Lemma 4.2] for the $\mathrm{K3}^{[n]}$ types, [Mon16a, Theorem 2.3] for the Kum_n types, [MR21, Theorem 1.4] for the deformation type OG6, and [Ono22, Theorem 5.4] for the deformation type OG10.

Since we chose $\mathcal{T}$ to be a known deformation type of IHS manifolds, we will simply denote $\mathrm{Mon}^2(\Lambda) := \mathrm{Mon}^2_\circ(\Lambda)$ from now on. We refer to Table 4 for a description of such groups.

We denote again by

$$\mathcal{W}^{pex}(\Lambda) \subseteq \mathcal{W}(\Lambda)$$

the sets of numerical stably prime exceptional and wall divisors of Λ (Definition 5.41). Recall that they are such that for any $[(X,\eta)] \in \mathcal{M}^\circ$, we have $\mathcal{W}(X) = \eta^{-1}(\mathcal{W}(\Lambda)) \cap H^{1,1}(X,\mathbb{R})$ (and similarly for $\mathcal{W}^{pex}(X)$).

Definition 6.6. Let $H \leq \mathrm{Mon}^2(\Lambda)$ be a finite subgroup.

(1) We define an *H-marked* IHS manifold to be a triple (X, η, G) consisting of a marked pair (X, η) with $X \sim \mathcal{T}$ and a finite subgroup $\ker \rho_X \leq G \leq \mathrm{Bir}(X)$ such that $\eta\rho_X(G)\eta^{-1} = H$.

(2) We say that a pair (X, G) as before is *H-markable* if there exists a marking η such that (X, η, G) is H-marked.

For $i = 1, 2$, we let (X_i, η_i, G_i) be a triple such that $[(X_i, \eta_i)] \in \mathcal{M}^\circ$ and $G_i \leq \mathrm{Bir}(X_i)$ is a finite subgroup. Then by definition, if we denote by $H_i := \eta_i\rho_{X_i}(G_i)\eta_i^{-1} \leq \mathrm{Mon}^2(\Lambda)$ for $i = 1, 2$, we have that (X_i, η_i, G_i) is H_i-marked (where ρ_{X_1} and ρ_{X_2} are defined in Equation (14)). Now suppose that there exists a birational isomorphism $f\colon X_1 \dashrightarrow X_2$ such that $\eta_2 = \eta_1 \circ f^*$ and $f^*\rho_{X_2}(G_2) = \rho_{X_1}(G_1)f^*$: we say that the triples (X_1, η_1, G_1) and (X_2, η_2, G_2) are *birational equivalent*. It follows that since f^* is a Hodge parallel transport operator (Definition 5.34), then the isometry $\eta_1 \circ f^* \circ \eta_2^{-1} \in \mathrm{Mon}^2(\Lambda)$, and it conjugates H_1 and H_2.

The notion of birational equivalent triples introduced before is natural, as soon as one wants to construct a moduli space of triples (X, η, G) for instance.

Remark 6.7. In Section 8.2.1 we give a more detailed account about this, where we actually introduce the moduli space of triples (X, η, G) for $G \leq \mathrm{Aut}(X)$ finite, following a presentation of Brandhorst and Cattaneo [BC23, §3].

We have just shown that two birational equivalent triples such that the underlying marked pairs lie in $\mathcal{M}^\circ$ are actually marked by two subgroups of $\mathrm{Mon}^2(\Lambda)$ which are $\mathrm{Mon}^2(\Lambda)$-conjugate. This is an important observation which structures the rest of the thesis.

Definition 6.8. Let $H \leq \mathrm{Mon}^2(\Lambda)$ be a finite subgroup. We say the group H is *effective* if there exists an H-marked IHS manifold (X, η, G). Moreover, if H is effective, we say

(1) H is *regular effective* if there exists an H-marked IHS manifold (X, η, G) so that $G \leq \mathrm{Aut}(X)$.

(2) H is *symplectic* if there exists an H-marked IHS manifold (X, η, G) so that $G \leq \mathrm{Bir}_s(X)$.

We say that H is *regular symplectic* if it is regular effective and symplectic.

The aim of this part of the thesis is to describe an approach for determining what the finite effective subgroups of $\mathrm{Mon}^2(\Lambda)$ are. As we have just seen a moment earlier, we can restrict ourselves into considering such groups only up to $\mathrm{Mon}^2(\Lambda)$-conjugacy, if we look at triples (X, η, G) up to the birational equivalence we have described. Note moreover that since conjugating with elements in $\mathrm{Mon}^2(\Lambda)$ is the same as changing marking while preserving the connected component $\mathcal{M}^\circ$, for two $\mathrm{Mon}^2(\Lambda)$-conjugate finite subgroups $H, H' \leq \mathrm{Mon}^2(\Lambda)$, we have that H is effective (resp. regular effective, resp. symplectic) if and only if so is H'.

In the rest of this section, we give numerical criteria to determine whether a finite subgroup of $\mathrm{Mon}^2(\Lambda)$ is effective. We start by the symplectic case: we refer to [GOV23, §2] for a more general approach, using *signed Hodge structures*. See also [Nik79, 4.2, 4.3] for the case of K3 surfaces.

Theorem 6.9. *Let $H \leq \mathrm{Mon}^2(\Lambda)$ be a finite subgroup.*

(1) *The group H is symplectic if and only if Λ_H is negative definite and $\Lambda_H \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$.*

(2) *The group H is regular symplectic if and only if Λ_H is negative definite and $\Lambda_H \cap \mathcal{W}(\Lambda) = \emptyset$.*

Proof. We have already seen in Proposition 6.5 that Λ_H being negative definite is a necessary condition for H to be symplectic.

(1) Suppose first that H is symplectic. Then there exists an H-marked IHS manifold (X,η,G) of deformation type $\mathcal{T}$ with $G \leq \mathrm{Bir}_s(X)$. Since G is finite, Proposition 6.4 tells us that $\rho_X(G)$ fixes a vector h in $\mathcal{FE}_X$. Moreover Proposition 6.5 tells us that $H^2(X,\mathbb{Z})_{\rho_X(G)}$ is negative definite. But now, since h is fixed by $\rho_X(G)$, we have that $H^2(X,\mathbb{Z})_{\rho_X(G)}$ and h are orthogonal to each other, and thus $H^2(X,\mathbb{Z})_{\rho_X(G)} \cap \mathcal{W}^{pex}(X) = \emptyset$. In particular

$$\Lambda_H \cap \mathcal{W}^{pex}(\Lambda) = \eta(H^2(X,\mathbb{Z})_{\rho_X(G)} \cap \mathcal{W}^{pex}(X)) = \emptyset.$$

Now suppose that Λ_H is negative definite and that $\Lambda_H \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$. Since Λ^H has signatures $(3,*)$, we can find a primitive vector $h \in \Lambda^H$ with positive norm. Now, $(h^\perp_\Lambda \otimes_\mathbb{Z} \mathbb{R})^H$ has signatures $(2,*)$, so we can find an element $\sigma \in (h^\perp_\Lambda \otimes_\mathbb{Z} \mathbb{C})^H$ such that $\sigma^2 = 0$, $\sigma.\overline{\sigma} > 0$ and $\sigma^\perp \cap \Lambda = \overline{\Lambda_H \oplus \mathbb{Z}h}$. By surjectivity of the period map $\mathcal{P}_\mathcal{T}$ (Theorem 5.30), there exists a marked IHS manifold (X,η) of deformation type $\mathcal{T}$ such that

$$\mathcal{P}_\mathcal{T}([(X,\eta)]) = \eta(H^{2,0}(X)) = \mathbb{C}\sigma.$$

By the assumption $\sigma^\perp \cap \Lambda = \overline{\Lambda_H \oplus \mathbb{Z}h}$, it follows that $\eta(\mathrm{T}(X)) = h^\perp_{\Lambda^H}$, and X is projective with $\eta(\mathrm{NS}(X)) = \overline{\Lambda_H \oplus \mathbb{Z}h}$ (Theorem 5.21). But now, by the assumption $\Lambda_H \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$, we have that $k := \eta^{-1}(h)$ lies in an exceptional chamber $\mathcal{E}_X$ in the decomposition of the positive cone C_X of X. Since the group $W_{pex}(X) \leq \mathrm{Mon}^2_{Hdg}(X)$ (see Proposition 5.47) acts transitively on the set of such exceptional chambers in C_X, we can find another marking η' of X such that $[(X,\eta)]$ and $[(X,\eta')]$ lie in the same connected component $\mathcal{M}^\circ$ of the moduli space $\mathcal{M}$, they have the same period σ, and $k' := (\eta')^{-1}(h) \in \mathcal{FE}_X \cap \mathrm{NS}(X)$. If one denotes $G' := (\eta')^{-1}H\eta'$, we have by construction that $G' \leq \mathrm{Mon}^2_{Hdg}(X)$ is a finite subgroup and it fixes k'. Hence G' preserves the fundamental exceptional chamber $\mathcal{FE}_X$ and $G' \leq \mathrm{Mon}^2_{bir}(X)$. By Torelli Theorem 5.48, this implies that $G' = \rho_X(G)$ for some finite subgroup $\ker \rho_X \leq G \leq \mathrm{Bir}(X)$. Note moreover that since $H^2(X,\mathbb{Z})_{G'} = (\eta')^{-1}(\Lambda_H)$ is negative definite, we have from Proposition 6.5 that G is symplectic. Hence (X,η',G) is H-marked and H is symplectic.

(2) For Item (2) the proof is similar. If H is regular symplectic, we let (X,η,G) be an H-marked IHS manifold of deformation type $\mathcal{T}$ with, this time, $G \leq \mathrm{Aut}_s(X)$. In particular, $\rho_X(G)$ preserves the Kähler cone $\mathcal{K}_X$ of X. Hence this time, we choose $h \in \mathcal{K}_X$, and we conclude similarly by remarking that h is not orthogonal to any element in $\mathcal{W}(X)$.

For the second part, we proceed also similarly. The only adjustment is that the vector $k' \in \mathcal{FE}_X \cap \mathrm{NS}(X)$ obtained for the H-marked IHS manifold (X,η',G) might not lie in $\mathcal{K}_X$. However, since it is contained in a Kähler-type chamber of the positive cone C_X, by the assumption $\Lambda_H \cap \mathcal{W}(\Lambda) = \emptyset$, we can find a birational model $f\colon X \dashrightarrow X'$ of X so that $(f^*)^{-1}(k) \in \mathcal{K}_{X'}$. Doing so, we would have that $(X', \eta' \circ f^*, fGf^{-1})$ is H-marked, and H is regular symplectic. □

Let now $H \leq \mathrm{Mon}^2(\Lambda)$ be a finite effective subgroup which is not symplectic. Then, for any H-markable pair (X,G), the action of G on $H^{2,0}(X) \simeq \mathbb{C}$ induces a nontrivial cyclic character $\chi\colon H \to \mathbb{C}^\times$ of finite order $n > 1$, the transcendental value of G.

Definition 6.10. A nontrivial cyclic character $\chi\colon H \to \mathbb{C}^\times$ defined on a finite subgroup $H \leq \mathrm{Mon}^2(\Lambda)$ is called *effective* if it arises as before. We call it moreover *regular effective* if the group H is regular effective too.

We have seen already in the proof of Proposition 6.5 that if (X,η,G) is H-marked, then there exists a nonsymplectic isometry $h \in H$ such that $\eta(\sigma_X)$ lies in the $\chi(h)$-eigenspace of $\Lambda \otimes_\mathbb{Z} \mathbb{C}$, and

$\chi(h) \in \mathbb{C}^\times$ is a primitive nth root of unity. Since $\eta(\mathrm{T}(X))$ is fixed by any symplectic isometry in H, we have that $\eta(\mathrm{T}(X)) \otimes_{\mathbb{Z}} \mathbb{C}$ is contained in the set

$$\{v \in \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \ :\ h(v) = \chi(h)v,\ \forall h \in H\}.$$

Definition 6.11. Let $H \leq O(\Lambda)$ be finite and let $\chi\colon H \to \mathbb{C}^\times$ be a cyclic character. We define

(1) the *generic transcendental lattice* $T(\chi)$ of χ to be the smallest primitive sublattice of Λ such that $\mathrm{T}(\chi) \otimes_{\mathbb{Z}} \mathbb{C}$ contains the eigenlattice

$$(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})^\chi := \{v \in \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \ :\ h(v) = \chi(h)v,\ \forall h \in H\}\,;$$

(2) the *generic Néron–Severi lattice* $\mathrm{NS}(\chi)$ to be the orthogonal complement of $\mathrm{T}(\chi)$ in Λ.

It follows that if $\chi\colon H \to \mathbb{C}^\times$ is effective and (X, η, G) is H-marked, then one has

$$\eta(\mathrm{T}(X)) \leq \mathrm{T}(\chi) \text{ and } \mathrm{NS}(\chi) \leq \eta(\mathrm{NS}(X)). \tag{15}$$

From the generic transcendental and Néron–Severi lattices of χ, it is possible to determine whether $\chi\colon H \to \mathbb{C}^\times$ is effective or not. The following proposition is a generalization of [BH23, Proposition 3.3]: its proof is essentially the same as the proof of Theorem 6.9.

Theorem 6.12. *Let $H \leq \mathrm{Mon}^2(\Lambda)$ be a finite subgroup, and let $\chi\colon H \to \mathbb{C}^\times$ be a nontrivial cyclic character of finite order $n \geq 2$. We denote $H_s := \ker \chi$. Then χ is effective if and only if*

(1) *H_s is symplectic;*

(2) *the signatures of Λ^H are $(1, *)$;*

(3) *the signatures of*

$$(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})^{\chi+\overline{\chi}} := \left\{v \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \ :\ (h + h^{-1})(v) = \chi(h)v + \overline{\chi(h)}v,\ \forall h \in H\right\}$$

are $(2, *)$;

(4) $\mathrm{NS}(\chi)_H \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$.

Moreover, if χ is effective, it is regular effective if and only if

(5) *H_s is regular symplectic;*

(6) $\mathrm{NS}(\chi)_H \cap \mathcal{W}(\Lambda) = \emptyset$.

Proof. First assume that χ is effective, and let (X, η, G) be an H-marked IHS manifold of deformation type $\mathcal{T}$ such that χ is induced by the action of $G \leq \mathrm{Bir}(X)$ on $H^{2,0}(X)$. We prove Items (1)–(6). Already note that the finite group $H_s = \ker \chi = \eta\rho_X(G_s)\eta^{-1}$, where G_s is the symplectic subgroup of G: hence H_s is symplectic, which proves Item (1). Now, we have seen that $\eta(\mathrm{T}(X)) \leq \mathrm{T}(\chi)$, so the complex quadratic space $\mathrm{T}(\chi) \otimes_{\mathbb{Z}} \mathbb{C}$ contains $\eta(\sigma_X)$. Hence $(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})^{\chi+\overline{\chi}}$ contains at least two nonproportional vectors, being $\eta(\sigma_X + \overline{\sigma_X})$ and $\eta(i(\sigma_X - \overline{\sigma_X}))$, which have both positive norm: in fact, they both have norm $2\sigma_X.\overline{\sigma_X} > 0$ (see Section 5.3). Now, we assume X to be projective, so as in the proof of Theorem 6.9, there exists a G-invariant vector $h \in \mathcal{FE}_X \cap \mathrm{NS}(X)$ of positive norm, so we know that Λ^H contains at least one vector of positive norm, being $\eta(h)$. Since $h \in \mathrm{NS}(X)$, we have that $\eta(h) \in \Lambda^H \cap ((\Lambda \otimes_{\mathbb{Z}} \mathbb{R})^{\chi+\overline{\chi}})^\perp$. In particular, we deduce that Λ^H has signatures $(1, *)$ and similarly $(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})^{\chi+\overline{\chi}}$ has signatures $(2, *)$, implying

(2) and (3). Notice that $\mathrm{NS}(\chi) \leq \eta(\mathrm{NS}(X))$. Since $h \in \mathcal{FE}_X$, we know that $h.v > 0$ for any $v \in \mathcal{W}^{pex}(X)$: since $\mathrm{NS}(X)_{\rho_X(G)} \leq h_\Lambda^\perp \cap \mathrm{NS}(X)$, we already know that $\mathrm{NS}(X)_{\rho_X(G)} \cap \mathcal{W}^{pex}(X) = \emptyset$. A fortiori, $\mathrm{NS}(\chi)_H \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$ because $\mathrm{NS}(\chi)_H \leq \eta(\mathrm{NS}(X)_{\rho_X(G)})$. This implies (4). Finally, if we suppose moreover that χ is regular symplectic, then $G \leq \mathrm{Aut}(X)$ and we can choose $h \in \mathcal{K}_X \cap \mathrm{NS}(X)$. In particular, we have that $H_s = \eta^{-1}\rho_X(G_s)\eta$ is regular symplectic and $\mathrm{NS}(\chi)_H \cap \mathcal{W}(\Lambda)$ since $h.v > 0$ for all $v \in \mathcal{W}(X)$, proving Items (5) and (6).

Now suppose that (1)—(4) hold. As in the second part of the proof of Theorem 6.9, since $(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})^{\chi + \overline{\chi}}$ has signatures $(2, *)$, one can find an element $\sigma \in (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})^\chi$ such that $\sigma^2 = 0$, $\sigma.\overline{\sigma} > 0$ and $\sigma^\perp \cap \mathrm{T}(\chi) = \{0\}$. By surjectivity of the period map $\mathcal{P}_\mathcal{T}$ (Theorem 5.30), there exists a marked IHS manifold (X, η) with $X \sim \mathcal{T}$ such that $\mathcal{P}_\mathcal{T}([(X,\eta)]) = \eta(H^{2,0}(X)) = \mathbb{C}\sigma$. By the assumption $\sigma^\perp \cap \mathrm{T}(\chi) = \{0\}$, it follows that $\eta(\mathrm{T}(X)) = \mathrm{T}(\chi)$. Therefore, since $\mathrm{T}(\chi)$ has signatures $(2, *)$, we have that X is projective with $\eta(\mathrm{NS}(X)) = \mathrm{NS}(\chi)$. Now, Λ^H has signatures $(1, *)$, so we can find an H-invariant vector $h \in \Lambda$ of positive norm and, up to replacing h by $-h$, we have that $k := \eta^{-1}(h) \in H^2(X, \mathbb{Z})^{\rho_X(G)} \leq \mathrm{NS}(X)$ (Proposition 6.5). By the assumption $\mathrm{NS}(\chi)_H \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$, we have that $h_\Lambda^\perp \cap \mathrm{NS}(\chi) \leq \Lambda_H \cap \mathrm{NS}(\chi) = \mathrm{NS}(\chi)_H$ contains no vectors of $\mathcal{W}^{pex}(\Lambda)$. In particular, the vector k lies in $\mathcal{E}_X \cap \mathrm{NS}(X)$ where $\mathcal{E}_X \subseteq C_X$ is an exceptional chamber. The subgroup $W_{pex}(X) \leq \mathrm{Mon}^2(X)$ generated by reflections in the vectors in $\mathcal{W}^{pex}(X)$ acts transitively on the set of exceptional chambers in C_X (Proposition 5.47): hence there exists a marking η' of X such that $(\eta')^{-1}k \in \mathcal{FE}_X \cap \mathrm{NS}(X)$. But now, if one denotes $G' := (\eta')^{-1}H\eta'$, we have by construction that $G' \leq \mathrm{Mon}^2_{Hdg}(X)$ preserves $\mathbb{C}\sigma_X$ and the fundamental exceptional chamber $\mathcal{FE}_X$. Hence $G' \leq \mathrm{Mon}^2_{bir}(X)$ is a finite group and by Torelli Theorem 5.48, $G' = \rho_X(G)$ for some finite subgroup $G \leq \mathrm{Bir}(X)$. Note that by construction, the character χ is induced by the action of G on $H^{2,0}(X)$. Thus χ is effective. Note if Items (5) and (6) hold too, then similarly to the first part of the proof and Item (2) of the proof of Theorem 6.9, we conclude that χ is regular effective. □

Remark 6.13. The previous statements do not necessarily hold similarly for actions of infinite order. In fact, the proof uses that the action is finite to find a fixed vector of positive norm from any vector of positive norm. For infinite order actions, this is not the case anymore and checking whether one preserves the fundamental exceptional chamber is done differently (see for instance [McM11, OY20] in the case of K3 surfaces).

Note that for an effective character $\chi\colon H \to \mathbb{C}^\times$, the definition of $H_s = \ker \chi$ does not depend on χ, and H_s can be defined as

$$H_s := \left\{ h \in H \ : \ \Lambda^h \text{ has signatures } (3, *) \right\} \tag{16}$$

(see Proposition 6.5). Moreover, [BH23, Lemma 3.4] tells that for a given effective finite subgroup $H \leq \mathrm{Mon}^2(\Lambda)$, there are at most two effective characters $\chi\colon H \to \mathbb{C}^\times$. They are complex conjugate. In particular, $\mathrm{T}(\chi)$ and $\mathrm{NS}(\chi)$ being defined over $\mathbb{Z}$ do not depend on χ, but only on H. Thus we can apply Theorems 6.9 and 6.12 to determine whether a given finite subgroup $H \leq \mathrm{Mon}^2(\Lambda)$ is effective, regular effective, symplectic or regular symplectic.

Remark 6.14. Let H_s be a finite symplectic subgroup of $\mathrm{Mon}^2(\Lambda)$. Note that the conditions in Theorem 6.9 only depend on the coinvariant sublattice of H_s. In particular, we obtain that the saturation G of H_s in $\mathrm{Mon}^2(\Lambda)$ is symplectic, and H_s is regular symplectic if and only if so is G. Therefore, for a classification, up to conjugacy, of finite effective subgroups of $\mathrm{Mon}^2(\Lambda)$, it is enough to determine representatives whose symplectic subgroup is saturated in $\mathrm{Mon}^2(\Lambda)$.

6.3. Classification problems

Let again $\mathcal{T}$ be a known deformation type of IHS manifolds, let again $\Lambda := \Lambda_{\mathcal{T}}$ be the associated abstract BBF form and let $\mathrm{Mon}^2(\Lambda)$ be the corresponding monodromy group (Table 4). From the setting of the previous section, we split the problem of classifying finite effective subgroups of $\mathrm{Mon}^2(\Lambda)$ into five different core classification problems. This gives the structure for the rest of the thesis. Let us start by recalling the following simple but convenient fact.

Lemma 6.15. *The orthogonal representation of* $\mathrm{Mon}^2(\Lambda)$ *on* D_Λ *has order 2, except for* $\mathcal{T} = \mathrm{K3}, \mathrm{K3}^{[2]}$ *where it has order 1.*

Proof. Follows from the monodromy computations of [Mar10, Lemma 4.2], [Mon16b, Theorem 2.3], [MR21, Theorem 1.4] and [Ono22, Theorem 5.4]. See Table 4 for an explicit description. □

Let H be a finite effective subgroup of $\mathrm{Mon}^2(\Lambda)$. Then we have seen in Section 6.2 that there is an exact sequence

$$1 \to H_s \to H \to \mathbb{C}^\times \tag{17}$$

where H_s, defined as in Equation (16), is the symplectic subgroup of H. We have moreover that $n := [H : H_s]$ is finite. Now, according to Lemma 6.15, there is another exact sequence

$$1 \to H_s^\# \to H_s \to \mu_2 \tag{18}$$

where $H_s^\# \trianglelefteq H_s$ is the stable subgroup of H_s: we call it the *stable symplectic* subgroup of H. If H is so that $n := [H : H_s] > 1$ and $[H_s : H_s^\#] = 2$, then we obtain the following generic diagram:

Figure 1: Classification problems for the known IHS manifolds

$$\begin{tikzcd} \text{(StS)}\; H_s^\# \arrow[d, hook, "\text{(S)}"'] & & \\ H_s \arrow[r, hook, "\text{(M)}"] \arrow[d, two heads, "\text{(SC)}"'] & H \arrow[r, two heads, "\text{(PNS)}"] & \mu_n. \\ \mu_2 & & \end{tikzcd}$$

Lemma 6.16. *Let* $H, H' \leq \mathrm{Mon}^2(\Lambda)$ *be finite effective subgroups. If the groups* H *and* H' *are conjugate in* $\mathrm{Mon}^2(\Lambda)$*, then so are* H_s *and* H'_s*, and so are* $H_s^\#$ *and* $(H'_s)^\#$*.*

Proof. If there exists $\psi \in \mathrm{Mon}^2(\Lambda)$ such that $\psi H \psi^{-1} = H'$, then for all $h \in H$ we have that Λ^h and $\Lambda^{\psi h \psi^{-1}}$ are isometric via ψ. Hence, $\psi H_s \psi^{-1} = H'_s$. The second assertion follows by similar arguments. □

Note moreover that we have seen in Lemma 1.57 that if H_s is saturated in $\mathrm{Mon}^2(\Lambda)$, then $H_s^\#$ is saturated in the stable subgroup $\mathrm{Mon}^2(\Lambda) \cap O^\#(\Lambda) \leq \mathrm{Mon}^2(\Lambda)$. Hence, in order to determine a complete set of representatives for the conjugacy classes of finite effective subgroups $H \leq \mathrm{Mon}^2(\Lambda)$ with H_s saturated, we propose the following strategy.

(StS) Construct and classify conjugacy classes of finite subgroups $H_s^\# \leq \mathrm{Mon}^2(\Lambda)$ consisting of stable symplectic isometries (see Section 7.2);

(S) Extend the groups $H_s^\#$ to finite groups H_s of symplectic isometries (see Section 9.2);

(M) Extend the groups H_s to finite groups H of effective isometries (see Section 9.1).

Note that for the *extension* steps (S) and (M), we treat separately the extension of the trivial groups $H_s^\# = 1$ and $H_s = 1$ respectively. In fact, in those cases, the groups we aim to recover are cyclic and the strategy to classify such cyclic actions is very well-understood. This gives rise to two extra classification problems to our previous list:

(SC) Construct and classify nonstable symplectic involutions (see Section 7.6);

(PNS) Construct and classify purely nonsymplectic effective isometries (see Section 8).

In the rest of this chapter, we describe procedures to tackle these problems computationally. As an application, we provide examples of known deformation types to which these procedures have been applied. Note that the solutions we present always come in two aspects:

(1) a constructive aspect, giving tools to reconstruct explicit lattice-theoretic group representations, in terms of matrices;

(2) a classifying aspect, giving ways to classify the previously constructed actions.

While the first aspect of our solution can be applied to all the known deformation types of IHS manifolds, the classification aspect is more subtle. First, the classification is done via the construction: this means that the two procedures are done simultaneously and not one after another. In fact, for indefinite $\mathbb{Z}$-lattices, testing whether two given isometries are conjugate is a hard problem. In particular, our approach is to construct directly representatives for the conjugacy classes of finite subgroups of $\mathrm{Mon}^2(\Lambda)$. However currently, this can only be done up to $O(\Lambda)$-conjugation, which is often coarser than a classification up to conjugacy in $\mathrm{Mon}^2(\Lambda)$ (Table 4). Therefore, and this will be made precise later, most of the procedures described in this chapter currently apply to the known deformation types $\mathcal{T}$ of IHS manifolds for which $\mathrm{Mon}^2(\Lambda_{\mathcal{T}}) = O^+(\Lambda_{\mathcal{T}})$.

Notation. For the rest of the thesis, if X is an IHS manifold and $G \leq \mathrm{Bir}(X)$ is a finite subgroup, we say G is *stable*, or *nonstable*, if so is $\rho_X(G) \leq O(H^2(X, \mathbb{Z}))$.

7. Finite groups of symplectic birational automorphisms

The study of symplectic actions of IHS manifolds plays an important role from a geometric point of view. For instance, some of the known examples of singular analogues of IHS manifolds are constructed as terminalization of quotients of known IHS manifolds by such symplectic automorphisms [Men22, BGMM24]. Another interesting property of such (birational) automorphisms is that the restriction to their fixed loci of the nowhere degenerate holomorphic 2-form of the ambient IHS manifolds they act on is still nowhere degenerate and holomorphic. For a long time, a hope was to construct new examples of IHS manifolds by studying fixed loci of symplectic automorphisms. Fixed loci of symplectic automorphisms on $\mathrm{K3}^{[2]}$-type IHS manifolds have been studied for instance in [Cam12, Mon12]. In the recent couple of works [KMO22, KMO25], Kamenova, Mongardi and Oblomkov show that the fixed loci of finite symplectic regular actions on IHS manifolds of deformation types $\mathrm{K3}^{[n]}$ and Kum_n consist of points and known IHS manifolds.

In recent years, there has been much progress in the lattice-theoretic classification of symplectic birational automorphisms of IHS manifolds. In [Nik79, Theorem 4.5], Nikulin classifies finite symplectic actions on K3 surfaces which are abelian. This was later extended in a celebrated paper of Mukai who shows that for a K3 surface S, a finite subgroup of symplectic automorphisms $G_s \leq \mathrm{Aut}(S)$ embeds into one among 11 groups [Muk88a, Theorems (0.3), (0.6)]. A decade later, Kondō gives a simpler proof of Mukai's result using a relation with automorphisms of the Niemeier lattices [Kon98, §3]. In [Xia96], Xiao brings back the geometry of such symplectic actions into the picture and gives a new aspect to Mukai's classification by studying the combinatorics of fixed points of such automorphisms. To complete this series of work, Hashimoto determines in [Has12] transcendental data associated to the action of these groups on the associated $\mathbb{Z}$-lattice Λ_{K3}.

Beyond the case of K3 surfaces, the classification of finite symplectic birational actions of IHS manifolds of the known deformation types is more intricate and it is still an open problem. There are three main reasons: there are infinitely many deformation types to cover, the associated BBF forms are not unimodular, and the associated monodromy groups are not always maximal. In this section, we review what is known about symplectic actions on the known deformation types of IHS manifolds, and which differences there are with the similar study on K3 surfaces. We develop a bit more around this classification problem and suggest a systematic approach for classifying stable symplectic birational automorphisms.

7.1. Known results and new challenges

Let us recall the recognized result of Mukai in the case of K3 surfaces.

Theorem 7.1 ([Muk88a, Theorems (0.3), (0.6)]). *Let G be a finite group. Then there exists a K3 surface S such that G embeds into $\mathrm{Aut}_s(S)$ if and only if G embeds into the Mathieu group M_{23} and decomposes $\{1, \ldots, 24\}$ into at least 5 orbits. In particular, all such G's embed into one of the following 11 groups:*

$$L_2(7),\ A_6,\ S_5,\ M_{20},\ F_{384},\ A_{4,4},\ T_{192},\ H_{192},\ N_{72},\ M_9,\ T_{48}.$$

These groups are often referred to as maximal *(for the inclusion).*

While there is Mukai's result for symplectic actions on K3 surfaces, we do not know yet a fine description of the symplectic actions of general IHS manifolds. However, some important results have been established for the known deformation types, allowing one to develop techniques to classify such actions. Let us review a part of the state of the art on this problem. We recall that any isometry f of an even $\mathbb{Z}$-lattice L is called stable if $D_f \in O(D_L)$ is trivial.

Theorem 7.2 ([Mon16b, Theorem 26], [MTW18, Lemma 5.1]). *Let $n \geq 2$, let $X \sim \mathrm{K3}^{[n]}$, Kum_n and let $f \in \mathrm{Aut}_s(X)$ be of finite order. Then $\rho_X(f)$ is stable.*

It was later shown that this stability phenomenon is also observed for the deformation types OG6 and OG10. Actually, for IHS manifolds of these two deformation types, symplectic birational automorphisms are much more restricted: we talk about *symplectic rigidity.*

Theorem 7.3 ([GOV23, Theorems 1.1 and 1.2]). *Let $X \sim$ OG6 and let $f \in \mathrm{Bir}_s(X)$ be of finite order. Then $\rho_X(f)$ is stable. If moreover $f \in \mathrm{Aut}_s(X)$, then $\rho_X(f)$ is trivial.*

Theorem 7.4 ([GGOV24, Theorem 1.1]). *Let $X \sim$ OG10 and let $f \in \mathrm{Aut}_s(X)$ be of finite order. Then $f = \mathrm{id}_X$.*

One sees that for all of the known cases, symplectic regular automorphisms of finite order have stable action on cohomology.

Remark 7.5. Note that nonstable isometries cannot exist for K3 surfaces and $\mathrm{K3}^{[2]}$-type IHS manifolds (see Table 4).

There are several questions that arise from the previous results:

(1) What about the stability of nonregular birational symplectic actions ?

(2) Are the previous finite order behaviours, or do we observe similar phenomena for infinite order symplectic automorphisms ?

(3) What can one say about symplectic regular actions of finite order for singular analogues of IHS manifolds?

Remark 7.6. Following their work on Morrison–Kawamata Cone Conjecture for IHS manifolds [AV17a], Amerik and Verbitsky show that for all the known deformation types, there exist examples of projective IHS manifolds equipped with an infinite order automorphism [AV17b, AV23]. In particular, this implies that the symplectic rigidity results for OG6 and OG10 is a finite order phenomenon, since automorphisms of projective IHS manifolds have finite transcendental value (Proposition 6.1). We refer to [Ouc18] for more explicit examples.

Remark 7.7. In the joint work [BMM24], the author together with Brandhorst and Menet study symplectic birational automorphisms on a deformation family of *irreducible holomorphic orbifolds*, see [Men20] for a definition. They show in particular that finite order symplectic automorphisms are nonstable [BMM24, Table 1], at the exception of certain *exceptional involutions* [MR22, §4.2] which are the only stable symplectic isometries.

7.2. Stable symplectic isometries

The content of this section is a generalization of results of the author of the thesis, which can be found in a joint work with Marquand [MM25b].

Let $\mathcal{T}$ be a known deformation type of IHS manifolds, let $\Lambda := \Lambda_{\mathcal{T}}$ be the associated even indefinite $\mathbb{Z}$-lattice as given in Table 4, and let $\mathrm{Mon}^2(\Lambda) \leq O^+(\Lambda)$ be the associated mondromy.

Notation. In what follows, for any primitive sublattice $C \leq \Lambda$, we let $S(C) := SO(C)$ if $\mathcal{T} = \mathrm{Kum}_n$ for $n \geq 2$ and $S(C) = O(C)$ otherwise. We moreover let $S^+(C) := S(C) \cap O^+(C)$, $S^\#(C) = S(C) \cap O^\#(C)$ and $S^{+,\#}(C) = S^+(C) \cap S^\#(C)$.

Note that for any known deformation type $\mathcal{T}$, we have that $S^{+,\#}(\Lambda) = \mathrm{Mon}^2(\Lambda) \cap O^{\#}(\Lambda)$ is the stable subgroup of $\mathrm{Mon}^2(\Lambda)$.

We aim to determine and classify saturated finite symplectic subgroups of $S^{+,\#}(\Lambda)$, up to conjugacy in $\mathrm{Mon}^2(\Lambda)$. We show in what follows that each such conjugacy class is uniquely determined by the isomorphism class, as primitive sublattice of Λ, of the corresponding coinvariant sublattice.

Theorem 7.8. *Let $H \leq \mathrm{Mon}^2(\Lambda)$ be a nontrivial finite subgroup. If H is stable and symplectic then Λ_H is negative definite, H embeds into $S^{\#}(\Lambda_H)$ and $S^{\#}(\Lambda_H)$ fixes no nontrivial vector in Λ_H. Moreover, H is stably saturated in $\mathrm{Mon}^2(\Lambda)$, if and only if $H = S^{\#}(\Lambda_H)$, seen as subgroups of $O(\Lambda_H)$.*

Proof. If H is symplectic, we already know that Λ_H is negative definite (Theorem 6.9). Moreover, if $\mathcal{T} = \mathrm{Kum}_n$ for some $n \geq 2$, we know that H stable implies that $H \leq S(\Lambda)$ (Table 4). Moreover, we know that H acts trivially on Λ^H, so H embeds in $O(\Lambda_H)$. Since H is stable, Corollary 2.21 and Remark 2.22 tell us that H actually embeds into $O^{\#}(\Lambda_H)$. We conclude by remarking that $\det(\mathrm{id}_{\Lambda^H}) = 1$ to obtain that H maps injectively into $S^{\#}(\Lambda_H)$, and since H fixes no nontrivial vector in Λ_H by definition, so does $S^{\#}(\Lambda_H)$.

Now for any element $h \in H$, we have that $\Lambda_h \leq \Lambda_H$ is negative definite. Since $h = \mathrm{id}_{\Lambda^h} \oplus h_{|\Lambda_h} \in O(\Lambda^h \oplus \Lambda_h)$, we have that the spinor norm of h is the same as the one of $g := h_{|\Lambda_h}$. But since Λ_h is negative definite we know that g, seen as an element of $O(\Lambda_h \otimes_{\mathbb{Z}} \mathbb{R})$, decomposes as a product of reflections in vectors of $\Lambda_h \otimes_{\mathbb{Z}} \mathbb{R}$ of negative norm (see also [GOV23, Lemma 2.3]). Hence g has positive spinor norm, and so does h. Therefore any element $h \in \{\mathrm{id}_{\Lambda^H}\} \times S^{\#}(\Lambda_H) \leq O(\Lambda)$ has positive spinor norm and $\det(h) = \det(h_{|\Lambda_H})$. Thus $\{\mathrm{id}_{\Lambda^H}\} \times S^{\#}(\Lambda_H) \leq S^{+,\#}(\Lambda)$ with equality if and only if H is saturated in $S^{+,\#}(\Lambda)$. □

We have seen in Theorem 7.8 that the coinvariant sublattice C of any finite stable symplectic subgroup of $\mathrm{Mon}^2(\Lambda)$ is negative definite and satisfies that $S^{\#}(C)$ fixes no nontrivial vector in C. This motivates the following definition.

Definition 7.9. A primitive sublattice $C \leq \Lambda$ is said to be *stable symplectic* if C is negative definite and $S^{\#}(C)$ fixes no nontrivial vector in C.

Definition 7.9 is inspired by the concept of Leech couples introduced by Mongardi [Mon13], based on Kondō's idea [Kon98] for the proof of Mukai's theorem [Muk88a]. We show in the next section how one can retrieve the abstract isometry class of such stable symplectic sublattices of Λ. Then, together with the next results, we obtain an explicit procedure on how to classify finite stable symplectic subgroups of $\mathrm{Mon}^2(\Lambda)$.

Two primitive sublattices $C, C' \leq \Lambda$ are said to be $\mathrm{Mon}^2(\Lambda)$-*isomorphic* if there exist $f \in \mathrm{Mon}^2(\Lambda)$ such that $f(C) = C'$.

Theorem 7.10. *The set of $\mathrm{Mon}^2(\Lambda)$-conjugacy classes of saturated finite symplectic subgroups of $S^{+,\#}(\Lambda)$ is in bijection with the set of $\mathrm{Mon}^2(\Lambda)$-isomorphism classes of stable symplectic sublattices of $C \leq \Lambda$ such that $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$.*

Proof. Recall from Theorem 6.9 that a finite subgroup $H \leq \mathrm{Mon}^2(\Lambda)$ is symplectic if and only if Λ_H is negative definite and $\Lambda_H \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$. Hence, we know from Theorem 7.8 that the set of saturated finite symplectic subgroups of $S^{+,\#}(\Lambda) = \mathrm{Mon}^2(\Lambda) \cap O^{\#}(\Lambda)$ is in bijection with the set of stable symplectic sublattices $C \leq \Lambda$ such that $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$. Indeed, to any of the former groups $H \leq S^{+,\#}(\Lambda)$ we associate $C := \Lambda_H$ and Theorem 7.8 tells us that $\mathrm{im}(H \to O(C)) = S^{\#}(C)$. Conversely, given a stable symplectic sublattice $C \leq \Lambda$ with $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$, then the finite subgroup $H := \{\mathrm{id}_{C^\perp}\} \times S^{\#}(C) \leq S^{+,\#}(\Lambda)$ is saturated and H is symplectic by the assumption on C.

(1) Let $H, H' \leq S^{+,\#}(\Lambda)$ be finite subgroups and suppose there exists $f \in \mathrm{Mon}^2(\Lambda)$ such that $H = f^{-1}H'f$. Then $f(\Lambda_H) = \Lambda_{H'}$.

(2) Suppose that $C, C' \leq \Lambda$ are stable symplectic sublattices such that there exists $f \in \mathrm{Mon}^2(\Lambda)$ satisfying $f(C) = C'$. Note that, in particular, conjugation by $g := f_{|C}$ maps any isometry $h \in S^\#(C)$ to an isometry $g \circ h \circ g^{-1} \in S^\#(C')$. Therefore, since we have that $f(C^\perp) = (C')^\perp$, we observe that $f[\{\mathrm{id}_{C^\perp}\} \times S^\#(C)]f^{-1} = \{\mathrm{id}_{(C')^\perp}\} \times S^\#(C')$.

This concludes the proof. $\square$

In practice, Theorem 7.10 is not effective. The main reason is that, in Algorithm 2 for instance, the classification of isomorphism classes of primitive sublattices of Λ is done at the level of gluings. From that point of view, we do not have much control on the properties of the resulting isometries of Λ which classify such primitive sublattices. However, since Λ is unique in its genus, Algorithm 2 tells us how to algorithmically classify such primitive sublattices up to the action of $O(\Lambda)$, at least. Hence the following direct corollary.

Corollary 7.11. *Suppose that $\mathcal{T}$ is so that $\mathrm{Mon}^2(\Lambda) = O^+(\Lambda)$. Then the set of $\mathrm{Mon}^2(\Lambda)$-conjugacy classes of saturated finite symplectic subgroups of $S^{+,\#}(\Lambda)$ is in bijection with the set of isomorphism classes of stable symplectic sublattices of $C \leq \Lambda$ such that $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$.*

Proof. Note that for all known deformation types $\mathcal{T}$, we have that Λ has signatures $(3, *)$ (Table 4). In particular, we have that $-\mathrm{id}_\Lambda$ has negative spinor norm (Remark 1.52), and $O(\Lambda)/O^+(\Lambda)$ is generated by the coset represented by the central involution $-\mathrm{id}_\Lambda$. This implies that $O(\Lambda)$-conjugacy classes of subgroups of $S^{+,\#}(\Lambda)$ are $\mathrm{Mon}^2(\Lambda)$-conjugacy classes. The proof then follows similarly as in Theorem 7.10. $\square$

From Theorem 7.10 and Corollary 7.11 we see that what remains to be understood is what is the isometry class for the stable symplectic sublattices of Λ.

Computational comments. The condition $C \cap \mathcal{W}^{pex}(\Lambda)$ in Theorem 7.10 and Corollary 7.11, for $C \leq \Lambda$ stable symplectic, can be computationally checked by enumerating vectors of a given norm in definite $\mathbb{Z}$-lattices, using for instance an algorithm of Fincke and Pohst (see for instance [FP85] and [Coh93, §2.7.3]). Given a definite $\mathbb{Z}$-lattice C, and given a positive integer n, such an algorithm can be used to enumerate all vectors of absolute norm n in C: it has been implemented under the name `short_vectors` on the computer algebra system Hecke [FHHJ17], for instance. Note that given three positive integers d, r and n, the number of vectors of norm n in a positive definite $\mathbb{Z}$-lattice C of rank r and determinant d grows as

$$\sqrt{\frac{n^r}{d}}$$

[FP85, §3]. Thus checking the condition $C \cap \mathcal{W}^{pex}(\Lambda)$ can be expensive if C has large rank or if some vectors in $\mathcal{W}^{pex}(\Lambda)$ have large norm in absolute value. However, we have seen that vectors in $\mathcal{W}^{pex}(\Lambda)$ are not only characterized by their norm, but also by their divisibility in Λ (see for instance Examples 5.43 and 5.46). This last point can allow us to actually improve our application of Fincke–Pohst algorithm by looking for n-vectors in a sublattice of C of larger determinant.

Lemma 7.12. *Let L be an even $\mathbb{Z}$-lattice and $C \leq L$ be a primitive sublattice. If $v \in C$ has divisibility $\gamma \geq 1$ in L, then v is a primitive vector in $C \cap \gamma L^\vee$.*

Proof. By definition of the divisibility, if $\mathrm{div}(v, L) = \gamma \geq 1$, then $v \in C \cap \gamma L^\vee$. It is moreover primitive in the intersection since γ is the largest integer such that $v \in \gamma L^\vee$. $\square$

In our setting, from the fact that $C = C \cap \Lambda^\vee$, we see that for any integer $\gamma > 1$ we have that C is an overlattice of $C \cap \gamma\Lambda^\vee$. In particular, $\det(C \cap \gamma\Lambda^\vee) \geq \det(C)$ (Lemma 2.6) and therefore, for a given pair (n, γ) of positive integers, it is generically more efficient to enumerate n-vectors in $C \cap \gamma\Lambda^\vee$, rather than enumerating n-vectors in C and then keeping the ones with divisibility γ.

As a final remark, let us note the following. Any representative H of a $\mathrm{Mon}^2(\Lambda)$-conjugacy class of finite stable symplectic subgroups of $\mathrm{Mon}^2(\Lambda)$, constructed via Theorem 7.10, is saturated in $S^{+,\#}(\Lambda)$. However, it might not be saturated in $\mathrm{Mon}^2(\Lambda)$.

Proposition 7.13. *Let $H \leq S^{+,\#}(\Lambda)$ be saturated, and let G be the saturation of H in $\mathrm{Mon}^2(\Lambda)$. Then $G^\# = H$ and $[G : H] \leq 2$.*

Proof. The fact that $G^\# = H$ follows from H being saturated in $S^{+,\#}(\Lambda)$, and that the lattter is the stable subgroup of $\mathrm{Mon}^2(\Lambda)$. We conclude by remarking that G/H embeds into $\mathrm{im}(\mathrm{Mon}^2(\Lambda) \to O(D_\Lambda))$ which has order at most equal to 2 (Lemma 6.15). □

The previous result, despite its simplicity, does not provide an effective way to determine whether a given saturated finite subgroup of $S^{+,\#}(\Lambda)$ is saturated in $\mathrm{Mon}^2(\Lambda)$. We describe now a general way to test the previous, which has immediate application whenever $\mathrm{Mon}^2(\Lambda) = O^+(\Lambda)$ is maximal.

For any primitive sublattice $C \leq \Lambda$, with orthogonal complement $F := C^\perp_\Lambda$, and for any normal subgroup $N \trianglelefteq O(\Lambda)$, let us denote by $P_N(C)$ the pointwise stabilizer of F in N: there is an embedding $P_N(C) \hookrightarrow O(C)$. By abuse of notation, for any subgroup $H \leq P_N(C)$, we denote again by H its image in $O(C)$

Definition 7.14. We call the group $P_N(C)$ the *pulse of C in N*.

Remark 7.15. Note that we have already seen that if $C \leq \Lambda$ is stable symplectic, then $P_{S^{+,\#}(L)}(C) = S^\#(C)$.

The following is a generalization of [MM25b, Lemma 4.21].

Proposition 7.16. *Let $C \leq \Lambda$ be a negative definite primitive sublattice and let $N \trianglelefteq O(\Lambda)$ be a normal subgroup. Suppose that the pulse $P_N(C) \leq N$ of C in N fixes no nontrivial vector in C. Then for any subgroup $H \leq P_N(C)$ fixing no nontrivial vector in C, we have that*

$$\mathrm{Sat}_N(H) = P_N(C).$$

Proof. Since H fixes no nontrivial vectors in C, we have that $L_H = C$ — hence $\mathrm{Sat}_N(H)$ is the pointwise stabilizer of $F := C^\perp_L$, which is $P_N(C)$ by definition. □

Remark 7.17. With the notation of Proposition 7.16 and its proof, let $N := S^+(\Lambda)$ and let $D_C \geq H_C \xrightarrow{\gamma} H_F \leq D_F$ be the gluing of C and F associated to the primitive extension $C \oplus F \leq \Lambda$. Let $\pi_C \colon S(C) \to O(D_C)$ be the discriminant representation of $S(C)$, whose image will be denoted by G_C here. We let $K_C \leq G_C$ be the pointwise stabilizer of H_C in G_C. Then $P_{S^+(\Lambda)}(C) = \pi_C^{-1}(K_C)$, seen as subgroups of $O(C)$. In particular,

$$\#\,\mathrm{Sat}_{S^+(\Lambda)}(H) = \#\pi_C^{-1}(K_C) = \#K_C \cdot \#\ker(\pi_C) = \frac{\#K_C \cdot \#S(C)}{\#G_C}.$$

7.3. Stable symplectic sublattices

For a known deformation type $\mathcal{T}$ of IHS manifolds, the stable symplectic sublattices of $\Lambda_{\mathcal{T}}$ have been studied by Mongardi in his PhD thesis [Mon13, §7]. He shows the following.

Lemma 7.18. *Let $\mathcal{T}$ be one of the known deformation types, and let $C \leq \Lambda_{\mathcal{T}}$ be a stable symplectic sublattice. Then C embeds primitively into an even unimodular $\mathbb{Z}$-lattice L of signatures*

(1) $(0,8)$ *if* $\mathcal{T} = \mathrm{Kum}_n$ *for some* $n \geq 2$*;*

(2) $(0,16)$ *if* $\mathcal{T} = \mathrm{OG6}$*;*

(3) $(0,24)$ *if* $\mathcal{T} = \mathrm{K3}^{[n]}$ *for some* $n \geq 1$*;*

(4) $(1,25)$ *if* $\mathcal{T} = \mathrm{OG10}$.

Proof. For the reader's convenience, we show how to prove this lemma in case (1), and we explain how to adapt it for the cases (2)–(4).

Let $n \geq 2$ and let $\Lambda := U^{\oplus 3} \oplus A_1(n+1)$ be the abstract $\mathbb{Z}$-lattice associated to the deformation type Kum_n: it has signatures $(3,4)$. If $C \leq \Lambda$ is stable symplectic, then it is in particular negative definite and therefore $\mathrm{rank}_{\mathbb{Z}}(C) \leq 4$. Now, according to Example 2.20 we have that $\Lambda \oplus A_1(-n-1)$ admits an even unimodular primitive extension L of signatures $(4,4)$, given by any glue map $D_\Lambda = D_{A_1(n+1)} \simeq D_{A_1(-n-1)}(-1)$. We get that the composite embedding

$$C \hookrightarrow \Lambda \hookrightarrow \Lambda \oplus A_1(-n-1) \hookrightarrow L$$

is primitive. The associated orthogonal complement $T \leq L$ satisfies that $D_T \simeq D_C(-1)$ and T has signatures $(4, 4-\mathrm{rank}_{\mathbb{Z}}(C))$. According to Remark 2.28, the existence of T is equivalent to the existence of an even $\mathbb{Z}$-lattice T' of signatures $(0, 8-\mathrm{rank}_{\mathbb{Z}}(C))$ and discriminant form $D_C(-1)$. Therefore, according to Proposition 2.27 we have that C embeds primitively into an even unimodular $\mathbb{Z}$-lattice of signatures $(0,8)$. For the remaining cases, we take instead:

(2) $\Lambda = U^{\oplus 3} \oplus A_1^{\oplus 2}$, the $\mathbb{Z}$-lattice L is even unimodular of signatures $(8,8)$, T has signatures $(8, 8-\mathrm{rank}_{\mathbb{Z}}(C))$ and T' has signatures $(0, 16-\mathrm{rank}_{\mathbb{Z}}(C))$;

(3) $\Lambda = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_1(n-1)$, the $\mathbb{Z}$-lattice L is even unimodular of signatures $(4,20)$, T has signatures $(4, 20-\mathrm{rank}_{\mathbb{Z}}(C))$ and T' has signatures $(0, 24-\mathrm{rank}_{\mathbb{Z}}(C))$;

(4) $\Lambda = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$, the $\mathbb{Z}$-lattice L is even unimodular of signatures $(5,21)$, T has signatures $(5, 21-\mathrm{rank}_{\mathbb{Z}}(C))$ and T' has signatures $(1, 25-\mathrm{rank}_{\mathbb{Z}}(C))$. □

Technically, in Item (4) of Lemma 7.18 one could have taken signatures $(0,32)$ instead. However, while the genera $\mathrm{II}_{(0,8)}$, $\mathrm{II}_{(0,16)}$ and $\mathrm{II}_{(0,24)}$ consists respectively of 1, 2, and 24 isometry classes, the genus $\mathrm{II}_{(0,32)}$ consists of at least one billion isometry classes [Kin03, Corollary 17], which makes it less convenient to work with. Moreover, the indefinite even unimodular $\mathbb{Z}$-lattice in $\mathrm{II}_{(1,25)}$, which is unique in its genus, is very well-studied.

In order to determine the stable symplectic sublattices $C \leq \Lambda_{\mathcal{T}}$, for a fixed deformation type $\mathcal{T}$, one sees that we need to study primitive sublattices of certain even unimodular $\mathbb{Z}$-lattices. Note already that the genus $\mathrm{II}_{(0,8)}$ consists of a unique isometry class, represented by the root lattice E_8. In particular, the following holds.

Proposition 7.19 ([Mon13, Proposition 7.1.9],[MTW18, Proposition 5.2])**.** *Let $n \geq 2$, let $X \sim \mathrm{Kum}_n$ and let $G \leq \mathrm{Bir}_s(X)$ be finite and such that $H := \rho_X(G)$ is stable. Then $H^2(X,\mathbb{Z})_H$ embeds primitively into E_8.*

Negative definite primitive sublattices $C \leq E_8$ such that $O^{\#}(C)$ fixes no nontrivial vector in C are known, and they have been classified [DPR13, Table 5] (see also [HM16, Theorem 3.6]).

Now the genus $\mathrm{II}_{(0,16)}$ consists of two isometry classes. From Lemma 7.18 it is not clear into which of these two $\mathbb{Z}$-lattices stable symplectic sublattices of Λ_{OG6} should embed. In her PhD thesis [Gro20], Grossi actually shows that in some cases, one can restrict to the following.

Proposition 7.20 ([Gro20, Proposition 5.0.10]). *Let $X \sim \mathrm{OG6}$ and let $G \leq \mathrm{Bir}_s(X)$ be finite, and such that $H := \rho_X(G)$ is stable. Denote $C := H^2(X,\mathbb{Z})_H$. If $\mathrm{rank}_{\mathbb{Z}}(C) + l((D_C)) \leq 8$, then C embeds primitively into E_8.*

Remark 7.21. In the case where $C \leq \Lambda_{\mathrm{OG6}}$ is the coinvariant sublattice associated to a prime order symplectic isometry, then the authors in [GOV23] show that C embeds primitively in E_8. But a priori, there could still be examples of stable symplectic sublattices $C \leq \Lambda_{\mathrm{OG6}}$ such that $\mathrm{rank}_{\mathbb{Z}}(C) + l(D_C) > 8$. One solution to determine such cases could be to classify negative definite primitive sublattices in the $\mathbb{Z}$-lattices in $\mathrm{II}_{(0,16)}$.

For the $\mathrm{K3}^{[n]}$ types, things start to be a bit more involved. According to Venkov [Ven80], the genus $\mathrm{II}_{(0,24)}$ consists of 24 isometry classes, represented by the so-called *Niemeier lattices* (see for instance [CS99, Chapter 18] or [Ebe02, §3.4] for some more details). These 24 classes are uniquely determined by the isometry class of their associated root sublattice. Among them, there is a unique class whose representatives have trivial root sublattice, i.e. they contain no (-2)-roots.

Definition 7.22. We define the *Leech lattice* $\mathbb{L}$ to be the unique, up to isometry, even unimodular $\mathbb{Z}$-lattice in the genus $\mathrm{II}_{(0,24)}$ which contains no (-2)-roots.

Remark 7.23. The Leech lattice is a very particular definite $\mathbb{Z}$-lattice which is known for some remarkable properties. For instance, it gives the densest sphere packing in dimension 24 [CKM+17, Theorem 1.1]. Note that the root lattice E_8 itself determines the densest sphere packing in dimension 8 [Via17, Theorem 1]. Both $\mathbb{L}$ and E_8 are very special, and we have just shown that the symmetries of E_8 are related to birational automorphisms of some IHS manifolds. We show later that it is also the case for the Leech lattice $\mathbb{L}$.

Another interesting remark is the existence of the "holy constructions" of the Leech lattice [CS99, Chapter 24]: starting from any other Niemeier lattice $L \in \mathrm{II}_{(0,24)}$, there is a procedure to reconstruct the Leech lattice $\mathbb{L}$. We will not enter into the details of these constructions, but we point out that these constructions allowed Mongardi to refine the result from Lemma 7.18 and obtain the following.

Theorem 7.24 ([Mon13, Theorem 7.2.4], [Mon16b, Theorem]). *Let $n \geq 1$[1], let $X \sim \mathrm{K3}^{[n]}$ and let $G \leq \mathrm{Bir}_s(X)$ be of finite order such that $H := \rho_X(G)$ is stable. Then $H^2(X,\mathbb{Z})_H$ embeds primitively into the Leech lattice.*

Remark 7.25. It is important to note that the previous theorem of Mongardi is originally stated for symplectic automorphisms, in which case the action on cohomology is always stable (Theorem 7.2). Nevertheless the proof still holds for symplectic birational automorphisms which are stable.

Remark 7.26. In the case of K3 surfaces, Theorem 7.24 actually goes back to Kondō's arguments [Kon98]. It has also been proved in a different context, such as in the study of automorphisms of nonlinear σ-models on the (topologial) K3 surface [GHV12, Theorem 1.1]. Huybrechts gives also a derived categorical equivalent of this proof in [Huy16, Theorem 0.1.]. See [Kon18] for a general survey on symplectic automorphisms of K3 surfaces.

[1] in this thesis, $\mathrm{K3}^{[1]} = \mathrm{K3}$ as deformation types

Similarly to the E_8 case, negative definite primitive sublattices $C \leq \mathbb{L}$ such that $O^\#(C)$ fixes no nontrivial vector in C have been classified by Höhn and Mason [HM16].

Theorem 7.24 and Proposition 7.19 highlights the importance of $\mathbb{L}$ and E_8 for studying stable symplectic actions of the two infinite families of known deformation types. Conceptually, such results are not as fine as Mukai's classification for symplectic actions on K3 surfaces, but they give enough restrictions to perform an actual explicit classification. The closest we can get today to a Mukai-like result is for the deformation type $\mathrm{K3}^{[2]}$, where all symplectic actions are stable.

Theorem 7.27 ([HM19, Theorem 8.7])**.** *Let G be a finite group. Then there exists an IHS manifold $X \sim \mathrm{K3}^{[2]}$ such that G embeds into $\mathrm{Bir}_s(X)$ if and only if*

(1) *either G embeds into M_{23} with at least four orbits in the natural action on 24 elements,*

(2) *or G is isomorphic to a subgroup of $3^{1+4} : 2.2^2$ or $3^4 : A_6$ associated to the corresponding $\mathcal{S}$-lattices (see [HM19, §3] for a definition).*

More recently, Laza and Zheng give in [LZ22] another proof of Theorem 7.24, in the context of studying *symplectic automorphisms* of smooth cubic fourfolds.

Notation. Given a cubic fourfold $V \subseteq \mathbb{P}^5$ and an automorphism $f \in \mathrm{Aut}(V)$, we call f *symplectic* if $f^*\colon H^4(V) \to H^4(V)$ is the identity on $H^{3,1}(V)$ (which is a 1-dimensional complex vector space). There is also a natural faithful orthogonal representation

$$\rho_V\colon \mathrm{Aut}(V) \to O(H^4_{prim}(V,\mathbb{Z}))$$

where $H^4_{prim}(V,\mathbb{Z}) := (h^2)^{\perp}_{H^4(V,\mathbb{Z})}$. Here $h^2 \in H^4(V,\mathbb{Z})$ is the square of the hyperplane class on V, and the lattice structure on $H^4(X,\mathbb{Z})$ is odd unimodular with real signatures $(21,2)$. Note that since any automorphism of V must preserve the class h, we obtain that $\mathrm{im}(\rho_V) \leq O^{+,\#}(H^4_{prim}(V,\mathbb{Z}))$ (see [Voi86, Huy23] for more details)

Their result on cubic fourfolds brings another classification of symplectic automorphisms in Mukai's fashion, i.e. they are able to determine a finite list of groups of symplectic automorphisms into which any other such group embeds. Moreover, they prove the following.

Theorem 7.28 ([LZ22, Lemma 4.2])**.** *Let V be a smooth cubic fourfold, let $G \leq \mathrm{Aut}_s(V)$ be finite and let $H := \rho_V(G)$. Then $H^4(V,\mathbb{Z})_H$ embeds primitively into the Leech lattice.*

From our description, it remains to study and understand the stable symplectic sublattices of Λ_{OG10}. As we have seen, these embed primitively into the even unimodular $\mathbb{Z}$-lattice $\mathbb{B}$ in $\mathrm{II}_{(1,25)}$ which is unique in its genus. Moreover, one observes the following:

Lemma 7.29. *Let $H \leq O^{+,\#}(\Lambda_{\mathrm{OG10}})$ be symplectic. Then $C := (\Lambda_{\mathrm{OG10}})_H$ contains no (-2)-roots, $O^\#(C)$ fixes no nontrivial vectors in C and there exists a subgroup $H \leq O^+(\mathbb{B})$ such that $\mathbb{B}_H \simeq C$.*

Proof. The first assertion follows from the fact that H is symplectic, Theorem 6.9 and Example 5.46. For the second assertion, we use the fact that H is stable and acts without fixing any nontrivial vector in $C = \Lambda_H$: in particular, the same holds for $O^\#(C)$. For the last assertion, we have that C is stable symplectic, and C embeds primitively into $\mathbb{B}$ according to Lemma 7.18. Following the same arguments as in the proof of Theorem 7.10, we have that $O^\#(C)$ extends to a subgroup $H \leq O^+(\mathbb{B})$ acting trivially on $C^\perp_{\mathbb{B}}$. Since $O^\#(C)$ fixes no nontrivial vector in C, we have that $\mathbb{B}_H \simeq C$. □

So we have changed our problem into studying finite subgroups of $O^+(\mathbb{B})$ whose coinvariant sublattice is negative definite and contains no (-2)-roots. The important point now is that the isometries of $\mathbb{B}$ are very well-studied [CS99, Chapter 27]. Let us review this in the next section.

7.4. Finite groups of isometries of Borcherds' lattice

The content of this section is a recollection of results of the thesis' author, which can be found in the joint work [MM25b]. The ideas of this section were suggested to the authors by Brandhorst, based on a first version of this thesis. In particular, some of the main results, such as Theorem 7.37 and Proposition 7.40, have been established after fruitful discussions between the thesis' author and Brandhorst.

Recall that we denote by $\mathbb{L}$ the Leech lattice. According to Theorem 2.29 we have that

$$\mathbb{B} := U \oplus \mathbb{L} \in \mathrm{II}_{(1,25)}$$

is unique up to isometry: we refer to it as *Borcherds' lattice* [Bor90]. Let us fix a basis $\{e, s\}$ for the first summand $U \leq \mathbb{B}$ with $e^2 = 0,\ s^2 = -2$ and $e.s = 1$. The group of isometries of $\mathbb{B}$ is known and it has been studied for instance by Conway in [CS99, Chapter 27]. For the reader's convenience, we recall Conway's results, following an exposition of Brandhorst and Mezzedimi [BM24]. Since $\mathbb{B}$ has signatures $(1, 25)$, we have that the set

$$\left\{x \in \mathbb{B} \otimes_{\mathbb{Z}} \mathbb{R} \ :\ x^2 > 0\right\}$$

has two connected components. Let $\mathcal{P}$ be one of these, and call it the *positive cone* of $\mathbb{B}$. Note that $O(\mathbb{B}) = \{\pm \mathrm{id}\} \times O^+(\mathbb{B})$ and $-\mathrm{id}$ does not preserve $\mathcal{P}$ — we can see the group $O^+(\mathbb{B})$ as the stabiliser of $\mathcal{P}$ in $O(\mathbb{B})$.

Let us denote by $\Delta := \{r \in \mathbb{B} \ :\ r^2 = -2\}$ the set of (-2)-roots of $\mathbb{B}$ and denote by $W(\mathbb{B}) \leq O^+(\mathbb{B})$ the subgroup generated by the reflections in the roots $r \in \Delta$. The group $W(\mathbb{B})$ is the so-called *Weyl group* of $\mathbb{B}$, and it acts simply transitively on the set of connected components, or *chambers*, of

$$\Gamma := \mathcal{P} \setminus \bigcup_{r\in\Delta} r^\perp.$$

For any chamber $D \subseteq \Gamma$, we let $P_D := \{v \in \mathbb{B} \ :\ v.x > 0,\ \forall x \in \overline{D}\}$ and moreover, we define

$$\Delta_D := \{r \in P_D \cap \Delta \ \mid\ r - r' \notin P_D,\ \forall r' \in P_D \cap \Delta\}$$

the set of *simple roots of D* [BM24, §2.6].

Lemma 7.30. *There exists a chamber $D_0 \subset \Gamma$ such that $e \in \overline{D_0} \cap \mathbb{B}$ and the set $\{|e.r| \ :\ r \in \Delta_{D_0}\}$ is bounded. Moreover, the vector e is the unique isotropic vector in $\overline{D_0} \cap \mathbb{B}$ satisfying this property, and $e.r = 1$ for all $r \in \Delta_{D_0}$*

Proof. Existence of an isotropic element with the required properties follows from [CS99, Chapter 27]. Uniqueness follows from [BM24, Theorem 3.7, Remark 3.8]. □

For the rest of the section, we refer to e and D_0 as a *Weyl vector* and a *Weyl chamber*, respectively. We moreover denote by $\mathcal{D} := \overline{D_0}$ the closure of D_0 in $\mathcal{P}$, and we let $\mathrm{Aut}(\mathcal{D})$ be the subgroup of isometries of $O^+(\mathbb{B})$ preserving $\mathcal{D}$.

Lemma 7.31 ([BM24, Theorem 3.7])**.** *The group $\mathrm{Aut}(\mathcal{D})$ is infinite and any element $h \in \mathrm{Aut}(\mathcal{D})$ fixes e.*

The relevance of $\mathrm{Aut}(\mathcal{D})$ for us follows from the following.

Lemma 7.32. *Let $C \leq \mathbb{B}$ be a negative definite primitive sublattice without (-2)-roots and such that $O^\#(C)$ fixes no nontrivial vector in C. Then the group $O^\#(C)$ is isomorphic to a subgroup $H \leq \mathrm{Aut}(\mathcal{D})$ so that $\mathbb{B}_H \simeq C$.*

Proof. According to Lemma 7.29 we can extend $O^{\#}(C)$ to a subgroup $\widetilde{H} \leq O^+(\mathbb{B})$ acting trivially on $N := C^{\perp}_{\mathbb{B}}$. In particular $\mathbb{B}_{\widetilde{H}} \simeq C$. Since C does not contain (-2)-roots, $N \otimes_{\mathbb{Z}} \mathbb{R}$ intersects a chamber D of the positive cone of $\mathbb{B}$. The group $\widetilde{H}$ acting trivially on N, we have that $\widetilde{H}$ fixes a vector in D and thus, $\widetilde{H}$ preserves the entire chamber D. Hence $\widetilde{H} \leq \mathrm{Aut}(\overline{D})$. Since $\mathrm{Aut}(\overline{D})$ and $\mathrm{Aut}(\mathcal{D})$ are $W(\mathbb{B})$-conjugate, we obtain that $\widetilde{H}$ is $O^+(\mathbb{B})$-conjugate to a subgroup $H \leq \mathrm{Aut}(\mathcal{D})$. In particular $\mathbb{B}_H \simeq \mathbb{B}_{\widetilde{H}} \simeq C$. □

Thus to pursue, we would like to study finite subgroups of $\mathrm{Aut}(\mathcal{D})$ in order to determine their coinvariant sublattices. For this, we will need to understand the structure of $\mathrm{Aut}(\mathcal{D})$.

Definition 7.33 (Eichler-Siegel transformation)**.** For any $\lambda \in \mathbb{L}$, we define

$$\psi_\lambda\colon \mathbb{B} \to \mathbb{B},\ x \mapsto x + (x.\lambda)e - (x.e)\lambda - \frac{1}{2}(x.e)\lambda^2 e.$$

For all $\lambda \in \mathbb{L}$, we have that $\psi_\lambda \in \mathrm{Aut}(\mathcal{D})$ by the proofs of [BM24, Proposition 3.2, Theorem 4.7]. Furthermore, the assignment

$$\psi\colon \mathbb{L} \to \mathrm{Aut}(\mathcal{D}),\ \lambda \mapsto \psi_\lambda$$

is an injective group homomorphism (where we view $\mathbb{L}$ as torsionfree abelian group of finite positive rank, under addition). By the definition of the Weyl vector e, which is isotropic, there is an exact sequence of $\mathbb{Z}$-lattices

$$0 \to \mathbb{Z}e \to e^\perp \to \mathbb{L} \to 0$$

inducing an isometry $\kappa\colon e^\perp/\mathbb{Z}e \xrightarrow{\simeq} \mathbb{L}$. Moreover, since any isometry in $\mathrm{Aut}(\mathcal{D})$ fixes e (Lemma 7.31), the isometry κ defines an orthogonal representation

$$\pi\colon\ \mathrm{Aut}(\mathcal{D}) \to O(\mathbb{L}) \tag{19}$$

which admits a section $\phi : O(\mathbb{L}) \to \mathrm{Aut}(\mathcal{D})$, given by extending an isometry of $\mathbb{L}$ to one of $\mathbb{B}$ acting as the identity on $\mathbb{Z}e + \mathbb{Z}s \simeq U$. In particular, π is surjective.

Lemma 7.34 ([CS99, Chapter 27])**.** *We have that* $\mathrm{Aut}(\mathcal{D}) = \mathbb{L} \rtimes_\phi O(\mathbb{L})$.

Proof. We have an exact sequence

$$0 \to \mathbb{L} \xrightarrow{\psi} \mathrm{Aut}(\mathcal{D}) \xrightarrow{\pi} O(\mathbb{L}) \to 1$$

which admits a splitting defined by ϕ. Since all nontrivial elements of $\mathbb{L}$ have infinite order and $O(\mathbb{L})$ is of finite order, we have that $\psi(\mathbb{L}) \cap \phi(O(\mathbb{L}))$ is trivial, as a subgroup of $\mathrm{Aut}(\mathcal{D})$. Hence, we have that

$$\mathrm{Aut}(\mathcal{D}) = \{\psi_\lambda \circ \phi(g)\ :\ \lambda \in \mathbb{L},\ g \in O(\mathbb{L})\}$$

is the semidirect product of $\mathbb{L}$ and $O(\mathbb{L})$ defined by ϕ, and the result follows. □

Hence, any element $h \in \mathrm{Aut}(\mathcal{D})$ can be uniquely written in the form $\psi_\lambda \circ \phi(g)$ for some $\lambda \in \mathbb{L}$ and some $g \in O(\mathbb{L})$: we write

$$h = (\lambda, g).$$

We record some properties of elements $h \in \mathrm{Aut}(\mathcal{D})$ for future use.

Proposition 7.35. *Let $\lambda \in \mathbb{L}$ and let $g \in O(\mathbb{L})$. The following hold:*

(1) $\phi(g) \circ \psi_\lambda = \psi_{g(\lambda)} \circ \phi(g)$*;*

(2) $\psi_{\lambda}^{-1} = \psi_{-\lambda}$;

(3) $h := (\lambda, g)$ *is of finite order if and only if* $\lambda \in \mathbb{L}_g$.

Proof.

(1) Follows from the description of $\mathrm{Aut}(\mathcal{D}) = \mathbb{L} \rtimes_{\phi} O(\mathbb{L})$ as a semidirect product.

(2) Follows from the fact that $\psi \colon \mathbb{L} \to \mathrm{Aut}(\mathcal{D})$ is a group homomorphism.

(3) Let n be the order of g. Then we know that h is of finite order if and only if there $m \geq 1$ such that
$$h^m = (\lambda + g(\lambda) + \cdots + g^{m-1}(\lambda),\ g^m) = (0, \mathrm{id}_{\mathbb{L}}).$$
In particular, the latter holds if and only if n divides m, and $\lambda + g(\lambda) + \cdots + g^{m-1}(\lambda) = 0$. By the definition of $\mathbb{L}_g$, we have that $\lambda + g(\lambda) + \cdots + g^{m-1}(\lambda) = 0$ with $m = kn$, $k \geq 1$, if and only if $k\lambda \in \mathbb{L}_g$. But since $\lambda \in \mathbb{L}$ and $\mathbb{L}_g \leq \mathbb{L}$ is primitive, the latter is equivalent to $\lambda \in \mathbb{L}_g$. Hence, h is of finite order if and only if $\lambda \in \mathbb{L}_g$. $\square$

Remark 7.36. A consequence of Proposition 7.35 (3) is that for any $h := (\lambda, g) \in \mathrm{Aut}(\mathcal{D})$ of finite order, then the order of h is the same as the order of g.

We continue this first part with the following criterion for finite subgroups of $\mathrm{Aut}(\mathcal{D})$.

Theorem 7.37. *Let $H \leq \mathrm{Aut}(\mathcal{D})$ be a subgroup, and let $G := \pi(H) \leq O(\mathbb{L})$ (Equation (19)). Then H is of finite order if and only if there exists $n \in \mathbb{Z}$ positive and $v \in \mathbb{L}$ such that for all $h = (\lambda, g) \in H$,*
$$g(v) - v = n\lambda.$$
Moreover, if H is of finite order then $H \cong G$, and n can be chosen to be $\#(H \cdot s)$.

Proof. First remark that if H is of finite order, then $H \cap \mathbb{L}$ is the trivial subgroup of $\mathrm{Aut}(\mathcal{D})$. In particular, π restricts to an isomorphism between H and $G := \pi(H)$. Note moreover that for any $h = H \leq \mathrm{Aut}(\mathcal{D})$, we have $h(e) = e$ (Lemma 7.31).

Suppose that H is of finite order, and denote by $n := \#(H \cdot s)$ the length of the orbit of the (-2)-root s under the action of H. Since H is finite, we have that $w := \sum_{r \in H \cdot s} r$ is fixed by H, and moreover $e.w = n(e.s) = n$. This implies that there exist $m \in \mathbb{Z}$ and $v \in \mathbb{L}$ such that
$$w = me + ns + v \in U \oplus \mathbb{L}.$$
Since H fixes e and w, we have that H fixes $ns + v$. In particular, for all $h = (\lambda, g) \in H$, we have
$$0 = h(ns + v) - (ns + v) = (g(v) - v - n\lambda) + \left(\lambda.g(v) - n\frac{\lambda^2}{2}\right) e \in \mathbb{L} \oplus \mathbb{Z}e. \tag{20}$$
The latter implies that $g(v) - v = n\lambda$, and since this holds for any $h = (\lambda, g) \in H$, we can conclude.

Conversely, suppose that such $n > 0$ and $v \in \mathbb{L}$ exist, and let $h = (\lambda, g) \in H$ be arbitrary. Note that we have that
$$n\lambda.(g(v) + v) = (g(v) - v).(g(v) + v) = 0.$$
In particular, since $g(v) - v = n\lambda$, we observe that
$$2g(v).\lambda = (g(v) - v + g(v) + v).\lambda = (g(v) - v).\lambda = n\lambda^2.$$

According to Equation (20), we obtain that $ns+v \in \mathbb{B}^H$. Since $e \in \mathbb{B}^H$ too, one can find $m \in \mathbb{Z}_{\geq 0}$ large enough such that $me+ns+v \in \mathbb{B}^H$ has positive norm. Since the lattice $\mathbb{B}$ is hyperbolic, we deduce that $\mathbb{B}_H$ is negative definite and the group H acting faithfully on such a lattice must be finite. $\square$

Remark 7.38. Let $H \leq \mathrm{Aut}(\mathcal{D})$ be of finite order. Then the pair $(n,v) \in \mathbb{Z}_{>0} \times \mathbb{L}$ as in the statement of Theorem 7.37 is not unique: in fact, one can rescale simultaneously n and v by any nonzero integer, and v can be replaced by any element of $v+\Lambda^G$. In particular, we can always assume that $n = \#H$.

Remark 7.39. Let $H \leq \mathrm{Aut}(\mathcal{D})$ be finite, and let $G := \pi(H) \leq O(\mathbb{L})$ where $\pi\colon \mathrm{Aut}(\mathcal{D}) \to O(\mathbb{L})$ is the representation defined by the isometry $\kappa\colon e^\perp/\mathbb{Z}e \xrightarrow{\simeq} \mathbb{L}$. Since H fixes e, the $\mathbb{Z}$-lattice $\mathbb{B}_H \leq e^\perp$ and it does not contain $\mathbb{Z}e$: we obtain therefore that $\mathbb{B}_H$ embeds into $\mathbb{L}_G$, and the two $\mathbb{Z}$-lattices have the same rank. However, this embedding is not necessarily primitive and, $\mathbb{B}_H$ and $\mathbb{L}_G$ are not always isometric.

Proposition 7.40. *Let $H \leq \mathrm{Aut}(\mathcal{D})$ be a finite subgroup and let $G := \pi(H) \cong H$. Then $\mathbb{B}_H$ and $\mathbb{L}_G$ are isometric if and only if there exists $v \in \mathbb{L}$ such that $g(v)-v = \lambda$ for all $(\lambda, g) \in H$. If the latter does not hold, then there exists $n > 1$ such that*

$$det(\mathbb{B}_H) = n^2 det(\mathbb{L}_G).$$

Proof. According to the proof of Theorem 7.37, we know that

$$\mathbb{Z}e \oplus \mathbb{L}^G \leq \mathbb{B}^H,$$

and there exists a pair $(n,v) \in \mathbb{Z}_{>0} \times \mathbb{L}$ such that $ns+v \in \mathbb{B}^H$. We remark the following: let $(n',v') \in \mathbb{Z}_{>0} \times \mathbb{L}$ be another pair such that $n's+v' \in \mathbb{B}^H$. If we let $d := \gcd(n,n')$ and $u_1, u_2 \in \mathbb{Z}$ be such that $u_1 n + u_2 n' = d$, we obtain that $ds + (u_1 v + u_2 v') \in \mathbb{B}^H$. Moreover, if we let $k \in \mathbb{Z}_{\geq 1}$ be such that $n = kd$, one computes, for all $(\lambda, g) \in H$

$$\begin{aligned} g(k(ds+(u_1v+u_2v')) - (ns+v)) &= (ku_1-1)g(v) + ku_2 g(v') \\ &= (ku_1-1)(v+n\lambda) + ku_2(v'+n'\lambda) \\ &= (ku_1-1)v + ku_2 v' \\ &= k(ds+(u_1v+u_2v')) - (ns+v). \end{aligned}$$

Hence, $ns+v \in \mathbb{Z}(ds+(u_1v+u_2v')) + \mathbb{L}^G$, and the same holds for $n's+v'$. This implies the following. Let now n be the positive generator of the $\mathbb{Z}$-ideal

$$\sum_{ms+w \in \mathbb{B}^H} m\mathbb{Z}$$

where the sum runs over all the elements of $\mathbb{B}^H$ of the form $ms+w$ for some pair $(m,w) \in \mathbb{Z}_{>0} \times \mathbb{L}$, and let $v \in \mathbb{L}$ be such that $ns+v \in \mathbb{B}^H$. Then for any element $ae+ms+w \in \mathbb{B}^H$, we know that $ms+w \in \mathbb{B}^H$ too and by the previous, the element $ms+w$ actually lies in $\mathbb{Z}(ns+v)+\mathbb{L}^G$. Therefore

$$\mathbb{B}^H = (\mathbb{Z}e \oplus \mathbb{L}^G) + \mathbb{Z}(ns+v).$$

We define $H_v := \psi_v^{-1}\phi(G)\psi_v$: by definition of $\phi\colon O(\mathbb{L}) \to \mathrm{Aut}(\mathcal{D})$ and the fact that $g(v)-v = n\lambda$ for all $(\lambda, g) \in H$, we see that elements of H_v are of the form $(n\lambda, g)$ where $(\lambda, g) \in H$. We therefore already note that if $n = 1$, i.e. $g(v)-v = \lambda$ for all $(\lambda, g) \in H$, then $H = H_v$ is conjugate to $\phi(G)$ in $\mathrm{Aut}(\mathcal{D})$ and thus

$$\mathbb{B}_H = \mathbb{B}_{H_v} \simeq \mathbb{B}_{\phi(G)} = \mathbb{L}_G.$$

Furthermore, since $g(v) - v = n\lambda$ and $n\lambda.g(v) = \frac{(n\lambda)^2}{2}$ for all $(\lambda, g) \in H$ (Equation (20)), applying Equation (20) to $(n\lambda, g)$ instead of (λ, g) and $s+v$ instead of $ns+v$ gives $\mathbb{B}^{H_v} = (\mathbb{Z}e \oplus \mathbb{L}^G) + \mathbb{Z}(s+v)$. By comparing the Gram matrices of $\mathbb{B}^H$ and $\mathbb{B}^{H_v}$, we therefore obtain the equality

$$\det(\mathbb{B}^H) = n^2 \det(\mathbb{B}^{H_v}).$$

Since $\mathbb{B}$ is unimodular, and $\mathbb{B}_{H_v} \simeq \mathbb{L}_G$, we also obtain the wanted equality

$$\det(\mathbb{B}_H) = n^2 \det(\mathbb{L}_G).$$

From this, it is clear that if $\mathbb{B}_H \simeq \mathbb{L}_G$, then $n = 1$ and $g(v) - v = \lambda$ for all $(\lambda, g) \in H$. □

Remark 7.41. Let $H \leq \mathrm{Aut}(\mathcal{D})$ be a nontrivial subgroup. According to the proof of Theorem 7.37, the group H is finite if and only if $\mathbb{B}_H$ is negative definite, $O^\#(\mathbb{B}_H)$ fixes no nontrivial vectors in $\mathbb{B}_H$, and the latter does not contain (-2)-roots.

Combining Lemma 7.29 and Remark 7.41, we see that as abstract $\mathbb{Z}$-lattices, we can view stable symplectic sublattices $C \leq \Lambda_{\mathrm{OG10}}$ without (-2)-roots as the coinvariant sublattice of a finite subgroup $H \leq \mathrm{Aut}(\mathcal{D})$. As already noted in Proposition 7.40, in some cases these are actually primitively embedded into the Leech lattice $\mathbb{L}$. Unfortunately, there exist finite subgroups $H \leq \mathrm{Aut}(\mathcal{D})$ whose coinvariant sublattice does not embed primitively into the Leech lattice $\mathbb{L}$ — we call such groups *exceptional*.

Example 7.42. Let C be the negative definite even $\mathbb{Z}$-lattice with Gram matrix

$$\begin{pmatrix}
-4 & 2 & 2 & -2 & -2 & 2 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -1 & 1 \\
2 & -4 & -2 & 0 & 1 & -1 & 2 & -2 & -2 & -1 & -1 & 1 & 1 & -1 & -1 & -2 & -1 & -2 \\
2 & -2 & -4 & 1 & 0 & -2 & 2 & -2 & -2 & -1 & -1 & -1 & 1 & -1 & 0 & -2 & 0 & -1 \\
-2 & 0 & 1 & -4 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 2 & 0 & -1 \\
-2 & 1 & 0 & -2 & -4 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 2 & 2 & -1 & 1 \\
2 & -1 & -2 & 0 & 1 & -4 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 1 & -1 \\
-2 & 2 & 2 & 0 & 0 & 2 & -4 & 2 & 2 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 & 2 \\
2 & -2 & -2 & 0 & 0 & -2 & 2 & -4 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 0 & -1 \\
2 & -2 & -2 & 0 & 0 & -2 & 2 & -1 & -4 & 0 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\
0 & -1 & -1 & 1 & 1 & 0 & -1 & 1 & 0 & -4 & -1 & -1 & 2 & -2 & 0 & -2 & -1 & 0 \\
0 & -1 & -1 & 1 & 1 & 0 & 1 & -1 & 1 & -1 & -4 & 2 & 2 & -1 & 1 & -1 & 0 & -1 \\
0 & 1 & -1 & 0 & -1 & 0 & -1 & 1 & -1 & -1 & 2 & -4 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & 2 & 2 & 0 & -4 & 2 & -1 & 1 & 0 & 1 \\
0 & -1 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & -2 & -1 & 0 & 2 & -4 & 1 & -1 & 0 & -1 \\
2 & -1 & 0 & 0 & 2 & -2 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & -4 & -1 & 1 & -1 \\
2 & -2 & -2 & 2 & 2 & -1 & 0 & -1 & -1 & -2 & -1 & 0 & 1 & -1 & -1 & -4 & 0 & -1 \\
-1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 \\
1 & -2 & -1 & -1 & 1 & -1 & 2 & -1 & -1 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -4
\end{pmatrix}$$

The $\mathbb{Z}$-lattice C lies in the genus $\mathrm{II}_{(0,18)}3^{-7}$, $O^\#(C) \cong C_3 \times C_3$ fixes no nontrivial vector in C, and C contains no (-2)-roots. Using Algorithm 2, we check in [MM25c, Notebook "Counterexample"] that C embeds primitively into Λ_{OG10}, with orthogonal complement isometric to $U(3)^{\oplus 3}$.

However, we remark that $\mathrm{rank}_{\mathbb{Z}}(C) + l(D_C) = 25$: this implies that C does not embed primitively into the Leech lattice (Remark 2.26), but it does embed primitively into $\mathbb{B}$.

We describe now a procedure to recover the abstract isometry class of the coinvariant sublattices associated to exceptional finite subgroups of $\mathrm{Aut}(\mathcal{D})$.

We know from Proposition 7.40 that in order to determine exceptional finite subgroups of $\mathrm{Aut}(\mathcal{D})$, we need to look for finite subgroups $H \leq \mathrm{Aut}(\mathcal{D})$ such that there does not exist any $v \in \mathbb{L}$ so that $g(v) - v = \lambda$ for all $(\lambda, g) \in H$. Note that since the isometry class of $\mathbb{B}_H$ is preserved under conjugation of H by any element of $O(\mathbb{B})$, we describe a procedure to recover at least one representative for each $\mathrm{Aut}(\mathcal{D})$-conjugacy class of finite exceptional subgroups of $\mathrm{Aut}(\mathcal{D})$.

Let $G \leq O(\mathbb{L})$ be a subgroup. We define a $\mathbb{Z}$-linear map

$$p_G \colon \mathbb{L} \to \prod_{g\in G} \mathbb{L}_g,\ v \mapsto (g(v) - v)_{g\in G},$$

whose kernel is exactly $\mathbb{L}^G$. We can see any element in the image of p_G as a map assigning to each $g \in G$ an element of $\lambda \in \mathbb{L}_g$: in particular, any finite subgroup $H \leq \mathrm{Aut}(\mathcal{D})$ such that $\pi(H) = G$ defines an element in $\mathrm{im}(p_G)$. Let us denote by m the order of G. We define moreover

$$\mathbb{L} \to \prod_{g\in G} \mathbb{L}_g/m\mathbb{L}_g,\ v \mapsto (g(v) - v + m\mathbb{L}_g)_{g\in G}$$

whose kernel is denoted by $K(G)$. We observe that $K(G)$ contains $\mathbb{L}^G + m\mathbb{L}$. We denote by $A(G) := K(G)/(m\mathbb{L} + \mathbb{L}^G)$ — it is a finite abelian group. We have seen in Theorem 7.37 and Remark 7.38 that to any finite subgroup $H \leq \mathrm{Aut}(\mathcal{D})$ such that $\pi(H) = G$ corresponds an element $v \in K(G)$: this element is unique up to translation by a vector in $\ker(p_G) = \mathbb{L}^G$.

Lemma 7.43. *Let $H, H' \leq \mathrm{Aut}(\mathcal{D})$ be finite such that $\pi(H) = \pi(H') = G$, and let $v, v' \in K(G)$ be associated vectors. Then H, H' are $\mathbb{L}$-conjugate in $\mathrm{Aut}(\mathcal{D})$ if and only if $v - v' \in \mathbb{L}^G + m\mathbb{L}$ if and only if v and v' define the same class in $A(G)$.*

Proof. Let us suppose that there exists $\mu \in \mathbb{L}$ such that $\psi_\mu H \psi_\mu^{-1} = H'$. Then, for all $g \in G$, we have

$$\psi_\mu \left(\frac{g(v) - v}{m}, g\right) \psi_\mu^{-1} = \left(\frac{g(v') - v'}{m}, g\right)$$

which is equivalent to

$$g(v - v') - (v - v') = m(g(\mu) - \mu).$$

Thus, we conclude that $v - v' - m\mu \in \mathbb{L}^g$ for all $g \in G$, meaning exactly that $v - v' \in \mathbb{L}^G + m\mathbb{L}$. The converse holds similarly, by reversing the order of the arguments. □

Corollary 7.44. *The group $A(G)$ is trivial if and only if for all $H \leq \mathrm{Aut}(\mathcal{D})$ finite such that $\pi(H) = G$, the $\mathbb{Z}$-lattices $\mathbb{B}_H$ and $\mathbb{L}_G$ are isometric.*

Proof. If $A(G)$ is trivial, we know from Lemma 7.43 that if $H \leq \mathrm{Aut}(\mathcal{D})$ is finite and such that $\pi(H) = G$, then H is $\mathbb{L}$-conjugate to $\phi(G)$. In particular $\mathbb{B}_H \simeq \mathbb{B}_{\phi(G)} = \mathbb{L}_G$.

Now, if $A(G)$ is nontrivial, then there is a vector $v \in \mathbb{L} \setminus (\mathbb{L}^G + m\mathbb{L})$ such that, for all $g \in G$, $\lambda_g := \frac{g(v)-v}{m} \in \mathbb{L}_g$ and, $\{\lambda_g\}_{g\in G}$ does not lie in the image of $\mathbb{L}$ by p_G. Therefore, according to Proposition 7.40, the finite subgroup $H := \{(\lambda_g, g) \ : \ g \in G\}$ satisfies that $\mathbb{B}_H \not\simeq \mathbb{L}_G$. □

In particular, the group $A(G)$, which only depends on G, measures to which extent one can find a lift of G in $\mathrm{Aut}(\mathcal{D})$ which is exceptional.

Remark 7.45. Let $H' \leq \mathrm{Aut}(\mathcal{D})$ be finite such that $G' := \pi(H')$ is $O(\mathbb{L})$-conjugate to G. Then it is not hard to see that there exists $H \leq \mathrm{Aut}(\mathcal{D})$ conjugate to H' such that $\pi(H) = G$.

Therefore, in order to construct at least one representative for each $\mathrm{Aut}(\mathcal{D})$-conjugacy class of finite exceptional subgroups H of $\mathrm{Aut}(\mathcal{D})$, we proceed as follows:

(1) we start by fixing a primitive sublattice $C \leq \mathbb{L}$ so that $O^{\#}(C)$ fixes no nontrivial vector in C [HM16];

(2) we compute a complete set $\mathcal{G}$ of representatives for the $O(\mathbb{L})$-conjugacy classes of finite subgroups G of $O^{\#}(C)$ such that $C^G = \{0\}$ (Remark 7.45);

(3) for any $G \in \mathcal{G}$, we compute $A(G)$: if it is trivial, we try a new group (Corollary 7.44);

(4) for every $[v] \in A(G)$ nontrivial, we define $H := \left\{ \left(\frac{g(v)-v}{\#G}, g \right) \ : \ g \in G \right\} \leq \mathrm{Aut}(\mathcal{D})$.

Note that according to Lemma 7.43, the $\mathrm{Aut}(\mathcal{D})$-conjugacy class of H in step (4) does not depend on a choice of a representative for the nontrivial class $[v] \in A(G)$.

Theorem 7.46. *Let $H \leq \mathrm{Aut}(\mathcal{D})$ be an exceptional finite subgroup so that $\mathbb{B}_H$ has rank at most 21. Then, $\mathbb{B}_H$ is abstractly isometric to one of the 101 $\mathbb{Z}$-lattices in [MM25c, exceptional].*

Proof. We apply the previous procedure to the list of primitive sublattices $C \leq \mathbb{L}$ such that $O^{\#}(C)$ fixes no nontrivial vector in C and $\mathrm{rank}_{\mathbb{Z}}(C) \leq 21$ (see [HM16, Table 2]). This returns at least one representative for every $\mathrm{Aut}(\mathcal{D})$-conjugacy class of exceptional finite subgroups of $\mathrm{Aut}(\mathcal{D})$ (Lemma 7.43, Remark 7.45). We compare the coinvariant sublattices of each of the groups we have obtained, using the Plesken–Souvignier algorithm [PS97], and we keep only one representative for each isometry class. We record information about such $\mathbb{Z}$-lattices in Appendix A, Table 14. □

One observes that for all the primitive sublattices $C \leq \mathbb{B}$ presented in Table 14, we have $\mathrm{rank}_{\mathbb{Z}}(C) + l(D_C) > 24$. In particular, the following holds.

Corollary 7.47. *Let C be a negative definite even $\mathbb{Z}$-lattice of rank at most 21, containing no (-2)-roots and such that $O^{\#}(C)$ fixes no nontrivial vector in C. Then C embeds primitively into the Leech lattice $\mathbb{L}$ with $\mathbb{L}_{O^{\#}(C)} \simeq C$ if and only if*

$$\mathrm{rank}(C) + l(D_C) \leq 24.$$

Proof. Already note that if C embeds primitively into the Leech lattice, then according to Remark 2.26 we have that $\mathrm{rank}_{\mathbb{Z}}(C) + l(D_C) \leq \mathrm{rank}_{\mathbb{Z}}(\mathbb{L}) = 24$.

Now suppose that $\mathrm{rank}_{\mathbb{Z}}(C) + l(D_C) \leq 24$. By this assumption, we know that C embeds primitively into $\mathbb{B}$ and any two such embeddings are $O(\mathbb{B})$-isomorphic (see Corollary 2.30). In particular, we can choose a primitive embedding $j \colon C \hookrightarrow \mathbb{B}$ such that the complement $N := j(C)^{\perp}_{\mathbb{B}}$ intersects the interior of the Weyl chamber $\mathcal{D}$. Therefore, as in the proof of Lemma 7.32, we have that the finite group $H \leq O^{+}(\mathbb{B})$ extending $O^{\#}(C)$ and acting trivially on N preserves $\mathcal{D}$. Thus C is the coinvariant sublattice of a finite subgroup of $\mathrm{Aut}(\mathcal{D})$. There are two cases: either H is exceptional, or C is isometric to $\mathbb{L}_{\pi(H)}$. However, the former is not possible since Table 14 tells us that the coinvariant sublattices N of exceptional finite subgroups of $\mathrm{Aut}(\mathcal{D})$ satisfy

$$\mathrm{rank}_{\mathbb{Z}}(N) + l(D_N) \geq 25.$$ □

A result equivalent to Corollary 7.47 has been used by Mongardi in his PhD thesis to prove the statement of Theorem 7.24. However the proof of Mongardi's results differs from the one given here, and uses the holy constructions of the Leech lattice: such a proof does not rely on the computation of exceptional subgroups of $\mathrm{Aut}(\mathcal{D})$.

7.5. Applications

In this section, we computationally classify groups of stable symplectic isometries for the deformation types $\mathrm{K3}^{[3]}$ and OG10. For these deformation types, we know that monodromy is maximal (Table 4) so we can apply Corollary 7.11. The abstract isometry classes for the stable symplectic sublattices associated to these deformation types have been characterized through Theorem 7.24 for the $\mathrm{K3}^{[3]}$ case, and in Section 7.4 for the OG10 case. The computations presented in this section were performed by the thesis' author for the collaborative works [BMW25] and [MM25b]. The associated data, given in terms of matrix representations, are available respectively in the datasets [BMW24] and [MM25c].

7.5.1. The $\mathrm{K3}^{[3]}$ case

According to Theorem 7.24, for the deformation type $\mathcal{T} = \mathrm{K3}^{[3]}$, the associated stable symplectic sublattices $C \leq \Lambda_{\mathcal{T}}$ are primitively embedded into the Leech lattice $\mathbb{L}$. Moreover, we recall that in that situation, the numerical stably prime exceptional and wall divisors are respectively given by the two sets

$$\mathcal{W}^{pex}(\Lambda_{\mathrm{K3}^{[3]}}) = \{v \in \Lambda_{\mathrm{K3}^{[3]}} \ : \ v \text{ has type } (-2,1), (-4,2), (-4,4)\}$$

and

$$\mathcal{W}(\Lambda_{\mathrm{K3}^{[3]}}) = \mathcal{W}^{pex}(\Lambda_{\mathrm{K3}^{[3]}}) \sqcup \{v \in \Lambda_{\mathrm{K3}^{[3]}} \ : \ v \text{ has type } (-12,2), (-36,4)\}$$

(Example 5.43). We obtain the following.

Theorem 7.48. *There are exactly 219 $O^+(\Lambda_{\mathrm{K3}^{[3]}})$-conjugacy classes of saturated finite symplectic subgroups $H \leq O^{+,\#}(\Lambda_{\mathrm{K3}^{[3]}})$. Among them, 68 classes are represented by regular symplectic subgroups. The associated numerical data is available in Appendix B, Table 15. Representatives for the conjugacy classes, in terms of matrices, are found in the folder "dataset/data" in the database [BMW24].*

Proof. It is a direct application of Corollary 7.11, using the implementation of Algorithm 2, by the author of the thesis, in [OSC25, QuadFormAndIsom]. According to Theorem 7.24, we apply such an algorithm to the complete list of representatives for the isomorphism classes of primitive sublattices $C \leq \mathbb{L}$ of the Leech lattice, such that $O^{\#}(C)$ fixes no nontrivial vector in C (see [HM19, Table 2]). Remark that the actual data computed by Höhn and Mason is available in a format readable on the computer algebra system Magma [BCP97] — it has been translated by the thesis's author into an OSCAR-readable format [DVJL24].

Given a finite symplectic subgroup $H \leq O^{+,\#}(\Lambda_{\mathrm{K3}^{[3]}})$, with coinvariant sublattice $C \leq \Lambda_{\mathrm{K3}^{[3]}}$, one decides whether H is regular symplectic by checking whether

$$C \cap \mathcal{W}(\Lambda_{\mathrm{K3}^{[3]}}) = \emptyset$$

(Theorem 6.9). We test such a condition similarly to what is explained in the computational comments at the end of Section 7.2. This concludes the proof. □

Theorem 7.49. *Let X be an IHS manifold of $\mathrm{K3}^{[3]}$-type and let $G \leq \mathrm{Bir}_s(X)$ be a finite stable subgroup. Then there exists a marking $\eta\colon H^2(X,\mathbb{Z}) \to \Lambda_{\mathrm{K3}^{[3]}}$ via which G embeds into one of the groups in [BMW24]. Moreover, for any finite subgroup $G \leq O^{+,\#}(\Lambda_{\mathrm{K3}^{[3]}})$ contained in [BMW24], there exists an IHS manifold X of $\mathrm{K3}^{[3]}$-type, a finite stable subgroup $G' \leq \mathrm{Bir}_s(X)$ and a marking η such that G is induced by G' via the marking η.*

Proof. Follows from Theorem 6.9 and Theorem 7.48. □

Theorem 7.48 gives a complete list of representatives for the conjugacy classes of regular symplectic finite subgroups of $O^+(\Lambda_{\mathrm{K3}^{[3]}})$. In fact, we recall from Theorem 7.2 that regular symplectic isometries are stable. However, this is not always the case for nonregular symplectic isometries: in particular, Theorem 7.48 only gives a partial classification for the finite symplectic subgroups of $O^+(\Lambda_{\mathrm{K3}^{[3]}})$ since we only classified the stable ones. In the next example, we show the existence of at least one finite symplectic subgroup of $O^+(\Lambda_{\mathrm{K3}^{[3]}})$ which is not stable.

Example 7.50. For the computational details of what is claimed in this example, please refer to the notebook "Nonstable" in [BMW24].

Let us consider the $\mathbb{Z}$-lattice C no. 77 from the Höhn–Mason database [HM19, Table 2]: the group $O^\#(C)$ is actually isomorphic to the finite simple group $L_2(7) := \mathrm{PSL}_2(\mathbb{F}_7)$. The associated coinvariant sublattice $\mathbb{L}_G$ has rank 19 and $\Lambda_{\mathrm{K3}^{[3]}}$ admits exactly two $O^+(\Lambda_{\mathrm{K3}^{[3]}})$-orbits of primitive sublattices $C \simeq \mathbb{L}_G$ with $C \cap \mathcal{W}^{pex}(\Lambda_{\mathrm{K3}^{[3]}}) = \emptyset$ (Table 15, entries 77a and 77b). One of such C's is so that $C \cap \mathcal{W}(\Lambda_{\mathrm{K3}^{[3]}}) = \emptyset$ meaning that the finite symplectic subgroup $O^\#(C) \leq O^{+,\#}(\Lambda_{\mathrm{K3}^{[3]}})$ can be realized as a finite group of symplectic automorphisms on an IHS manifold of $\mathrm{K3}^{[3]}$-type. Let us fix such a primitive sublattice C, and let $G \leq O^{+,\#}(\Lambda_{\mathrm{K3}^{[3]}})$ be defined as the identity on $F := C^\perp$ and restricts to $O^\#(C)$ on C (Corollary 2.21). The invariant sublattice $F = \Lambda^G_{\mathrm{K3}^{[3]}}$ is isometric to

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 28 \end{pmatrix}.$$

It admits an involution with negative definite coinvariant sublattice which can be extended to an isometry $g \in O^+(\Lambda_{\mathrm{K3}^{[3]}})$ whose square lies in G. We show, by applying Theorem 6.12, that such an isometry is symplectic, and it is moreover nonstable. Hence, the group $G' := \langle G, g\rangle$ is symplectic and nonstable. One has moreover

$$\Lambda^{G'}_{\mathrm{K3}^{[3]}} \simeq \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 10 \end{pmatrix}$$

and the associated coinvariant sublattice lies in the genus $\mathrm{II}_{(0,20)}4_2^2 7^1$.

An extra step is therefore needed in order to complete Table 15 and get a complete classification of finite symplectic subgroups of $O^+(\Lambda_{\mathrm{K3}^{[3]}})$, up to conjugacy. We explain in more details in the chapter about the extension approaches (Section 9) how one can proceed.

7.5.2. The OG10 case

According to Lemma 7.32, if $C \leq \Lambda_{\mathrm{OG10}}$ is a stable symplectic sublattice without (-2)-roots, then C embeds primitively into Borcherds' lattice $\mathbb{B}$ and there exists a finite subgroup $G \leq \mathrm{Aut}(\mathcal{D})$ such that $\mathbb{B}_G \simeq C$. We recall that $\mathcal{D}$ denotes the Weyl chamber in the positive cone of $\mathbb{B}$. Note that since C is negative definite and Λ_{OG10} has signatures $(3, 21)$ (Table 4) then Corollary 7.47 tells us that under the assumption

$$\mathrm{rank}_{\mathbb{Z}}(C) + l(D_C) \leq 24$$

we have that C embeds primitively into the Leech lattice. In particular, the abstract isometry class of C is known [HM16, Table 2]. Moreover, the following holds.

Proposition 7.51. *Let $C \leq \Lambda_{\mathrm{OG10}}$ be a primitive sublattice. Then for all prime number p we have that*

$$\mathrm{rank}_{\mathbb{Z}}(C) + l(D_p) \leq 24 + \delta_{3,p}$$

where D_p is the p-primary part of D_C and $\delta_{3,p}$ is 1 if and only if $p = 3$, and 0 otherwise.

Proof. Let p be a prime number. Note that $\Lambda_p := \Lambda_{\mathrm{OG10}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is unimodular whenever $p \neq 3$. In particular, remarking that $C \leq \Lambda_{\mathrm{OG10}}$ being primitive implies that $C_p \leq \Lambda_p$ is primitive (i.e. Λ_p/C_p is torsionfree), then we can apply a similar reasoning as in Remark 2.26 to deduce that for $p \neq 3$, we have

$$\mathrm{rank}_{\mathbb{Z}}(C) + l(D_p) = \mathrm{rank}_{\mathbb{Z}_p}(C_p) + l(D_{C_p}) \leq \mathrm{rank}_{\mathbb{Z}_p}(\Lambda_p) = 24.$$

Now, similarly to what we have done in Proposition 2.27 for embeddings of even $\mathbb{Z}$-lattices into even unimodular $\mathbb{Z}$-lattices, we can apply [Nik80, Theorem 1.16.5] to show that Λ_{OG10} embeds primitively into an odd unimodular $\mathbb{Z}$-lattice M of signatures $(3, 22)$. In particular, so does C and globally

$$\mathrm{rank}_{\mathbb{Z}}(C) + l(D_C) \leq \mathrm{rank}_{\mathbb{Z}}(M) = 25.$$

We conclude by remarking that $l(D_3) \leq l(D_C)$. $\square$

From this we can conclude the following.

Lemma 7.52. *Let $C \leq \Lambda_{\mathrm{OG10}}$ be stable symplectic without (-2)-vectors. Then either C embeds primitively into the Leech lattice, or it is abstractly isometric to one of the three even $\mathbb{Z}$-lattices given in Appendix A.1.*

Proof. From Proposition 7.51, we know that for all prime number p, we have

$$\mathrm{rank}_{\mathbb{Z}}(C) + l(D_p) \leq 25$$

with equality possible only if $p = 3$. In the cases where this inequality is strict for all prime number p, we therefore have that $\mathrm{rank}_{\mathbb{Z}}(C) + l(D_C) \leq 24$ and Corollary 7.47 tells us that C embeds primitively into the Leech lattice. Otherwise, from Lemma 7.32 and Proposition 7.51 we have that C is isometric to the coinvariant sublattice of a finite exceptional subgroup $H \leq \mathrm{Aut}(\mathcal{D})$ so that

$$\mathrm{rank}_{\mathbb{Z}}(\mathbb{B}_H) + l((D_{\mathbb{B}_H})_3) = 25.$$

By investigating Table 14, we see that there are only three possible isometry classes for such a $\mathbb{Z}$-lattice C. We give a representative for the isometry class of each such even $\mathbb{Z}$-lattices in Appendix A.1. $\square$

Now that we know the possible abstract isometry classes for the stable symplectic sublattices of Λ_{OG10}, we can apply Corollary 7.11 and classify $O^+(\Lambda_{\mathrm{OG10}})$-orbits of primitive sublattices $C \leq \Lambda_{\mathrm{OG10}}$ which are isometric to such even $\mathbb{Z}$-lattices and such that C satisfies

$$C \cap \mathcal{W}^{pex}(\Lambda_{\mathrm{OG10}}) = \emptyset.$$

Recall from Example 5.46 that for the deformation type OG10, one has that

$$\mathcal{W}^{pex}(\Lambda_{\mathrm{OG10}}) = \{v \in \Lambda_{\mathrm{OG10}} \;:\; v \text{ has type } (-2,1) \text{ or } (-6,3)\}.$$

Note that none of the negative definite even $\mathbb{Z}$-lattices determined in Lemma 7.52 contains (-2)-vectors. So we only have to check for the presence of vectors of type $(-6, 3)$. We actually show now that it is not necessary: if a stable symplectic sublattice $C \leq \Lambda_{\mathrm{OG10}}$ contains no (-2)-vectors, then it does not contain vectors of type $(-6, 3)$. The following lemma is originally due to Marquand [MM25b, Lemma 4.6], based on an argument of Laza and Zheng [LZ22, Theorem 4.5]. We condense the previously mentioned argument of Laza–Zheng into a shorter proof.

Lemma 7.53. *Let $C \leq \Lambda$ be a stable symplectic sublattice without (-2)-vectors. Then $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$.*

Proof. Suppose that C has a vector v of norm -6 such that $\operatorname{div}(v, \Lambda) = 3$. Since the $\mathbb{Z}$-lattice C is stable symplectic, there exists an isometry $g \in O^\#(C)$ such that $gv \neq v$. Suppose that $g(v) = -v$: since g is stable, we obtain that

$$\frac{v}{3} + C = D_g\left(\frac{v}{3} + C\right) = \frac{g(v)}{3} + C = \frac{-v}{3} + C \in D_C.$$

This implies that $\frac{2v}{3} \in C$: however this vector has norm $\frac{-8}{3} \notin \mathbb{Z}$, which contradicts the fact that C is integral. Hence $v' := g(v)$ is not proportional to v, and moreover v' has again divisibility 3 in Λ because g is stable (Corollary 2.21). In fact, we actually have that $\frac{v}{3} + \Lambda = \frac{v'}{3} + \Lambda$, and in particular $\frac{v.v'}{9} + \mathbb{Z} = \frac{v^2}{9} + \mathbb{Z} = \frac{1}{3} + \mathbb{Z}$ for the torsion bilinear form b_Λ on D_Λ. But now, since $\mathbb{Z}v + \mathbb{Z}v' \leq C$ is negative definite, with $v^2 = (v')^2 = -6$, $v - v' \in 3C$ and $v.v' \in 3 + 9\mathbb{Z}$, we deduce that $v.v' = 3$. Hence, $\frac{v-v'}{3} \in \Lambda \cap C^\vee = C$ has norm -2, which is a contradiction. Thus $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$. □

We can therefore conclude.

Theorem 7.54. *There are exactly 192 $O^+(\Lambda_{\mathrm{OG10}})$-conjugacy classes of saturated finite symplectic subgroups $H \leq O^{+,\#}(\Lambda_{\mathrm{OG10}})$. The associated numerical data is available in Appendix C, Table 16. Representatives for the conjugacy classes, in terms of matrices, are found in the folder "dataset/data" in the database [MM25c].*

Proof. According to Corollary 7.11 and Lemma 7.53, it suffices to compute representatives for the isomorphism classes of stable symplectic sublattices $C \leq \Lambda$ which do not contain (-2)-vectors. According to Lemma 7.52, such an even $\mathbb{Z}$-lattice is abstractly isometric either to a stable symplectic sublattice of the Leech lattice $\mathbb{L}$, or to one of the three $\mathbb{Z}$-lattices given in Appendix A.1. □

Theorem 7.55. *Let X be an IHS manifold of* OG10*-type and let $G \leq \operatorname{Bir}_s(X)$ be a finite stable subgroup. Then there exists a marking $\eta \colon H^2(X, \mathbb{Z}) \to \Lambda_{\mathrm{OG10}}$ via which G embeds into one of the groups in [MM25c]. Moreover, for any finite subgroup $G \leq O^{+,\#}(\Lambda_{\mathrm{OG10}})$ contained in [MM25c], there exists an IHS manifold X of OG10-type, a finite stable subgroup $G' \leq \operatorname{Bir}_s(X)$ and a marking η such that G is induced by G' via the marking η.*

Proof. Follows from Theorem 6.9 and Theorem 7.54. □

To conclude this section about symplectic actions, we determine representatives for the conjugacy classes of nonstable symplectic involutions in $O^+(\Lambda_{\mathrm{OG10}})$. The goal is to show how one can solve the classification problem (SC), by applying an approach of Nikulin to the case of OG10-type IHS manifolds. Moreover, by combining with the results of the current subsection, we can actually give a classification for the conjugacy classes of prime order symplectic isometries of Λ_{OG10}.

7.6. Nonstable symplectic involutions

The content of this section is adapted from the published joint work [MM25a] with Marquand. A major part of this work is due to Marquand, who originally presented it as single-authored. The author of the thesis joined the project at a later stage. His main contribution is about enumerating the genera needed for Proposition 7.65.

As we have seen in Section 6.3, the problem of classifying finite symplectic actions on IHS manifolds can be divided into three parts. In the previous section, we have seen how to classify stable symplectic actions (problem (StS)), and in Section 9 we will see how to extend such nontrivial stable symplectic actions to general symplectic actions (problem (S)). In this section, we cover the extension of the trivial group, i.e. the classification of nonstable symplectic involutions (problem (SC)). We recall that nonstable symplectic isometries only exist for the deformation types different from K3 and $\mathrm{K3}^{[2]}$ (see Table 4). For the deformation type OG6, it is known from Theorem 7.3 that symplectic isometries are all stable. Moreover, by Theorems 7.2 and 7.4, nonstable symplectic isometries for the deformation types $\mathrm{K3}^{[n]}$ ($n \geq 3$), Kum_n ($n \geq 2$) and OG10 are induced by nonregular birational automorphisms on the associated IHS manifolds.

In this subsection, we review the ideas of Nikulin to classify involutions on even unimodular $\mathbb{Z}$-lattices, and we show how it can be applied in the OG10 case to construct nonstable symplectic involutions for this deformation type.

In his major work about *2-reflexive hyperbolic $\mathbb{Z}$-lattices* [Nik83], Nikulin studies and classifies involutions on K3 surfaces (section 4, 2° of the aforementioned paper). An important tool for the study of Nikulin are (hyperbolic) 2-elementary $\mathbb{Z}$-lattices. Indeed, since the BBF form $\Lambda := \Lambda_{\mathrm{K3}}$ for a K3 surface S is unimodular (Table 9), given an involution $i \in \mathrm{Aut}(S)$ the invariant and coinvariant sublattices of $h := \rho_S(i)$, denoted respectively by F and C, are 2-elementary (Corollary 2.42). In particular, since Λ is unimodular, F and C glue along their respective discriminant groups (see the proof of Lemma 2.25), meaning that D_F and D_C are $\mathbb{F}_2$-vector spaces. In particular, $-\mathrm{id}_C$ induces the identity on D_C, and according to Lemma 2.19, any primitive extension $F \oplus C \leq \Lambda$ is $(\mathrm{id}_F, -\mathrm{id}_C)$-equivariant.

A strategy arising from Nikulin's work is to determine the potential genera for F and C, and then construct the respective equivariant primitive extensions to Λ. Note that since Λ is unimodular, the genus of C determines the one of F (Lemma 2.25). Hence the complete classification goes back to determining all isomorphism classes of even 2-elementary primitive sublattices $C \leq \Lambda$ with given signatures.

Remark 7.56. This strategy actually applies for any even $\mathbb{Z}$-lattice L: in this case, the two sublattices F and C might not be 2-elementary anymore, but they will still glue $(\mathrm{id}_F, -\mathrm{id}_C)$-equivariantly to L along $\mathbb{F}_2$-vector spaces. On a case-by-case basis, depending on the genus of L, it is possible to determine the potential genera of F and C. See for instance [CCC21, Proposition 2.8] for the case of nonsymplectic involutions for $\mathrm{K3}^{[n]}$-type IHS manifolds, [MTW18] for involutions on Kum_n-type IHS manifolds, [Gro22a, §4] and [GOV23] for involutions in the OG6 case, and [BG25, Appendix A] for nonsymplectic involutions on OG10-type IHS manifolds.

In [MM25a], Marquand applies this strategy to determine the genus of the coinvariant sublattice associated to a nonstable symplectic isometry of Λ_{OG10}. Together with the thesis' author, they prove the following theorem.

Theorem 7.57. *There are exactly 4 $\mathrm{Mon}^2(\Lambda_{\mathrm{OG10}})$-conjugacy classes of nonstable symplectic involutions in $\mathrm{Mon}^2(\Lambda_{\mathrm{OG10}})$. For each representative f of such a conjugacy class, the pair $(\Lambda_{\mathrm{OG10}}^f, (\Lambda_{\mathrm{OG10}})_f)$ is given in Table 5.*

Table 5: Type of nonstable symplectic involutions — OG10 case

$\Lambda^f_{\mathrm{OG10}}$	$U^{\oplus 3}\oplus D_4^{\oplus 3}$	$U^{\oplus 2}\oplus E_8(2)\oplus\langle -2,2\rangle$	$\langle 2\rangle^{\oplus 3}\oplus\langle -2\rangle^{\oplus 9}$	$\langle 2\rangle^{\oplus 3}\oplus\langle -2\rangle^{\oplus 5}$
$(\Lambda_{\mathrm{OG10}})_f$	$E_6(2)$	M	G_{12}	G_{16}

Notation. We define the following integral $\mathbb{Z}$-lattices:

(1) M is the unique, up to isometry, index 2 overlattice of $D_9(2)\oplus A_1(12)$ [Mar23, Proposition 5.6] — it lies in the genus $\mathrm{II}_{(0,10)}2_4^{-10}3^{+1}$;

(2) G_{12} is the unique, up to isometry, $\mathbb{Z}$-lattice without (-2)-roots in the genus $\mathrm{II}_{(0,12)}2_2^{-12}3^{+1}$;

(3) G_{16} is the unique, up to isometry, $\mathbb{Z}$-lattice without (-2)-roots in the genus $\mathrm{II}_{(0,16)}2_6^{-8}3^{+1}$.

Remark 7.58. Note that $E_6(2)$, M and G_{12} are all 2-divisible (Definition 1.11). The even $\mathbb{Z}$-lattice G_{16} is the only possible coinvariant sublattice which is indivisible.

We explain now how to prove Theorem 7.57, by splitting nonstable symplectic involutions of Λ_{OG10} into two different cases. For the rest of this section, we denote $\Lambda := \Lambda_{\mathrm{OG10}} \simeq U^{\oplus 3}\oplus E_8^{\oplus 2}\oplus A_2$. Let us first prove the following.

Lemma 7.59. *Let L be an even $\mathbb{Z}$-lattice such that D_L has odd prime order $p\geq 3$, and let $h\in O(L)$ be an involution such that $D_h = -\mathrm{id}_{D_L}$. Then L^h is 2-elementary and $D_{L_h}\cong D_L\oplus D_{L^h}(-1)$.*

Proof. Let us denote $F := L^h$ and $C := L_h$. Since h has order 2, we know from Proposition 2.40 that $2L\leq F\oplus C\leq L$. Hence, since p is an odd prime number, the previous sequence of inclusions gives us that $L_p = F_p\oplus C_p$. In particular we observe that, as finite abelian groups,

$$\mathbb{Z}/p\mathbb{Z}\cong D_L = (D_L)_p \cong (D_F)_p\oplus (D_C)_p.$$

This implies in particular that either $(D_F)_p\cong D_L$ and $(D_C)_p=\{0\}$, or that $(D_C)_p\cong D_L$ and $(D_F)_p=\{0\}$. But, the primitive extension $F\oplus C\leq L$ is $(\mathrm{id}_F, -\mathrm{id}_C)$-equivariant, and the action of h on $D_L = (D_L)_p$ coincides with the action of $\mathrm{id}_F\oplus(-\mathrm{id}_C)$ on $(D_F)_p\oplus(D_C)_p$. Since we assume h nonstable, the latter implies that $(D_C)_p\cong D_L$ and F is 2-elementary (Corollary 2.42). □

In particular, from this, we conclude the following.

Corollary 7.60. *The set of $\mathrm{Mon}^2(\Lambda)$-conjugacy classes of nonstable symplectic involutions of Λ is in bijection with the set of isomorphism classes of negative definite primitive sublattices $C\leq\Lambda$ so that $C\cap\mathcal{W}^{pex}(\Lambda)=\emptyset$ and $C^\perp_\Lambda$ is 2-elementary.*

Proof. The proof is similar to the proof of Corollary 7.11, by using Theorem 6.9 and Lemma 7.59. This time we do not impose restrictions on $O^\#(C)$ since we extend $-\mathrm{id}_C$ with the identity on $F := C^\perp_\Lambda$ which is always possible if F is 2-elementary. □

Remark 7.61. The statement of Corollary 7.60 can actually be adapted for stable symplectic involutions, where this time one would require that C is 2-elementary.

As for the case of Nikulin, one sees that we have therefore reduced the problem to classifying primitive sublattices of Λ satisfying some given conditions. In [MM25a], Marquand could prove the following proposition which determines the potential isometry classes for the coinvariant sublattices of nonstable symplectic involutions in $O(\Lambda)$.

Proposition 7.62 ([MM25a, Corollary 4.7, Proposition 5.1])**.** *Let $f \in O^+(\Lambda)$ be a nonstable symplectic involution. Then one of the following two holds:*

(1) *either* $\mathrm{rank}_{\mathbb{Z}}(\Lambda_f) < 12$ *and* Λ_f *is isometric to* $E_6(2)$ *or* M *(with the notation in Table 5);*

(2) *or* $\mathrm{rank}_{\mathbb{Z}}(\Lambda_f) \geq 12$ *and* Λ_f *lies in one of the following genera:*

   (a) $\mathrm{II}_{(0,18)} 2_{\mathrm{II}}^{+6} 3^{+1}$*;*

   (b) $\mathrm{II}_{(0,14)} 2_{\mathrm{II}}^{-8} 3^{+1}$*;*

   (c) $\mathrm{II}_{(0,r)} 2_{\delta}^{-(24-r)} 3^{+1}$ *for* $12 \leq r \leq 21$ *and* $\delta \equiv 6 - r \mod 8$.

In order to prove such a result, Marquand separated the cases into whether Λ^f contains a hyperbolic plane U or not — case (1) corresponds to the case where such an observation is made. Let us comment on that part. Let us denote again F and C for the respective invariant and coinvariant sublattice of the nonstable symplectic involution $f \in O^+(\Lambda)$. Recall from Corollary 7.60 that F is 2-elementary of signatures $(3, *)$ and C is negative definite, $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$ and $D_C \simeq D_F(-1) \oplus D_\Lambda$ as torsion quadratic modules. Suppose that $\mathrm{rank}_{\mathbb{Z}}(C) < 12$: then in particular, $l(D_C) < 12$ and $\mathrm{rank}_{\mathbb{Z}}(F) \geq 13$. But now, D_F is isomorphic, as finite abelian group, to the 2-Sylow subgroup of D_C which has length at most equal to 11. Hence there are two cases:

(1) either $\mathrm{rank}_{\mathbb{Z}}(F) \geq 3 + l(D_F)$ in which case [Nik80, Corollary 1.13.5] tells us that $F \simeq U \oplus \Gamma$ for an even 2-elementary $\mathbb{Z}$-lattice Γ;

(2) or $\mathrm{rank}_{\mathbb{Z}}(F) = 13$ and $l(D_F) = 11$, in which case $F \in \mathrm{II}_{(3,10)} 2_{\delta}^{\epsilon 11}$ where $\delta \in \{1, 3, 5, 7\}$ and $\epsilon = \pm 1$ satisfy that
$$\delta + 2 - 2\epsilon \equiv 1 \mod 8$$
(Theorem 1.49 (c)). But now, using Theorem 1.32, up to replacing δ by $\delta \pm 4$, we can assume $\epsilon = +1$. Hence $F \in \mathrm{II}_{(3,10)} 2_{\delta}^{+11}$ where $\delta \equiv 1 \mod 8$ and moreover by Theorem 1.49, Item (c), there exists a $\mathbb{Z}$-lattice $\Gamma \in \mathrm{II}_{(2,9)} 2_{\delta}^{+11}$. The two $\mathbb{Z}$-lattices F and $U \oplus \Gamma$ lie in the same genus, and since they are indefinite Theorem 2.29 tells us that they are isometric.

Hence there exists an even $\mathbb{Z}$-lattice Γ such that $F \simeq U \oplus \Gamma$. Now, considering $-f$ instead of f, we have that $-f$ is stable, we exchange the roles of F and C, and up to rescaling Λ by -1, we have that the equivariant primitive extension

$$(F(-1), -\mathrm{id}_{F(-1)}) \oplus (C(-1), \mathrm{id}_{C(-1)}) \leq (\Lambda(-1), -f)$$

gives rise to an equivariant primitive extension

$$(\Gamma(-1), -\mathrm{id}_{\Gamma(-1)}) \oplus (C(-1), \mathrm{id}_{C(-1)}) \leq (\Lambda_0, g)$$

where $\Lambda_0 \simeq U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$. Note that since $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$ by assumption, we have in particular that $C(-1)$ has no 2-vectors and all its vectors of norm 6 have divisibility 1 in Λ_0. Hence $g \in O(\Lambda_0)$ is a stable involution whose invariant sublattice is positive definite and without 2-vectors or vectors of type $(6, 3)$ in Λ_0. Such involutions have been classified by Marquand in [Mar23, Theorem 1.1] and they are the isometries associated to nonsymplectic involutions on cubic fourfolds. In particular, in this situation, we know that $C(-1)$ is isometric to $E_6(-2)$ or $M(-1)$, and $\Gamma(-1) \simeq U^{\oplus 2} \oplus D_4(-1)^{\oplus 3}$ or $U \oplus E_8(-2) \oplus A_1 \oplus A_1(-1)$ respectively.

Remark 7.63. Note that in [MM25a, Theorem 3.1] Marquand also shows that if $f \in O^+(\Lambda)$ is a stable symplectic involution, then either $\Lambda_f \simeq E_8(2)$ or $\Lambda_f \simeq D_{12}^+(2)$, which is consistent with what we have obtained in Theorem 7.54 (see Table 16 entries 2 and 5 respectively).

For determining the genera in Proposition 7.62, Item (2), Marquand shows that if $f \in O^+(\Lambda)$ is a nonstable symplectic involution so that Λ_f does not contain a copy of U, then it has rank at least 12 and Λ^f is isometric to one of the following

(a) $U(2)^{\oplus 3}$;

(b) $U(2)^{\oplus 3} \oplus D_4$;

(c) $A_1(-1)^{\oplus 3} \oplus A_1^{\oplus 21-r}$ where $12 \leq r \leq 21$

which are all unique in their respective genera. From this, Marquand could apply a result similar to Lemma 7.59 to determine the corresponding genus for Λ_f. In particular, for any of the $\mathbb{Z}$-lattices C contained in one of the genera given in Proposition 7.62 (2), there exists a primitive embedding $C \hookrightarrow \Lambda$ with associated orthogonal complement given as above, by construction (Proposition 2.17).

What remains to be done is to enumerate the 12 genera given in Proposition 7.62 (2). We apply the procedure described in Section 1.5. The actual computations for this project took around 4 months, especially because of the genera of large ranks. The result of this enumeration is available in [MM24], and we refer to Table 6 for some details about this result. For each possible genus, we give the number N of isometry classes of $\mathbb{Z}$-lattices it contains. In the column WITH (-2)-ROOTS we record the number of classes that have a representative with a vector of norm -2. In the column WITHOUT (-2)-ROOTS, **BUT** WITH $(-6,3)$ we record how many classes have a representative without any vector of norm -2 but with at least one vector of type $(-6,3)$. If a primitive sublattice $C \leq \Lambda$ has no vectors of norm -6 and divisibility 3 in C, then it has no vectors of norm -6 and divisibility 3 in Λ. The converse does not hold in general, but in our case, we can show the following.

Lemma 7.64. *Let $f \in O(\Lambda)$ be a nonstable involution, and let $C := \Lambda_f$. Then for a vector $v \in C$ we have*

$$(\mathrm{div}(v,\Lambda) = 3) \iff (\mathrm{div}(v,C) \in 3\mathbb{Z}).$$

Proof. One implication is already clear: if $\mathrm{div}(v,\Lambda) = 3$, then $v.C \subseteq v.L \subseteq 3\mathbb{Z}$.

Now let $v \in C$ be such that $\mathrm{div}(v,C) \in 3\mathbb{Z}$ and let $w \in \Lambda$. Since f has order 2, we already know that $2\Lambda \subseteq F \oplus C$ where $F := C^\perp_\Lambda$. Thus there exist $w_F, w_F' \in F$ and $w_C, w_C' \in C$ such that

$$w = w_F + w_C + \frac{w_F' + w_C'}{2}.$$

Now, since F and C are in orthogonal direct sum in Λ, we have that $v.w_F = v.w_F' = 0$ and moreover, $(v.w_C), (v.w_C') \in 3\mathbb{Z}$ by assumption. Since $v \in C \leq \Lambda$ with Λ integral, we have that

$$v.\left(\frac{w_F' + w_C'}{2}\right) = \frac{1}{2}(v.w_C')$$

is an integer. Hence, $v.w_C' \in 2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$, and we deduce that $v.w \in 3\mathbb{Z}$. Hence, since we chose $w \in \Lambda$ arbitrary, we conclude that $\mathrm{div}(v,\Lambda) = 3$. $\square$

Therefore, in our case, for $C \leq \Lambda$ coinvariant sublattice of a nonstable symplectic involution, checking whether $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$ is the same as testing whether C has no vectors of norm -2 and no vectors of norm -6 whose divisibility in C is divisible by 3. This can be done again using the Fincke–Pohst algorithm [FP85]. Altogether, we obtain the following.

Proposition 7.65. *Let $f \in O^+(\Lambda)$ be a nonstable symplectic involution such that $C := \Lambda_f$ has rank at least 12. Then the isometry class of C is given in the last column of Table 6.*

Table 6: Genus enumeration and geometric cases of Proposition 7.65

Case	genus	N	with $(-2,1)$	without $(-2,1)$, **but** with $(-6,3)$	geometric cases
(1)	$\mathrm{II}_{(0,18)}2_{\mathrm{II}}^{+6}3^{+1}$	430	430	0	None
(2)	$\mathrm{II}_{(0,14)}2_{\mathrm{II}}^{-8}3^{+1}$	21	21	0	None
(3)	$\mathrm{II}_{(0,12)}2_{2}^{-12}3^{+1}$	5	4	0	1: G_{12}
	$\mathrm{II}_{(0,13)}2_{1}^{-11}3^{+1}$	23	22	1	None
	$\mathrm{II}_{(0,14)}2_{0}^{-10}3^{+1}$	70	70	0	None
	$\mathrm{II}_{(0,15)}2_{7}^{-9}3^{+1}$	211	211	0	None
	$\mathrm{II}_{(0,16)}2_{6}^{-8}3^{+1}$	617	616	0	1: G_{16}
	$\mathrm{II}_{(0,17)}2_{5}^{-7}3^{+1}$	1291	1291	0	None
	$\mathrm{II}_{(0,18)}2_{4}^{-6}3^{+1}$	2524	2524	0	None
	$\mathrm{II}_{(0,19)}2_{3}^{-5}3^{+1}$	3682	3682	0	None
	$\mathrm{II}_{(0,20)}2_{2}^{-4}3^{+1}$	3375	3375	0	None
	$\mathrm{II}_{(0,21)}2_{1}^{-3}3^{+1}$	1316	1316	0	None

Remark 7.66. For the reader's convenience, we describe the $\mathbb{Z}$-lattice G_{12} and G_{16} from Theorem 7.57 in terms of their respective Gram matrices.

$$G_{12} = \begin{pmatrix} -4 & 2 & -2 & -2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 & -2 \\ 2 & -4 & 2 & 0 & 2 & 0 & -2 & 2 & -2 & -2 & 2 & 2 \\ -2 & 2 & -4 & -2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 & -2 \\ -2 & 0 & -2 & -4 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & -2 & -2 & -4 & -2 & 2 & -2 & 2 & 2 & -2 & -2 \\ -2 & 0 & -2 & -2 & -2 & -4 & 2 & -2 & 2 & 2 & -2 & -2 \\ 2 & -2 & 2 & 0 & 2 & 2 & -4 & 2 & -2 & -2 & 2 & 2 \\ -2 & 2 & -2 & 0 & -2 & -2 & 2 & -6 & 4 & 4 & -2 & -4 \\ 2 & -2 & 2 & 0 & 2 & 2 & -2 & 4 & -6 & -2 & 4 & 4 \\ 2 & -2 & 2 & 0 & 2 & 2 & -2 & 4 & -2 & -6 & 2 & 2 \\ -2 & 2 & -2 & 0 & -2 & -2 & 2 & -2 & 4 & 2 & -6 & -2 \\ -2 & 2 & -2 & 0 & -2 & -2 & 2 & -4 & 4 & 2 & -2 & -6 \end{pmatrix} \quad G_{16} = \begin{pmatrix} -4 & 2 & -2 & 2 & -1 & 1 & 2 & 1 & 2 & 1 & -2 & 1 & -2 & -2 & 1 & -2 \\ 2 & -4 & 0 & -1 & 2 & -2 & 0 & -2 & 0 & -2 & 0 & -2 & 1 & 2 & 1 & 1 \\ -2 & 0 & -4 & 2 & 1 & 1 & 1 & -1 & 2 & -1 & -1 & 1 & -1 & -1 & 2 & -2 \\ 2 & -1 & 2 & -4 & -1 & -2 & -2 & -1 & 0 & 1 & 1 & 0 & 2 & 0 & -2 & 1 \\ -1 & 2 & 1 & -1 & -4 & 1 & -1 & 2 & 1 & 2 & 1 & 2 & 1 & -1 & -2 & 0 \\ 1 & -2 & 1 & -2 & 1 & -4 & -1 & -2 & 0 & -1 & -1 & -2 & 1 & 1 & -1 & 1 \\ 2 & 0 & 1 & -2 & -1 & -1 & -4 & -1 & 0 & 0 & 2 & 1 & 2 & 1 & -2 & 1 \\ 1 & -2 & -1 & -1 & 2 & -2 & -1 & -4 & 0 & -2 & -1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & -4 & -1 & 0 & -1 & 1 & 1 & -1 & 2 \\ 1 & -2 & -1 & 1 & 2 & -1 & 0 & -2 & -1 & -4 & 0 & -2 & 0 & 2 & 1 & 0 \\ -2 & 0 & -1 & 1 & 1 & -1 & 2 & -1 & 0 & 0 & -4 & -1 & -1 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 & 2 & -2 & 1 & -1 & -1 & -2 & -1 & -4 & -1 & 2 & 1 & 1 \\ -2 & 1 & -1 & 2 & 1 & 1 & 2 & 1 & 1 & 0 & -1 & -1 & -4 & 0 & 2 & -2 \\ -2 & 2 & -1 & 0 & -1 & 1 & 1 & 0 & 1 & 2 & -1 & 2 & 0 & -4 & 0 & -1 \\ 1 & 1 & 2 & -2 & -2 & -1 & -2 & 1 & -1 & 1 & 1 & 1 & 2 & 0 & -4 & 1 \\ -2 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 1 & -2 & -1 & 1 & -4 \end{pmatrix}$$

Proof of Theorem 7.57. From Proposition 7.62 and Proposition 7.65, we know that the coinvariant sublattice of a nonstable symplectic involution of Λ is isometric to $E_6(2)$, M, G_{12} or G_{16}. Moreover, each of these $\mathbb{Z}$-lattices admits a primitive embedding into Λ and none of them contains vectors of type $(-2,1)$ or $(-6,3)$. Therefore, by Corollary 7.60 and Lemma 7.64, we only have to determine the number of isomorphism classes of primitive sublattices of Λ which are isometric to one of the previous 4 $\mathbb{Z}$-lattices and whose complement is 2-elementary. To do so, we apply again Algorithm 2 and obtain the wanted result. $\square$

To conclude, let us observe the following. In Section 7.5.2 we have determined the possible stable symplectic actions on OG10-type IHS manifolds, and in the current section we have determined representatives for the conjugacy classes of nonstable symplectic involutions. We can therefore conclude on the classification of prime order symplectic isometries in $O^+(\Lambda)$.

Theorem 7.67. *There are exactly 17* $\mathrm{Mon}^2(\Lambda)$*-conjugacy classes of symplectic isometries of* Λ *of prime order. See Table 7 for the associated numerical data.*

Proof. In the case $p = 2$, we have determined representatives for the conjugacy classes of nonstable symplectic involutions in Theorem 7.57. For stable symplectic involutions, we know from Remark 7.61 that it goes back to classifying 2-elementary stable symplectic sublattices of Λ: we know that there are exactly two isomorphism classes of such (see Table 16, entries 2 and 5), concluding the proof for the case $p = 2$.

Suppose now that p is odd. In particular, we have that any symplectic isometry $f \in O^+(\Lambda)$ of order p is stable, and moreover $C := \Lambda_f$ is p-elementary. In fact, since f is stable of order p, the restriction $h \in O^\#(C)$ of f to C is also stable of minimal polynomial Φ_p. Hence, on D_C, we have an equality

$$0 = \Phi_p(D_h) = \Phi_p(\mathrm{id}_{D_C}) = p\,\mathrm{id}_{D_C}$$

and D_C is a p-elementary abelian group. Hence, by Proposition 7.51, we see that either C embeds primitively into the Leech lattice, or $p = 3$ and C is isometric to the $\mathbb{Z}$-lattice defined in Example 7.42 (which corresponds to the $\mathbb{Z}$-lattice E_{18} from Appendix A.1).

(1) In the former case, we know that C embeds primitively into the Leech lattice, so we can use the classification of conjugacy classes of prime order isometries of the Leech lattice in [HM16]. For each class of prime order isometries g of the Leech lattice, we have that $(\mathbb{L}_g, g_{|\mathbb{L}_g})$ is actually unique up to isomorphism of lattices with isometry. Hence, we can infer from Table 16 whether $\mathbb{L}_g$ embeds primitively into Λ. If it does, then we know that Λ has a prime order isometry $f \in O^{+,\#}(\Lambda)$ with $\Lambda_f \simeq \mathbb{L}_g$, and Table 16 also tells us what is the genus $F := \Lambda^f$. Note that one can apply [BH23, Algorithm 2] to the triple of lattices with isometry $((F, \mathrm{id}), (\mathbb{L}_g, g_{|\mathbb{L}_g}), (\Lambda, \mathrm{id}))$, with $k = p$, to obtain the number of $O^+(\Lambda)$-conjugacy classes of such isometries f.

(2) In the latter case, one can use an algorithm of Plesken–Souvignier [PS97] to show that the negative definite $\mathbb{Z}$-lattice C defined in Example 7.42 admits a unique conjugacy class of fixed-point free isometry g of order 3. By applying again [BH23, Algorithm 2] to the triple of lattices with isometry $((U(3)^{\oplus 3}, \mathrm{id}), (C, g), (\Lambda, \mathrm{id}))$ with $p = k = 3$, and we obtain that there are two $O^+(\Lambda)$-conjugacy classes of cyclic subgroups of $O^{+,\#}(\Lambda)$ generated by an isometry f of order 3 such that $\Lambda_f \simeq C$.

The representatives f of the conjugacy classes of stable isometries of prime order in $O(\Lambda)$ we have constructed are symplectic, according to Corollary 7.11. So we can conclude. We refer to the Notebook "Prime" in the folder "verification" of [MM25c] for more details about the previous computations. □

Remark 7.68. We define the two following rank 2 even $\mathbb{Z}$-lattices:

$$K_7 := \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}, \qquad H_5 := \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

which are respectively positive definite and indefinite, of respective determinant 7 and -5.

Table 7: Prime order symplectic isometries — OG10 case

ord(f)	Λ^f	$g(\Lambda_f)$	# conjugacy classes
2	$U^{\oplus 3} \oplus D_4^{\oplus 3}$	$\mathrm{II}_{(0,6)}2^{-6}3^1$	1
2	$U^{\oplus 3} \oplus A_2 \oplus E_8(2)$	$\mathrm{II}_{(0,8)}2^8$	1
2	$U^{\oplus 2} \oplus A_1 \oplus A_1(-1) \oplus E_8(2)$	$\mathrm{II}_{(0,10)}2_4^{-10}3^1$	1
2	$U(2)^{\oplus 2} \oplus A_1 \oplus A_1(-1) \oplus E_6(2)$	$\mathrm{II}_{(0,12)}2_4^{12}$	1
2	$A_1^9 \oplus A_1(-1)^{\oplus 3}$	$\mathrm{II}_{(0,12)}2_2^{-12}3^1$	1
2	$A_1^5 \oplus A_1(-1)^{\oplus 3}$	$\mathrm{II}_{(0,16)}2_6^{-8}3^1$	1
3	$U \oplus U(3)^{\oplus 2} \oplus E_6$	$\mathrm{II}_{(0,12)}3^6$	1
3	$U(3)^{\oplus 3} \oplus E_6$	$\mathrm{II}_{(0,12)}3^6$	1
3	$U(3)^{\oplus 3} \oplus A_2$	$\mathrm{II}_{(0,16)}3^8$	1
3	$U \oplus U(3)^{\oplus 2}$	$\mathrm{II}_{(0,18)}3^5$	1
3	$U(3)^{\oplus 3}$	$\mathrm{II}_{(0,18)}3^5$	1
3	$U(3)^{\oplus 3}$	$\mathrm{II}_{(0,18)}3^{-7}$	2
5	$U \oplus U(5)^{\oplus 2} \oplus A_2$	$\mathrm{II}_{(0,16)}5^4$	1
5	$H_5 \oplus A_2(-5)$	$\mathrm{II}_{(0,20)}5^3$	1
7	$U(7) \oplus K_7 \oplus A_2$	$\mathrm{II}_{(0,18)}7^3$	1
11	$U \oplus A_2(-11)$	$\mathrm{II}_{(0,20)}11^2$	1

8. Cyclic actions

The content of this section is adapted from the author's article [Mul25].

Now that we have answered the question about classifying finite symplectic actions of IHS manifolds (together with the extension approach from Section 9), let us focus on purely nonsymplectic cyclic actions. In this section, we have two main goals:

(1) give a theoretical answer to problem (PNS) in a *generic* situation (to be made precise later);

(2) give evidence that using the so-called *Mukai lattices* to classify birational automorphisms in the $\mathrm{K3}^{[n]}$ and Kum_n cases is promising.

The most understood purely nonsymplectic actions of IHS manifolds are the one of prime order, which have been thoroughly studied for all the known deformation types. In the case of antisymplectic involutions, this has been covered by [Nik83] for K3 surfaces, [Bea11, Jou16, CCC21] for the deformation types $\mathrm{K3}^{[n]}$, [Tar15, MTW18] for the deformation types Kum_n, and [Gro22a, BG25] for the sporadic deformation types OG6 and OG10 respectively.

For odd prime order automorphisms, the case of K3 surfaces has been completely covered by [AST11], with further developments in [MO98, Bra19, BF25]. In higher dimension, an important part of the work has been carried by [BCMS16, BCS16, CC19] for the $\mathrm{K3}^{[n]}$ types ($n \geq 2$), by [BNWS11, Tar15, MTW18] for the Kum_n types ($n \geq 2$) and [Gro22a] who covers the case of OG6-type IHS manifolds. All of the latter have been recovered by [BC23] who also treated the case of OG10-type IHS manifolds (see also the recent paper [BG25] for the OG10 case).

Few is known yet for general order, except in the case of K3 surfaces. Among the standard literature in this case, one could cite the works [Sch10, AS15, ATST16, ATS18, ATGS21, ACV22] for 2-power orders, [Tak10, ACV20] for 3-power orders, [Dil12] for order 6, [BCL+24] for orders divisible by 7, and [Bra19, ACV22] for orders n with $\varphi(n) \geq 12$ and $\varphi(n) = 8, 10$ respectively. Moreover, nonsymplectic isometries acting trivially on the associated Néron–Severi lattice have been determined in [Kon92, Tak12] (see also [BF24]). The complete lattice classification, up to conjugacy, of purely nonsymplectic isometries in the case of K3 surfaces is due to Brandhorst and Hofmann in [BH23].

In [BC23], Brandhorst and Cattaneo describe an approach to study and classify, up to conjugacy, nonsymplectic automorphisms of odd prime order in a uniform way for each known deformation type of IHS manifolds, by the mean of classifying isometries of even unimodular $\mathbb{Z}$-lattices. Later in this section, we review their approach and we apply it to classify *algebraically trivial* actions of the known IHS manifolds. But before that, we do some preliminary lattice-theoretic work on classifying conjugacy classes of finite order isometries of even indefinite unimodular $\mathbb{Z}$-lattices.

8.1. Isometries of even unimodular $\mathbb{Z}$-lattices

As already mentioned in the introduction of this thesis, we know necessary and sufficient conditions for the existence of finite order isometries of even unimodular $\mathbb{Z}$-lattices with given characteristic polynomial. We would like to go beyond these existence conditions, and classify finite order isometries with given type (Definition 1.61), up to conjugacy. This is however a hard problem in general, so one might need to restrict the types of lattices with isometry to study. In [BC23], the authors solve this problem in the case of odd prime order isometries of integral unimodular $\mathbb{Z}$-lattices. They suspected their approach could be applicable to classify finite order isometries of the known BBF-forms with minimal polynomial $\Phi_1\Phi_m$ for some $m \geq 3$. As we will show later in this section, their work indeed extend to studying such isometries. In particular (see Remark 8.52), the adaptation of their approach to this context translates into a classification of pairs (M, g) where M is even unimodular and $g \in O(M)$ is a finite order isometry satisfying either

(1) g has minimal polynomial $\Phi_1\Phi_m$ for some $m \geq 3$; or

(2) g has minimal polynomial $\Phi_1\Phi_2\Phi_m$ for some $m \geq 4$ even and M^{-g} has rank 1.

This motivates the first part of this section where we study and classify such pairs (M, g).

8.1.1. Type study

In this section, we study isometries of even unimodular $\mathbb{Z}$-lattices, of order $m \geq 3$ and with minimal polynomial dividing $\Phi_1\Phi_2\Phi_m$. We start by investigating their type in order to have more information on the genera of their kernel sublattices.

The first case to consider is a generalization of the odd prime order case studied in [BC23, §2]. The following is a standard result — see for instance [BF24, Proposition 4.8].

Proposition 8.1. *Let (M, g) be an even unimodular $\Phi_1\Phi_m^*$-lattice where $m \geq 2$. The following hold:*

(1) *D_{g_1} and D_{g_m} are trivial;*

(2) *if $m = p^k$ is a prime power, then both M^g and M_g are p-elementary;*

(3) *if m is not a prime power, then both M^g and M_g are unimodular.*

Proof. Let us denote by $F := M^g$ and $C := M_g$ the associated kernel sublattices.

(1) Since M is unimodular, the $\mathbb{Z}$-lattices F and C glue along their respective discriminant groups. By Equation (EGC), we know that $D_{g_1} = \mathrm{id}_{D_F}$ and D_{g_m} agree on D_C (via the gluing), and they are thus both trivial.

(2) Follows from Corollary 2.42 applied to $g^{m/p}$ and with $n = 1$.

(3) If m is not a prime power, then there exist two distinct prime numbers $p \neq q$ dividing m. Applying (1) and (2) to suitable powers of g_m, we obtain that F and C are simultaneously p-elementary and q-elementary. The previous being possible if and only if D_F and D_C are both trivial, we can conclude. □

In Figures 2 and 3, we summarize the *assembly diagram* [McM11, §8] for even unimodular $\Phi_1\Phi_m^*$-lattices depending on whether m is a prime power. Such assembly diagram is made as follows:

(1) each box represents a kernel sublattice;

(2) each edge represents a gluing:

   (a) either the kernel sublattices are in orthogonal direct sum and the edge is decorated with a symbol "$\oplus$";

   (b) or they glue along $\mathbb{F}_p$-vector spaces for some prime number p and the edge is decorated with the isomorphism class of the associated glue domains, as vector spaces.

In the case m is not a prime power, the kernel sublattices are indeed in orthogonal direct sum since their respective discriminant groups are trivial (Proposition 8.1).

Figure 2: Assembly diagram for $m = p^k$

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Invariant sublattice}\\ F\end{array}} \arrow[r, dash, "\mathbb{F}_p^l"] & \boxed{\begin{array}{c}\Phi_{p^k}\text{-kernel sublattice}\\ C\end{array}} \end{tikzcd}$$

Figure 3: Assembly diagram for $m \neq p^k$

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Invariant sublattice}\\ F\end{array}} \arrow[r, dash] & \oplus \arrow[r, dash] & \boxed{\begin{array}{c}\Phi_{m}\text{-kernel sublattice}\\ C\end{array}} \end{tikzcd}$$

Similarly to Proposition 8.1, we want to obtain information about the kernel sublattices of a unimodular $\Phi_1\Phi_2\Phi_m^*$-lattice for $m \geq 3$ even and how they glue.

Proposition 8.2. *Let (M, g) be an even unimodular $\Phi_1\Phi_2\Phi_m^*$-lattice where $m \geq 3$ is even. The following hold:*

(1) *if $m = 2^k$ is a power of 2, then $M^{\Phi_m(g)}$ is 2-elementary, and M^g and M^{-g} are both 4-elementary;*

(2) *if $m = 2p^k$ is twice an odd prime power, then $M^{\Phi_m(g)}$ is p-elementary, M^{-g} is $2p$-elementary and M^g is 2-elementary;*

(3) *otherwise, $M^{\Phi_m(g)}$ is unimodular, and M^g and M^{-g} are both 2-elementary.*

Proof. For what follows, we denote $C := M^{\Phi_m(g)}$ and $F := (C)^{\perp}_M = M^{g^2-1}$. Note that (M, g^2) is a $\Phi_1\Phi_{m/2}^*$-lattice, and F and C are the associated kernel sublattices — we can therefore obtain information on the latter two by applying Proposition 8.1 to g^2.

(1) Since $m = 2^k \geq 4$, Proposition 8.1 (1) gives that F and C are both 2-elementary. Moreover $g_{|F}$ has order 2, so Corollary 2.42 tells us that both M^g and M^{-g} are 4-elementary.

(2) We have now that g^2 has odd order p^k so Proposition 8.1 tells us that F and C are both p-elementary. Thus, according to Corollary 2.42 we have that M^g and M^{-g} are both $2p$-elementary. Note moreover that since g has minimal polynomial $\Phi_1\Phi_2\Phi_m$, we have that $M^g = M^{g^{p^k}}$: since g^{p^k} has order 2, we deduce that M^g is actually 2-elementary (Proposition 8.1 (2)).

(3) In that case, the order of g^2 is not a prime power and by Proposition 8.1 (3) we have that both F and C are unimodular. By Proposition 8.1 (2), we conclude that M^g and M^{-g} are 2-elementary. □

Similarly as before, we keep track in Figures 4 to 6 of the assembly diagrams for even unimodular $\Phi_1\Phi_2\Phi_m^*$-lattices depending on m. Together with Proposition 8.2, this provides us a good understanding of the type of such lattices with isometry.

Figure 4: Assembly diagram for $m = 2^k$

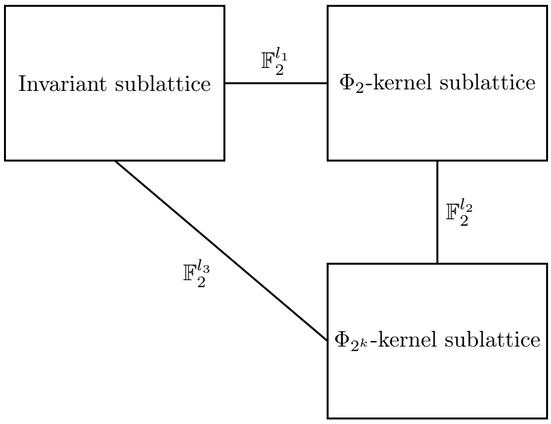


Figure 5: Assembly diagram for $m = 2p^k$

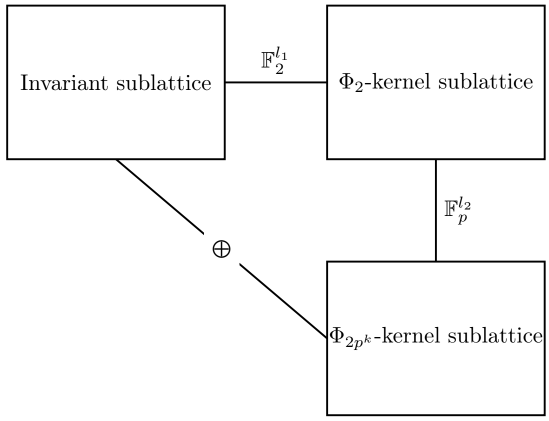
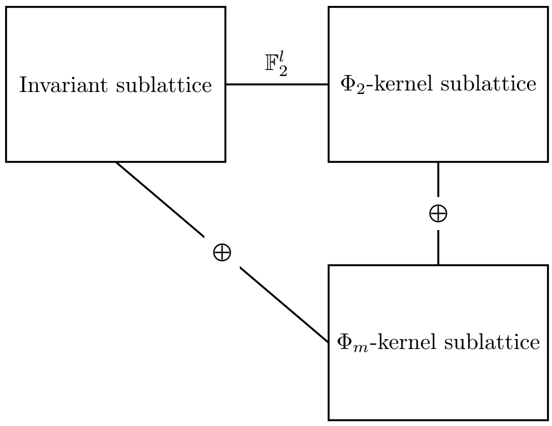


Figure 6: Assembly diagram for $m \neq 2^k, 2p^k$



8.1.2. Constructing isometries using hermitian lattices

Let us recall the following results.

Theorem 8.3 ([GM02, Theorem 1.2], [BFT20, Theorem A])**.** *Let $r, s \in \mathbb{Z}_{\geq 0}$ be such that $r \equiv s \mod 8$, let P be a monic irreducible polynomial and let S be a power of P. Assume that S has degree $r+s$ and let us denote by $m(S)$ the number of complex roots of S of norm bigger than 1. Suppose moreover that*

(C1) $t^{r+s}S(1/t) = S(t)$;

(C2) $m(S) \leq r$, $m(S) \leq s$ and $m(S) \equiv r \equiv s \mod 2$;

(C3) $|S(1)|$, $|S(-1)|$ and $(-1)^{(r+s)/2}S(1)S(-1)$ are squares.

Then there exists an even unimodular S-lattice (L, f) of signatures (r, s) and $\det(f) = +1$.

Theorem 8.4 ([BF24, Theorem 1.7])**.** *Let m, d, c be integers with $m \geq 3$, $d \geq 1$ and $0 \leq c \leq d\varphi(m)$ even, and let $C := \Phi_m^d$. Let moreover $r, s \geq 0$ be such that*

(1) *$r \equiv s \mod 8$;*

(2) $r \geq c$, $s \geq d\varphi(m) - c$ *and* $N := r + s > d\varphi(m)$.

Then there exists an even unimodular lattice with isometry (M, g) *with* $M \in \mathrm{II}_{(r,s)}$, $\chi_g(X) = C(X)(X-1)^{N-d\varphi(m)}$ *and* $M^{C(g)}$ *has signatures* $(c, d\varphi(m) - c)$ *if and only if*

(1) $C(-1)$ *is a square;*

(2) *if* $C(1) = 1$, *then* $d\varphi(m) \equiv 2c \mod 8$.

Remark 8.5. The previous theorems already give necessary and sufficient conditions for the existence of a unimodular $\Phi_1\Phi_m$-lattice (M, g) (see also [BF24, Lemma 4.2, Theorem 4.5]). In this section, we complement these existence conditions in order to classify these lattices with isometry up to isomorphism.

8.1.2.1. Local information

Let p be a prime number, let $m = p^k \geq 3$ be a power of p and let $\zeta := \zeta_m \in \mathbb{C}$ be a primitive mth root of unity. Let us denote moreover $E := \mathbb{Q}(\zeta)$, $K := \mathbb{Q}(\zeta + \zeta^{-1})$, $\pi := 1 - \zeta$, $\mathfrak{P} := \pi\mathcal{O}_E$ and $\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_K$. According to Proposition 3.2, $\mathfrak{P}$ (resp. $\mathfrak{p}$) is the unique prime $\mathcal{O}_E$-ideal (resp. $\mathcal{O}_K$-ideal) which divides $p\mathcal{O}_E$ (resp. $p\mathcal{O}_K$). Let (L, b, f) be an even Φ_m-lattice and suppose that D_f has order $p^l \leq p^{k-1}$.

Proposition 8.6. *Let* (L, b, f) *be an even* Φ_{p^k}*-lattice for some prime number* p *and* $k \geq 1$, *and suppose that* D_f *has order at most* p^{k-1}. *Then* (L, b) *is* p*-elementary.*

Proof. We know that $\Phi_{p^k}(D_f) = 0_{D_L}$ and moreover $D_f^{p^{k-1}} = \mathrm{id}_{D_L}$. Hence, we obtain that

$$0 = \Phi_{p^k}(D_f) = \Phi_p(D_f^{p^{k-1}}) = \Phi_p(\mathrm{id}_{D_L}) = p\,\mathrm{id}_{D_L}.$$

Therefore (L, b) is p-elementary. □

Remark 8.7. A consequence of Proposition 8.6 is that for a Φ_m-lattice (L, f) with m not a prime power, then L is not unimodular only if there exists a prime number p dividing m such that

$$\mathrm{val}_p(m) = \mathrm{val}_p(\mathrm{ord}(D_f)).$$

Example 8.8. Let $L := A_2$, and let $\{u, v\}$ be its basis with $u^2 = v^2 = -2$ and $u.v = 1$. We define an isometry $f \in O(L)$ by $f(u) = -v$ and $f(v) = u + v$. The lattice with isometry (L, f) is a Φ_6-lattice. The lattice L is known to be 3-elementary with $D_L \cong \mathbb{Z}/3\mathbb{Z}$ as abelian groups. Hence D_f is nontrivial, and since D_L only has one nontrivial isometry, we obtain that $D_f = -\mathrm{id}_{D_L}$ has order 2. Note that (L, f^2) is a Φ_3-lattice with $D_{f^2} = \mathrm{id}_{D_L}$.

We denote by (L, h) the hermitian structure of (L, b, f) (Section 4.3). We recall that by the trace construction, $L^\vee = \mathfrak{D}_{E/\mathbb{Q}}^{-1} L^\#$ (Table 2). Since by assumption (L, b) is even, we moreover have that $\mathfrak{n}(L) \subseteq \mathfrak{D}_{K/\mathbb{Q}}^{-1}$ (Lemma 4.24). Following the transfer construction, the $\mathcal{O}_E$-module structure on L is given by f. Hence, since $\mathrm{ord}(D_f) = p^l$, we have that

$$(1 - \zeta^{p^l})L^\vee \leq L \leq L^\vee.$$

Now, if we replace $L^\vee$ by $\mathfrak{D}_{E/\mathbb{Q}}^{-1} L^\#$, we can use the fact that $\mathfrak{D}_{E/\mathbb{Q}}$ is principal generated by π^α where $\alpha := p^{k-1}(pk - k - 1)$ (Proposition 3.2) and that $(1 - \zeta^{p^l})\mathcal{O}_E = \pi^{p^l}\mathcal{O}_E$ (Lemma 3.11) to translate the previous equation into

$$\pi^{p^l - \alpha} L^\# \leq L \leq \pi^{-\alpha} L^\#. \tag{21}$$

Proposition 8.9. *For all finite places $\mathfrak{q} \neq \mathfrak{p}$ of K, the hermitian $\mathcal{O}_{E_\mathfrak{q}}$-lattice $L_\mathfrak{q}$ is unimodular and uniquely determined by its rank. Moreover, any Jordan decomposition of $L_\mathfrak{p}$ consists of at most $p^l + 1$ pairwise orthogonal Jordan constituents which are respectively $\mathfrak{P}^{j-\alpha}$-modular, for $j = 0, \dots, p^l$.*

Proof. We have that $\pi\mathcal{O}_E = \mathfrak{P}$ by definition, and $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_K$ represents the only bad finite place of K (Lemma 3.1). From Equation (21), we have that for all (good) finite place $\mathfrak{q} \neq \mathfrak{p}$ of K, the lattice $L_\mathfrak{q}$ is unimodular and uniquely determined by its rank [Kir16, Proposition 3.3.5]. The statement about the Jordan decompositions at $\mathfrak{p}$ follows from the same equation. □

Therefore, in order to study the genus of (L, h), we only need to understand the Jordan constituents of $L_\mathfrak{p}$ (Proposition 8.9). Those have already been described in Proposition 4.17.

Remark 8.10. Note that since $\alpha = p^{k-1}(pk - k - 1)$ and p have the same parity, for all $j = 0, \dots, p^l$ such that $j + p$ is odd, the rank of a $\mathfrak{P}^{j-\alpha}$-modular hermitian $\mathcal{O}_{E_\mathfrak{p}}$-lattice is even according to Proposition 4.17.

We now state and prove a couple of results about the trace lattice of a hermitian $\mathcal{O}_E$-lattice. The first of these results is analogous to something we have already observed for plain integral $\mathbb{Z}$-lattices (see Remark 1.7 and Lemma 1.39).

Lemma 8.11. *Let (L, h) be a hermitian $\mathcal{O}_E$-lattice with trace lattice (L, b, f). We assume that $\mathfrak{s}(L, h) \subseteq \mathfrak{D}_{E/\mathbb{Q}}^{-1}$ and that D_f has order $p^l \leq p^{k-1}$. Then (L, b) is even if and only if p is odd, or the $\mathfrak{P}^{-\alpha}$-modular Jordan constituent of any Jordan decomposition of $L_\mathfrak{p}$ is isometric to $H(\pi^{-\alpha})^{\oplus r}$ for some $r \in \mathbb{Z}_{\geq 0}$.*

Proof. By Lemma 4.24 (see also [BH23, Proposition 6.6]), we know that (L, b) is even if and only if $\mathfrak{n}(L, h)\mathcal{O}_E \subseteq \mathfrak{D}_{K/\mathbb{Q}}^{-1}\mathcal{O}_E$. In the case where p is odd, we know that $\mathfrak{D}_{E/K} = \pi\mathcal{O}_E$ (Proposition 3.2) and in particular, $\mathfrak{D}_{E/\mathbb{Q}} \cap K = \mathfrak{D}_{K/\mathbb{Q}}$. This implies that

$$\mathfrak{n}(L, h) \subseteq \mathfrak{s}(L, h) \cap K \subseteq \mathfrak{D}_{K/\mathbb{Q}}^{-1}$$

by the assumption $\mathfrak{s}(L, h) \subseteq \mathfrak{D}_{E/\mathbb{Q}}^{-1}$. Hence the result follows in that case (see the proof of [BC23, Lemma 2.4] which uses the same argument in the case $k = 1$ and p is odd).

Now suppose that $p = 2$: in this case, we have that $\mathfrak{D}_{K/\mathbb{Q}}^{-1}\mathcal{O}_E = \pi^{2-\alpha}\mathcal{O}_E$ (Proposition 3.2). Since for all prime $\mathcal{O}_K$-ideals $\mathfrak{q}$ not dividing $2\mathcal{O}_K$ we have that $\mathfrak{D}_{E_\mathfrak{q}/K_\mathfrak{q}} = \mathcal{O}_{E_\mathfrak{q}}$, we already know that locally at $\mathfrak{q}$

$$\mathfrak{n}(L_\mathfrak{q})\mathcal{O}_{E_\mathfrak{q}} \subseteq \mathfrak{s}(L_\mathfrak{q}) \subseteq \mathfrak{D}_{E_\mathfrak{q}/\mathbb{Q}_q}^{-1} = \mathfrak{D}_{K_\mathfrak{q}/\mathbb{Q}_q}^{-1}\mathcal{O}_{E_\mathfrak{q}}$$

where $q\mathbb{Z} := \mathfrak{q} \cap \mathbb{Z} \neq 2\mathbb{Z}$. According to Equation (21), $L_\mathfrak{p}$ admits a Jordan decomposition of the form $L_\mathfrak{p} = \bigoplus_{i=0}^{p^l} N_i$ where N_i is $\mathfrak{P}^{i-\alpha}$-modular of rank $n_i \geq 0$. By definition, for all $i \geq 2$,

$$\mathfrak{n}(N_i)\mathcal{O}_{E_\mathfrak{p}} \subseteq \mathfrak{s}(N_i) = \mathfrak{P}^{i-\alpha} \subseteq \mathfrak{P}^{2-\alpha} = \pi^{2-\alpha}\mathcal{O}_{E_\mathfrak{p}}.$$

Moreover, if $n_1 \neq 0$, since $1 - \alpha$ is odd, Remark 4.18 gives us that $\mathfrak{n}(N_1)\mathcal{O}_{E_\mathfrak{p}} = \pi^{2-\alpha}\mathcal{O}_{E_\mathfrak{p}}$. So the only obstruction for (L, b) to be even should come from N_0. According to Proposition 4.17 either n_0 is odd and $\mathfrak{n}(N_0)\mathcal{O}_{E_\mathfrak{p}} = \mathfrak{s}(N_0) = \pi^{-\alpha}\mathcal{O}_{E_\mathfrak{p}}$, or n_0 is even and $\pi^{2-\alpha}\mathcal{O}_{E_\mathfrak{p}} \subseteq \mathfrak{n}(N_0)\mathcal{O}_{E_\mathfrak{p}} \subseteq \pi^{-\alpha}\mathcal{O}_{E_\mathfrak{p}}$. We conclude by remarking that since π is not invertible in $\mathcal{O}_{E_\mathfrak{p}}$, then $\mathfrak{n}(N_0) \subseteq \pi^{2-\alpha}\mathcal{O}_{E_\mathfrak{p}}$ if and only if $\mathfrak{n}(N_0) = \pi^{2-\alpha}\mathcal{O}_{E_\mathfrak{p}}$ if and only if n_0 is even and $N_0 \simeq H(\pi^{-\alpha})^{\oplus n_0/2}$. □

We are now equipped to prove the following result, generalizing the necessary conditions from [BC23, Proposition 2.15].

Proposition 8.12. *Let p be a prime number, let $l_+, l_- \geq 0$ be such that $l_+ + l_- \geq 1$ and let $k \geq \gcd(2,p)$ be positive. If there exists an even Φ_{p^k}-lattice (L,b,f) of signatures (l_+,l_-) such that D_f has order $p^l < p^k$, then there exist integers $n_0, \ldots, n_{p^l}$, with n_0 even and n_i even if $i+p$ is odd, such that*

(1) *(L,b) is p-elementary of absolute determinant $p^{\sum_{i=0}^{p^l} in_i}$;*

(2) *$l_+ + l_- = \varphi(p^k)\sum_{i=0}^{p^l} n_i$;*

(3) *$l_+, l_- \in 2\mathbb{Z}$;*

(4) *if $l > 0$, one among $n_{p^{l-1}+1}, \ldots, n_{p^l}$ is nonzero.*

Proof. Let ζ be a primitive p^kth root of unity, and let us denote again $E := \mathbb{Q}(\zeta)$ and $K := \mathbb{Q}(\zeta+\zeta^{-1})$. Recall that $\mathfrak{P}$ and $\mathfrak{p}$ are the unique prime ideals of $\mathcal{O}_E$ and $\mathcal{O}_K$ respectively lying above $p\mathbb{Z}$. Let (L,b,f) be an even Φ_{p^k}-lattice of signatures (l_+,l_-) and such that D_f has order p^l for some $0 \leq l < k$. Let moreover (L,h) be its hermitian structure. By Equation (10), we know that the rank of L as $\mathbb{Z}$-module is even divisible by $\varphi(p^k)$, and l_+ and l_- are both even according to Equation (13). Moreover, since $l < k$ we know that (L,b) is p-elementary (Proposition 8.6). According to Proposition 8.9, any Jordan decomposition of $L_\mathfrak{p}$ consists of at most p^l+1 Jordan constituents $N_0, \ldots, N_{p^l}$, and

$$L_\mathfrak{p} = \bigoplus_{0\leq i\leq p^l} N_i.$$

The $\mathfrak{P}$-adic valuations of their respective scales are $-\alpha, 1-\alpha, \ldots, p^l-\alpha$, where $\alpha := p^{k-1}(pk-k-1)$. For all $0 \leq i \leq p^l$, let us denote $n_i := \operatorname{rank}_{\mathcal{O}_{E_\mathfrak{p}}}(N_i)$. By Proposition 4.17 and Lemma 8.11 we have that N_0 is hyperbolic, and thus of even rank. The fact that n_i is even whenever $i+p$ is odd follows from Remark 8.10. Now, since the rank of L, as $\mathcal{O}_E$-module, is equal to the rank of $L_\mathfrak{p}$, as $\mathcal{O}_{E_\mathfrak{p}}$-module, we have that

$$\operatorname{rank}_\mathbb{Z}(L) = l_+ + l_- = \varphi(p^k)\sum_{0\leq i\leq p^l} n_i.$$

Finally, one observes that $L^\vee/L = \mathfrak{D}^{-1}_{E/\mathbb{Q}}L^\#/L \cong \bigoplus_{0\leq i\leq p^l}\mathfrak{D}^{-1}_{E_\mathfrak{p}/\mathbb{Q}_p}N_i^\#/N_i$ since $L_\mathfrak{q}$ is unimodular for all finite places of K outside of $\mathfrak{p}$. If we denote $\pi := 1-\zeta$, since for all $0 \leq i \leq p^l$ the hermitian lattice N_i is $\pi^i\mathfrak{D}^{-1}_{E_\mathfrak{p}/\mathbb{Q}_p}$-modular, one has that for all $0 \leq i \leq p^l$, as abelian groups,

$$\mathfrak{D}^{-1}_{E_\mathfrak{p}/\mathbb{Q}_p}N_i^\#/N_i = \pi^{-i}N_i/N_i \cong (\mathcal{O}_E/\mathfrak{P}^i)^{\oplus n_i}$$

where the latter is an $\mathcal{O}_E/\mathfrak{P}$-vector space of dimension in_i, with $\mathcal{O}_E/\mathfrak{P} \simeq \mathbb{F}_p$ (Proposition 3.2). Therefore, $\operatorname{val}_p(\det(L,b)) = \sum_{i=0}^{p^l} in_i$. Finally, if $l \geq 1$ and if $n_j = 0$ for all $p^{l-1}+1 \leq j \leq p^l$, we would have that $\pi^{p^{l-1}-\alpha}L_\mathfrak{p}^\# \leq L_\mathfrak{p}$, contradicting the fact that D_f has order p^l. ☐

Finally, for a 2-elementary integral $\mathbb{Z}$-lattice L, one defines

$$\delta_L := \begin{cases} 0 & \text{if} \quad n(L^\vee) \subseteq \mathbb{Z} \\ 1 & \text{else} \end{cases}.$$

In other words, since L is 2-elementary, any Jordan decompositions of L_2 consists of at most two constituents: a unimodular one, and a 2-modular one which is of the form $L'(2)$ with L' unimodular. Then $\delta_L = 0$ if and only if L' is even. Such a result translates as follows.

Lemma 8.13. *Let $p = 2$, and let $m = 2^k$ for some $k \geq 2$. Let (L, h) be a hermitian $\mathcal{O}_E$-lattice with trace lattice (L, b, f). We assume that $\mathfrak{s}(L, h) \subseteq \mathfrak{D}_{E/\mathbb{Q}}^{-1}$ and that D_f has order $2^l \leq 2^{k-1}$. Then $\delta_L = 0$ if and only if the $\mathfrak{P}^{2^{k-1}-\alpha}$-modular Jordan constituent of any Jordan decomposition of $L_\mathfrak{p}$ is isometric to $H(\pi^{2^{k-1}-\alpha})^{\oplus r}$ for some $r \in \mathbb{Z}_{\geq 0}$.*

Proof. According to Lemma 4.24, $n(L^\vee) \subseteq \mathbb{Z}$ if and only if $2\mathfrak{n}(\mathfrak{D}_{E/\mathbb{Q}}^{-1}L^\#)\mathcal{O}_E \subseteq \mathfrak{P}^{2-\alpha}$. Now $2\mathcal{O}_E = \mathfrak{P}^{2^{k-1}}$ and therefore

$$2\mathfrak{n}(\mathfrak{D}_{E/\mathbb{Q}}^{-1}L^\#)\mathcal{O}_E = \mathfrak{n}(\pi^{2^{k-2}}\mathfrak{D}_{E/\mathbb{Q}}^{-1}L^\#)\mathcal{O}_E.$$

But, since the order of D_f is at most 2^{k-1}, we know that (L, b) is 2-elementary and $L_\mathfrak{q}$ is unimodular for all finite places $\mathfrak{q} \neq \mathfrak{p}$ of K. In particular, $\mathfrak{n}(\pi^{2^{k-2}}\mathfrak{D}_{E/\mathbb{Q}}^{-1}L^\#) = \mathfrak{n}(\pi^{2^{k-2}}\mathfrak{D}_{E_\mathfrak{p}/\mathbb{Q}_2}^{-1}L_\mathfrak{p}^\#)$. By Proposition 8.9, we have that

$$\pi^{2^{k-2}}\mathfrak{D}_{E_\mathfrak{p}/\mathbb{Q}_2}^{-1}L_\mathfrak{p}^\# \simeq \bigoplus_{j=0}^{2^{k-1}} \pi^{-j+2^{k-2}}N_j$$

where N_j is $\mathfrak{P}^{j-\alpha}$-modular. The rest of the proof follows similarly as for the proof of Lemma 8.11 after remarking that for all $0 \leq j \leq 2^{k-1}$, since $\alpha = (k-1)2^{k-1}$,

$$\mathfrak{s}(\pi^{-j+2^{k-2}}N_j) = \pi^{-2j+2^{k-1}}\mathfrak{s}(N_j) = \mathfrak{P}^{2^{k-1}-j-\alpha} \subseteq \mathfrak{P}^{-\alpha}$$

with equality if and only if $j = 2^{k-1}$. □

In particular, we can prove the following.

Corollary 8.14. *Let (L, b, f) be an even Φ_{2^k}-lattice for some $k \geq 2$ and suppose that D_f has order at most 2^{k-2}. Then $\delta_L = 0$.*

Proof. In that case, the Jordan decomposition of the hermitian structure of (L, b, f) at the prime ideal $\mathfrak{p}$ has constituents whose scales have $\mathfrak{P}$-adic valuation at most $2^{k-2} - \alpha$. Hence by Lemma 8.13, we have that $n(L^\vee) \subseteq \mathbb{Z}$ and $\delta_L = 0$. □

Remark 8.15. This generalizes the result of Taki [Tak12, Proposition 2.4] for the case $k = 2$.

In the next paragraph, we prove that Proposition 8.12 admits a converse if one makes further assumption on the genus of (L, b).

8.1.2.2. Existence of Φ_m-lattices with given invariants

Let (L, b, f) be an even Φ_m-lattice with $m \geq 3$ arbitrary, and suppose that D_f has order at most 2. In regard to Propositions 8.1 and 8.2, we have to consider the following cases:

(1) m is composite and D_f is trivial;

(2) m is a prime power and D_f is trivial;

(3) m is twice an odd prime power and $D_f = -\mathrm{id}_{D_L}$;

(4) m is a power of 2 and D_f has order 2.

In the first case, (L, b) is unimodular and the existence conditions are already well-known [BF84, McM15, BFT20, BF84]. For (2), (3) and (4), we know that (L, b) is p-elementary for some prime number p and we can use the results of the previous paragraph.

Remark 8.16. For an odd prime number p and some integer $k \geq 1$, if (L, b, f) is a non-unimodular Φ_{p^k}-lattice with D_f trivial, then $(L, -f)$ is a Φ_{2p^k}-lattice where $D_{-f} = -\mathrm{id}_{D_L}$ has order 2. And the converse also holds. Hence the existence of a Φ_{2p^k}-lattice (L, f) with $D_f = -\mathrm{id}_{D_L}$ is equivalent to the existence of a Φ_{p^k}-lattice with stable isometry. Therefore the case of twice an odd prime power in (3) is already covered by (2).

For (1), the existence conditions can be reduced to the following proposition:

Proposition 8.17. *Let $m \geq 3$ be a composite integer and let $l_+, l_- \geq 0$ with $l_+ + l_- \geq 1$. There exists an even unimodular Φ_m-lattice (L, b, f) of signatures (l_+, l_-) if and only if there exists a positive integer $d > 0$ such that:*

(1) $l_+ \equiv l_- \mod 8$, *and* $l_+ + l_- = d\varphi(m)$;

(2) $l_+, l_- \equiv 0 \mod 2$;

(3) *if $m = 2p^k$ for some odd prime number p and positive integer $k > 0$, then $d \equiv 0 \mod 2$.*

Moreover, up to fixing signatures, the genus of the hermitian structure of any such (L, b, f) is uniquely determined by (l_-, m, d).

Proof. The existence part follows from Theorem 1.46, Equation (13) and [BF24, Theorem 4.5]. Now, let (L, b, f) be an even Φ_m-lattice and let (L, h) be its hermitian structure over the field $E := \mathbb{Q}(\zeta)$ where $\zeta := \zeta_m$ is a primitive mth root of unity. If m is not twice a prime power, then Lemma 3.1 tells us that the different ideal $\mathfrak{D}_{E/\mathbb{Q}} = \mathcal{O}_E$ is trivial. In particular, since (L, b) is unimodular, $L^\# = L$ and the rank of L uniquely determines the isometry class of $L_\mathfrak{p}$ for all finite places $\mathfrak{p}$ of $K := \mathbb{Q}(\zeta + \zeta^{-1})$. Otherwise, if $m = 2p^k$ is twice an odd prime power, the fact that (L, b) is unimodular is equivalent to

$$\mathfrak{P}^{-\alpha} L^\# = L \tag{22}$$

where $\mathfrak{P}$ is the unique prime $\mathcal{O}_E$-ideal above $p\mathbb{Z}$, and $\alpha := \mathrm{val}_\mathfrak{P}(\mathfrak{D}_{E/\mathbb{Q}}) = p^{k-1}(pk - k - 1)$. In that case, the isometry class of $L_\mathfrak{q}$ for all finite places $\mathfrak{q} \neq \mathfrak{p} := N_K^E(\mathfrak{P})$ of K is uniquely determined by the rank of L. Furthermore, according to Equation (22), we have that $\mathfrak{P}^{-\alpha} L_\mathfrak{p}^\# = L_\mathfrak{p}$ meaning that $L_\mathfrak{p}$ is $\mathfrak{P}^{-\alpha}$-modular by definition. Since α is odd, Proposition 4.17 and Lemma 8.11 tell us that $d = \mathrm{rank}_{\mathcal{O}_E}(L)$ is indeed even and $L_\mathfrak{p} \simeq H(\pi^{-\alpha})^{\oplus d/2}$ where $\pi := 1 - \zeta$. Hence, the isometry class of $L_\mathfrak{p}$ is uniquely determined too. According to Lemma 4.29 and the following discussion, we have that replacing f by another generator of $\langle f \rangle \leq O(L, b)$ will not change the isometry class of $L_\mathfrak{p}$ but only the signatures of (L, h). Hence, for fixed signatures, we have that the genus of (L, h) is uniquely determined by the isometry class of $L_\mathfrak{p}$, which itself is determined by the conditions of the statement. $\square$

We now need to settle the prime power cases. For this, we adapt and extend the proof of [BC23, Proposition 2.15]. We recall that if (L, b, f) is an even Φ_m-lattice for some $m \geq 3$, then by the trace construction, there exists a hermitian $\mathbb{Z}[\zeta_m]$-lattice (L, h) whose trace lattice is (L, b, f), and we have determined a list of local invariants for such a hermitian lattice. We now find sufficient conditions to prove the converse. We treat separately the case where D_f is trivial, and the case where D_f has order 2.

Proposition 8.18. *Let p be a prime number, let $l_+, l_- \geq 0$ be such that $l_+ + l_- \geq 1$ and let $k \geq \gcd(2, p)$ be positive. Suppose that there exist two integers $n_0 \in 2\mathbb{Z}_{\geq 0}$ and $n_1 \in \mathbb{Z}_{\geq 0}$, with n_1 even if $p = 2$, such that:*

(1) $l_+ + l_- = \varphi(p^k)(n_0 + n_1)$;

(2) $l_- \in 2\mathbb{Z}$;

(3) $l_+ \equiv l_- \mod 8$ *if* $n_1 = 0$.

Then there exists an even p-elementary Φ_{p^k}-lattice (L, b, f) of signatures (l_+, l_-), absolute determinant p^{n_1} and $\delta_L = 0$ if $p = 2$, such that D_f is trivial. Moreover, up to fixing signatures, the genus of the hermitian structure of any such (L, b, f) is uniquely determined by (l_-, p, k, n_0, n_1).

Proof. Let ζ be a primitive p^kth root of unity, and let us denote again $E := \mathbb{Q}(\zeta)$ and $K := \mathbb{Q}(\zeta + \zeta^{-1})$. Recall that $\mathfrak{P}$ and $\mathfrak{p}$ are the unique respective prime ideals of $\mathcal{O}_E$ and $\mathcal{O}_K$ lying above $p\mathbb{Z}$. We moreover denote $\Omega_\infty(K) \subseteq \Omega(K)$ the set of (real) infinite places, respectively the set of all places, of K. In Proposition 8.12, we have seen that if (L, b, f) is an even p-elementary Φ_{p^k}-lattice satisfying the conditions in the statement, then it is the trace lattice of a hermitian $\mathcal{O}_E$-lattice which is locally unimodular at all finite places of K different from $\mathfrak{p}$. Moreover, we know the invariants of its local isometry class at $\mathfrak{p}$. Therefore, we need to prove that conditions (1)-(3) are sufficient for the existence of such a hermitian $\mathcal{O}_E$-lattice, and that up to fixing the signatures at the real places of K, the genus of this lattice is uniquely determined.

For $i = 0, 1$, let N_i be a $\mathfrak{P}^{i-\alpha}$-modular $\mathcal{O}_{E_\mathfrak{p}}$-hermitian lattice of rank n_i where $\alpha := p^{k-1}(pk - k - 1)$. Moreover, with respect to Lemma 8.11, we suppose that $N_0 \simeq H(\pi^{-\alpha})^{\oplus n_0/2}$ where $\pi := 1 - \zeta$. Now, the existence of a hermitian $\mathcal{O}_E$-lattice (L, h) with given local structures $\{L_\mathfrak{q}\}_{\mathfrak{q}\in\Omega(K)}$ is equivalent to the finite set

$$S := \left\{\mathfrak{q} \in \Omega(K) : \ \det(L_\mathfrak{q}) \neq N^{E_\mathfrak{q}}_{K_\mathfrak{q}}(E_\mathfrak{q}^\times)\right\}$$

being of even cardinality [Kir16, Remark 3.4.2 (3) and Algorithm 3.5.6]. Note that an infinite place $\mathfrak{q} \in \Omega_\infty(K)$ lies in S if and only if $n(\mathfrak{q})$ is odd. Hence, for fixed signatures $\{n(\mathfrak{q})\}_{\mathfrak{q}\in\Omega_\infty(K)}$, there exists a hermitian $\mathcal{O}_E$-lattice (L, h) such that $L_\mathfrak{q}$ is unimodular for all finite places $\mathfrak{q}$ of K different from $\mathfrak{p}$ and such that $L_\mathfrak{p} \simeq N_0 \oplus N_1$ if and only if the following holds

$$\sum_{\mathfrak{q}\in\Omega_\infty(K)} n(\mathfrak{q}) \equiv \begin{cases} 0 & \text{if} \quad \det(N_0)\det(N_1) = N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times) \\ 1 & \text{else} \end{cases} \mod 2. \tag{23}$$

By condition (2) l_- is even and according to Equation (13), the lefthand side $\sum_{\mathfrak{q}\in\Omega_\infty(K)} n(\mathfrak{q})$ must be equal to $l_-/2$. If n_1 is nonzero, then Equation (23) and the congruence class $(n_0 \mod 4)$ uniquely determines $\det(N_1) \in K^\times/N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times)$. In particular, there exists a hermitian lattice (L, h) with the local structures as previously fixed. Now, if $n_1 = 0$, we have that $L_\mathfrak{p} \simeq N_0$ is a direct sum of $\frac{n_0}{2}$ copies of $H(\pi^{-\alpha})$ whose determinant is $-N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times)$ (Proposition 4.17). Hence $\det(L_\mathfrak{p}) = N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times)$ if and only if n_0 is divisibile by 4 or -1 is a local norm at $\mathfrak{p}$. By Theorem 3.3, this is equivalent to saying that $\det(L_\mathfrak{p}) = N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times)$ if and only if $n_0\varphi(p^k)$ is divisible by 8. Thus Equation (23) is equivalent to

$$2l_- \equiv \varphi(p^k)n_0 \mod 8.$$

But condition (1) enforces that $l_+ + l_- = \varphi(p^k)n_0$. Thus, in this situation, such a (L, h) exists if and only if

$$l_+ - l_- \equiv 0 \mod 8.$$

Therefore, condition (3) ensures that also in the case where $n_1 = 0$ there exists a hermitian lattice (L, h) with the local structures as previsouly fixed. In both of the previous cases, up to fixing the signatures $\{n(\mathfrak{q})\}_{\mathfrak{q}\in\Omega_\infty(K)}$, the genus of (L, h) is uniquely determined by conditions (1)-(3). $\square$

By using the same kind of arguments, we can also state a similar theorem for nonstable isometries in the case $p = 2$.

Proposition 8.19. *Let $l_+, l_- \geq 0$ be such that $l_+ + l_- \geq 1$ and let $k \geq 2$ be positive. Suppose that there exist four integers n_0, n_1, n_2 and δ, with $n_0, n_1 \in 2\mathbb{Z}_{\geq 0}$, $n_2 \in \mathbb{N}$ and $\delta \in \{0, 1\}$, such that*

(1) $l_+ + l_- = \varphi(2^k)(n_0 + n_1 + n_2)$;

(2) $l_- \in 2\mathbb{Z}$;

(3) *if $k = 2$ and n_2 is odd, then $\delta = 1$ and $l_+ - l_- \equiv \pm 2 \mod 8$;*

(4) *if $k = 2$ and n_2 is even, then $l_+ - l_- \equiv 0, 4 \mod 8$ with $l_+ - l_- \equiv 0 \mod 8$ if $\delta = n_1 = 0$;*

(5) *if $k \geq 3$, then $\delta = 0$ and $l_+ - l_- \equiv 0, 4 \mod 8$.*

Then there exists an even 2-elementary Φ_{2^k}-lattice (L, b, f) of signatures (l_+, l_-), absolute determinant $p^{n_1 + 2n_2}$ and $\delta_L = \delta$, such that D_f has order 2. Moreover, up to fixing signatures, the genus of the hermitian structure of any such (L, b, f) is uniquely determined by $(l_-, k, n_0, n_1, n_2, \delta)$, except in the case where $k \geq 3$, n_2 is even, and either $n_1 \neq 0$ or $l_+ - l_- \equiv 0 \mod 8$, where there are two possibilities.

Proof. We fix the same notations as in the proof of Proposition 8.18. Moreover, we let $u \in K_{\mathfrak{p}}^{\times} \setminus N_{K_{\mathfrak{p}}}^{E_{\mathfrak{p}}}(E_{\mathfrak{p}}^{\times})$, whose existence is ensured by [Joh68, Proposition 6.1]. This scalar u satisfies that $u \in 1 + \mathfrak{p}$ and $\mathcal{O}_{K_{\mathfrak{p}}}^{\times} = N_{K_{\mathfrak{p}}}^{E_{\mathfrak{p}}}(\mathcal{O}_{E_{\mathfrak{p}}}^{\times}) \sqcup u N_{K_{\mathfrak{p}}}^{E_{\mathfrak{p}}}(\mathcal{O}_{E_{\mathfrak{p}}}^{\times})$ (see for instance [Kir16, Corollary 3.3.17]).

Since we want D_f to have order 2, we let N_i be $\mathfrak{P}^{i-\alpha}$-modular of rank n_i for $i = 0, 1, 2$, with $n_2 \neq 0$. This time, the existence condition Equation (23) can be rewritten as

$$2l_- \equiv \begin{cases} 0 & \text{if} \quad \det(N_0)\det(N_1)\det(N_2) = N_{K_{\mathfrak{p}}}^{E_{\mathfrak{p}}}(E_{\mathfrak{p}}^{\times}) \\ 4 & \text{else} \end{cases} \mod 8. \tag{24}$$

(i) We first consider the case where $k = 2$, and $u = -1$ is not a local norm at $\mathfrak{p}$ (Theorem 3.3). Hence, $\det(N_0) = (-1)^{n_0/2} N_{K_{\mathfrak{p}}}^{E_{\mathfrak{p}}}(E_{\mathfrak{p}}^{\times})$ and there exists $\epsilon_1 \in \{0, 1\}$ such that $\det(N_1) = (-1)^{n_1/2 + \epsilon_1} N_{K_{\mathfrak{p}}}^{E_{\mathfrak{p}}}(E_{\mathfrak{p}}^{\times})$, with $\epsilon_1 = 0$ for $n_1 = 0$ (Proposition 4.17).

(a) If n_2 is odd, then there exists $\epsilon_2 \in \{0, 1\}$ such that $\det(N_2) = (-1)^{(n_2-1)/2 + \epsilon_2} N_{K_{\mathfrak{p}}}^{E_{\mathfrak{p}}}(E_{\mathfrak{p}}^{\times})$. The existence condition Equation (24) can be formulated as

$$2l_- \equiv 2(n_0 + n_1 + n_2 - 1) + 4(\epsilon_1 + \epsilon_2) \mod 8.$$

Since condition (1) imposes that $l_+ + l_- = 2(n_0 + n_1 + n_2)$, the latter is equivalent to

$$l_+ - l_- \equiv 2 - 4(\epsilon_1 + \epsilon_2) \equiv \pm 2 \mod 8.$$

Hence condition (3) ensures the existence of (L, h).

Fixing ($l_+ - l_- \mod 8$) actually fixes the parity $\epsilon_1 + \epsilon_2$: this gives rise in particular to two possible Jordan decompositions for $L_{\mathfrak{p}} = N_0 \oplus N_1 \oplus N_2$. These Jordan decompositions have the same invariants, except for the determinants of N_1 and N_2 which are both simultaneously changed (so that $\det(L_{\mathfrak{p}})$ is fixed). According to Theorem 4.16, these two decompositions define isometric $\mathcal{O}_{E_{\mathfrak{p}}}$-hermitian lattices if and only if

$$u\mathcal{O}_{K_{\mathfrak{p}}} \subseteq 1 + \mathfrak{n}(N_1)\mathfrak{n}(N_2)\mathfrak{s}(N_1)^{-2}.$$

By Proposition 4.17 and Remark 4.18, one can check that $\mathfrak{n}(N_1) = \mathfrak{n}(N_2) = \mathfrak{p}^{1-\alpha/2}$, while by definition of N_1, we know that $\mathfrak{s}(N_1)^{-2} = \mathfrak{p}^{\alpha-1}$. Hence

$$\mathfrak{n}(N_1)\mathfrak{n}(N_2)\mathfrak{s}(N_1)^{-2} = \mathfrak{p}^{1-\alpha/2}\mathfrak{p}^{1-\alpha/2}\mathfrak{p}^{\alpha-1} = \mathfrak{p} \supseteq (u-1)\mathcal{O}_{K_\mathfrak{p}}$$

and thus the two previous Jordan decompositions define isometric $\mathcal{O}_{E_\mathfrak{p}}$-lattices. Therefore, the genus of (L, h) is uniquely determined, up to fixing $\{n(\mathfrak{q})\}_{\mathfrak{q}\in\Omega_\infty(K)}$.

(b) If n_2 is even, according to Lemma 8.13 we have that either $\delta = 0$ and $\det(L_2) = (-1)^{n_2/2}N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times)$, or there exists $\epsilon_2 \in \{0,1\}$ such that $\det(L_2) = (-1)^{n_2/2+\epsilon_2}N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times)$. In this case, Equation (24) is equivalent to

$$l_+ - l_- \equiv 4(\epsilon_1 + \delta\epsilon_2) \mod 8.$$

Thus, condition (4) ensures the existence of (L, h).

Now, if $\delta = 0$, then ϵ_1 is fixed by $(l_+ - l_- \mod 8)$ and in particular, the genus of (L, h) is uniquely determined. Otherwise, the parity of $\epsilon_1 + \epsilon_2$ is completely determined by $(l_+ - l_- \mod 8)$: as in the case n_2 odd, since $\mathfrak{n}(N_2) = \mathfrak{p}^{1-\alpha/2}$ for $\delta = 1$, we have that the two possible Jordan decompositions for a fixed parity of $\epsilon_1 + \epsilon_2$ give rise to isometric hermitian lattices.

(ii) Now, in the case where $k \geq 3$, we have that -1 is a local norm at $\mathfrak{p}$ and therefore, $\det(N_0) = N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times)$ and there exists $\epsilon_1 \in \{0,1\}$ such that $\det(N_1) = u^{\epsilon_1}N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times)$, with $\epsilon_1 = 0$ for $n_1 = 0$. Moreover, there exists $\epsilon_2 \in \{0,1\}$ such that $\det(N_2) = u^{\epsilon_2}N^{E_\mathfrak{p}}_{K_\mathfrak{p}}(E_\mathfrak{p}^\times)$, with $\epsilon_2 = 0$ if $\mathfrak{n}(N_2) = \mathfrak{p}^{2-\alpha/2}$ (recall that $n_2 \neq 0$). In this setting, the existence condition Equation (24) is equivalent to

$$l_+ - l_- \equiv 2^{k-2}(n_0 + n_1 + n_2) - 4(\epsilon_1 + \epsilon_2) \equiv 0, 4 \mod 8.$$

Hence condition (5) ensures the existence of (L, h).

As before, fixing $(l_+ - l_- \mod 8)$ and the parity of n_2 fixes the parity of $\epsilon_1 + \epsilon_2$. Furthermore:

(a) for $n_1 = 0$ and n_2 odd, the congruence class $(l_+ - l_- \mod 8)$ uniquely determines ϵ_2 and there is a unique possible genus for (L, h).

(b) for $n_1 = 0$ and n_2 even, we have that $l_+ - l_- \equiv 4\epsilon_2 \mod 8$. If the latter congruence class is $(4 \mod 8)$ then the genus of (L, h) is again uniquely determined, and if it is $(0 \mod 8)$, then $\epsilon_2 = 0$ and there are two nonisometric choices for N_2 depending on $\mathfrak{n}(N_2) \in \{\mathfrak{p}^{1-\alpha/2}, \mathfrak{p}^{2-\alpha/2}\}$ (Remark 4.18), giving rise to two possible genera for (L, h).

(c) for $n_1 \neq 0$, then either $\mathfrak{n}(N_2) = \mathfrak{p}^{2-\alpha/2}$ in which case $\epsilon_2 = 0$, n_2 is even and the congruence class $(l_+ - l_- \mod 8)$ uniquely determines ϵ_1, or the congruence class $(l_+ - l_- \mod 8)$ and the parity of n_2 determines the parity of $\epsilon_1 + \epsilon_2$. Since we suppose now that $\mathfrak{n}(N_2) \neq \mathfrak{p}^{2-\alpha/2}$, in both the cases n_2 even or odd, we have that

$$\mathfrak{n}(N_1)\mathfrak{n}(N_2)\mathfrak{s}(N_1)^{-2} = \mathfrak{p}.$$

Hence, by applying again Theorem 4.16, we know that the parity of $\epsilon_1 + \epsilon_2$ determines two possible Jordan decompositions of $L_\mathfrak{p}$ which are isometric.

To summarize, when $n_1 \neq 0$ and n_2 is odd, there is only one possible genus for (L, h), but if n_2 is even, the two choices for $\mathfrak{n}(N_2) \in \{\mathfrak{p}^{1-\alpha/2}, \mathfrak{p}^{2-\alpha/2}\}$ give rise to two possible genera for (L, h). □

Remark 8.20. Let (L, b, f) be a Φ_{2^k}-lattice satisfying the assumptions of Proposition 8.19. According to the proof of the proposition, since f corresponds to multiplication by ζ on the $\mathcal{O}_E$-module L, we know that f acts trivially on $\mathfrak{D}^{-1}_{E_\mathfrak{p}/\mathbb{Q}_2} N_1^\# / N_1$. Now, if $n_2 = 1$, we have that as $\mathcal{O}_{E_\mathfrak{p}}$-modules,

$$\mathfrak{D}^{-1}_{E_\mathfrak{p}/\mathbb{Q}_2} N_2^\# / N_2 \simeq (\mathcal{O}_{E_\mathfrak{p}} \oplus \zeta\mathcal{O}_{E_\mathfrak{p}})/(1-\zeta^2)\mathcal{O}_{E_\mathfrak{p}}.$$

The latter is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as abelian group, where an isomorphism is given by $1 \mapsto (1, 0)$ and $\zeta \mapsto (0, 1)$. In particular, since for all $x \in L$ we have that $h(x, x) = h(\zeta x, \zeta x)$, the torsion quadratic form on the copy $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in D_L corresponding to $\mathfrak{D}^{-1}_{E_\mathfrak{p}/\mathbb{Q}_2} N_2^\# / N_2$ takes the same value for $(1, 0)$ and $(0, 1)$. Moreover, multiplication by ζ corresponds to exchanging those two generators, meaning that D_f which has order 2, is represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on this summand of D_L. The previous argument can be generalized for every direct summand of $\mathfrak{D}^{-1}_{E_\mathfrak{p}/\mathbb{Q}_2} N_2^\# / N_2$ when $n_2 > 1$.

8.1.3. Cyclotomic hermitian Miranda–Morrison theory

In [BC23, Proposition 2.20] the authors determine whenever two isometries $f, g \in O(M)$ of an even unimodular $\mathbb{Z}$-lattice M, with minimal polynomial $\Phi_1\Phi_p$ for some odd prime number p, are conjugate. They show, thanks to strong approximation, that for an indefinite p-elementary Φ_p-lattice (L, f) with D_f trivial, the discriminant representation $O(L, f) \to O(D_L)$ is surjective. In this section, we show that their result holds in greater generality using a refinement of their argument which one can find in [BH23, §6], based on a hermitian analog of Miranda–Morrison theory (Remark 1.53). We actually aim to prove the following theorem.

Theorem 8.21. *Let $m = p^k \geq 3$ for some prime number p and let (L, f) be an even p-elementary Φ_{p^k}-lattice. Assume that D_f has order at most 2. If L has rank $\varphi(p^k)$ or is indefinite then*

$$\pi_{L,f} : O(L, f) \to O(D_L, D_f)$$

is surjective.

Before going through the proof of this theorem, let us remark the following easy cases.

Lemma 8.22. *Let m, L, f be as in the statement of Theorem 8.21. If L is unimodular or has rank $\varphi(m)$, then $\pi_{L,f} : O(L, f) \to O(D_L, D_f)$ is surjective.*

Proof. If L is unimodular, it is necessarily true. Now let us assume that L is not unimodular. By Proposition 8.12, either D_f is trivial and $D_L \cong \mathbb{Z}/p\mathbb{Z}$ as abelian groups, or $p = 2$ and $D_L \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. In the former case, $O(D_L, D_f) = O(D_L)$ is generated by $-\mathrm{id}_{D_L} = \pi_{L,f}(-\mathrm{id}_L)$. In the latter case, when $p = 2$ and D_f has order 2, the isometry D_f generates $O(D_L)$ and it acts by exchanging the generators of both copies of $\mathbb{Z}/2\mathbb{Z}$ in D_L (Remark 8.20). Again, we obtain that $O(D_L, D_f) = O(D_L)$ and it is generated by $\pi_{L,f}(f)$. In both cases $\pi_{L,f}$ is surjective. □

Now let $m = p^k$ and let (L, b, f) be an even p-elementary Φ_m-lattice as in the statement of Theorem 8.21. Let us assume now that (L, b) is indefinite, not unimodular and of rank larger than $\varphi(p^k)$. We let (L, h) be the associated hermitian structure and we denote again $\zeta := \zeta_m$, $E := \mathbb{Q}(\zeta)$, $K := \mathbb{Q}(\zeta + \zeta^{-1})$ and $\pi := 1 - \zeta$. Moreover, we let $\mathfrak{P}$ and $\mathfrak{p}$ be the unique prime ideals of $\mathcal{O}_E$ and $\mathcal{O}_K$ respectively lying above $p\mathbb{Z}$. We recall that $\mathcal{O}_E := \mathbb{Z}[\zeta]$ and $\mathcal{O}_K := \mathbb{Z}[\zeta + \zeta^{-1}]$ are respective maximal orders of E and K. We denote again by $\iota \in \mathrm{Gal}(E/K) \cong \mathrm{Gal}(E_\mathfrak{p}/K_\mathfrak{p})$

a generator. For the reader's convenience, in order to be introduced to the hermitian Miranda–Morrison theory as described in [BH23, §6], we fix some further notations (see [BH23, §6.5] for more details). We denote by $\mathcal{O}_{\mathbb{A}_K}$ the ring of integral finite adeles of K. We let $U^\#(L,h)$ be the kernel of the map $U(L,h) = O(L,b,f) \to O(D_L, D_f)$, and we define $U^\#(L_\mathfrak{q})$ similarly for each finite place $\mathfrak{q}$ of K. For $i \geq 1$, we define the following groups:

- $\mathcal{F}(E) := \{e \in \mathcal{O}_E^\times :\ e\iota(e) = 1\}$;
- $\mathcal{F}(E_\mathfrak{p}) := \{e \in \mathcal{O}_{E_\mathfrak{p}}^\times :\ e\iota(e) = 1\}$;
- $\mathcal{F}(L,h) := \det(U((L,h) \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathbb{A}_K}))$;
- $\mathcal{F}^\#(L,h) := \det(U^\#((L,h) \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathbb{A}_K}))$;
- $\mathcal{F}_i(E) := \{e \in \mathcal{F}(E) :\ e \equiv 1 \mod \mathfrak{P}^i\}$;
- $\mathcal{F}_i(E_\mathfrak{p}) := \{e \in \mathcal{F}(E_\mathfrak{p}) :\ e \equiv 1 \mod \mathfrak{P}^i\}$;
- $\mathcal{F}(L_\mathfrak{p}) := \det(U(L_\mathfrak{p})) \subseteq \mathcal{F}(E_\mathfrak{p})$;
- $\mathcal{F}^\#(L_\mathfrak{p}) := \det(U^\#(L_\mathfrak{p}))$.

In what follows, by abuse of notation, we sometimes identify $\mathcal{F}(E)$ with its image along the map $E \to \prod_{\mathfrak{q} \text{ finite}} E_\mathfrak{q}$, $e \mapsto (e)_\mathfrak{q}$. In [BH23, Theorem 6.15], the authors prove that the following sequence is exact

$$O(L,f) \xrightarrow{\pi_{L,f}} O(D_L, D_f) \xrightarrow{\delta} \mathcal{F}(L,h)/(\mathcal{F}(E) \cap \mathcal{F}(L,h)) \cdot \mathcal{F}^\#(L,h) \to 1, \tag{25}$$

where δ is induced by the determinant morphism $\det\colon U((L,h) \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathbb{A}_K}) \to \prod_{\mathfrak{q} \text{ finite}} E_\mathfrak{q}$. In order to show that $O(L,f) \to O(D_L, D_f)$ is surjective, it is therefore equivalent to show that δ is trivial. Since by assumption (L,b) is p-elementary, and $\mathfrak{p}$ is the unique prime $\mathcal{O}_K$-ideal lying above $p\mathbb{Z}$, [BH23, Remark 6.16] gives us the following isomorphism of groups

$$\mathcal{F}(L,h)/(\mathcal{F}(E) \cap \mathcal{F}(L,h)) \cdot \mathcal{F}^\#(L,h) \cong \mathcal{F}(L_\mathfrak{p})/(\mathcal{F}(E) \cap \mathcal{F}(L_\mathfrak{p})) \cdot \mathcal{F}^\#(L_\mathfrak{p}) =: I_\mathfrak{p}.$$

Hence, the surjectivity of $\pi_{L,f}$ is equivalent to $I_\mathfrak{p}$ being trivial, by exactness of Equation (25). Following the idea of [BH23, §6.8], we actually show in what follows that the quotient

$$Q_\mathfrak{p} := \mathcal{F}(E_\mathfrak{p})/\mathcal{F}(E) \cdot \mathcal{F}^\#(L_\mathfrak{p})$$

is trivial (as long as (L,h) satisfies the assumptions of Theorem 8.21). Since $I_\mathfrak{p}$ can be identified with a subgroup of $Q_\mathfrak{p}$, we then conclude.

According to [BH23, Theorem 6.25], there exists an integer $j_0 \geq 0$ such that $\mathcal{F}^\#(L_\mathfrak{p}) = \mathcal{F}_{j_0}(E_\mathfrak{p})$. Note that for a fixed $j \geq 0$, the quotient $\mathcal{F}(E_\mathfrak{p})/\mathcal{F}_j(E_\mathfrak{p})$ is naturally isomorphic [BH23, §6.8] to the kernel of the norm map

$$N_{K_\mathfrak{p}}^{E_\mathfrak{p}} :\ \mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}^j) \to \mathcal{O}_{K_\mathfrak{p}}^\times/(1+\mathfrak{p}^{\phi_p(j)}), \tag{26}$$

where we define $\phi_p(j) := \frac{j+1}{2}$ if j is odd, and $\phi_p(j) := \frac{j+(2\mathrm{val}_\mathfrak{P}(D_{E/K})-2)}{2}$ if j is even. We recall that by Proposition 3.2, we know that $e := \mathrm{val}_\mathfrak{P}(\mathfrak{D}_{E/K}) = \gcd(2,p)$.

Lemma 8.23 ([Ser04, Chapter V, §3, Corollaries 1, 2 & 5])**.** *For all $j \geq 0$, the norm map in Equation (26) induces*

$$N_j\colon (1+\mathfrak{P}^j)/(1+\mathfrak{P}^{j+1}) \to (1+\mathfrak{p}^{\phi_p(j)})/(1+\mathfrak{p}^{\phi_p(j)+1})$$

which is bijective except if $j = e-1$ in which case there is an exact sequence

$$1 \to C_2 \to (1+\mathfrak{P}^{e-1})/(1+\mathfrak{P}^e) \to (1+\mathfrak{p}^{e-1})/(1+\mathfrak{p}^e) \to C_2 \to 1.$$

Note that we define $1+\mathfrak{P}^0 := \mathcal{O}_{E_\mathfrak{p}}^\times$, and similarly $1+\mathfrak{p}^0 := \mathcal{O}_{K_\mathfrak{p}}^\times$.

Remark 8.24 ([Ser04, Chapter IV, §2, Proposition 6])**.** As abelian groups, $\mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}) \cong \mathbb{F}_p^{\times}$ and for all $j \geq 1$, $(1+\mathfrak{P}^j)/(1+\mathfrak{P}^{j+1}) \cong \mathbb{F}_p$.

The upshot now is that $Q_{\mathfrak{p}}$ is trivial if and only if the sequence of abelian groups

$$\mathcal{F}(E) \to \mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}^j) \to \mathcal{O}_{K_{\mathfrak{p}}}^{\times}/(1+\mathfrak{p}^{\phi_p(j)}) \tag{27}$$

is exact for $j = j_0$. Therefore if we can show that Equation (27) is exact for given values of $m = p^k$ and $j = j_0$, then we would have proven Theorem 8.21.

Since we work with finite groups, our plan is to show that Equation (27) is exact for given m and j by showing that the image of the first map and the kernel of the second map have the same order. Note that for all values of m and j, the kernel $\mathcal{F}_j(E)$ of the first map is known thanks to Corollary 3.12, and so we know the size of its image. In order to pursue now, we need to determine from the assumptions of Theorem 8.21 which values for j_0 we have to consider.

According to [BH23, Theorem 6.25], the value of $j_0 \geq 0$ for which $\mathcal{F}^{\#}(L_{\mathfrak{p}}) = \mathcal{F}_{j_0}(E_{\mathfrak{p}})$ is given by $j_0 = 2n + \alpha$ where $\alpha := \mathrm{val}_{\mathfrak{P}}(\mathfrak{D}_{E/\mathbb{Q}}) = p^{k-1}(pk-k-1)$ and $n := \mathrm{val}_{\mathfrak{p}}(\mathfrak{n}(L,h))$. Since by assumption the trace form (L,b) of (L,h) is even, we have that $i \leq 2n \leq i+e$ where $i := \mathrm{val}_{\mathfrak{P}}(\mathfrak{s}(L))$ and $e = \mathrm{val}_{\mathfrak{P}}(\mathfrak{D}_{E/K})$ [BH23, §6.6]. According to Proposition 8.9, since we assumed that D_f has order at most 2, we have that $-\alpha \leq i \leq \gcd(2,p) - \alpha$. In particular, in the setting of Theorem 8.21, we have either that p is odd and $0 \leq j_0 \leq 2$ or $p = 2$ and $0 \leq j_0 \leq 4$.

Lemma 8.25. *Recall that $m = p^k$ is a prime power and let $j \geq 0$ be such that*

(1) $0 \leq j \leq 2$ *if p is odd;*

(2) $0 \leq j \leq 4$ *if $p = 2$.*

Then the sequence in Equation (27) is exact.

Proof. First of all, note that if $j = 0$, then $\mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}^j)$ is trivial so there is nothing to prove.

(1) Now suppose that p is odd and let $1 \leq j \leq 2$. By Corollary 3.12, we know that the image of $\mathcal{F}(E) \to \mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}^j)$ has order 2 for $j = 1$ and $2p$ for $j = 2$.

   (a) For $j = 1$, we have $\phi_p(1) = 1$ and according to Lemma 8.23, the kernel of

   $$N_0 : \mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}) \to \mathcal{O}_{K_{\mathfrak{p}}}^{\times}/(1+\mathfrak{p})$$

   has order 2 and thus Equation (27) is exact.

   (b) For $j = 2$, remark that $\phi_p(2) = 1$ too, and that the norm map $\mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}^2) \to \mathcal{O}_{K_{\mathfrak{p}}}^{\times}/(1+\mathfrak{p})$ factors as

   $$\mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}^2) \to \mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}) \xrightarrow{N_0} \mathcal{O}_{K_{\mathfrak{p}}}^{\times}/(1+\mathfrak{p})$$

   where the first arrow is surjective and with kernel $(1+\mathfrak{P})/(1+\mathfrak{P}^2)$. Hence, by Lemma 8.23 and Remark 8.24, we have that the kernel of $\mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}^2) \to \mathcal{O}_{K_{\mathfrak{p}}}^{\times}/(1+\mathfrak{p})$ has order $2p$.

(2) Let us assume now that $p = 2$ and let $1 \leq j \leq 4$. Again, by Corollary 3.12, we know that the image of $\mathcal{F}(E) \to \mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}^j)$ is trivial for $j = 1$, it has order 2 for $j = 2$ and it has order 4 for $j = 3, 4$.

   (a) For $j = 1$, we have that $\phi_2(1) = 1$ and by Lemma 8.23, when $p = 2$, the map $N_0 : \mathcal{O}_{E_{\mathfrak{p}}}^{\times}/(1+\mathfrak{P}) \to \mathcal{O}_{K_{\mathfrak{p}}}^{\times}/(1+\mathfrak{p})$ is an isomorphism.

(b) For $j=2$, we have that $\phi_2(2)=2$ and there is a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & (1+\mathfrak{P})/(1+\mathfrak{P}^2) \arrow[r] \arrow[d, "N_1"] & \mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}^2) \arrow[r] \arrow[d] & \mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}) \arrow[r] \arrow[d, "N_0", "\cong"'] & 1 \\
1 \arrow[r] & (1+\mathfrak{p})/(1+\mathfrak{p}^2) \arrow[r] & \mathcal{O}_{K_\mathfrak{p}}^\times/(1+\mathfrak{p}^2) \arrow[r] & \mathcal{O}_{K_\mathfrak{p}}^\times/(1+\mathfrak{p}) \arrow[r] & 1
\end{tikzcd}.$$

According to Lemma 8.23, the vertical map N_0 is an isomorphism and $\ker N_1$ has order 2. Therefore, the kernel of the middle vertical map is isomorphic to $\ker N_1$ and it has order 2.

(c) For the case $j=3$, we have again that $\phi_2(3)=2$ and following the idea for the odd prime order case (1)(a), we have that the norm map $\mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}^3)\to\mathcal{O}_{K_\mathfrak{p}}^\times/(1+\mathfrak{p}^2)$ factors as

$$\mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}^3)\to\mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}^2)\to\mathcal{O}_{K_\mathfrak{p}}^\times/(1+\mathfrak{p}^2).$$

By similar arguments, we deduce that the kernel of $\mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}^3)\to\mathcal{O}_{K_\mathfrak{p}}^\times/(1+\mathfrak{p}^2)$ has order 4.

(d) Finally, when $j=4$ we have that $\phi_2(4)=3$, and there is a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & (1+\mathfrak{P}^3)/(1+\mathfrak{P}^4) \arrow[r] \arrow[d, "N_3", "\cong"'] & \mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}^4) \arrow[r] \arrow[d] & \mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}^3) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & (1+\mathfrak{p}^2)/(1+\mathfrak{p}^3) \arrow[r] & \mathcal{O}_{K_\mathfrak{p}}^\times/(1+\mathfrak{p}^3) \arrow[r] & \mathcal{O}_{K_\mathfrak{p}}^\times/(1+\mathfrak{p}^2) \arrow[r] & 1
\end{tikzcd}.$$

According to Lemma 8.23, the vertical map N_3 is an isomorphism and by the Snake lemma, we have that the kernels of the two other vertical maps are isomorphic. Hence, the kernel of $\mathcal{O}_{E_\mathfrak{p}}^\times/(1+\mathfrak{P}^4)\to\mathcal{O}_{K_\mathfrak{p}}^\times/(1+\mathfrak{p}^3)$ has order 4.

Therefore, for $p=2$ and $1\le j\le 4$, the sequence in Equation (27) is exact, and that concludes the proof. $\square$

Hence, Lemma 8.25 together with the prior discussions proves Theorem 8.21.

8.1.4. Conjugacy classes of isometries of even unimodular $\mathbb{Z}$-lattices

Given an isometry g of an even unimodular $\mathbb{Z}$-lattice M with minimal polynomial $\Phi_1\Phi_m$, or $\Phi_1\Phi_2\Phi_m$, for some integer $m\ge 3$, we have determined necessary and sufficient conditions for the existence of the Φ_m-kernel sublattice of (M,g) (Section 8.1.2). We would like now to find sufficient conditions for the existence of such isometries, and classify them up to conjugacy.

8.1.4.1. Existence of isometries of given type

Let $m\ge 3$ and let $l_+,l_-\ge 0$ be such that $l_++l_-\ge 1$. We first cover the existence of even unimodular $\Phi_1\Phi_m^*$-lattices of signatures (l_+,l_-). Note that this is actually already understood [BF24, Theorem 4.5]. Nonetheless, we provide alternative statements which include local invariants of the associated invariant and coinvariant sublattices. In the case where m is not a prime power, we have already seen in Proposition 8.1 that the two latter lattices are unimodular.

Theorem 8.26. *Let $m \geq 3$ be composite. Let $l_+, l_-, s_+, s_- \geq 0$ be such that $l_+ + l_- > s_+ + s_- \geq 1$. There exists an even unimodular $\Phi_1\Phi_m^*$-lattice (M, g) where $M \in \mathrm{II}_{(l_+,l_-)}$ such that $M_g \in \mathrm{II}_{(s_+,s_-)}$ if and only if there exists an integer $d \geq 1$ such that*

(1) $l_\pm \geq s_\pm$;

(2) $l_+ - l_- \equiv 0 \mod 8$;

(3) $s_\pm \equiv 0 \mod 2$;

(4) $s_+ + s_- = d\varphi(m)$;

(5) $s_+ - s_- \equiv 0 \mod 8$;

(6) *if $m = 2p^k$ is twice an odd prime power, then $d \equiv 0 \mod 2$.*

Proof. Follows from [BF24, Theorem 4.5]: in particular condition (6) ensures that $\Phi_m^d(-1)$ is a square and, conditions (4) and (5) together give that $d\varphi(m) = \deg(\Phi_m^d) \equiv 2s_+ \mod 8$. Note that if conditions (1)-(6) hold, we have that the genera $\mathrm{II}_{(l_+,l_-)}$, $\mathrm{II}_{(s_+,s_-)}$ and $\mathrm{II}_{(l_+-s_+,l_--s_-)}$ are nonempty, and there exists a Φ_m-lattice (L, f) in $\mathrm{II}_{(s_+,s_-)}$ by Proposition 8.17. Moreover, for any representative F of an isometry class in $\mathrm{II}_{(l_+-s_+,l_--s_-)}$, we have that $(F, \mathrm{id}_F) \oplus (L, f)$ is an even unimodular $\Phi_1\Phi_m^*$-lattice in the genus $\mathrm{II}_{(l_+,l_-)}$ whose coinvariant sublattice is (L, f). □

For the case where $m = p^k$ is a prime power, we follow the same proof as [BC23, Theorem 1.1]. We recall again that according to Proposition 8.1, the invariant and coinvariant sublattices of (M, g) are p-elementary.

Theorem 8.27. *Let $m = p^k \geq 3$ for some prime number p and some positive integer $k \geq \gcd(2, p)$. Let $l_+, l_-, s_+, s_-, n_1 \geq 0$ be such that $l_+ + l_- \geq s_+ + s_- + n_1 \geq n_1 + 1$. There exists an even unimodular $\Phi_1\Phi_m^*$-lattice (M, g) where $M \in \mathrm{II}_{(l_+,l_-)}$ such that M_g is p-elementary of absolute determinant p^{n_1} and signatures (s_+, s_-) if and only if there exists an even integer $n_0 \geq 0$ such that*

(1) $l_\pm \geq s_\pm$;

(2) $l_+ - l_- \equiv 0 \mod 8$;

(3) $s_\pm \equiv 0 \mod 2$;

(4) $s_+ + s_- = \varphi(p^k)(n_0 + n_1)$;

(5) $s_+ - s_- \equiv 0 \mod 8$ *if* $n_1 = 0$ *or* $n_1 = (l_+ - s_+) + (l_- - s_-)$;

(6) n_1 *is even if* $p = 2$.

Proof. By Theorem 1.46, Theorem 1.49 and Proposition 8.18 we have that if such a pair (M, g) exists, then conditions (1)-(6) hold. Conversely, suppose that conditions (1)-(6) hold. In particular, the genus $\mathrm{II}_{(l_+,l_-)}$ is nonempty and there exists an even Φ_{p^k}-lattice (L, f) of absolute determinant p^{n_1}, signatures (s_+, s_-) and $\delta_L = 0$ if $p = 2$, such that D_f is trivial. Moreover, according to Proposition 2.27 and Theorem 1.49, we have that conditions (1) and (5) imply that there exists an even $\mathbb{Z}$-lattice with discriminant form $D_L(-1)$ and signatures $(l_+ - s_+, l_- - s_-)$. Hence for any such $\mathbb{Z}$-lattice F, the equivariant gluing condition Equation (EGC) tells us that any glue map $\gamma\colon\ D_F \to D_L$ is (id_F, f)-equivariant and gives rise to an equivariant primitive extension $(F, \mathrm{id}_F) \oplus (L, f) \leq (M, g)$. By construction, the lattice with isometry (M, g) satisfies the first part of the statement. □

Remark 8.28. Note that as for Theorem 8.26, parts of the conditions in Theorem 8.27 can be recovered from [BF24, Theorem 4.5]. In particular, since for any power $m \geq 4$ of 2 we have that $\Phi_m(-1) = 2$, condition (6) of the previous statement is equivalent to $\Phi_m^{n_0+n_1}(-1) = 2^{n_0+n_1}$ being a square.

We now aim to prove similar results in the case where $m = 2p^k$ is twice a nontrivial prime power for some prime number p, the isometry $g \in O(M)$ has minimal polynomial $\Phi_1\Phi_2\Phi_m$ and the kernel sublattice M^{-g} is isometric to $\langle 2p\rangle$ (Remark 8.52). Let us first assume that p is odd. We prove the following lemma.

Lemma 8.29. *Let p be an odd prime number and let $m = 2p^k$ for some $k \geq 1$. Let (M, g) be an even unimodular $\Phi_1\Phi_2\Phi_m^*$-lattice such that $K := M^{-g} \simeq \langle 2p\rangle$. Then there exist 3 positive even integers s_+, s_-, n_0 satisfying*

1. $s_+ + s_- \leq \operatorname{rank}_{\mathbb{Z}}(M) - 2$;
2. $s_+ + s_- = \varphi(p^k)(n_0 + 1)$;
3. $s_+ - s_- \equiv 2\left(\frac{-2}{p}\right) - 1 - p \mod 8$;

and such that the Φ_m-kernel sublattice (L, f) of (M, g) lies in the genus $\mathrm{II}_{(s_+,s_-)}p^{\left(\frac{-2}{p}\right)}$ with $D_f = -\mathrm{id}_{D_L}$.

Proof. As abelian groups, $D_K \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and the torsion quadratic form on D_K is $\begin{pmatrix} p/2 & 0 \\ 0 & 2/p \end{pmatrix}$. Note that $-\mathrm{id}_K$ induces the identity on the 2-primary part of D_K, and $-\mathrm{id}$ on the p-primary part. Moreover, we have that K lies in the genus $\mathrm{II}_{(1,0)}2^{\epsilon}_{\delta}p^{\epsilon}$ where $\delta \in \{1, 3, 5, 7\}$ is congruent to p modulo 8, and $\epsilon = \left(\frac{2}{p}\right)$. Let $(L, f) := (M^{\Phi_m(g)}, g_m)$ be the Φ_m-kernel sublattice of (M, g). By the proof of Proposition 8.2 and the equivariant gluing condition Equation (EGC), we have that L has absolute determinant p, the discriminant group D_L of L is equipped with the discriminant form $\left(-2/p\right)$ and $D_f = -\mathrm{id}_{D_L}$. Let us denote by (s_+, s_-) the signatures of L: note that since g admits at least one fixed vector in M and M^{-g} has rank 1, we must have $s_+ + s_- \leq \operatorname{rank}_{\mathbb{Z}}(M) - 2$. Moreover, we have that L lies in the genus $\mathrm{II}_{(s_+,s_-)}p^{\left(\frac{-1}{p}\right)\epsilon}$. Finally, since $s_+ + s_- \geq \varphi(p^k) > 1$, the former genus is nonempty if and only if $s_+ - s_- \equiv 2\left(\frac{-2}{p}\right) - 1 - p$ mod 8. Note that (2) follows from Proposition 8.12. ☐

We keep the notations of Lemma 8.29 and its proof, and let us denote further $F := M^g$. If we denote by (l_+, l_-) the signatures of M, we have that $i_+ := l_+ - s_+ - 1$ and $i_- := l_- - s_-$ define the signatures of F. By similar arguments as in the proof of Lemma 8.29, we have that D_F is equipped with the torsion quadratic form $\left(-p/2\right)$. We then deduce that F lies in the genus $\mathrm{II}_{(i_+,i_-)}2^{\epsilon}_{8-\delta}$, where we recall that $\delta \in \{1, 3, 5, 7\}$ is congruent to p modulo 8, and $\epsilon = \left(\frac{2}{p}\right)$. Now if $i_+ + i_- > 1$, then the genus $\mathrm{II}_{(i_+,i_-)}2^{\epsilon}_{8-\delta}$ is nonempty if and only if

$$s_- - s_+ - 1 \equiv i_+ - i_- \equiv -p + 2 - 2\left(\frac{2}{p}\right) \mod 8$$

(Theorem 1.49). Put together, we have that the genera of F and L are simultaneously nonempty if and only if

$$2\left(\frac{-2}{p}\right) - 1 - p \equiv 2\left(\frac{2}{p}\right) - 3 + p \mod 8$$

and one can easily check that the latter is always true (it follows from the fact that for any odd prime number p, $\left(\frac{-1}{p}\right) \equiv p \mod 4$). In the case where F has rank 1, either $F \in \mathrm{II}_{(1,0)}2_1^{+1} = \mathrm{II}_{(1,0)}2_5^{-1}$ or $F \in \mathrm{II}_{(0,1)}2_7^{+1} = \mathrm{II}_{(0,1)}2_3^{-1}$. In the former case, we must have that $p \equiv 3, 7 \mod 8$, and $p \equiv 1, 5 \mod 8$ in the latter case. By straightforward computations, we see that the existence conditions for F and (L, f) are still equivalent in both of the previous cases. Therefore, if p has the appropriate residue class modulo 8 when $l_+ - s_+ + l_- - s_- = 2$, we have that the existence of (L, f) ensures the existence of F.

Theorem 8.30. *Let $m = 2p^k \geq 3$ for some odd prime number p and some positive integer $k \geq 1$. Let $l_+, l_-, s_+, s_- \geq 0$ be such that $l_+ \geq 1$ and $l_+ + l_- \geq s_+ + s_- + 2 \geq 3$. There exists an even unimodular $\Phi_1\Phi_2\Phi_m^*$-lattice (M, g) where $M \in \mathrm{II}_{(l_+,l_-)}$ such that $M^{-g} \simeq \langle 2p\rangle$ and $M^{\Phi_m(g)}$ is p-elementary of absolute determinant p and signatures (s_+, s_-) if and only if there exists an even integer $n_0 \geq 0$ such that*

(1) $l_\pm \geq s_\pm$;

(2) $l_+ - l_- \equiv 0 \mod 8$;

(3) $s_\pm \equiv 0 \mod 2$;

(4) $s_+ + s_- = \varphi(p^k)(n_0 + 1)$;

(5) $s_+ - s_- \equiv 2\left(\frac{-2}{p}\right) - 1 - p \mod 8$;

(6) $p \equiv 1 \mod 4$ *if* $(l_+ - s_+,\ l_- - s_-) = (1, 1)$ *and* $p \equiv 3 \mod 4$ *if* $(l_+ - s_+,\ l_- - s_-) = (2, 0)$.

Proof. Let (M, g) be as in the first part of the statement. Then according to Theorem 1.46, condition (2) holds. Now, Theorem 1.49 and Proposition 8.18, together with Remark 8.16 for the case where p is odd, give us that conditions (3) and (4) holds too. Conditions (1), (5) and (6) follow directly from Lemma 8.29 and the follow-up discussions. Now suppose that conditions (1)-(6) hold. In particular, the genus $\mathrm{II}_{(l_+,l_-)}$ is nonempty, and there exists a Φ_{2p^k}-lattice (L, f) of absolute determinant p and signatures (s_+, s_-) such that $D_f = -\mathrm{id}_{D_L}$. By conditions (4) and (5), the torsion quadratic form on D_L is $\left(-2/p\right)$. Moreover, by conditions (5) and (6), there exists a $\mathbb{Z}$-lattice F with signatures $(l_+ - s_+ - 1, l_- - s_-)$ and absolute determinant 2, such that the torsion quadratic form on D_F is $\left(-p/2\right)$. Hence, if we let $K := \langle 2p\rangle$, since $D_{F\oplus L} \simeq D_F \oplus D_L \simeq D_K(-1)$ as torsion quadratic modules, we obtain that $F \oplus K \oplus L$ has an overlattice M in $\mathrm{II}_{(l_+,l_-)}$. Moreover, by the equivariant gluing condition Equation (EGC), since $\mathrm{id}_F \oplus f$ and $-\mathrm{id}_K$ agree along any glue map $D_{F\oplus L} \to D_K$, we have that $\mathrm{id}_f \oplus (-\mathrm{id}_K) \oplus f \in O(F \oplus K \oplus L)$ extends to an isometry $g \in O(M)$ with minimal polynomial $\Phi_1\Phi_2\Phi_m$, such that $M^{-g} = K$ and $(M^{\Phi_m(g)}, g_m) = (L, f)$. □

Let us conclude with the case of powers of 2. Let $m = 2^k$ for some $k \geq 2$, let (M, g) be an even unimodular $\Phi_1\Phi_2\Phi_{2^k}^*$-lattice such that $K := M^{-g} \simeq \langle 4\rangle$. Let us denote moreover (l_+, l_-) the signatures of M, and let $F := M^g$ of signatures (i_+, i_-). The Φ_m-kernel sublattice (L, f) of (M, g) has signatures $(s_+, s_-) := (l_+ - i_+ - 1, l_- - i_-)$.

By Proposition 8.19, there exist three integers $n_0, n_1, n_2 \geq 0$ with $n_0, n_1 \in 2\mathbb{Z}$ and $n_2 \neq 0$ such that $\mathrm{rank}_{\mathbb{Z}}(L) = \varphi(2^k)(n_0 + n_1 + n_2)$. According to Proposition 8.2, we know moreover that L is 2-elementary, the $\mathbb{Z}$-lattice F is 4-elementary, and F and K glue along elementary abelian 2-groups $H_F \leq D_F$ and $H_K \leq D_K$ (Figure 4). Remark that as abelian groups, $D_K \cong \mathbb{Z}/4\mathbb{Z}$ and it is equipped with the torsion quadratic form $\left(1/4\right)$. Hence, $H_K \cong \mathbb{Z}/2\mathbb{Z}$ and the torsion quadratic form on H_K is $\left(1\right)$. By [BH23, Proposition 4.10], the glue map $H_F \to H_K$ maps isomorphically

$2D_F$ to $2D_K = H_K$. Therefore, there exists $n \geq 0$ such that as abelian groups $D_F \cong D_{F,4} \oplus D_{F,2}$ where $D_{F,4} \cong \mathbb{Z}/4\mathbb{Z}$ and $D_{F,2} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus n}$, and $H_F \leq D_{F,4}$ is the subgroup generated by twice a generator. Seeing $D_{F,4} \cong \mathbb{Z}/4\mathbb{Z}$ as the discriminant group of the 4-modular Jordan constituent of F_2, we know that there exists $\alpha \in \{1,3,5,7\}$ such that the torsion quadratic form on $D_{F,4} \leq D_F$ is $\begin{pmatrix}\alpha/4\end{pmatrix}$ (Corollary 1.22). Now, let us denote $N := L_M^\perp$ which is a primitive extension of $F \oplus K$ in M.

Claim 8.31. *As abelian groups $D_N \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus D_{F,2}$, and the torsion quadratic form on the first two summands is*

$$\begin{pmatrix}(\alpha+1)/4 & (\alpha+3)/4 \\ (\alpha+3)/4 & (\alpha+1)/4\end{pmatrix}. \tag{28}$$

Moreover, $\mathrm{id}_F \oplus (-\mathrm{id}_K)$ extends to an isometry h of N such that D_h acts by $\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$ on the first two summands of D_N and by the identity on $D_{F,2}$. In particular, $(n_1, n_2) = (n, 1)$ and the torsion quadratic form on $D_{F,2}$ takes integer values.

Proof of Claim 8.31. Let $\gamma\colon H_F \to H_K$ be the glue map of the primitive extension $F \oplus K \leq N$, and let $\Gamma \leq D_{F,4} \oplus D_K$ be the graph of γ. As abelian groups we have that $D_N \cong \Gamma^\perp/\Gamma \oplus D_{F,2}$. Now, by straightforward computations, we obtain that $\Gamma^\perp/\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ is generated by the classes $(1,1)+\Gamma$ and $(1,3)+\Gamma$. Moreover, since the torsion quadratic form on $D_{F,4} \oplus D_K$ is $\begin{pmatrix}\alpha/4 & 0 \\ 0 & 1/4\end{pmatrix}$, it follows that the torsion quadratic form on $\Gamma^\perp/\Gamma$ is as stated. We remark that $\mathrm{id}_F \oplus (-\mathrm{id}_K)$ extends to an isometry $h \in O(N)$ such that D_h acts by exchanging the two generators on $\Gamma^\perp/\Gamma$, and D_h is trivial on $D_{F,2}$. For the final statements, since N and L glue equivarantly along their respective discriminant groups, we have that $n_1 = n$ and $n_2 = 1$ by the equivariant gluing condition Equation (EGC) and Remark 8.20. The fact that the torsion quadratic form on $D_{F,2}$ takes integer values follows from the proofs of Lemmas 8.11 and 8.13. □

Let $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ be such that $F \in \mathrm{II}_{(i_+,i_-)}2_{\mathrm{II}}^{\epsilon_1 n_1}4_\alpha^{\epsilon_2}$. By Proposition 1.50, we know that n_1 must indeed be even, $\epsilon_2 = (\frac{\alpha}{2})$ and $\epsilon_1 = 1$ if $n_1 = 0$. Moreover, we have that $i_+ + i_- \leq n_1 + 1$ with equality only if $\epsilon_1 = \epsilon_2$, and finally, that

$$i_+ - i_- \equiv \alpha + 2 - 2\epsilon_1 \mod 8. \tag{29}$$

We aim to show that the genus of F uniquely determines the one of L (similarly to what has been done in Theorem 8.30). We explain our argument for the case $\alpha = 1$, the other cases follow similarly.

If $\alpha = 1$, then the torsion quadratic form Equation (28) is given by $\begin{pmatrix}1/2 & 0 \\ 0 & 1/2\end{pmatrix}$. Hence, according to Theorem 1.49 and Equation (29), we have that $N \in \mathrm{II}_{(i_++1,i_-)}2_2^{\epsilon_1(n_1+2)}$. Since N and L glue to a unimodular $\mathbb{Z}$-lattice, we have that $L \in \mathrm{II}_{(s_+,s_-)}2_6^{\epsilon_L(n_1+2)}$ where $\epsilon_L \in \{\pm 1\}$ satisfies that $s_+ - s_- \equiv -2\epsilon_L \mod 8$ and $\epsilon_L = 1$ for $s_+ + s_- = n_1 + 2$ (Theorem 1.49). Since $(s_+, s_-) = (l_+ - i_+ - 1, l_- - i_-)$, Equation (29) tells us moreover that $\epsilon_L = \epsilon_1$. Analogously, one can check that for all $\alpha \in \{1,3,5,7\}$, the genus of F determines the ones of N and L, which are given in Table 8.

Since the genus of L determines the one of N (because they glue to a unimodular lattice), it is not hard to see that the genus of L actually also determines the one of F, from the description

Table 8: Genera of F, N and L depending on $\alpha \in \{1,3,5,7\}$

α	Equation (28)	$g(F)$	$g(N)$	$g(L)$
1	$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$	$\mathrm{II}_{(i_+,i_-)}2_{\mathrm{II}}^{\epsilon_1 n_1}4_1^1$	$\mathrm{II}_{(i_++1,i_-)}2_2^{\epsilon_1(n_1+2)}$	$\mathrm{II}_{(s_+,s_-)}2_6^{\epsilon_1(n_1+2)}$
3	$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$	$\mathrm{II}_{(i_+,i_-)}2_{\mathrm{II}}^{\epsilon_1 n_1}4_3^{-1}$	$\mathrm{II}_{(i_++1,i_-)}2_{\mathrm{II}}^{-\epsilon_1(n_1+2)}$	$\mathrm{II}_{(s_+,s_-)}2_{\mathrm{II}}^{-\epsilon_1(n_1+2)}$
5	$\begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$	$\mathrm{II}_{(i_+,i_-)}2_{\mathrm{II}}^{\epsilon_1 n_1}4_5^{-1}$	$\mathrm{II}_{(i_++1,i_-)}2_6^{\epsilon_1(n_1+2)}$	$\mathrm{II}_{(s_+,s_-)}2_2^{\epsilon_1(n_1+2)}$
7	$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$	$\mathrm{II}_{(i_+,i_-)}2_{\mathrm{II}}^{\epsilon_1 n_1}4_7^1$	$\mathrm{II}_{(i_++1,i_-)}2_{\mathrm{II}}^{\epsilon_1(n_1+2)}$	$\mathrm{II}_{(s_+,s_-)}2_{\mathrm{II}}^{\epsilon_1(n_1+2)}$

given in Claim 8.31.

Remark 8.32. Note that if $i_+ + i_- = n_1 + 1$, then $\epsilon_1 = 1$ by Theorem 1.49. In this case, Proposition 1.50 gives that $\epsilon_2 = \epsilon_1 = 1$. Since $\epsilon_2 = \left(\frac{\alpha}{2}\right)$, we therefore know that $i_+ + i_- = n_1 + 1$ only if $\alpha \in \{1,7\}$. Moreover, from the symbol of the genus of L, we know that $\delta_L = 1$ for $\alpha = 1,5$ and $\delta_L = 0$ otherwise. Therefore, Proposition 8.19 tells us that $\alpha \equiv 1 \mod 4$ if and only if $m = 4$.

Theorem 8.33. *Let $m = 2^k \geq 3$ for some positive integer $k \geq 2$. Let $l_+, l_-, s_+, s_-, n_1 \geq 0$ be such that $l_+ \geq 1$, $n_1 \in 2\mathbb{Z}$ and $l_+ + l_- \geq s_+ + s_- + n_1 + 2 \geq n_1 + 3$. There exists an even unimodular $\Phi_1\Phi_2\Phi_m^*$-lattice (M,g) where $M \in \mathrm{II}_{(l_+,l_-)}$ such that $M^{-g} \simeq \langle 4 \rangle$ and $M^{\Phi_m(g)}$ is 2-elementary of absolute determinant 2^{n_1+2} and signatures (s_+, s_-) if and only if there exists an even integer $n_0 \geq 0$, and two integers $\alpha \in \{1,3,5,7\}$ and $\epsilon \in \{\pm 1\}$ such that*

(1) $l_\pm \geq s_\pm$;

(2) $l_+ - l_- \equiv 0 \mod 8$;

(3) $s_\pm \equiv 0 \mod 2$;

(4) $s_+ + s_- = \varphi(2^k)(n_0 + n_1 + 1)$;

(5) $s_+ - s_- \equiv 1 - \alpha - 2\epsilon \mod 8$;

(6) $\alpha \equiv 1 \mod 4$ *if and only if* $m = 4$;

(7) *if* $(l_+ - s_+) + (l_- - s_-) = n_1 + 2$, *then* $\alpha \equiv \pm 1 \mod 8$ *and* $\epsilon = +1$;

(8) *if* $s_+ + s_- = n_1 + 2$ *then* $\epsilon = +1$ *and* $\alpha \in \{1,5\}$.

Proof. The proof is similar to the proof of Theorem 8.30 by using Claim 8.31 and the follow-up discussions. Note that $s_+ + s_- = n_1 + 2$ can only happen if and only if $k = 2$ and $n_0 = n_1 = 0$. In that situation, we know that we must have $\epsilon = +1$ by Proposition 1.50 and $\alpha \in \{1,5\}$ by condition (6) of the statement. Hence condition (8) is necessary. □

8.1.4.2. Classification up to conjugacy

Now that we have given necessary and sufficient conditions for the existence of even unimodular $\mathbb{Z}$-lattices equipped with an isometry of a given type, we aim to show how one can classify such lattices with isometry, up to isomorphism. We moreover give the number of conjugacy classes of cyclic subgroups generated by such isometries.

Theorem 8.34. *Let M be an even indefinite unimodular $\mathbb{Z}$-lattice, let $g \in O(M)$ be an isometry of finite order $m \geq 3$ and let $E := \mathbb{Q}(\zeta_m)$. Suppose that the minimal polynomial of g divides $\Phi_1\Phi_m$ and that M_g is indefinite or of rank $\varphi(m)$. Then*

(1) *the isometry class of the invariant sublattice M^g; and*

(2) *the isometry class of the hermitian structure of (M_g, g_m)*

form a complete set of invariants for the isomorphism class of (M, g). In particular, the number of isomorphism classes in the type of (M, g) is given by $c \times h^-(E)$ where c is the number of isometry classes in the genus of M^g.

Proof. The proof is similar to the proof of [BC23, Theorem 1.2]. Let us assume that we are given two isometries $g, h \in O(M)$ such that $M^g \simeq M^h$, and (M_g, g_m) and (M_h, h_m) are isomorphic. Now, g and h are conjugate in $O(M)$, if and only if (M, g) and (M, h) are isomorphic, if and only if the equivariant primitive extensions

$$(M^g, \mathrm{id}) \oplus (M_g, g_m) \leq (M, g) \quad \text{and} \quad (M^h, \mathrm{id}) \oplus (M_h, h_m) \leq (M, h) \tag{30}$$

are isomorphic. If we denote by $\gamma_g : D_{M^g} \to D_{M_g}$ and $\gamma_h : D_{M^h} \to D_{M_h}$ the respective equivariant glue maps, Lemma 2.19 tells us that the primitive extensions in Equation (30) are isomorphic if and only if there exist an isometry $\psi_1 : M^g \to M^h$ and an isomorphism $\psi_2 : (M_g, g_m) \to (M_h, h_m)$ such that $\gamma_h \circ \overline{\psi_1} = \overline{\psi_2} \circ \gamma_g$. Let us fix an isometry $\psi_1 : M^g \to M^h$ and an isomorphism $\psi_2 : (M_g, g_m) \to (M_h, h_m)$, and let $\kappa := \gamma_h \circ \overline{\psi_1} \circ \gamma_g^{-1} \circ \overline{\psi_2}^{-1} \in O(D_{M_h})$. If m is not a prime power, then we know from Proposition 8.1 that M_h is unimodular, meaning that $\kappa = \mathrm{id}_{D_{M_h}}$ and g, h are conjugate in $O(M)$. Otherwise, if $m = p^k$ is a prime power, the same proposition tells us that M_h is an even p-elementary Φ_{p^k}-lattice with D_{h_m} trivial. Therefore, by Theorem 8.21, there exists $\psi \in O(M_h, h_m)$ such that $\kappa = D_\psi$. In particular, up to replacing ψ_2 by $\psi \circ \psi_2 : (M_g, g_m) \to (M_h, h_m)$, we have that the primitive extensions in Equation (30) are isomorphic, and hence $g, h \in O(M)$ are conjugate. The last assertion of the statement follows from Proposition 4.19. □

Remark 8.35. According to [Was97, Tables §3], the relative class number in the cyclotomic case $E := \mathbb{Q}(\zeta_m)$ is 1 for all $3 \leq m \leq 66$ with $\varphi(m) \leq 22$, except for $m = 23, 46$ where $h^{-1}(E) = 3$.

Theorem 8.36. *Let M be an even indefinite unimodular $\mathbb{Z}$-lattice, let $g \in O(M)$ be an isometry of even order $m = 2p^k$ where p is a prime number and $k \geq 1$, and let $E = \mathbb{Q}(\zeta_m)$. Suppose that the minimal polynomial of g is $\Phi_1\Phi_2\Phi_m$, that the kernel sublattice $M^{-g} \simeq \langle 2p \rangle$, and that $M^{\Phi_m(g)}$ is indefinite or of rank $\varphi(m)$. Then*

(1) *the isometry class of the invariant sublattice M^g; and*

(2) *the isometry class of the hermitian structure of $(M^{\Phi_m(g)}, g_m)$*

form a complete set of invariants for the isomorphism class of (M, g). In particular, the number of isomorphism classes in the type of (M, g) is given by $c \times h^-(E)$ where c is the number of isometry classes in the genus of M^g.

Proof. This time, by Proposition 8.2, we have that M^g and M^{-g} glue along abelian groups of order 2, which therefore have no nontrivial isomorphism. Hence, for two isometries $g, h \in O(M)$ such that $M^g \simeq M^h$ and $M^{-g} \simeq M^{-h} \simeq \langle 2p \rangle$, we obtain that $(M^{g^2-1}, g_{|M^{g^2-1}})$ and $(M^{h^2-1}, h_{|M^{h^2-1}})$ are actually isomorphic too. If p is odd, we have that $-D_{g_m}$ is the identity, the centralizers $O(M^{\Phi_m(g)}, g_m)$ and $O(M^{\Phi_m(g)}, -g_m)$ coincide, and $O(D_{M^{\Phi_m(g)}}, D_{g_m}) = O(D_{M^{\Phi_m(g)}})$. Otherwise, if $p = 2$, we know that $(M^{\Phi_m(g)}, g_m)$ is an even 2-elementary Φ_{2^k}-lattice with D_{g_m} of order at most 2. Hence, in both cases, by Theorem 8.21, we know that $O(M^{\Phi_m(g)}, g_m) \to O(D_{M^{\Phi_m(g)}}, D_{g_m})$ is surjective, and we conclude by applying similar arguments as in the proof of Theorem 8.34. Note that we have already seen that in such a situation, the genus of $M^{\Phi_m(g)}$ together with $M^{-g} \simeq \langle 2p \rangle$ uniquely determines the genus of M^g, hence the last statement holds also. □

From these theorems, we can describe a procedure to compute representatives of isomorphism classes of such isometries. Up to fixing the signatures, one starts by determining the possible genera for the hermitian structure of $(M^{\Phi_m(g)}, g_m)$ using Propositions 8.17 to 8.19. For each such genus, one constructs the trace lattice associated to each representative of an isometry class in this genus. Each of the previous genera actually determines the genus G of the candidates for M^g. We then conclude using Theorems 8.34 and 8.36 that one has to enumerate G and compute the corresponding equivariant primitive extensions. In particular, if $h^-(\mathbb{Q}(\zeta_m)) = 1$ and G consists of a unique isometry class, one obtains a unique pair (M, g) for a given set of signatures of $(M^{\Phi_m(g)}, g_m)$.

8.2. Algebraically trivial nonsymplectic automorphisms

Let X be an IHS manifold of known deformation type. Let $\rho_X\colon \mathrm{Bir}(X) \to O(H^2(X, \mathbb{Z}))$ be the natural orthogonal representation and let $G \leq \mathrm{Bir}(X)$ be finite.

Definition 8.37. We call the group G *algebraically trivial* if $\rho_X(G)$ fixes pointwise the Néron–Severi lattice $\mathrm{NS}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ of X.

Proposition 8.38. *Let X be a projective IHS manifold. The subgroup of* $\mathrm{Bir}(X)$ *consisting of algebraically trivial symplectic birational automorphisms coincides with* $\ker \rho_X$.

Proof. Since any symplectic symmetry of X acts trivially on the transcendental lattice $\mathrm{T}(X) = \mathrm{NS}(X)^{\perp}_{H^2(X,\mathbb{Z})}$ of X, we have that algebraically trivial symplectic birational automorphisms act trivially on $H^2(X, \mathbb{Z})$. Conversely, any automorphisms in $\ker \rho_X$ is symplectic and algebraically trivial. □

Let us suppose now that G is algebraically trivial and that $G \neq G_s$. Let $g \in G \setminus G_s$ be nonsymplectic such that the coset gG_s generates the finite cyclic factor group G/G_s of order $m := [G : G_s]$. Since G is algebraically trivial, so is g and if we denote $h := \rho_X(g)$, one has in particular that by definition $H^2(X, \mathbb{Z})^h = \mathrm{NS}(X)$ and $H^2(X, \mathbb{Z})_h = \mathrm{T}(X)$ (Proposition 6.5). According to Proposition 6.1, we have that the minimal polynomial of $h_{|\mathrm{T}(X)}$ is cyclotomic. Consequently, the characteristic polynomial of h is of the form $\Phi_1^{\rho}(X)\Phi_m^k(X)$ where $\rho := \mathrm{rank}_{\mathbb{Z}}(\mathrm{NS}(X))$ and k are positive integers.

Remark 8.39. Note that G, being algebraically trivial, consists of regular automorphisms of X (see for instance [Deb22, Proposition 4.1]). Therefore, in the rest of this section, we will always assume X to be projective, and $G \leq \mathrm{Aut}(X)$.

The authors in [BC23] suspected that their methods to classify odd prime order nonsymplectic automorphisms on the known IHS manifolds could be applied to classifying automorphisms of nonprime order. The purpose of this section is to show that their results extend to algebraically trivial nonsymplectic automorphisms of finite order $m > 2$. In particular we show that, up to deformation and birational conjugacy (defined as in Section 8.2.1), there are finitely many pairs (X, f) where X is of known deformation type and $f \in \mathrm{Aut}(X)$ is an algebraically trivial purely nonsymplectic automorphism of X such that $\rho_X(f)$ is nonstable.

Theorem 8.40. *Let X be a projective IHS manifold of known deformation type $\mathcal{T}$, and let $\ker \rho_X < G \leq \mathrm{Aut}(X)$ be algebraically trivial and nonsymplectic. Let us denote $\Lambda := H^2(X, \mathbb{Z})$ and suppose that the cyclic group $H := \rho_X(G) \leq \mathrm{Mon}^2(\Lambda)$ is nonstable of order $m \geq 3$. Then, up to deformation and birational conjugacy, the pair (X, G) is uniquely determined by $(\mathcal{T}, m, \Lambda^H, \Lambda_H)$, except in the case $(\mathcal{T}, m) = (\mathrm{K3}^{[24]}, 46)$ where there are 3 such pairs. The corresponding tuples $(\mathcal{T}, m, \Lambda^H, \Lambda_H)$ are given in Table 18.*

8.2.1. A moduli classification

In [BC23], the authors classify pairs (X, G) consisting of a (projective) IHS manifold X of known deformation type, and of a finite subgroup $G \leq \mathrm{Aut}(X)$ of automorphisms whose action on $H^2(X, \mathbb{Z})$ is cyclic of odd prime order, generated by the image of a nonsymplectic automorphism. We recall their result and show that it extends to the case where G is algebraically trivial and not symplectic.

For the rest of this section, let us fix $\mathcal{T}$ a deformation type of IHS manifolds.

Definition 8.41. Let $X, X' \sim \mathcal{T}$ be two projective IHS manifolds of the same deformation type, and let $G \leq \mathrm{Aut}(X)$ and $G' \leq \mathrm{Aut}(X')$ be finite. We call the pairs (X, G) and (X', G') *birational conjugate* if there exists a birational map $f \colon X \dashrightarrow X'$ such that $fGf^{-1} = G'$.

Let X be an IHS manifold, and let $\mathcal{X} \to \mathrm{Def}(X)$ be the universal family. In [BC23, §3.1], Brandhorst and Cattaneo show that for any finite subgroup $G \leq \mathrm{Aut}(X)$, there exists a universal deformation family of the pair (X, G) whose base is $\mathrm{Def}(X)^G$. This motivates the following definition.

Definition 8.42. Let $X, X' \sim \mathcal{T}$ be two projective IHS manifolds of the same deformation type, and let $G \leq \mathrm{Aut}(X)$ and $G' \leq \mathrm{Aut}(X')$ be finite. We call the pairs (X, G) and (X', G') *deformation equivalent* if there exists a connected family $\pi : \mathcal{X} \to B$ of IHS manifolds, a group $\mathcal{G} \leq \mathrm{Aut}(\mathcal{X}/B)$ and two points $b, b' \in B$ such that $(\mathcal{X}, \mathcal{G})_b = (X, G)$ and $(\mathcal{X}, \mathcal{G})_{b'} = (X', G')$.

Here, we say $\pi : \mathcal{X} \to B$ is a connected family of IHS manifolds if the base B is connected, π is a smooth proper holomorphic map and the fiber over every closed point in B is an IHS manifold.

Let $\mathcal{M}^\circ$ be a connected component of the moduli space $\mathcal{M}$ of marked IHS manifolds of deformation type $\mathcal{T}$. Let Λ be the abstract $\mathbb{Z}$-lattice associated to the deformation type $\mathcal{T}$ and let $\mathrm{Mon}^2(\Lambda) := \eta \mathrm{Mon}^2(X) \eta^{-1}$ for some $[(X, \eta)] \in \mathcal{M}^\circ$. We denote again by $\mathcal{W}^{pex}(\Lambda) \subseteq \mathcal{W}(\Lambda)$ the sets of numerical prime exceptional and wall divisors of Λ, and we define $C_\Lambda \subseteq \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ to be such that for all $[(X, \eta)] \in \mathcal{M}^\circ$, the positive cone $C_X = \eta^{-1}(C_\Lambda) \cap H^{1,1}(X, \mathbb{R})$.

Remark 8.43. Note that the definitions of $\mathrm{Mon}^2(\Lambda)$, $\mathcal{W}^{pex}(\Lambda)$, $\mathcal{W}(\Lambda)$ and C_Λ do not depend on a choice of $[(X, \eta)] \in \mathcal{M}^\circ$. Hence, these objects are well-defined as long as we fix a connected component of $\mathcal{M}$.

Let now $H \leq \mathrm{Mon}^2(\Lambda)$ be cyclic of finite order $m \geq 3$ generated by h with minimal polynomial $\Phi_1 \Phi_m$. We fix a primitive mth root of unity ζ_m such that

$$\ker(h + h^{-1} - \zeta_m - \zeta_m^{-1})$$

has real signatures $(2,*)$ (Section 4.4), and we define the corresponding character $\chi\colon H \to \mathbb{C}^\times$, $\chi(h) = \zeta_m$. The choice of χ gives rise to a moduli space $\mathcal{M}_H^\chi$ parametrizing H-marked IHS manifolds (X, η, G) of deformation type $\mathcal{T}$, such that for all $g \in G$

$$\chi(\eta\rho_X(g)\eta^{-1})\sigma_X = (g^*)^{-1}\sigma_X$$

where σ_X generates $H^{2,0}(X)$. We recall that a triple (X, η, G) is said to be H-marked if $X \sim \mathcal{T}$, $\ker \rho_X \leq G \leq \mathrm{Aut}(X)$ and $\eta\rho_X(G)\eta^{-1} = H$. According to [BC23, Proposition 3.8], the forgetful map

$$\phi\colon \mathcal{M}_H^\chi \to \mathcal{M},\ (X, \eta, G) \mapsto (X, \eta)$$

is a closed embedding. In particular, the period map

$$\mathcal{P}^\chi\colon \mathcal{M}_H^\chi \to \Omega^\chi := \{[\omega] \in \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C}) \ : \ \omega^2 = 0,\ \omega.\overline{\omega} > 0 \text{ and } h(\omega) = \chi(h)\omega,\ \forall h \in H\}$$

is a local isomorphism. In what follows, we denote $\mathcal{M}_H^{\chi;\circ} := \phi^{-1}(\mathcal{M}^\circ \cap \phi(\mathcal{M}_H^\chi))$.

We denote by $N := \Lambda^H$ the corresponding invariant sublattice, and $M = N_\Lambda^\perp$ its orthogonal complement. For later purpose, we let $C_N := C_\Lambda \cap (N \otimes_{\mathbb{Z}} \mathbb{R})$. We call *Kähler-type chamber* in C_N any connected component of $C_N \setminus \bigcup_{v \in \mathcal{W}(\Lambda)} v^\perp$.

Definition 8.44 ([BC23, Definition 3.10])**.** Let (X, η, G) be an H-marked IHS manifold of deformation type $\mathcal{T}$.

(1) We say (X, η, G) is (H, N)*-polarized* if $C_N \cap \eta(\mathcal{K}_X) \neq \emptyset$.

Suppose now (X, η, G) is (H, N)-polarized.

(2) For a Kähler-type chamber $K(N)$ in C_N preserved by H, we say (X, η, G) is $K(N)$*-general* if $\eta(\mathcal{K}_X) \cap (N \otimes_{\mathbb{Z}} \mathbb{R}) = K(N)$.

(3) We say (X, η, G) is H*-general* if it is $K(N)$-general for some Kähler-type chamber $K(N)$ in C_N preserved by H.

For an H-marked IHS manifold (X, η, G) of deformation type $\mathcal{T}$, we denote $C_X^G := C_X \cap H^2(X, \mathbb{R})^{\rho_X(G)}$ and $\mathcal{FE}_X^G := \mathcal{FE}_X \cap H^2(X, \mathbb{R})^{\rho_X(G)}$. Recall that the fundamental exceptional chamber $\mathcal{FE}_X$ of X is a chamber of

$$C_X \setminus \bigcup_{v \in \mathcal{W}^{pex}(X)} v^\perp$$

where $\mathcal{W}^{pex}(X) \subseteq \mathrm{NS}(X)$ is the set of stably prime exceptional divisors of X. The following is a straightforward generalization of [BC23, Lemma 3.12].

Lemma 8.45. *If an (H, N)-polarized IHS manifold (X, G, η) of deformation type $\mathcal{T}$ is H-general, then $\mathcal{FE}_X^G$ is a chamber of*

$$C_X^G \setminus \bigcup_{v \in \mathcal{W}^{pex}(X)} v^\perp.$$

Proof. By the generality assumption, we observe that

$$C_X^G \setminus \bigcup_{v \in \mathcal{W}^{pex}(X)} v^\perp = C_X^G \setminus \bigcup_{v \in \mathcal{W}^{pex}(X) \cap H^2(X, \mathbb{Z})^{\rho_X(G)}} v^\perp.$$

Hence, the prime exceptional divisors of X are fixed under the action of G, so we can conclude (see also [BCS19, Lemma 5.2]). $\square$

We define

$$\Delta := \bigcup_{v \in \mathcal{W}(M)} \mathbb{P}(v^\perp) \subseteq \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})$$
$$\Delta' := \bigcup_{v \in \mathcal{W}'} \mathbb{P}(v^\perp) \subseteq \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})$$

where $\mathcal{W}(M) := \mathcal{W}(\Lambda) \cap M$ and

$$\mathcal{W}' := \{v \in \mathcal{W}(\Lambda) \ : \ \exists (v_N, v_M) \in N^\vee \times M^\vee, \ v = v_N + v_M \text{ and } v_N^2 < 0, \ v_M^2 < 0\}.$$

In [BC23, Proposition 3.11, 1.] the authors show that the restriction of the period map $\mathcal{P}^\chi$ to $\mathcal{M}_H^{\chi,\circ}$ surjects onto $\Omega^\chi \setminus \Delta$. Moreover, for a fixed Kähler-type chamber $K(N) \subseteq C_N$, if one denotes by $\mathcal{M}_{K(N)}^{\chi,\circ}$ the subset of $\mathcal{M}_H^{\chi,\circ}$ consisting of $K(N)$-general (H, N)-polarized IHS manifolds of deformation type $\mathcal{T}$, then $\mathcal{P}^\chi$ induces a bijection

$$\mathcal{P}_{K(N)}^\chi \colon \mathcal{M}_{K(N)}^{\chi,\circ} \to \Omega^\chi \setminus (\Delta \cup \Delta')$$

([BC23, Proposition 3.11, 2.], [BCS19, Theorem 5.6]). Moreover, the following holds.

Proposition 8.46. *Let $w \in \Omega^\chi \setminus (\Delta \cup \Delta')$ and let (X_1, η_1, G_1), (X_2, η_2, G_2) be two $K(N)$-general elements in the fiber of $\mathcal{P}^\chi$ over the period point w. Then (X_1, G_1) and (X_2, G_2) are birational conjugate.*

Proof. The proof is similar to the one of [BC23, Proposition 3.11, 3.]. Let us denote by $W^{G_2}(X_2) \leq \mathrm{Mon}(X_2)$ the subgroup generated by reflections in G_2-invariant stably prime exceptional divisors.

Since $(X_1, \eta_1) = \phi(X_1, \eta_1, G_1) \in \mathcal{M}^\circ$ and $(X_2, \eta_2) = \phi(X_2, \eta_2, G_2) \in \mathcal{M}^\circ$ lie in the same connected component of $\mathcal{M}$, Proposition 5.29 tells us that $\psi := \eta_1^{-1} \circ \eta_2 \colon H^2(X_2, \mathbb{Z}) \to H^2(X_1, \mathbb{Z})$ is a Hodge parallel transport operator. In particular, $\psi(\mathcal{FE}_{X_2})$ is an exceptional chamber of C_{X_1}. Now since both triples are H-general, Lemma 8.45 tells us that $\mathcal{FE}_{X_2}^{G_2}$ and $\psi^{-1}(\mathcal{FE}_{X_1}^{G_1})$ are preserved under the action of G_2 on C_{X_2}. Hence, there exists $w \in W^{G_2}(X_2)$ such that $\psi \circ w(\mathcal{FE}_{X_2}^{G_2}) = \mathcal{FE}_{X_1}^{G_1}$. But now, since $\psi \circ w(\mathcal{FE}_{X_2})$ and $\mathcal{FE}_{X_1}$ are two exceptional chambers in C_{X_1} with nonempty intersection, we deduce that there are equal. Hence, $\psi \circ w$ is a Hodge parallel transport operator and it sends $\mathcal{FE}_{X_2}$ to $\mathcal{FE}_{X_1}$. By Torelli Theorem 5.48, there exists a birational map $f \colon X_1 \dashrightarrow X_2$ such that $f^* = \psi \circ w$. We conclude by remarking that since $w\rho_{X_2}(G_2) = \rho_{X_2}(G_2)w$, we have that $fG_1f^{-1} = G_2$: this means exactly that (X_1, G_1) and (X_2, G_2) are birational conjugate. □

We can now state the following theorem, which is a generalization of [BC23, Theorem 3.13]. It relates a classification of pairs (X, G) as before, up to deformation and birational conjugacy, with a classification of conjugacy classes of finite cyclic subgroups $H \leq \mathrm{Mon}^2(\Lambda)$.

Theorem 8.47. *Fix a known deformation type $\mathcal{T}$ of IHS manifolds and a connected component $\mathcal{M}^\circ$ of the moduli of marked IHS manifolds of deformation type $\mathcal{T}$. Let $h_1, \ldots, h_n$ be a complete set of representatives for the conjugacy classes of elements in $\mathrm{Mon}^2(\Lambda)$ of order $m \geq 3$ whose minimal polynomial equals $\Phi_1\Phi_m$. Suppose that the h_i's are chosen such that the real quadratic space $\ker(h_i + h_i^{-1} - \zeta_m - \zeta_m^{-1}) \leq \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ has signatures $(2, *)$ for all i. Let $H_i = \langle h_i \rangle \leq \mathrm{Mon}^2(\Lambda)$ and $\chi_i \colon H_i \to \mathbb{C}^\times$ the character defined by $\chi_i(h_i) := \zeta_m$. For each H_i, choose a point $(X_i, G_i, \eta_i) \in \mathcal{M}_{H_i}^{\chi_i,\circ}$. Then, the pairs $(X_1, G_1), \ldots, (X_n, G_n)$ form a complete set of representatives of pairs (X, G) up to deformation and birational conjugacy where $X \sim \mathcal{T}$, the group $G \leq \mathrm{Aut}(X)$ is not symplectic with $G_s = \ker(\rho_X)$, and $\rho_X(G)$ is cyclic of order m generated by an isometry h with minimal polynomial $\Phi_1\Phi_m$.*

Proof. The monodromy conjugacy class of $H_i = \langle h_i \rangle$ is invariant under deformation and birational conjugacy, therefore the minimal polynomial of a generator stays unchanged too. Finally, since χ_i is not real, the period domains are connected and the proof follows similarly as in [BC23, Theorem 3.13]. We omit the details. □

Remark 8.48. For $H_i = \langle h_i \rangle$ as in the theorem, all the generators h_i' of H_i have order m and minimal polynomial $\Phi_1\Phi_m$. However the signatures of the real quadratic space $\ker(h_i + h_i^{-1} - \zeta_m - \zeta_m^{-1})$ are $(2, *)$, while the ones of $\ker(h_i' + h_i'^{-1} - \zeta_m - \zeta_m^{-1})$ are $(0, *)$ for any other generator $h_i' \neq h_i,\ h_i^{-1}$. Since we aim to classify cyclic groups of nonsymplectic isometries, this extra condition avoids duplication as we could have two representatives of different conjugacy classes generating the same cyclic subgroup of $O(\Lambda)$. This actually boils down to considering different genera of hermitian $\mathbb{Z}[\zeta_m]$-lattices, up to the action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m + \zeta_m^{-1})/\mathbb{Q})$ (see Section 4.4).

By [Cam16, Lemma 3.7, Theorem 3.9], the component $\mathcal{M}_H^{\chi,\circ}$ contains a Hausdorff subspace consisting of those polarized IHS manifolds such that $\eta(\mathrm{NS}(X)) = N$, i.e. for which the corresponding group G is algebraically trivial. According to [Cam16, Theorem 3.9], the image of this subspace under the period map is dense and connected in the associated period domain Ω^χ.

Remark 8.49. Note that this makes sense because $m \geq 3$: in that case, the character χ is complex not real and therefore, Ω^χ is itself connected [DK07].

In particular, for any effective finite subgroup $H \leq \mathrm{Mon}^2(\Lambda)$ which is cyclic of order $m \geq 3$ generated by h with minimal polynomial $\Phi_1\Phi_m$, and for any H-marked IHS manifold (X, η, G), the deformation family of the pair (X, G) contains a pair (X', G') with G' algebraically trivial. Therefore, for a classification of pairs up to birational conjugacy and deformation as in Theorem 8.47, we can always assume the group G to be algebraically trivial. In what follows, we aim to connect the results from Section 8.1 to Theorem 8.47, and give a proof to Theorem 8.40.

8.2.2. Lattice-theoretic approach

We want to study the existence of isometries of the known BBF forms with minimal polynomial $\Phi_1\Phi_m$, and classify them up to conjugacy in the respective monodromy groups (Theorem 8.47). For each known deformation type $\mathcal{T}$ of IHS manifolds, with reference $\mathbb{Z}$-lattice $\Lambda_\mathcal{T}$, there exists an even unimodular $\mathbb{Z}$-lattice $M_\mathcal{T}$ into which $\Lambda_\mathcal{T}$ embeds primitively with positive definite complement (the so-called Mukai lattices). In [BC23], Brandhorst and Cattaneo use this fact to transport the lattice classification of isometries for each $\mathcal{T}$ to a study of isometries with given minimal polynomial in certain even unimodular $\mathbb{Z}$-lattices. The important point here is that this allows one to unify the study into a given common problem, mainly performing a classification at the level of unimodular $\mathbb{Z}$-lattices.

Remark 8.50. This approach is nontrivial and complicated to set up in general. However, it offers a very practical framework to classify birational automorphisms for the two infinite families $\mathrm{K3}^{[n]}$ and Kum_n of deformation types in a uniform way. We expect that with enough care, one could be able to exploit this to classify all finite symplectic actions for such deformation types, for instance (despite the associated monodromy groups being nonmaximal).

8.2.2.1. From monodromies to isometries of unimodular $\mathbb{Z}$-lattices

Let $\mathcal{T}$ be a known deformation type of IHS manifolds, let $\Lambda := \Lambda_\mathcal{T}$ and let $M := M_\mathcal{T}$ be the corresponding unimodular $\mathbb{Z}$-lattice (see Table 9).

Table 9: Known deformation types and unimodular data

$\mathcal{T}$	Λ	$\mathrm{Mon}^2(\Lambda)$	M	V	v^2	K
$\mathrm{Kum}_n,\ n \geq 2$	$U^{\oplus 3} \oplus A_1(n+1)$	$\mathcal{N}^+(\Lambda)$	$U^{\oplus 4}$	$\langle 2n+2 \rangle$	0	$\langle 2n+2 \rangle$
OG6	$U^{\oplus 3} \oplus A_1^{\oplus 2}$	$O^+(\Lambda)$	$U^{\oplus 5}$	$\langle 2 \rangle^{\oplus 2}$	4	$\langle 4 \rangle$
K3	$U^{\oplus 3} \oplus E_8^{\oplus 2}$	$O^+(\Lambda)$	$U^{\oplus 3} \oplus E_8^{\oplus 2}$	$\{0\}$	0	$\{0\}$
$\mathrm{K3}^{[n]},\ n \geq 2$	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_1(n-1)$	$\mathcal{W}^+(\Lambda)$	$U^{\oplus 4} \oplus E_8^{\oplus 2}$	$\langle 2n-2 \rangle$	0	$\langle 2n-2 \rangle$
OG10	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$	$O^+(\Lambda)$	$U^{\oplus 5} \oplus E_8^{\oplus 2}$	$A_2(-1)$	2	$\langle 6 \rangle$

Notation. Following [MS17, §4], given an even $\mathbb{Z}$-lattice L we denote

$$\mathcal{W}^+(L) := \{f \in O^+(L) \ : \ D_f = \pm \mathrm{id}_{D_L}\}$$
$$\mathcal{N}^+(L) := \{f \in O^+(L) \ : \ \det(f) D_f = \mathrm{id}_{D_L}\}.$$

We denote $V := \Lambda^\perp_M$. For $\mathcal{T} = \mathrm{K3}^{[n]}, \mathrm{Kum}_n$, we let $h_V := -\mathrm{id}_V$. For $\mathcal{T} = \mathrm{OG6}, \mathrm{OG10}$, we let h_V be represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on a basis of V such that

$$V = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ if } \mathcal{T} = \mathrm{OG6}$$
$$V = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ if } \mathcal{T} = \mathrm{OG10}.$$

Note that in all cases, $h_V \in O(V)$ is nonstable, except for $\mathcal{T} = \mathrm{K3}, \mathrm{K3}^{[2]}$ where the discriminant group has no nontrivial automorphisms. For $\mathcal{T} = \mathrm{K3}^{[n]}, \mathrm{Kum}_n$, we let $v := 0$ be the zero vector, and for $\mathcal{T} = \mathrm{OG6}, \mathrm{OG10}$ we let v be the sum of the basis vectors of V, for the given bases. In all cases, note that v generates the invariant sublattice of (V, h_V). Finally, we let $O(M, V, v)$ be the joint stabilizer. The following lemma is known from [BC23]; we give a proof for completeness.

Lemma 8.51. *There exists a well-defined restriction map $O(M, V, v) \to O(\Lambda)$ which admits a section*

$$\gamma\colon \mathrm{Mon}^2(\Lambda) \to O(M, V, v),\ h \mapsto \widehat{\chi}(h) \oplus h$$

where $\widehat{\chi}(h) = id_V$ if h is stable, and $\widehat{\chi}(h) = h_V$ otherwise.

Proof. By definition of $O(M, V, v)$, since Λ is the orthogonal complement of V in M, we have that

$$r\colon O(M, V, v) \to O(\Lambda),\ g \mapsto g_{|\Lambda}$$

is well-defined. Moreover, by Lemma 6.15, we know that any $h \in \mathrm{Mon}^2(\Lambda)$ acts with order at most 2 on D_Λ. In fact, for $\mathcal{T} = \mathrm{K3}^{[n]}$, Kum_n, OG10, we have that $D_h = \pm \mathrm{id}_{D_\Lambda}$, and if $\mathcal{T} = \mathrm{OG6}$, then D_h is either the identity or the map exchanging two generators. It follows from the definition of $\widehat{\chi}(h)$ that in both cases D_h trivial or not, $D_h \in O(D_\Lambda)$ and $D_{\widehat{\chi}(h)} \in O(D_V)$ agree along the glue map $D_\Lambda \to D_V$: hence $\widehat{\chi}(h) \oplus h$ extends along the primitive extension $V \oplus \Lambda \leq M$ to an integral isometry preserving V and fixing v. $\square$

Notation. As in the section about stable symplectic isometries, for a sublattice $N \leq M$ we define $S(N) := O(N)$ if $\mathcal{T} = \mathrm{K3}^{[n]}, \mathrm{OG6}, \mathrm{OG10}$ and $S(N) := SO(N)$ if $\mathcal{T} = \mathrm{Kum}_n$. Moreover, we let $S(M, V, v) := S(M) \cap O(M, V, v)$.

According to [BC23, Lemma 4.6] the image of γ is given by the kernel G of

$$\vartheta \cdot \chi_V : S(M, V, v) \to \{\pm 1\}$$

where ϑ is the character induced by the spinor norm morphism $\sigma_{\mathbb{Q}}$ on $O(M \otimes_{\mathbb{Z}} \mathbb{Q})$, and χ_V is the natural character induced by the composite morphism $S(M, V, v) \to O(V) \to O(D_V)$.

Remark 8.52. For a known deformation type $\mathcal{T} \neq \mathrm{K3}$, the $\mathbb{Z}$-lattice $K := v_V^\perp$ has rank 1. If $h \in \mathrm{Mon}^2(\Lambda)$ is stable, then the image of K in M is contained in M^g where $g := \gamma(h)$. Otherwise, since K is the (-1)-sublattice of (V, h_V), we have that K is contained in M^{-g}. In particular, if $m_h(1) \neq 0$ where m_h is the minimal polynomial of h, then M^{-g} has rank 1 and it is equal to K. The isometry class of K can be read from Table 9.

Lemma 8.53. *Let $g \in G$. Then $\chi_V(g) = +1$ if and only if $K \leq M^g$, and $\chi_V(g) = -1$ if and only if $K \leq M^{-g}$.*

Proof. By definition of G and χ_V, we have that $\chi_V(g) = +1$ if and only if $g = \mathrm{id}_V \oplus h$ for some $h \in \mathrm{Mon}^2(\Lambda)$ with D_h trivial. Otherwise, since K is the (-1)-kernel sublattice of (V, h_V), we have that $\chi_V(g) = -1$ if and only if $g = h_V \oplus h$ for some $h \in \mathrm{Mon}^2(\Lambda)$ with D_h nontrivial. We conclude with Remark 8.52. $\square$

8.2.2.2. Case restrictions

By Proposition 8.2, there are strong restrictions on the local invariants of the Φ_2-kernel sublattice of any even unimodular $\Phi_1\Phi_2\Phi_m^*$-lattice (M, g). In particular, in the case where $M \simeq M_{\mathcal{T}}$ for a known deformation type $\mathcal{T}$ of IHS manifolds, and g is induced by a nonstable isometry of $\Lambda_{\mathcal{T}}$, we have seen in Remark 8.52 that the kernel sublattice M^{-g} has rank 1 and it is uniquely determined. We can therefore already conclude on the possible orders such an isometry could have.

Proposition 8.54. *Let X be a projective IHS manifold of known deformation type $\mathcal{T}$. Let $f \in \mathrm{Bir}(X)$ be such that the minimal polynomial of $h := \rho_X(f)$ is $\Phi_1\Phi_m$ for some positive integer $m \geq 3$ and suppose that D_h is nontrivial. Then the pair $(\mathcal{T}, m)$ appears in Table 10.*

Table 10: Deformation types and orders for nonstable purely nonsymplectic isometries

$\mathcal{T}$	OG10	OG6	$\mathrm{K3}^{[3]}$	$\mathrm{K3}^{[4]}$	$\mathrm{K3}^{[6]}$	$\mathrm{K3}^{[p+1]}$, $7 \leq p \leq 23$	Kum_2	Kum_{p-1}, $5 \leq p \leq 7$
m	6, 18, 54	4, 8	4, 8, 16, 32	6, 18, 54	10, 50	$2p$	6, 18	$2p$

Proof. We follow the notation of Paragraph 8.2.2.1, and we let $g := \gamma(h)$. Recall that according to Remark 8.52, since D_h is nontrivial, in that situation we have that the minimal polynomial of g is $\Phi_1\Phi_2\Phi_m$.

(1) First suppose that $\mathcal{T} = \mathrm{OG10}$. Then we have that $M^{-g} \simeq \langle 6\rangle$ (Table 9). By Proposition 8.2, M^{-g} is 6-elementary if and only if $m = 2\cdot 3^k$ for some $k \geq 1$. Note that in this case $3\varphi(m) = m$. Since $H^2(X,\mathbb{Z})$ has rank 24 and $H^2(X,\mathbb{Z})^h$ has rank at least 1, we have $2 \leq \varphi(m) \leq 22$ and therefore $6 \leq m \leq 66$.

(2) Now let $\mathcal{T} = \mathrm{K3}^{[n]}$ for some $n \geq 2$ — the following can directly be adapted to cover the cases where $\mathcal{T} = \mathrm{Kum}_n$. This time, $M^{-g} = \langle 2n-2\rangle$ is either unimodular, 4-elementary or its discriminant group is of the form $(\mathbb{Z}/2\mathbb{Z})^{\oplus\alpha} \oplus (\mathbb{Z}/p\mathbb{Z})^{\oplus\beta}$ for some $\alpha, \beta \geq 0$, and p an odd prime integer. The unimodular case is clearly not possible. The 4-elementary case occurs only when m is a power of 2, and in that situation we know that M^g and M^{-g} are both 4-elementary (Proposition 8.2). In particular, $n-1 = 2$ and $4 \leq m \leq 32$ since $\Lambda_{\mathrm{K3}^{[n]}}$ has rank 23. For the remaining case, since the discriminant group of M^{-g} is $\mathbb{Z}/(2n-2)\mathbb{Z}$ with $n \geq 3$, then necessarily $\alpha = \beta = 1$. In particular $n - 1 = p$ for an odd prime number p, $m \leq 2p^3$ with possible equality only for $p = 3$, and $m = 2p^2$ is possible only for $p = 3$ or $p = 5$.

(3) Finally let us assume that $\mathcal{T} = \mathrm{OG6}$. In a same way as before, we have that $M^{-g} \simeq \langle 4\rangle$ is 4-elementary. According to Proposition 8.2, this is possible only if m is a power of 2. Since $H^2(X,\mathbb{Z})^h$ has rank at least 1 and Λ_{OG6} has rank 8, we see that $4 \leq m \leq 8$ as $\varphi(m) = m/2 < 8$. □

Remark 8.55. We later show that some of the cases in Table 10 cannot occur if we require moreover that the coinvariant sublattice $(H^2(X,\mathbb{Z}), q_X)_h$ has signatures $(2,*)$ (Proposition 8.68).

We give in Section 8.2.3 examples for some of the previous cases. These are very particular birational automorphisms since we know very few examples of geometric constructions of IHS manifolds giving rise to nonstable birational automorphisms.

8.2.2.3. From isometries of unimodular $\mathbb{Z}$-lattices to monodromies

We have seen in Paragraph 8.2.2.1 how to transport the problem of classifying monodromy operators to a classification of isometries of unimodular $\mathbb{Z}$-lattices. In Section 8.1, we have studied isometries of even unimodular $\mathbb{Z}$-lattices with given minimal polynomial, and we have shown how to classify them. It remains now to bring back this classification to a classification, up to conjugacy, of monodromy operators. In their paper [BC23], Brandhorst and Cattaneo show how to relate conjugacy classes in $G := \ker(\vartheta\cdot\chi_V)$ with conjugacy classes in $\mathrm{Mon}^2(\Lambda)$ corresponding to odd prime order nonsymplectic automorphisms [BC23, Theorem 4.7]. We show now that one can generalize their result in the case of purely nonsymplectic automorphisms of any order $m \geq 3$ and minimal polynomial $\Phi_1\Phi_m$.

Let $\mathcal{T}$ be a known deformation type of IHS manifolds, and let Λ, M, V, v and K as given in Table 9. We recall that there is an injective homomorphism $\gamma\colon\ \mathrm{Mon}^2(\Lambda) \to S(M,V,v)$, $h \mapsto \widehat{\chi}(h) \oplus h$ whose image is $G := \ker(\vartheta\cdot\chi_V\colon\ S(M,V,v) \to \{\pm 1\})$.

Remark 8.56. Eventually, we will be interested in the case where $h \in \mathrm{Mon}^2(\Lambda)$ is effective and nonsymplectic. In particular Λ_h has signatures $(2,*)$ (Proposition 6.5). Since we assume $m > 2$, we have that Λ_h is also the Φ_m-kernel sublattice of $(M,\gamma(h))$. Moreover, Λ_h is either indefinite or $m = 3,4,6$ and $\mathrm{rank}(\Lambda_h) = \varphi(m) = 2$.

Notation. As in [BC23, §4.2], for a group H and for two elements $g, h \in H$, we denote by ${}^hg := hgh^{-1}$ the conjugation of g by h, and ${}^Hg$ denotes the conjugacy class of g in H. We let moreover $\mathrm{cl}(H)$ the set of conjugacy classes in H.

The following theorem is already known in the case of odd prime order isometries in G [BC23, Theorem 4.7]. We omit the proof of the theorem since it is the same as the one of Brandhorst–Cattaneo: we refer to the proof of Theorem 8.59 for more details.

Theorem 8.57. *Let $\psi\colon \mathrm{cl}\,(G) \to \mathrm{cl}(S(M))$ be the natural map. Let $g \in G$ be of order $m \geq 3$ where $m_g(X) = \Phi_1(X)\Phi_m(X) \in \mathbb{Q}[X]$, the coinvariant sublattice M_g is of signatures $(2,*)$ and $V \leq M^g$. Then, the map*

$$\begin{aligned} \phi:\ \psi^{-1}\left({}^{S(M)}g\right) &\to S(M^g)\backslash\left\{W\ :\ W \leq M^g \textit{ primitive},\ W \simeq V\right\} \\ {}^{G}h &\mapsto S(M^g)fV \textit{ where } g = {}^{f}h,\ f \in S(M) \end{aligned}$$

is a bijection.

Remark 8.58. For $g \in G$ as in the statement of Theorem 8.57, since $V \leq M^g$, we have that $\chi_V(g) = 1$ (Lemma 8.53). In particular, the fact that $g \in G$ is equivalent to $\vartheta(g) = 1$. In the statement and the proof of [BC23, Theorem 4.7], the authors dropped the condition for g to be in G since odd order isometries have positive spinor norm.

Note that the previous theorem is actually constructive: g has a well-defined restriction to $W_M^{\perp}$ for all $W \leq M^g$ isometric to V. The corresponding sublattice with isometry is isomorphic to (Λ, h) where $\gamma(h)$ and g are $S(M)$-conjugate in G.

Now let $h \in \mathrm{Mon}^2(\Lambda)$ be nonstable of even order $m \geq 3$ and such that $m_h = \Phi_1\Phi_m$. The minimal polynomial of $g := \gamma(h)$ is $\Phi_1\Phi_2\Phi_m$ and according to Remark 8.52, we have a succession of primitive sublattices $K = M^{-g} \leq V \leq M^{g^2-1}$. We can adapt the statement of Theorem 8.57 to this case.

Theorem 8.59. *Let $\psi\colon \mathrm{cl}\,(G) \to \mathrm{cl}(S(M))$ be the natural map. Let $g \in G$ be of even order $m \geq 3$ where $m_g(X) = \Phi_1(X)\Phi_2(X)\Phi_m(X) \in \mathbb{Q}[X]$, the kernel sublattice $M^{\Phi_m(g)}$ is of signatures $(2,*)$ and $K = M^{-g} \leq V \leq M^{g^2-1}$. Then, the map*

$$\begin{aligned} \phi\colon \psi^{-1}\left({}^{S(M)}g\right) &\to S(M^{g^2-1}, M^{-g})\backslash\left\{W\ :\ M^{-g} \leq W \leq M^{g^2-1} \textit{ primitive},\ W \simeq V\right\} \\ {}^{G}h &\mapsto S(M^{g^2-1}, M^{-g})fV \textit{ where } g = {}^{f}h,\ f \in S(M) \end{aligned}$$

is a bijection.

Proof. We follow the same steps as in the proof of [BC23, Theorem 4.7]. Let us denote $\Gamma := S(M^{g^2-1}, M^{-g})$.

(1) We start by proving that ϕ is well defined. Let $h \in G$ and $f \in S(M)$ be such that $g = fhf^{-1}$. Since g and h are conjugate, their kernel sublattices are isometric via f and their spinor norms are equal. The former implies that $M^{-h} = f^{-1}(M^{-g})$ has rank 1, and the latter implies that $\chi_V(h) = \chi_V(g)$ by the assumption $g, h \in G$. Since $K = M^{-g}$, Lemma 8.53 tells us that $\chi_V(h) = \chi_V(g) = -1$, meaning that $M^{-g} = K \leq M^{-h}$ too. Therefore, since $\mathrm{rank}(M^{-g}) = \mathrm{rank}(M^{-h})$, we get that $M^{-h} = M^{-g} (= K)$. Now, we also have that $gfv = fhv = fv$ and $gfV = fhV = fV$ since $h \in G \leq S(M, V, v)$. In particular, $fv \in M^g$ and we have already seen that $fM^{-g} = M^{-g}$. Hence, using the fact that V is an overlattice of $\mathbb{Z}v \oplus M^{-g}$ in M^{g^2-1}, we obtain $M^{-g} \leq fV \leq M^{g^2-1}$.

Now let $h_1, h_2, s \in G$ be such that $h_2 = {}^{s}h_1$. Let moreover $f_1, f_2 \in S(M)$ be such that $g = {}^{f_1}h_1 = {}^{f_2}h_2$. We need to show that $\Gamma f_1 V = \Gamma f_2 V$. For that, let $t := f_2 s f_1^{-1} \in S(M)$: by straightforward computations, we have that $gt = tg$ and in particular, t preserves the kernel sublattices of g. According to [BC23, Lemma 2.21], since $O(M^{\Phi_m(g)}, g_m) = SO(M^{\Phi_m(g)}, g_m)$ and $t_{|M^{\Phi_m(g)}}$ commutes with g_m, we have that $\det(t) = \det(t_{|M^{\Phi_m(g)}})\det(t_{|M^{g^2-1}}) = \det(t_{|M^{g^2-1}})$ and therefore $t_{|M^{g^2-1}} \in \Gamma$. Moreover, $tf_1V = f_2sV = f_2V$ since s preserves V. Hence, $\Gamma f_1 V = \Gamma f_2 V$, and ϕ is well defined.

(2) We now want to prove the surjectivity of ϕ. Let $M^{-g} \leq W \leq M^{g^2-1}$ be a succession of primitive sublattices and suppose that $W \simeq V$. By [Nik80, Proposition 1.6.1] and Table 9, the primitive embedding of $K = M^{-g}$ into V is unique up to isometry. Hence, one can extend the identity on M^{-g} to an isometry $\overline{f} \colon V \xrightarrow{\simeq} W$. Similarly to [BC23, Theorem 4.7], one can actually show that $\overline{f}$ extends to an isometry $f \in S(M)$ such that $W = fV$ and $f_{|M^{-g}} = \mathrm{id}_{M^{-g}}$. Let $k \in M$ be such that $M^{-g} = \mathbb{Z}k$. Note that f maps $v \in V \cap M^g$ to a generator w of $(M^{-g})^{\perp}_W \leq M^g$. If we let $h := f^{-1}gf$, we have that $hk = f^{-1}gfk = f^{-1}gk = -f^{-1}k = -k$ because $\chi_V(g) = -1$ and f is the identity on M^{-g}. Moreover, $hv = v$. Hence h restricts to $h_V \in O(V)$, meaning that $h \in S(M, V, v)$. Moreover, $(\vartheta \cdot \chi_V)(h) = (\vartheta \cdot \chi_V)(g) = 1$. Therefore $h \in G$, and it satisfies $\phi({}^G h) = \Gamma f V = \Gamma W$.

(3) Finally, we prove the injectivity of ϕ. In order to do so, we let $h_1, h_2 \in G$ and $f_1, f_2 \in S(M)$ be such that $g = {}^{f_1}h_1 = {}^{f_2}h_2$, and we suppose that there exists $t \in \Gamma = S(M^{g^2-1}, M^{-g})$ such that $tf_1V = f_2V$. We aim to show that $h_2 \in {}^G h_1$.

Recall from the proof of well-definedness of ϕ that $g, h_1, h_2 \in G$ with $g = {}^{f_1}h_1 = {}^{f_2}h_2$ implies that $f_1M^{-g} = M^{-g} = f_2M^{-g}$. In particular, since $tf_1V = f_2V$ and $tf_1M^{-g} = f_2M^{-g}$, we have that $tf_1v = \pm f_2v$. By similar argument as in the proof of [BC23, Theorem 4.7], we can actually assume that $tf_1v = f_2v$, up to composing t with an appropriate element of the joint stabilizer $S(M^{g^2-1}, M^{-g}, V)$. According to Proposition 8.2, we have that $M^{\Phi_m(g)}$ is unimodular, or p-elementary for some prime number p, and D_{g_m} has order at most 2. Hence, according to Remark 8.56, Theorem 8.21 and [BC23, Lemma 2.21], the map $SO(M^{\Phi_m(g)}, g_m) = O(M^{\Phi_m(g)}, g_m) \to O(D_{M^{\Phi_m(g)}}, D_{g_m})$ is surjective. Since M is unimodular, M^{g^2-1} and $M^{\Phi_m(g)}$ glue along their respective discriminant groups, and for any given glue map $\eta : D_{M^{g^2-1}} \to D_{M^{\Phi_m(g)}}$ there exists $t' \in SO(M^{\Phi_m(g)}, g_m)$ such that $\eta \circ D_t \circ \eta^{-1} = D_{t'} \in O(D_{M^{\Phi_m(g)}})$. Hence, by the equivariant gluing condition Equation (EGC), $t \oplus t' \in S(M^{g^2-1} \oplus M^{\Phi_m(g)})$ extends to an isometry $\widetilde{t} \in S(M)$ commuting with g. Therefore $u := f_2^{-1} \circ \widetilde{t} \circ f_1 \in S(M, V, v)$ and $h_2 = {}^u h_1$. Finally, we conclude by remarking that if $(\vartheta \cdot \chi_V)(u) \neq 1$, we can always compose u by $\widehat{\chi}(-\mathrm{id}_\Lambda) \oplus (-\mathrm{id}_\Lambda)$ to ensure that $u \in G$, where $\widehat{\chi}(-\mathrm{id}_\Lambda) = \mathrm{id}_V$ for $\mathcal{T} = \mathrm{OG6}$ and h_V otherwise. □

For the rest of this section, let g satisfy the assumptions of Theorem 8.59: we would like to describe a constructive way to obtain representatives for the classes in $\psi^{-1}({}^{S(M)}g)$ using ϕ.

Let us first suppose that $\mathcal{T} = \mathrm{K3}^{[n]}, \mathrm{Kum}_n$: according to Table 9, we have that V has rank 1. Thus, there exists a succession of primitive sublattices

$$M^{-g} \leq W \leq M^{g^2-1}$$

with $W \simeq V$ if and only if $W = M^{-g}$. In particular, the following holds.

Corollary 8.60. *Suppose that $\mathcal{T} = \mathrm{K3}^{[n]}, \mathrm{Kum}_n$, and let $m \geq 3$ even. The set of $\mathrm{Mon}^2(\Lambda)$-conjugacy classes of nonstable isometries $h \in \mathrm{Mon}^2(\Lambda)$ with minimal polynomial $\Phi_1\Phi_m$ such that Λ_h has signatures $(2, *)$ is in bijection with the set of $S(M)$-conjugacy classes of isometries $g \in G$ of minimal polynomial $\Phi_1\Phi_2\Phi_m$ such that $M^{\Phi_m(g)}$ has signatures $(2, *)$ and $M^{-g} \simeq V$. For any such isometry $g \in G$ corresponds the restriction h of g to $(M^{-g})^{\perp}_M \simeq \Lambda$.*

Proof. For $\mathcal{T} = \mathrm{K3}^{[n]}, \mathrm{Kum}_n$, Table 9 tells us that $v = 0$ and in particular $K = V$. Hence, for each $g \in G$ satisfying the assumptions of Theorem 8.59, we have that $V \simeq M^{-g}$ and the codomain of ϕ has cardinality 1. $\square$

Now suppose that $\mathcal{T} = \mathrm{OG6}$. In this case, $M^{-g} \simeq \langle 4 \rangle$ (Table 9). If we let $k \in M$ be such that $M^{-g} = \mathbb{Z}k$, we know that $V = \mathbb{Z}v + \mathbb{Z}\frac{v+k}{2}$ is a primitive sublattice of M^{g^2-1}. In fact, if $h \in \mathrm{Mon}^2(\Lambda)$ is so that $g = \gamma(h)$, by construction and the fact that $\Lambda_h = M^{\Phi_m(g)}$, we know that M^{g^2-1} is a primitive extension of $\Lambda^h \oplus V$. But now, $\frac{v+k}{2} \notin M^g \oplus M^{-g}$ and Figure 4 tells us that $M^{g^2-1}/(M^g \oplus M^{-g})$ has order 2. In particular, since $D_{M^{-g}} \cong \mathbb{Z}/4\mathbb{Z}$ as abelian groups, we infer that

$$M^{g^2-1} = (M^g \oplus M^{-g}) + \mathbb{Z}\frac{v+k}{2}. \tag{31}$$

Note that $v/2 \in (M^g)^\vee$ and $v^2 = 4$, so we have that the divisibility d of v in M^g is either 2 or 4.

Similarly, suppose that $\mathcal{T} = \mathrm{OG10}$. This time, $M^{-g} \simeq \langle 6 \rangle$ (Table 9) and the proof of Theorem 8.30 tells us that $M^{g^2-1}/(M^g \oplus M^{-g})$ has order 2. Similarly as before, we deduce that

$$M^{g^2-1} = (M^g \oplus M^{-g}) + \mathbb{Z}\frac{v+k}{2}. \tag{32}$$

This time, we observe that $v^2 = \mathrm{div}_{M^g}(v) = 2$.

Lemma 8.61. *Let L be an even lattice and let $v \in L$ be a primitive vector. We denote $I := v_L^\perp$, and we let d be the divisibility of v in L. Then, the index of $I \oplus \mathbb{Z}v$ in L is v^2/d.*

Proof. We have a succession of inclusions

$$\mathbb{Z}v \oplus I \leq L \leq L^\vee \leq (\mathbb{Z}v)^\vee \oplus I^\vee.$$

Let us denote by $\pi\colon L \to (\mathbb{Z}v)^\vee$ the first projection. We know that $[L\colon \mathbb{Z}v \oplus I]$ is equal to the order h of the glue domains for the primitive extension $\mathbb{Z}v \oplus I$. From Section 2 we know that such finite abelian groups are isomorphic to $\pi(L)/\mathbb{Z}v = (\mathbb{Z}(v/h))/\mathbb{Z}v$. Moreover, we observe

$$d\mathbb{Z} = v.L = v.\pi(L) = v.(\mathbb{Z}(v/h)) = (v^2/h)\mathbb{Z}.$$

Hence $h = v^2/d$. $\square$

Remark 8.62.

(1) For $\mathcal{T} = \mathrm{OG6}$ we know, according to the discussion prior to Claim 8.31, that $D_{M^g} = D_4 \oplus D_2$ where, as abelian groups, $D_4 \cong \mathbb{Z}/4\mathbb{Z}$ and $D_2 \cong (\mathbb{Z}/2\mathbb{Z})^n$ for some $n \geq 0$.

   (a) If v has divisibility 4 in M^g, Lemma 8.61 tells us that $M^g = v^\perp \oplus \mathbb{Z}v$ and $v^\perp$ is 2-elementary. Indeed, without loss of generality, we may assume that $v/4 + M^g$ generates D_4, in which case $D_{v^\perp} \simeq D_2$;

   (b) If v has divisibility 2 in M^g, we know that $2M^g \leq v^\perp \oplus \mathbb{Z}v$ and, $\mathbb{Z}v$ and $v^\perp$ glue along elementary abelian 2-groups. Since $D_{\mathbb{Z}v} \cong \mathbb{Z}/4$ as abelian groups (Table 9), we know that $\mathbb{Z}v$ and $v^\perp$ glue along order 2 subgroups of their respective discriminant groups.

(2) For $\mathcal{T} = \mathrm{OG10}$ we know, according to Theorem 8.30, that $D_{M^g} \cong \mathbb{Z}/2\mathbb{Z}$ as abelian groups. Since $v^2 = 2$ and $\mathrm{div}_{M^g}(v) = 2$, Lemma 8.61 tells us that $M^g = v^\perp \oplus \mathbb{Z}v$, and in particular, $v^\perp$ is even unimodular.

It is hard in general to compute representatives for the cosets in the codomain of ϕ (Theorem 8.59). In the next proposition, we would like to use the description given in Remark 8.62 in order to find an alternative way to describe such cosets, and thus use ϕ in a more explicit way.

Proposition 8.63. *Suppose that $\mathcal{T} = \mathrm{OG6}, \mathrm{OG10}$ and let $g \in G$ satisfy the assumptions of Theorem 8.59. We denote by $d \geq 2$ the divisibility of v in M^g. Then the sets of cosets*

$$C_1 := O(M^{g^2-1}, M^{-g}) \backslash \left\{ W \ : \ M^{-g} \leq W \leq M^{g^2-1} \text{ primitive},\ W \simeq V \right\}$$

and

$$C_2 := O\left(M^g, \tfrac{v}{2} + M^g\right) \backslash \left\{ \mathbb{Z}w \ : \ w \in M^g \text{ primitive},\ w^2 = v^2,\ \operatorname{div}(w, M^g) = d,\ w + 2M^g = v + 2M^g \right\}$$

are in bijection.

Proof. Let $I := v^{\perp}_{M^g}$, let again $k \in M$ be such that $M^{-g} = \mathbb{Z}k$, and let us define

$$\kappa \colon C_1 \to C_2,\ O(M^{g^2-1}, M^{-g}) \cdot W \mapsto O\left(M^g, \frac{v}{2} + M^g\right) \cdot (M^{-g})^{\perp}_{W}.$$

Let us first remark the following: since the gluing of $M^g \oplus M^{-g} \leq M^{g^2-1}$ is given by

$$v/2 \oplus M^g \mapsto k/2 + M^{-g}$$

(Equations (31) and (32)), we see that any isometry in $O(M^{g^2-1}, M^{-g})$ restricts to an isometry of M^g preserving $v/2 + M^g$ (Equation (EGC)), and vice-versa, any isometry in $O(M^g, v/2 + M^g)$ can be extended to an isometry in $O(M^{g^2-1}, M^{-g})$. More precisely, restriction to M^g induces a surjective group homomorphism

$$\pi \colon O(M^{g^2-1}, M^{-g}) \xrightarrow{f \mapsto f_{|M^g}} O(M^g, v/2 + M^g) \tag{33}$$

whose kernel is generated by $\mathrm{id}_{M^g} \oplus (-\mathrm{id}_{M^{-g}}) \cong O(M^{-g})$.

(1) Let us prove that κ is well-defined. Let $M^{-g} \leq W \leq M^{g^2-1}$ be a succession of primitive sublattices with $W \simeq V$. We let moreover $\mathbb{Z}w := (M^{-g})^{\perp}_{W} \leq M^g$ and $J := w^{\perp}_{M^g}$. Following the proof of surjectivity in Theorem 8.59, one can find an isometry $\overline{f} \colon V \to W$ preserving M^{-g}. In particular, $\overline{f}$ restricts to an isometry $\widetilde{f} \colon \mathbb{Z}v \to \mathbb{Z}w$, and

$$v^2 = w^2.$$

According to Remark 8.62, we know that either I and $\mathbb{Z}v$ are in orthogonal direct sum in M^g, or they glue along order 2 subgroups of their respective discriminant groups, which therefore have no nontrivial automorphisms. Similarly for J and $\mathbb{Z}w$. In particular, in both cases, we know that we can extend $\widetilde{f}$ to an isometry $\widehat{f} \in O(M^g)$ such that $\widehat{f}(v) = w$ and $\widehat{f}(I) = J$ (Lemma 2.19). Since M^g and M^{-g} also glue along subgroups of order 2 (see prior discussions), we have moreover that $f := \widehat{f} \oplus \overline{f}_{|M^{-g}}$ defines an isometry of M^{g^2-1} and it is an extension of $\overline{f}$ to M^{g^2-1} satisfying that $f(I) = J$. As a consequence, since f preserves M^g and $f(v) = w$, we obtain that

$$\operatorname{div}(w, M^g) = \operatorname{div}(v, M^g).$$

Now recall that $M^{g^2-1}/(M^g \oplus M^{-g})$ has order 2, generated by $\frac{v+k}{2} + (M^g \oplus M^{-g})$. Since f preserves $(M^g \oplus M^{-g})$, we know that the latter quotient is also generated by

$$f\left(\frac{v+k}{2} + (M^g \oplus M^{-g})\right) = \frac{w+k}{2} + (M^g \oplus M^{-g}).$$

This implies in particular that $\frac{v-w}{2} \in (M^g \oplus M^{-g}) \cap (M^g)^\vee = M^g$, and

$$v - w \in 2M^g.$$

Hence κ is well-defined.

(2) We show now that κ is surjective. Let $w \in M^g$ be a primitive vector so that $w^2 = v^2$, $\mathrm{div}(w, M^g) = d$ and $w + 2M^g = v + 2M^g$. We define $W := \mathbb{Z}w + \mathbb{Z}\frac{w+k}{2} \leq (M^g \oplus M^{-g})^\vee$: it is a primitive extension of $\mathbb{Z}w \oplus M^{-g}$, and we observe that W is a primitive sublattice of $M^{g^2-1} = (M^g \oplus M^{-g}) + \mathbb{Z}\frac{w+k}{2}$. Moreover, we have that $W \simeq V$. Hence, κ is surjective.

The injectivity of κ follows from the surjectivity of π (Equation (33)). □

Following Proposition 8.63 we see how to make Theorem 8.59 constructive even in the OG6 and OG10 cases. Indeed, the problem reduces to finding orbits of primitive vectors with given norm and divisibility which generate the glue domain of M^g for the primitive extension $M^g \oplus M^{-g} \leq M^{g^2-1}$. For each such an orbit of vectors $O\left(M^g, \frac{v}{2} + M^g\right) \cdot \mathbb{Z}w$, one reconstructs a representative for the corresponding monodromy conjugacy class by restricting g to the orthogonal complement of $\mathbb{Z}w \oplus M^{-g}$ in M.

8.2.3. Classification results

In this section, we prove Theorem 8.40, we make some comments about the geometric aspects of the classification of algebraically trivial automorphisms of IHS manifolds, and give some explicit examples.

8.2.3.1. Stable cohomological action

Using Proposition 4.19, Theorems 8.26, 8.27 and 8.34, we can establish a list of genera for the invariant and coinvariant sublattices associated to representatives of conjugacy classes of isometries g of any finite order $m \geq 3$, with minimal polynomial $\Phi_1\Phi_m$ and coinvariant sublattice of signatures $(2, *)$, on the Mukai lattice $M_{\mathcal{T}}$ for each known deformation type $\mathcal{T}$ of IHS manifolds. Before describing the tables of results, we remark the following.

Lemma 8.64. *Let $\mathcal{T}$ be one of the known deformation types. There are no IHS manifolds $X \sim \mathcal{T}$ admitting a nonsymplectic birational automorphism whose action on $H^2(X,\mathbb{Z})$ has order $m \in \{15, 20, 24, 30, 40, 48, 60\}$, and with minimal polynomial $\Phi_1\Phi_m$.*

Proof. Let $X \sim \mathcal{T}$ and let us suppose there exists $f \in \mathrm{Bir}(X)$ nonsymplectic such that $h := \rho_X(f)$ has order $m \in \{15, 20, 24, 30, 40, 48, 60\}$, and with minimal polynomial $\Phi_1\Phi_m$. Since f is nonsymplectic, we know that $H^2(X,\mathbb{Z})^{\Phi_m(h)}$ has signatures $(2, k)$ for some $k \geq 0$ so that $\varphi(m)$ divides $k + 2$ (see the proof of Proposition 6.5). Let $g := \gamma(h) \in O(M_{\mathcal{T}})$ be defined as in Lemma 8.51, where we replace $\Lambda_{\mathcal{T}}$ by $H^2(X,\mathbb{Z}) \simeq \Lambda_{\mathcal{T}}$. In particular, $(M_{\mathcal{T}}, g)$ is either a $\Phi_1\Phi_m^*$-lattice or a $\Phi_1\Phi_2\Phi_m^*$-lattice, and $M_{\mathcal{T}}^{\Phi_m(g)} = H^2(X,\mathbb{Z})^{\Phi_m(h)}$. According to Proposition 8.1 (3) and Proposition 8.2 (3), since m is neither a prime power nor twice a prime power, we have that $M_{\mathcal{T}}^{\Phi_m(g)}$ is even unimodular, of signatures $(2, k)$. Hence, Theorem 1.46 tells us in particular that $2 - k$ is divisible by 8, and so is $2 + k$ since $\varphi(m) \equiv 0 \mod 8$ by assumption. But this is absurd, since the two previous conditions would imply that 4 is divisible by 8. Hence such a birational automorphism $f \in \mathrm{Bir}(X)$ cannot exist. □

Lemma 8.65. *Let $\mathcal{T}$ be one of the known deformation types. There are no IHS manifolds $X \sim \mathcal{T}$ admitting a nonsymplectic birational automorphism whose action on $H^2(X,\mathbb{Z})$ is stable of order $m \in \{10, 26, 34, 38, 46, 50, 54\}$, and with minimal polynomial $\Phi_1\Phi_m$.*

Proof. We suppose existence, and we follow the same notation as in the proof of Lemma 8.64. This time, the isometry $g \in O(M_{\mathcal{T}})$ has minimal polynomial $\Phi_1\Phi_m$ by the stability assumption, its coinvariant sublattice $M_{\mathcal{T}}^{\Phi_m(g)}$ is still unimodular, of signatures $(2, k)$ for some $k \geq 0$. Now, note that for any known deformation type $\mathcal{T}$, the Mukai lattice $M_{\mathcal{T}}$ has negative signature strictly less than 22 (Table 9). Therefore, if $m = 26, 34, 38, 46, 50$ or 54 is twice an odd prime power with $\varphi(m) \geq 12$, we would need by Proposition 8.17 (3) that the rank of the previously mentioned coinvariant sublattice is a nontrivial multiple of $2\varphi(m) \geq 24$, meaning that $k \geq 22$, giving rise to a contradiction. Similarly, if $m = 10$, we would have that $2 + k$ is divisible by $2\varphi(10) = 8$: by similar arguments as in the proof of Lemma 8.64, we conclude that this is absurd. □

Remark 8.66. Following the arguments of the proof of Lemmas 8.64 and 8.65, it follows that if $X \sim \mathrm{Kum}_n$ for some $n \geq 2$ or $X \sim \mathrm{OG6}$, then X does not admit any nonsymplectic birational automorphism f such that $\rho_X(f)$ has order $m \in \{8, 14, 16, 18\}$ and minimal polynomial $\Phi_1\Phi_m$. Moreover, as a consequence of Proposition 8.18, there does not exist any IHS manifold X of known deformation type admitting a nonsymplectic birational automorphism f such that $\rho_X(f)$ is stable of order 32 and minimal polynomial $\Phi_1\Phi_{32}$.

Theorem 8.67. *Let (X, η) be a marked projective IHS manifold of known deformation type $\mathcal{T}$, and let $f \in \mathrm{Aut}(X) \setminus \ker \rho_X$ be purely nonsymplectic and algebraically trivial. Let $M := M_{\mathcal{T}}$ be the corresponding Mukai lattice. Suppose $h := \rho_X(f) \in \mathrm{Mon}^2(X)$ is stable of order $m \geq 3$ nonprime, and let $g := h \oplus \mathrm{id}_{\Lambda_M^\perp} \in O(M)$. Then the $O(M)$-conjugacy class of $\langle g \rangle$ is uniquely determined by the order of g, the genus of M^g and the genus of M_g.*

Proof. We use Lemmas 8.64 and 8.65, as well as Remark 8.66 to restrict the values of m depending on $\mathcal{T}$. We then apply Theorem 8.26 and Theorem 8.27 to the remaining orders to determined the genera of M^g and M_g. For each known deformation type $\mathcal{T} \neq \mathrm{K3}$ and each possible orders m, we record the previous pairs of genera in Tables 17a to 17d. Together with Remark 8.48, Theorem 8.34 and Proposition 4.19 we are able to conclude that given m, the genus of M^g and the genus of M_g, there exists a unique $O(M)$-conjugacy class of cyclic subgroup $\langle g' \rangle \leq O(M)$ such that g' has minimal polynomial $\Phi_1\Phi_m$, and $M^{g'}$ and $M_{g'}$ are in the same genera as M^g and M_g respectively. □

For the case of K3 surfaces, we do not display our result in a proper table since their algebraically trivial automorphisms are known (see for instance [AST11] and the reference therein for the case of prime order, and [Kon92, Tak12] for the other orders).

8.2.3.2. Nonstable cohomological action

We recall that according to Proposition 8.54, there are only finitely many deformation types $\mathcal{T}$ for which there exists an algebraically trivial nonsymplectic automorphism whose action on cohomology is nonstable of finite order m. The possible pairs $(\mathcal{T}, m)$ were given in Table 10. Furthermore, as for the case of stable cohomological actions, we can already discard some pairs $(\mathcal{T}, m)$.

Proposition 8.68. *Let $(\mathcal{T}, m)$ be one of*

$$(\mathrm{OG10}, 54),\ (\mathrm{K3}^{[4]}, 54),\ (\mathrm{K3}^{[12]}, 22),\ (\mathrm{K3}^{[18]}, 34),\ (\mathrm{K3}^{[20]}, 38),\ (\mathrm{Kum}_2, 6),\ (\mathrm{Kum}_4, 10),\ \mathrm{Kum}_6, 14).$$

There exists no IHS manifold $X \sim \mathcal{T}$ with a nonsymplectic birational automorphism whose action on $H^2(X, \mathbb{Z})$ is nonstable of finite order m, and with minimal polynomial $\Phi_1\Phi_m$.

Proof. Let $(\mathcal{T}, m)$ be one of the pairs given in the statement. Let $X \sim \mathcal{T}$ and suppose that there exists $f \in \mathrm{Bir}(X)$ nonsymplectic such that $h := \rho_X(f)$ is nonstable of finite order m and minimal polynomial $\Phi_1\Phi_m$. First note the following: the restriction of h to $\Lambda_{\mathcal{T}}^{\Phi_m(h)}$ has determinant 1 according to [BC23, Lemma 2.21], meaning that $\det(h) = +1$. In that case, we know that $\det(h) \cdot D_h = -\mathrm{id}_{D_{\Lambda_{\mathcal{T}}}}$: thus $h \notin \mathrm{Mon}^2(\Lambda_{\mathcal{T}})$ for $\mathcal{T} = \mathrm{Kum}_2, \mathrm{Kum}_4, \mathrm{Kum}_6$ (Table 9). Hence, for the three last pairs $(\mathcal{T}, m)$ of the statement, such a pair (X, f) cannot exist.

Now, as in the proof of Lemma 8.64, we let $g := \gamma(h) \in O(M_{\mathcal{T}})$: the pair $(M_{\mathcal{T}}, g)$ is a $\Phi_1\Phi_2\Phi_m^*$-lattice, and the $\mathbb{Z}$-lattice $M_{\mathcal{T}}^{\Phi_m(g)} = H^2(X, \mathbb{Z})^{\Phi_m(h)}$ has signatures $(2, k)$ for some $k \geq 1$ (because f is nonsymplectic). Now according to Theorem 8.30, since $m = 2p^l$ is twice an odd prime power, we have that there exists $n_0 \geq 0$ such that

$$4 \equiv \varphi(p^l)(n_0 + 1) + 2\left(\frac{-2}{p}\right) - 1 - p \mod 8. \tag{34}$$

Since moreover $\varphi(m) = \varphi(p^l)$ divides $2 + k \leq 22$ (Table 9), we see that $n_0 = 0$ except when $(\mathcal{T}, m) = (\mathrm{K3}^{[12]}, 22)$ where $n_0 \in \{0, 1\}$. In particular, it follows that in all cases, the righthand side of Equation (34) is $(0, 2 \mod 8)$, which is absurd. Hence such a pair (X, f) cannot exist. □

Proof of Theorem 8.40. According to Theorem 8.47 and Remark 8.48, it is enough to classify $\mathrm{Mon}^2(\Lambda_{\mathcal{T}})$-conjugacy classes of nonstable isometries $h \in \mathrm{Mon}^2(\Lambda_{\mathcal{T}})$ of finite order m and minimal polynomial $\Phi_1\Phi_m$ such that $\Lambda_{\mathcal{T}}^{\Phi_m(h)}$ has signatures $(2, *)$. For this, we proceed as in the proof of Theorem 8.67. We use first Propositions 8.54 and 8.68 to restrict to the pairs $(\mathcal{T}, m)$ for which such an isometry h can exist, and let us fix such a pair $(\mathcal{T}, m)$ for the sake of the proof. Let $h \in \mathrm{Mon}^2(\Lambda_{\mathcal{T}})$ be a nonstable isometry of order m, and minimal polynomial $\Phi_1\Phi_m$. We let moreover $g := \gamma(h) \in \mathrm{Mon}^2(\Lambda_{\mathcal{T}})$. Using Theorems 8.30 and 8.33, we determine the possible genera for the kernel sublattices $M_{\mathcal{T}}^g$ and $M_{\mathcal{T}}^{\Phi_m(g)}$, based on the fact that the isometry class of $M_{\mathcal{T}}^{-g}$ is fixed by $\mathcal{T}$. We observe in particular that in all cases, $M_{\mathcal{T}}^g$ is unique in its genus, up to isometry (see Table 18 for a description of the genus of $M_{\mathcal{T}}^g$ in all cases, by the means of symbols). Moreover, the hermitian structure of $M_{\mathcal{T}}^{\Phi_m(g)}$ is indefinite or of rank 1 by the assumption on the real signatures of $M_{\mathcal{T}}^{\Phi_m(g)}$. Therefore, together with Theorem 8.36 and Proposition 4.19, we infer that there is exactly one $O(M)$-conjugacy class of finite order isometries $g' \in O(M)$ such that $M_{\mathcal{T}}^g$ and $M_{\mathcal{T}}^{g'}$ are in the same genus, and such that the hermitian structures of $(M_{\mathcal{T}}^{\Phi_m(g)}, g_m)$ and $(M_{\mathcal{T}}^{\Phi_m(g')}, g'_m)$ are in the same genus, except when $m = 46$ where there are 3 such classes (Remark 8.35). Using the fact that $O(M_{\mathcal{T}}^{\Phi_m(g)}, g_m) = SO(M_{\mathcal{T}}^{\Phi_m(g)}, g_m)$ [BC23, Lemma 2.21], one can actually show in the proof of Theorem 8.36 that the $O(M)$-conjugacy class and the $SO(M)$-conjugacy class of g are the same. We conclude by applying Theorem 8.59 together with Corollary 8.60 and Proposition 8.63. The results are presented in Table 18. □

Notation. The $\mathbb{Z}$-lattices described in Table 18 are given in terms of a representative of their isometry class. We try as much as possible to choose a representative which is a direct sum of (rescaled) ADE root lattices and (rescaled) hyperbolic plane lattices U. However, we sometimes have to resort to other well known $\mathbb{Z}$-lattices which do not fit in the previous list. In particular, we fix

$$H_5 := \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad K_7 := \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}, \quad K_{23} := \begin{pmatrix} -12 & 1 \\ 1 & -2 \end{pmatrix}.$$

Moreover, we denote L_8^5 and L_8^{13} negative definite rank 8 $\mathbb{Z}$-lattices of determinant 5 and 13 respectively.

Remark 8.69. The square of the isometries representing the entries of Table 18 are known already, as well as the involutions they induce. However, it is more difficult to construct geometric examples realizing any of these cases.

8.2.3.3. About induced actions

For every known deformation type $\mathcal{T}$, one can construct an example of IHS manifold $X \sim \mathcal{T}$ by considering certain moduli space of stable sheaves on some projective K-trivial surface (see for instance [PR13] and the references therein). Let us review briefly such a construction for $\mathcal{T} = \mathrm{K3}^{[n]}$, Kum_n, and let us comment on a natural way of defining *induced actions* through such constructions.

Let S be a projective K3 surface, or an abelian surface. Let us denote $\widetilde{H}(S,\mathbb{Z}) := H^0(S,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$, and write any element $v \in \widetilde{H}(S,\mathbb{Z})$ as (v_0, v_1, v_2) where $v_1 \in H^2(S,\mathbb{Z})$ and $v_0, v_2 \in \mathbb{Z}$. Recall that for a projective K3 surface, $H^2(S,\mathbb{Z})$ is equipped with a even unimodular quadratic form q_S of real signatures $(3,21)$ (Table 4). Similarly, we have that if S is an abelian surface, then $H^2(S,\mathbb{Z})$ has also an indivisible even quadratic form q_S turning $(H^2(S,\mathbb{Z}), q_S)$ into an even $\mathbb{Z}$-lattice, isometric to $U^{\oplus 3}$. In both cases S a projective K3 surface or an abelian surface, we have that $H^0(S,\mathbb{Z})$ and $H^4(S,\mathbb{Z})$ are 1-dimensional. We can endow $\widetilde{H}(S,\mathbb{Z})$ with a nondegenerate symmetric bilinear pairing b defined as

$$b((v_0, v_1, v_2), (v_0', v_1', v_2')) := q_S(v_1, v_1') - v_0 v_2' - v_2 v_0'.$$

The free $\mathbb{Z}$-module $\widetilde{H}(S,\mathbb{Z})$ equipped with the form b is an indivisible even $\mathbb{Z}$-lattice, and

$$(\widetilde{H}(S,\mathbb{Z}), b) \simeq \begin{cases} U^{\oplus 4} \oplus E_8^{\oplus 2} & \text{if} \quad S \text{ is a projective K3 surface} \\ U^{\oplus 4} & \text{if} \quad S \text{ is an abelian surface} \end{cases}.$$

Remark 8.70. These are often called *Mukai lattices*, and in fact, their description coincides with the definitions of Mukai lattices we have given for the deformation types $\mathrm{K3}^{[n]}$ and Kum_n (Section 8.2.2, Table 9).

There is a pure Hodge structure of weight 2 on $\widetilde{H}(S,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ defined as

$$\widetilde{H}^{2,0}(S) := H^{2,0}(S), \ \widetilde{H}^{0,2}(S) := H^{0,2}(S), \ \text{and } \widetilde{H}^{1,1}(S) := H^0(S,\mathbb{C}) \oplus H^{1,1}(S) \oplus H^4(S,\mathbb{C}).$$

This Hodge structure is polarized with respect to the form q.

Definition 8.71. Let $v = (v_0, v_1, v_2) \in \widetilde{H}(S,\mathbb{Z})$. We call v a *Mukai vector* if one of the following holds

(1) $v_0 > 0$;

(2) $v_0 = 0$ and $0 \neq v_1 \in \mathrm{NS}(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$ is the first Chern class of an effective divisor;

(3) $v_0 = v_1 = 0$ and $v_2 > 0$.

Let $v \in \widetilde{H}(S,\mathbb{Z})$ be a primitive Mukai vector such that $v^2 := b(v,v) \geq 2$, and let τ be a v-generic Bridegland stability condition on $D^b(S)$ [BM14, Definition 2.4]. Then, according to [BM14, Theorem 5.9], the coarse moduli space $M_\tau(v)$ of τ-stable objects F in $D^b(S)$ so that

$$(\mathrm{rank}(F), c_1(F), \mathrm{ch}_2(F) + \epsilon \, \mathrm{rank}(F)) = v,$$

where $\epsilon = 1$ for S a K3 surface and $\epsilon = 0$ for S an abelian surface, is a smooth quasiprojective variety of dimension $v^2 + 2$. Moreover, if S is a projective K3 surface, then $M_\tau(v)$ is an IHS manifold of $\mathrm{K3}^{[\frac{v^2+2}{2}]}$-type. If S is an abelian surface, similarly to the generalized Kummer constructions (Paragraph 5.2.1.2), there is an Albanese map

$$a_v \colon M_\tau(v) \to S \times \mathrm{Pic}^0(S)$$

such that the fiber $K_\tau(v) := a_v^{-1}(0_S, \mathcal{O}_S)$ is an IHS manifold of $\mathrm{Kum}_{\frac{v^2-2}{2}}$-type, provided $v^2 \geq 6$ [Yos16, Theorem 1.13].

Definition 8.72 ([MW15, Definition 4.1])**.** Let $n \geq 2$, let $X \sim \mathrm{K3}^{[n]}$ and let $G \leq \mathrm{Aut}(X)$. The group G is said to be an *induced group of automorphisms* if there exists a projective K3 surface S with $G \leq \mathrm{Aut}(S)$, a G-invariant primitive Mukai vector $v \in \widetilde{H}(S, \mathbb{Z})$ and a v-generic stability condition τ such that X is isomorphic to the moduli space $M_\tau(v)$ of τ-stable sheaves on S, and the action of G induced on $M_\tau(v)$ by pullback of sheaves coincides with that of G on X. A similar definition hold for $X \sim \mathrm{Kum}_n$ and S an abelian surface.

Note that given a projective K3 or abelian surface S, a primitive Mukai vector $v \in \widetilde{H}(S, \mathbb{Z})$ and a v-generic stability condition τ, one has an integral Hodge isometry

$$H^2(M_\tau(v), \mathbb{Z}) \to v^\perp \leq \widetilde{H}(S, \mathbb{Z}) \ \text{ (resp. } H^2(K_\tau(v), \mathbb{Z}) \to v^\perp \leq \widetilde{H}(S, \mathbb{Z})).$$

This Hodge isometry is equivariant with respect to induced actions, and in particular, any induced action must restrict to the identity on the discriminant group of $H^2(M_\tau(v), \mathbb{Z})$ (resp. of $H^2(K_\tau(v), \mathbb{Z})$) since the chosen vector v is invariant in $\widetilde{H}(S, \mathbb{Z})$ [MW15, §2]. Moreover:

Proposition 8.73 ([MW15, Theorems 4.4 & 4.5],[BC23, Proposition 4.8])**.** *Let $n \geq 2$, let $\mathcal{T} := \mathrm{K3}^{[n]}, \mathrm{Kum}_n$, and let $g \in O(M_\mathcal{T})$ satisfy the assumptions of Theorem 8.57. Then the monodromy classes in the fiber $\psi^{-1}({}^{S(M_\mathcal{T})}g)$ (see notation Theorem 8.57) admit a geometric realization as actions of induced automorphisms (in the sense of Definition 8.72) if and only if $M_\mathcal{T}^g$ contains a copy of U as a direct summand.*

Remark 8.74. There is a generalization of this criterion where one considers a primitive embedding of a rescaled copy $U(k)$ of U in $M_\mathcal{T}^g$ instead, for some $k \geq 2$. These correspond to induced actions on moduli spaces of k-twisted stable sheaves [CKKM19, §3].

Remark 8.75. One can extend the previous definition and proposition for detecting induced actions on IHS manifolds of deformation type OG6 and OG10 (see [Gro22b, §3.2] for the OG6 case and [MW15, §5] for the OG10 case). In both cases, the induced groups of Hodge monodromies are again stable.

Besides the previous notion of induced automorphisms, there are other ways of constructing geometric examples of birational automorphisms of IHS manifolds through given constructions (see for instance [Bea83b, §6], [O'G06, §4], [OW13, §4], [CC19, §4.1], [IKKR19, §3], [Gro22b, §4] or [Sac23, §3.1]). For most of these actions though, we do not have numerical criteria to decide whether certain lattice data correspond to such actions. Moreover, for most of the cases the induced action on the discriminant group of the associated BBF form is trivial. Nonetheless, there are known geometric examples of nonstable birational automorphisms for which it is not known whether they can be realized as induced in any meaningful way (see for instance [Fer12, §4], [MW15, Corollary 5.11] or [CCL22, Theorems 1.1 and 1.2]).

Remark 8.76. Studying induced actions is interesting for geometric reasons. In order to study the fixed loci of certain automorphisms of IHS manifolds, it is often easier to first consider the case of induced actions as one can expect to relate the geometry of the fixed locus of the induced action and the geometry of the fixed locus of the original action. We refer for instance to [BCS16, Examples 6.4–6.7] where the authors study the fixed loci of actions on Fano varieties of lines on cubic fourfolds, induced from automorphisms of the underlying cubic fourfolds. Moreover, in dimension higher than 2, it is often hard to construct geometric examples of automorphisms of IHS manifolds. Therefore, being able to deduce that certain Hodge monodromies of the associated BBF forms can be realized by some automorphisms induced in a meaningful way is an interesting result on its own.

For the Mukai lattice $U^{\oplus 4}$ (Table 17a), we specify the smallest integer $k \geq 1$ for which the action can be realized as induced on a moduli space of k-twisted sheaves on an abelian surface (see Paragraph 8.2.3.3). If $k = 1$, then the action can also be realized as induced on an OG6-type IHS manifolds, and we set $k = 0$ if the action cannot be induced (in the sense of Paragraph 8.2.3.3). Similarly for the Mukai lattice $U^{\oplus 4} \oplus E_8^{\oplus 2}$ (Table 17c). Note that for these two tables, whenever $k = 1$, then the action can also be realized by *natural automorphisms* (see [Boi12, Définition 1], [BS12] and [BNWS11, §3.1]). Finally, in the cases of order 4 for which $k = 2$ in Table 17c, the first in the list is induced for $\mathrm{K3}^{[n]}$-type IHS manifolds if and only if n is odd, while the other one is induced for all $n \geq 2$.

Remark 8.77. In the case of isometries of composite order m (i.e. m is not a prime power), we observe that all the cases in Table 17c can be realized as natural automorphisms for all $\mathcal{T} = \mathrm{K3}^{[n]}$ with $n \geq 2$. In particular, we recover the classification of Kondō for algebraically trivial nonsymplectic automorphisms of composite order, in the case of K3 surfaces [Kon92].

8.2.4. Geometric examples

In Section 8.2, we have seen how to classify, at the level of periods, algebraically trivial nonsymplectic actions for the known IHS manifolds. In this section, we give two geometric examples realizing two of the nonstable cases in Table 18.

Example 8.78 (($\mathrm{K3}^{[4]}, 6$) with $\Lambda^g \simeq \langle 2 \rangle$)**.** Let $\mathcal{C}$ be the 10-dimensional family of smooth cubic fourfolds $V \subseteq \mathbb{P}^5_{\mathbb{C}}$ of the form

$$x_5^3 + F_3(x_0, x_1, x_2, x_3, x_4) = 0$$

where F_3 is a homogeneous polynomial of degree 3 in 5 variables. Any V in this family is preserved by the projective change of coordinates

$$g : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : x_1 : x_2 : x_3 : x_4 : \zeta_3 x_5)$$

where ζ_3 is a primitive third root of unity.

Now, let $V \in \mathcal{C}$ be a general element of this family not containing a plane. The authors in [CC19, §4.1] show that the LLSvS eightfold $Z_V \sim \mathrm{K3}^{[4]}$ associated to V (see [LLSvS17, Theorems A and B] for a definition) has an induced nonsymplectic automorphism $\widetilde{g} \in \mathrm{Aut}(Z_V)$ whose action on $H^2(Z_V, \mathbb{Z})$ only fixes the vector δ representing the polarization of the projective manifold Z_V. Their argument actually shows that $\mathrm{NS}(Z_V) = H^2(Z_V, \mathbb{Z})^{\rho_{Z_V}(\widetilde{g})} = \mathbb{Z}\delta \simeq \langle 2 \rangle$. Note that the vector $\delta \in 2H^2(Z_V, \mathbb{Z})^\vee$ and in particular $H^2(Z_V, \mathbb{Z}) = \mathrm{NS}(Z_V) \oplus \mathrm{T}(Z_V)$ where $\mathrm{T}(Z_V) \simeq U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$. Following [LPZ23, §5.2], the involution τ defined by

$$\tau \colon H^2(Z_V, \mathbb{Z}) \to H^2(Z_V, \mathbb{Z}),\ v \mapsto -v + (v.\delta)\delta$$

is a nonsymplectic Hodge monodromy and it commutes with the induced action of $\widetilde{g}$ on $H^2(Z_V, \mathbb{Z})$. By its description, we also have that D_τ is nontrivial. Hence, the composition $\tau \circ \rho_{Z_V}(\widetilde{g})$ is a Hodge monodromy and it is realized as a nonstable algebraically trivial purely nonsymplectic automorphism of order 6 on $Z_V \sim \mathrm{K3}^{[4]}$ with associated invariant sublattice $\langle 2 \rangle$. This also shows that the involution of [LLMS18, Lemma 3.7] whose cohomological action coincides with τ actually commutes with the induced automorphism $\widetilde{g}$.

Example 8.79 ($(\mathrm{OG6}, 8)$ with $\Lambda^g \simeq U \oplus \langle -2\rangle^{\oplus 2}$)**.** In [Gro22b, Theorem 1.4], Grossi proves the existence of an OG6-type IHS manifold X equipped with a nonsymplectic involution τ such that $H^2(X,\mathbb{Z})^{\rho_X(\tau)} \simeq U \oplus \langle -2\rangle^{\oplus 2}$ and $H^2(X,\mathbb{Z})_{\rho_X(\tau)} \simeq U \oplus U(2)$. Grossi also shows, using some numerical criteria, that X is a *numerical moduli space* [Gro22b, Definition 3.2] and the involution τ can be geometrically realized as induced from an associated abelian surface. Moreover, she shows that X admits an *MRS birational model* [MRS18, Gro22b], which means that X is birational to the quotient of an IHS manifold $Y \sim \mathrm{K3}^{[3]}$ by a birational symplectic involution i, after contracting the 256 $\mathbb{P}^3$'s along which i is not defined.

It turns out that on this birational model, the involution τ can be realized as induced from a nonsymplectic involution on Y. Since the numerical criteria from [Gro22b, Theorem 1.3] are independent on the order and the action on the discriminant group, we obtain that up to deformation τ is the fourfold iterate of an algebraically trivial nonsymplectic automorphim of order 8, with the same invariant and coinvariant sublattices. Note that in Table 18 there exists a pair $(\mathrm{K3}^{[3]}, 8)$ with the same coinvariant sublattice $U \oplus U(2)$; however there is no reason for this to be linked with the previous automorphism of order 8 on X. In fact, since we know the action on cohomology for the birational symplectic involution on $Y \sim \mathrm{K3}^{[3]}$ from the MRS model, one can actually show that Y cannot admit an algebraically trivial nonsymplectic automorphism of order 8 with such a lattice action.

Remark 8.80. From Table 18, we observe that the cases

$$(\mathcal{T}, m, \Lambda^g) = (\mathrm{K3}^{[3]}, 4, \langle 4\rangle),\ (\mathrm{K3}^{[24]}, 46, \langle 2\rangle)$$

could deserve some attention. In fact, in such cases, there is a canonical invariant vector of positive norm, smaller than 4. Moreover, the general element of the associated deformation family in moduli has Picard rank 1. In Example 11.13 we explain how one could possibly realize the case $(\mathrm{K3}^{[3]}, 4, \langle 4\rangle)$ working with a special 9-dimensional family of *double EPW-cubes*. Moreover, in Example 12.4 we show how to realize geometrically the case $(\mathrm{OG10}, 6, U)$ from Table 18.

Computational comments. As we have seen in Section 8.1, a theoretical classification of conjugacy classes of isometries for even $\mathbb{Z}$-lattices, with given characteristic polynomial, is quite complicated. Even for isometries of finite order whose minimal polynomial has few irreducible factors, no existence results are known for general even $\mathbb{Z}$-lattices. In [BH23], Brandhorst and Hofmann suggest a new partial solution to the problem of classifying conjugacy classes of finite order isometries of even $\mathbb{Z}$-lattices, by transforming it into a recursive problem. Given an even $\mathbb{Z}$-lattice L which is unique in its genus, the authors describe algorithms to enumerate representatives of $O(L)$-conjugacy classes of isometries of finite order. We have already seen an application of the Brandhorst–Hofmann algorithms to enumerate prime order isometries of even definite $\mathbb{Z}$-lattices in a given genus (see Section 1.5). The primary purpose of these algorithms though is the application to the extension approach described in the next section of this thesis. In fact, this was part of the technical support to the classification of finite group actions on K3 surfaces in [BH23, Theorem 1.2]. A particular case in which one can observe the potential of this enumeration procedure is for the classification of cyclic actions on varieties fitting in a Torelli setting similarly to Section 6.2. Examples of such varieties are *irreducible symplectic varieties* (see [Per20] for a survey), cubic fourfolds or Enriques surfaces. See [BMM24] for a more recent application to the deformation family of Nikulin-type orbifolds.

9. Extension procedures

In Sections 7 and 8, we have seen how to solve the classification problems (StS), (SC) and (PNS) described in Section 6.3. In order to complete our answer to the classification of finite groups of birational automorphisms for IHS manifolds, we would need to cover the classification problems (S) and (M). We treat this two problems similarly, using some extension procedure as introduced by Brandhorst and Hashimoto in [BH21]. Applying such a procedure, first for the problem (S) and then to the problem (M), brings an end to the computational solution described in this thesis.

In [BH21], Brandhorst and Hashimoto classified, up to conjugacy, pairs (S, G) consisting of a projective K3 surface S and a finite subgroup $G \leq \mathrm{Aut}(S)$ such that the normal subgroup $G_s \trianglelefteq G$ of symplectic automorphisms is one of Mukai's maximal group (Theorem 7.1). They show in particular.

Theorem 9.1 ([BH21, Theorem 1.1])**.** *There exist exactly 42 isomorphism classes of pairs* (S, G) *such that* S *is a projective K3 surface and* $G \leq \mathrm{Aut}(S)$ *is so that* $1 \neq G_s < G$ *is a proper subgroup isomorphic to one of the maximal groups from Mukai's list (Theorem 7.1).*

Their techniques have been revisited by the independent works of [CDQM25] and [Waw23] in the case of maximal mixed actions on IHS manifolds of deformation type $\mathrm{K3}^{[2]}$. After the series of technical works [BH21, BHM22, BV24], this classification procedure culminated in the recent major algorithmic progress of Brandhorst and Hofmann [BH23] where the authors completed a long-standing classification effort for finite mixed actions on K3 surfaces.

Theorem 9.2 ([BH23, Theorem 1.2])**.** *Up to birational conjugacy, there exist exactly 4167 deformation classes of pairs* (S, G) *where* S *is a projective K3 surface and* $G \leq \mathrm{Aut}(S)$ *is mixed and finite, and such that* G_s *is saturated.*

From a computational point of view, as it is currently described, this procedure applies for all known deformation types of IHS manifolds, and related varieties, with maximal monodromy. This is the case for the deformation types OG6, OG10 and $\mathrm{K3}^{[p^k+1]}$ where p is prime and $k \geq 0$, for cubic fourfolds, for Enriques surfaces, and for Nikulin-type orbifolds [BMM24]. Nonetheless, we give a more general description which only rely on monodromy being normal in the full group of isometries of the associated $\mathbb{Z}$-lattices. This is the case for all the known examples of IHS manifolds and related varieties for which monodromy has been explicitly computed.

Throughout, let us fix $\mathcal{T}$ a known deformation type of IHS manifolds, and let $\Lambda := \Lambda_{\mathcal{T}}$ be the associated even $\mathbb{Z}$-lattice (which is unique in its genus).

Notation. For a $\mathbb{Z}$-lattice $\Lambda' \simeq \Lambda$, fix an isometry $f\colon \Lambda' \to \Lambda$ and define $\mathrm{Mon}^2(\Lambda') := f^{-1}\,\mathrm{Mon}^2(\Lambda) f$. This definition does not depend on f since $\mathrm{Mon}^2(\Lambda) \trianglelefteq O(\Lambda)$ is normal. Similarly, we let $\mathcal{W}(\Lambda') := f^{-1}\mathcal{W}(\Lambda)$ and $\mathcal{W}^{pex}(\Lambda') := f^{-1}\mathcal{W}^{pex}(\Lambda)$.

In the next section, we review the original procedure of Brandhorst–Hashimoto–Hofmann (abbreviated BHH-procedure) for recovering finite effective subgroups of $\mathrm{Mon}^2(\Lambda)$ from symplectic ones, in a more general setting than the one of K3 surfaces. We later explain how their approach has been adapted in [MM25b] to obtain such symplectic finite subgroups of $\mathrm{Mon}^2(\Lambda)$ from stable symplectic ones. We apply the latter to the classification of stable symplectic groups for the deformation type OG10, obtained in Section 7.5.2: we describe in particular two simple algorithms to perform the actual computations.

9.1. Brandhorst–Hashimoto–Hofmann procedure

The content of this section draws from the methods developed in [BH23].

Let $H \leq \mathrm{Mon}^2(\Lambda)$ be a finite effective subgroup, and let us denote by

$$H_s := \{h \in H \ : \ \Lambda^h \text{ has signatures } (3, *)\}$$

the associated symplectic subgroup. For a classification purpose, we assume that H_s is saturated in $\mathrm{Mon}^2(\Lambda)$.

Suppose that $H \neq H_s$ and we denote $n := [H : H_s] \geq 2$ (see the Torelli setting from Section 6). The quotient H/H_s is cyclic generated by the coset represented by a nonsymplectic isometry $h \in H \setminus H_s$. The restriction a of h to Λ^{H_s} has order n, and $a \in O(\Lambda^{H_s})$ does not depend on a choice of a representative of hH_s. Remark that any choice of a representative $h' \in hH_s$ determines an isometry $b \in O(\Lambda_{H_s})$ and an equivariant primitive extension $(\Lambda^{H_s}, a) \oplus (\Lambda_{H_s}, b) \leq (\Lambda, h')$ with associated glue map γ.

Remark 9.3. Even though the equivariant primitive extension

$$(\Lambda^{H_s}, a) \oplus (\Lambda_{H_s}, b) \leq (\Lambda, h')$$

and $b \in O(\Lambda_{H_s})$ depend on the choice of h', the associated glue map γ does not depend on any $h' \in hH^\#$ or even on a. Indeed, γ is the glue map of the primitive extension $\Lambda^{H_s} \oplus \Lambda_{H_s} \leq \Lambda$.

Definition 9.4. Let $H \leq \mathrm{Mon}^2(\Lambda)$ be an effective finite subgroup such that $H_s < H$ is saturated in $\mathrm{Mon}^2(\Lambda)$. We define

(1) the *heart* of H to be the primitive sublattice $\Lambda_{H_s} \leq \Lambda$;

(2) the *head* of H to be the pair (Λ^{H_s}, a);

(3) the *spine* of H to be the glue map γ.

Recall that given an even $\mathbb{Z}$-lattice L and a primitive sublattice $C \leq L$, we define the pulse of C in a normal subgroup $N \trianglelefteq O(L)$, denoted $P_N(C)$, to be the pointwise stabilizer of $C^\perp$ in N. Its restriction to C is the largest subgroup of $O(C)$ which can be extended with $\mathrm{id}_{C^\perp}$ to a subgroup of N. By abuse of notation, we often denote this pulse by $P_N(C) \leq O(C)$ too.

Remark 9.5. Note that our definition of hearts in Definition 9.4 agrees with the one given in [BH23, Definition 3.17]. In fact, recall that $\mathrm{Mon}^2(\Lambda)$ is normal in $O(\Lambda)$ for all the known deformation types of IHS manifolds. Hence, since as $H_s \leq \mathrm{Mon}^2(\Lambda)$ in Definition 9.4 is saturated, we have that it is equal to the pulse $P_{\mathrm{Mon}^2(\Lambda)}(\Lambda_{H_s})$ of the primitive sublattice $\Lambda_{H_s} \leq \Lambda$ in $\mathrm{Mon}^2(\Lambda)$. In particular, the $\mathrm{Mon}^2(\Lambda)$-conjugacy class of the group H_s is completely determined by the $\mathrm{Mon}^2(\Lambda)$-isomorphism class of the primitive sublattice $\Lambda_{H_s} \leq \Lambda$.

Proposition 9.6. *Let $H, H' \leq \mathrm{Mon}^2(\Lambda)$ be two effective finite subgroups such that H_s and H'_s are both saturated in $\mathrm{Mon}^2(\Lambda)$, and let $\psi \in \mathrm{Mon}^2(\Lambda)$ conjugate H and H'. Then:*

(1) *ψ restricts to an isometry between the respective hearts of H and H';*

(2) *ψ induces an isomorphism between the respective heads of H and H'.*

Proof. The first claim follows by definition of ψ, and the second point follows from the fact that the definition of the heads does not depend on the choice of a representative for the generators of H/H_s and H'/H'_s respectively. $\square$

From Proposition 9.6 one sees that conjugate finite effective subgroups of $\mathrm{Mon}^2(\Lambda)$ with saturated symplectic subgroups share the same hearts and heads. We would like to measure to which extent the converse holds, and describe a procedure to compute a complete list of representatives of conjugacy classes of groups with given heart and head. This is the content of what we call the *extension procedure*.

The strategy behind the extension procedure is to first make the definitions of heads and hearts independent of a given finite subgroup of $\mathrm{Mon}^2(\Lambda)$. Then starting from a potential heart $C \leq \Lambda$, we first classify pairs of potential heads (F, a) with $F \simeq C^{\perp}_{\Lambda}$ and $a \in O(F)$ of given order n. The next step is to classify, for each candidate head (F, a), equivariant primitive extensions $(F, a) \oplus (C, b) \leq (\Lambda, h)$ where $b \in O(C)$ and h is effective. This is done by constructing representatives for the potential spines. Eventually, one reconstructes a finite subgroup $H := \langle h, P_{\mathrm{Mon}^2(\Lambda)}(C)\rangle$ such that $H_s = P_{\mathrm{Mon}^2(\Lambda)}(C)$ is saturated in $\mathrm{Mon}^2(\Lambda)$ and H is equipped with a nontrivial character χ whose kernel is H_s. One concludes by applying Theorem 6.12 to the resulting group H with its character χ in order to check whether it is effective.

Definition 9.7. Let $C \leq \Lambda$ be a negative definite primitive sublattice. We call C a *heart* if $C \cap \mathcal{W}^{pex}(\Lambda) = \emptyset$ and if the pulse $P_{\mathrm{Mon}^2(\Lambda)}(C)$ fixes no nontrivial vector in C.

The hearts we have just defined play the role of the starting groups we aim to extend. In fact, for an effective finite subgroup $H \leq \mathrm{Mon}^2(\Lambda)$, we have that $\Lambda_{H_s} \leq \Lambda$ is a heart.

Definition 9.8. Let $C \leq \Lambda$ be a heart and let $F := C^{\perp}_{\Lambda}$. A *head* of C is a lattice with isometry (F, a) where $a \in O(F)$ has finite order $n \geq 2$, the invariant sublattice F^a has signatures $(1, *)$ and the kernel sublattice $F^{\Phi_n(a)}$ has signatures $(2, *)$. We call n the *order* of the head (F, a).

Notation. We denote by $\mathcal{F}_C(n)$ the set of heads of C of order $n \geq 2$.

Remark 9.9. If $C_1 \leq \Lambda$ and $C_2 \leq \Lambda$ are two hearts which are isomorphic as sublattices of Λ, then $\mathcal{F}_{C_1}(n)$ and $\mathcal{F}_{C_2}(n)$ are in bijection for all $n \geq 2$.

For each head (F, a) of a given heart $C \leq \Lambda$, of order $n \geq 2$, we would like to classify isomorphism classes of equivariant primitive extensions

$$(F, a) \oplus (C, b) \leq (\Lambda_\gamma, h) \text{ where } \Lambda_\gamma \simeq \Lambda,\ h \leq \mathrm{Mon}^2(\Lambda_\gamma)$$

and γ is the glue map associated to the primitive extension $F \oplus C \leq \Lambda_\gamma$. If such an equivariant primitive extension exists, let us define

$$H_{\gamma,h} := \langle h, P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C)\rangle \leq \mathrm{Mon}^2(\Lambda_\gamma). \tag{35}$$

Lemma 9.10. *The group $P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C)$ is normal in $H_{\gamma,h}$, and the quotient is cyclic of order n.*

Proof. By the definitions of the head (F, a) and the heart C, we have that $P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C)$ is the symplectic subgroup of $H_{\gamma,h}$ and it is therefore normal. Since the quotient $H_{\gamma,h}/P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C)$ is generated by the coset represented by h, it suffices to show that n is the smallest positive integer such that $h^n \in P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C)$. But since a has order n, we observe that h^m for $m \geq 1$ acts trivially on F if and only if n divides m. The result follows then by definition of $P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C)$. □

We can therefore define a character $\chi_{\gamma,h} \colon H_{\gamma,h} \to \mathbb{C}^\times$ as follows:

$$\chi_{\gamma,h}(P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C)) = \{1\} \text{ and } \chi_{\gamma,h}(h) := \zeta_n \tag{36}$$

where ζ_n is a complex nth root of unity such that the real quadratic space $\ker(a + a^{-1} - \zeta_n - \overline{\zeta_n})$ has signatures $(2, *)$ [BH23, §3.5] (see also Section 4.4 and Remark 8.48).

Proposition 9.11. *The definitions of $H_{\gamma,h}$ and $\chi_{\gamma,h}$ do not depend on the choice of $b \in O(C)$ for the given glue map γ. Moreover, up to complex conjugation, the character $\chi_{\gamma,h}$ is unique.*

Proof. Let $b' \in O(C)$ be such that γ is (a, b')-equivariant and $h' := a \oplus b'$ lies in $\mathrm{Mon}^2(\Lambda_\gamma)$. Then, $h^{-1}h' = \mathrm{id}_F \oplus b^{-1}b'$ fixes F pointwise, meaning that $h^{-1}h' \in P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C) \leq \mathrm{Mon}^2(\Lambda_\gamma)$ by definition of the pulse. Hence $h' \in H_{\gamma,h}$.

Now, if $n = 2$ we see that χ_γ is uniquely determined by the image -1 of h. Suppose that $n \geq 3$. Since the kernel sublattice $F^{\Phi_n(a)}$ has signatures $(2, *)$, we know that there exists a primitive nth root of unity ζ_n such that the real quadratic space

$$\ker(a + a^{-1} - \zeta_n - \overline{\zeta_n})$$

has signatures $(2, *)$ and such that

$$\ker(a + a^{-1} - \zeta_n' - \overline{\zeta_n'})$$

is negative definite for every other primitive nth root of unity $\zeta_n' \neq \zeta_n$, $\overline{\zeta_n}$ (Sections 4.3 and 4.4). Hence the choice of the primitive nth of unity in the definition of $\chi_{\gamma,h}$ is uniquely determined, up to complex conjugation. □

Notation. According to Proposition 9.11, we have that $H_{\gamma,h}$ and $\chi_{\gamma,h}$ only depend on γ and a, so we may as well denote by $H_{\gamma,a}$ and $\chi_{\gamma,a}$ the group and its character previously defined.

Definition 9.12. Let $C \leq \Lambda$ be a heart and let (F, a) be a head of C of order $n \geq 2$. A *spine* between C and (F, a) is a glue map

$$D_F \geq H_F \xrightarrow{\gamma} H_C \leq D_C$$

such that

(1) $D_a H_F \leq H_F$;

(2) there exists $b \in O(C)$ such that $D_b H_C \leq H_C$ and γ is (a, b)-equivariant;

(3) the equivariant primitive extension $(F, a) \oplus (C, b) \leq (\Lambda_\gamma, h)$ is such that
   (a) $\Lambda_\gamma \simeq \Lambda$;
   (b) $h \in \mathrm{Mon}^2(\Lambda_\gamma)$;
   (c) $\chi_{\gamma,a} \colon H_{\gamma,a} \to \mathbb{C}^\times$ is effective (Theorem 6.12)

   where $H_{\gamma,a}$ and $\chi_{\gamma,a}$ are defined as before.

Given a finite symplectic subgroup $H \leq \mathrm{Mon}^2(\Lambda)$ such that H_s is saturated in $\mathrm{Mon}^2(\Lambda)$, it is clear that the heart, the head and the spine of H are respectively a heart, a head and a spine for the definitions prevlsouly given.

Recall that given a $\mathbb{Z}$-lattice $\Lambda' \simeq \Lambda$, and two finite subgroups $H \leq \mathrm{Mon}^2(\Lambda)$ and $H' \leq \mathrm{Mon}^2(\Lambda')$ we say the pairs (Λ, H) and (Λ', H') are conjugate if there exists an isometry $f \colon \Lambda' \to \Lambda$ such that $H' = f^{-1}Hf$.

Remark 9.13. Since $\mathrm{Mon}^2(\Lambda)$ is normal in $O(\Lambda)$, we see that this notion of conjugacy makes sense, and it defines an equivalence relation on the set of pairs (Λ', H') as before. Note that (Λ', H') is conjugate to (Λ, H) if and only if there exists an isometry $f \colon \Lambda \to \Lambda'$ such that H and $f^{-1}H'f$ are $O(\Lambda)$-conjugate in $\mathrm{Mon}^2(\Lambda)$. In particular, one can recover a complete set of representatives for the $O(\Lambda)$-conjugacy classes of finite subgroups of $\mathrm{Mon}^2(\Lambda)$ by computing a complete set of representatives for the conjugacy classes of pairs (Λ', H') where $\Lambda' \simeq \Lambda$ and $H' \leq \mathrm{Mon}^2(\Lambda')$. This is where we also observe the current limitations of the extension procedure described in this section. The way we describe it, in what follows, cannot guarantee that we get a complete set of representatives for the $\mathrm{Mon}^2(\Lambda)$-conjugacy classes of subgroups of $\mathrm{Mon}^2(\Lambda)$, but only for the $O(\Lambda)$-conjugacy classes. Nonetheless, in the cases where $\mathrm{Mon}^2(\Lambda) = O^+(\Lambda)$ is maximal, we have that $O(\Lambda)/O^+(\Lambda)$ is generated by the coset represented by $-\mathrm{id}_\Lambda$ which is a central involution, and therefore $O(\Lambda)$-conjugacy classes are $\mathrm{Mon}^2(\Lambda)$-conjugacy classes.

We prove the following.

Theorem 9.14. *Let $C \leq \Lambda$ be a heart, let (F, a) be a head of C of order $n \geq 2$, let γ be a spine between C and (F, a), and let $b \in O(C)$ be as in Definition 9.21 (2). Then the conjugacy class of $(\Lambda_\gamma, H_{\gamma,a})$ does not depend on the choice of a representative in the double coset $\overline{O(C)} \cdot \gamma \cdot \overline{O(F,a)}$.*

Proof. Let us denote by $h := a \oplus b \in \mathrm{Mon}^2(\Lambda)$ be the isometry such that $H_{\gamma,a} = \langle P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C), h\rangle$. Let moreover $H_C \leq D_C$ and $H_F \leq D_F$ be the glue domains of γ.

Let $\psi \in O(F, a)$ and let $\phi \in O(C)$. Then, we have that $\gamma' := D_\phi \circ \gamma \circ D_\psi$ is a glue map between $D_{\psi^{-1}} H_F \leq D_F$ and $D_\phi H_C \leq D_C$, and it defines a primitive extension

$$F \oplus C \leq \Lambda_{\gamma'}.$$

According to Lemma 2.19, we have that $f := \psi \oplus \phi$ defines an isometry from Λ_γ to $\Lambda_{\gamma'}$. Moreover, since γ' is $(a, \phi b \phi^{-1})$-equivariant, we have that $fh = h'f$ where $h' := a \oplus (\phi b \phi^{-1}) \leq O(\Lambda_{\gamma'})$. Since f maps C to itself, it follows that $f P_{\mathrm{Mon}^2(\Lambda_{\gamma'})}(C) f^{-1} = P_{\mathrm{Mon}^2(\Lambda_\gamma)}(C)$. Hence $f H_{\gamma',a} f^{-1} = H_{\gamma,a}$ and according to Theorem 6.12, we have that $\chi_{\gamma',a} \colon H_{\gamma',a} \to \mathbb{C}^\times$ is also effective. This concludes the proof. $\square$

The following classification theorem is one of the main results regarding the BHH extension procedure, and it is adapted from [BH23, Theorem 3.25]. Its proof follows directly from Lemma 9.10, Proposition 9.11 and Theorem 9.14.

Theorem 9.15. *Let $C \leq \Lambda$ be a heart, let (F, a) be a head of C and let S be the set of spines between C and (F, a). Then the double cosets in*

$$\overline{O(C)} \backslash S / \overline{O(F, a)}$$

are in bijection with the conjugacy classes of pairs $(\Lambda_\gamma, H_{\gamma,a})$ where $\Lambda_\gamma \simeq \Lambda$, $H_{\gamma,a} \leq \mathrm{Mon}^2(\Lambda_\gamma)$ is a nonsymplectic finite effective subgroup whose symplectic subgroup is saturated in $\mathrm{Mon}^2(\Lambda)$, and such that C and (F, a) are the respective heart and head of $H_{\gamma,a}$.

Based on Proposition 9.6 and Theorem 9.15, we can make explicit the steps of the extension procedure allowing us to recover a complete set of representatives for the $O(\Lambda)$-conjugacy classes of finite effective subgroups $H \leq \mathrm{Mon}^2(\Lambda)$ such that H_s is saturated in $\mathrm{Mon}^2(\Lambda)$ and $[H : H_s] = n \geq 2$.

(1) Determine a complete set of representatives for the isomorphism classes of hearts of $C \leq \Lambda$.

(2) For each such $C \leq \Lambda$, determine a complete set $\mathcal{F}_C(n)$ of representatives of isomorphism classes of heads (F, a) of C of order n.

(3) For each such head $(F, a) \in \mathcal{F}_C(n)$, determine a complete set of representatives for the double cosets of spines between C and (F, a) as in Theorem 9.15.

For each spine obtained in step (3), one reconstructs the associated subgroup $H \leq \mathrm{Mon}^2(\Lambda)$ and its associated character similarly as what was done before Definition 9.12.

Computational comments. Let us fix the same notation as in Theorem 9.15.

(1) A complete set of representatives for the double cosets in $\overline{O(C)}\backslash S/\overline{O(F,a)}$ is effectively computable. In fact, the $\mathbb{Z}$-lattice C being negative definite, one can compute explicitly the representation $O(C) \to O(D_C)$ using algorithms of Plesken–Souvignier [PS97] and Brandhorst–Veniani [BV24]. Computing the subgroup $\overline{O(F,a)} \leq O(D_F, D_a)$ is also computationally accessible thanks to the hermitian analog of Miranda–Morrison theory by Brandhorst–Hofmann [BH23, §6]. Similarly to what was explained regarding Algorithm 1, one can relate representatives for the double cosets in $\overline{O(C)}\backslash S/\overline{O(F,a)}$ with representatives of double cosets of isometries in the glue domain $H_C \leq D_C$ of γ. Hence, Step (3) above is computationally accessible.

(2) Step (2) can be carried out explicitly computationally thanks to an algorithm of Brandhorst–Hofmann [BH23, §4]. Their algorithm has been implemented in the computer algebra system OSCAR [OSC25, QuadFormAndIsom] by the thesis' author. We refer to Sections 1, 2 and 4 and computational comments therein for an overview of the infrastructure required for the implementation of such a code. Note that it is a crucial step of the extension procedure.

What remains to be covered is Step (1). We have seen in Section 7 how to computationally determine a complete set of representatives for the $O(\Lambda)$-conjugacy classes of finite symplectic subgroups $H_s \leq \mathrm{Mon}^2(\Lambda)$ which are stable. However, in general, not all effective subgroups of $\mathrm{Mon}^2(\Lambda)$ have to be stable (see Section 6.3 and Section 7.6). In the next section, we show how one can adapt the extension procedure of Brandhorst–Hashimoto–Hofmann in order to recover finite symplectic subgroups of $\mathrm{Mon}^2(\Lambda)$ from the stable ones.

9.2. Classifying hearts

The next three sections are adapted from [MM25b, §5]: their content is mostly due to the author of the thesis, with some suggestions of improvements by collaborator Marquand and an anonymous referee.

Let $H_s \leq \mathrm{Mon}^2(\Lambda)$ be a saturated finite symplectic subgroup and suppose that $H_s^\# \neq H_s$ — we have that $[H_s : H_s^\#] = 2$ (Lemma 6.15). The quotient $H_s/H_s^\#$ is cyclic generated by the coset represented by a nonstable symplectic isometry $h \in H_s \setminus H_s^\#$. As in the previous section, the restriction a of h to $\Lambda^{H_s^\#}$ has order 2, except when H_s is the saturation of $H_s^\#$ in $\mathrm{Mon}^2(\Lambda)$, in which case $a = \mathrm{id}$. In both cases though, we observe that the definition of $a \in O(\Lambda^{H_s})$ does not depend on a choice of a representative of $hH_s^\#$. Again, we remark that any choice of a representative $h' \in hH_s^\#$ determines an isometry $b \in O(\Lambda_{H_s^\#})$ and an equivariant primitive extension $(\Lambda^{H_s^\#}, a) \oplus (\Lambda_{H_s^\#}, b) \leq (\Lambda, h')$. The associated glue map γ is defined by the primitive extension $\Lambda^{H_s^\#} \oplus \Lambda_{H_s^\#} \leq \Lambda$

Definition 9.16. Let $H_s \leq \mathrm{Mon}^2(\Lambda)$ be a saturated finite symplectic subgroup. We define

(1) the *heart* of H_s to be the stable symplectic sublattice $\Lambda_{H_s^\#} \leq \Lambda$;

(2) the *head* of H_s to be the pair $(\Lambda^{H_s^\#}, a)$;

(3) the *spine* of H_s to be the glue map γ.

Recall that for a sublattice $C \leq \Lambda$, we define $S(C) := SO(C)$ if $\mathcal{T} = \mathrm{Kum}_n$, $n \geq 2$, and $S(C) := O(C)$ otherwise. Similarly to Remark 9.5, for $H_s \leq \mathrm{Mon}^2(\Lambda)$ a saturated finite symplectic subgroup, the group $H_s^{\#}$ is saturated in $\mathrm{Mon}^2(\Lambda) \cap O^{\#}(\Lambda)$ (Lemma 1.57). In particular, the group $H_s^{\#}$ is completely characterized by the primitive sublattice $\Lambda_{H_s^{\#}} \leq \Lambda$, and it is equal to $S^{\#}(\Lambda_{H_s^{\#}})$ (Theorem 7.8).

Remark 9.17. Note that $H_s^{\#}$ is saturated in $\mathrm{Mon}^2(\Lambda) \cap O^{\#}(\Lambda)$ but it might not be saturated in $\mathrm{Mon}^2(\Lambda)$ itself. A priori, since H_s and $H_s^{\#}$ may have distinct associated invariant sublattices, nothing tells us that H_s is the saturation of $H_s^{\#}$ in $\mathrm{Mon}^2(\Lambda)$. However, thanks to Proposition 7.13 we know that $H_s^{\#}$ has index 2 in its saturation in $\mathrm{Mon}^2(\Lambda)$.

We therefore see that the heart of a saturated finite symplectic subgroup of $\mathrm{Mon}^2(\Lambda)$ is a stable symplectic sublattice of Λ (Definition 7.9). Stable symplectic sublattices of Λ are a special case of hearts, defined as in Definition 9.7.

Proposition 9.18. *Let $H_s, H_s' \leq \mathrm{Mon}^2(\Lambda)$ be two saturated symplectic finite subgroups and let $\psi \in \mathrm{Mon}^2(\Lambda)$ conjugate H_s and H_s'. Then:*

1. *ψ restricts to an isomorphism between the respective hearts of H_s and H_s';*
2. *ψ induces an isomorphism between the respective heads of H_s and H_s'.*

Proof. Similar to the proof of Proposition 9.6. □

We would like to describe an extension procedure to compute a complete list of representatives of conjugacy classes of saturated finite symplectic subgroups of $\mathrm{Mon}^2(\Lambda)$ with given heart and head. Let us follow the same structure as in the previous section.

Let $C \leq \Lambda$ be stable symplectic. Recall from Table 4 that for all the known deformation types of IHS manifolds, we have that $S^{+,\#}(\Lambda) \leq \mathrm{Mon}^2(\Lambda)$: hence we can see $S^{\#}(C)$ as a subgroup of $\mathrm{Mon}^2(\Lambda)$ be extending with the identity on $C^{\perp}$. By abuse of notation, we will therefore denote by $S^{\#}(C)$ both the subgroup of $O(C)$ and the corresponding subgroup of $\mathrm{Mon}^2(\Lambda)$.

Definition 9.19. Let $C \leq \Lambda$ be stable symplectic sublattice and let $F := C^{\perp}$. A *symplectic head* of C is a lattice with isometry (F, a) where $a \in O(F)$ has order $n = 1, 2$ and the coinvariant sublattice F_a is negative definite or trivial. Again, we call $n \in \{1, 2\}$ the order of the symplectic head (F, a).

For each symplectic head (F, a) of the stable symplectic sublattice $C \leq \Lambda$, we would like to determine representatives for the isomorphism classes of equivariant primitive extensions

$$(F, a) \oplus (C, b) \leq (\Lambda_\gamma, h), \text{ where } \Lambda_\gamma \simeq \Lambda \text{ and } D_h \neq \mathrm{id}, \tag{37}$$

such that $b \notin S^{\#}(C)$ if $a = \mathrm{id}_F$. Indeed, we know that $S^{\#}(C) \leq \mathrm{Mon}^2(\Lambda)$ is already stable, so D_h would be trivial if $b \in S^{\#}(C)$ when $a = \mathrm{id}_F$. Given an equivariant primitive extension as before, we define $H_{\gamma,h} := \langle S^{\#}(C), h \rangle$.

Lemma 9.20. *The following hold:*

1. *the group $S^{\#}(C)$ is normal in $H_{\gamma,h}$ and the quotient is cyclic of order 2;*
2. *the definition of $H_{\gamma,h}$ is independent on the choice of $b \in O(C)$ for the given glue map γ.*

In particular, from now on, we denote $H_{\gamma,a} := \langle S^{\#}(C), h \rangle$.

Proof.

(1) By definition, we know that any isometry in $S^{\#}(C) \leq H_{\gamma,h}$ is stable while h is not. Since the image of the representation $\mathrm{Mon}^2(\Lambda) \to O(D_\Lambda)$ has order 2, we deduce that $S^{\#}(C)$ is the stable subgroup of $H_{\gamma,h}$ and it has index 2. This proves the statement.

(2) The proof follows similarly as the proof of Proposition 9.11. □

Now that we have adapted the definitions of heads to this new context, let us adapt the definition of spines as well. Remark that by definition of symplectic heads, the groups $H_{\gamma,a}$ constructed above have negative definite coinvariant sublattices.

Definition 9.21. Let $C \leq \Lambda$ be a stable symplectic sublattice and let (F, a) be a symplectic head of C of order $n \in \{1, 2\}$. A *spine* between C and (F, a) is a glue map

$$D_F \geq H_F \xrightarrow{\gamma} H_C \leq D_C$$

such that

(1) $D_a H_F \leq H_F$;

(2) there exists $b \in O(C)$ such that $b \notin S^{\#}(C)$ if $a = \mathrm{id}_F$, $D_b H_C \leq H_C$ and γ is (a, b)-equivariant

(3) the equivariant primitive extension $(F, a) \oplus (C, b) \leq (\Lambda_\gamma, h)$ is such that
   (a) $\Lambda_\gamma \simeq \Lambda$;
   (b) $h \in \mathrm{Mon}^2(\Lambda_\gamma) \setminus O^{\#}(\Lambda_\gamma)$;
   (c) $(\Lambda_\gamma)_{H_{\gamma,a}} \cap \mathcal{W}^{pex}(\Lambda_\gamma) = \emptyset$

   where $H_{\gamma,a}$ is defined as before.

With the previous being settled, we can now state the analogues of Theorems 9.14 and 9.15 to the context of symplectic groups. Note that for $(\Lambda_\gamma, H_\gamma)$ as in Definition 9.21, condition (3)(c) actually implies that $H_{\gamma,a} \leq \mathrm{Mon}^2(\Lambda_\gamma)$ is symplectic (Theorem 6.9).

Theorem 9.22. *Let $C \leq \Lambda$ be stable symplectic, let (F, a) be a symplectic head of C of order $n \in \{1, 2\}$, let γ be a spine of between C and (F, a), and let $b \in O(C)$ be as in Definition 9.21 (2). Then the conjugacy class of $(\Lambda_\gamma, H_{\gamma,a})$ does not depend on the choice of a representative in the double coset $\overline{O(C)} \cdot \gamma \cdot \overline{O(F, a)}$.*

Proof. The proof is identical to the one of Theorem 9.14. □

We conclude with a parallel theorem to Theorem 9.15, whose proof follows from Lemma 9.20 and Theorem 9.22.

Theorem 9.23. *Let $C \leq \Lambda$ be stable symplectic, let (F, a) be a symplectic head of C of order $n \in \{1, 2\}$ and let S be the set of spines between C and (F, a). Then the double cosets in*

$$\overline{O(C)} \backslash S / \overline{O(F, a)}$$

are in bijection with the conjugacy classes of pairs $(\Lambda_\gamma, H_{\gamma,a})$ where $\Lambda_\gamma \simeq \Lambda$, $H_{\gamma,a} \leq \mathrm{Mon}^2(\Lambda_\gamma)$ is a nonstable finite symplectic subgroup whose stable subgroup is saturated in $\mathrm{Mon}^2(\Lambda) \cap O^{\#}(\Lambda)$, and such that C and (F, a) are the respective heart and head of $H_{\gamma,a}$.

Note that Remark 9.13 also applies for this extension procedure. In particular, from Proposition 9.18 and Theorem 9.23 one can describe a procedure to determine a complete set of representatives for the $O(\Lambda)$-conjugacy classes of saturated finite symplectic subgroups of $\mathrm{Mon}^2(\Lambda)$. Moreover, such a classification is up to monodromy conjugation if $\mathrm{Mon}^2(\Lambda) = O^+(\Lambda)$ is maximal. One proceeds as follows.

(1) Determine a complete set of representatives for the isomorphism classes of stable symplectic sublattices $C \leq \Lambda$;

(2) For each such $C \leq \Lambda$, determine a complete set of representatives of isomorphism classes of symplectic heads (F, a) of C;

(3) For each such symplectic head (F, a), determine a complete set of representatives for the double cosets of spines γ between C and (F, a) as in Theorem 9.23;

(4) For each such spine γ, compute the subgroup $H_{\gamma,a} \leq \mathrm{Mon}^2(\Lambda_\gamma)$.

We have already seen that Steps (2) and (3) above are computationally accessible (see the computational comments from the previous section). Moreover, we explain in Section 7 how Step (1) can be carried out explicitly for the known deformation types of IHS manifolds. The groups we obtain in Theorem 9.23 might not be saturated in $\mathrm{Mon}^2(\Lambda)$. Hence, after Step (4), one can apply Proposition 7.16 to determine which of these groups are saturated. Therefore, the previously described procedure can be implemented. In the next section, we apply it to the classification of symplectic stable subgroups for the deformation type OG10 we have performed earlier in this thesis (Section 7.5.2).

9.3. Application to symplectic actions in the OG10 case

We apply the extension procedure from the previous section to the classification of stable symplectic sublattices of the OG10 $\mathbb{Z}$-lattice

$$\Lambda := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2.$$

Recall in that situation that $\mathrm{Mon}^2(\Lambda) = O^+(\Lambda)$ and

$$\mathcal{W}^{pex}(\Lambda) = \{v \in \Lambda \; : \; v \text{ has type } (-2, 1) \text{ or } (-6, 3)\}.$$

In Section 7.5.2 we have determined a complete set of representatives for the isomorphism classes of stable symplectic sublattices $C \leq \Lambda$. We now apply the extension procedure of the previous section to obtain a classification of saturated finite symplectic subgroups of $O^+(\Lambda)$. In order to do so, we show how to simplify our approach in this particular case, using the fact that $D_\Lambda \cong \mathbb{Z}/3\mathbb{Z}$ is a finite simple group.

9.3.1. Simplified algorithms

Let us observe the following.

Proposition 9.24. *Let $C \leq \Lambda$ be a stable symplectic sublattice, and let (F, a) be a symplectic head of C. One of the following two holds:*

(1) *either D_C embeds into D_F, as abelian groups;*

(2) *or D_F embeds into D_C, as abelian groups.*

Proof. Let us see $C \leq \Lambda$ as the image of a primitive embedding $i\colon C \hookrightarrow \Lambda$. According to [Nik80, Proposition 1.15.1], the primitive embedding i determines an isomorphism between a subgroup $I_C \leq D_C$ and a subgroup $I_\Lambda \leq D_\Lambda$. Now, since $D_\Lambda \cong \mathbb{Z}/3\mathbb{Z}$ as abelian groups, then either I_C is the trivial group, or $I_C \cong D_\Lambda$. In the former case, [Nik80, Proposition 1.15.1] tells us that D_C is the glue domain of i, and thus D_C is identified with a subgroup of D_F. Similar arguments apply in the other case by exchanging the role of F and C. □

Given a symplectic stable sublattice $C \leq \Lambda$, and given a symplectic head (F, a) of C, we can easily decide in which case of Proposition 9.24 the pair (C, F) fits, by comparing the determinants of C and F. In particular, we can already conclude the following.

Corollary 9.25. *Let $C \leq \Lambda$ be a stable symplectic sublattice, and let (F, a) be a symplectic head of C. If $\det(C)$ divides $\det(F)$, then the group $O^\#(C)$, seen as a finite subgroup of $O^+(\Lambda)$, is saturated.*

Proof. According to Proposition 9.24, we know that the glue map associated to $F \oplus C \leq \Lambda$ identify D_C with a proper subgroup of D_F. Thus the pulse of C in $O^+(\Lambda)$ is exactly $O^\#(C)$ and the result follows from Proposition 7.16. □

Remark 9.26. With the notation of Corollary 9.25, if $\det(C) \mid \det(F)$, then a must have order 2. In the case where $\det(F)$ divides $\det(C)$, we cannot conclude similarly.

In what follows, we make Theorem 9.23 more explicit by separating these two cases from Proposition 9.24. Indeed, we prove the following lemma.

Lemma 9.27. *Let $C \leq \Lambda$ be a stable symplectic sublattice, and let (F, a) be a symplectic head of C. Let $b \in O(C)$ and let γ be an (a, b)-equivariant glue map. Then, $a \oplus b \in O(\Lambda_\gamma)$ is nonstable if and only if*

(1) *$\det(C) \mid \det(F)$ and D_a restricts to negative identity on the orthogonal complement of the glue domain of $F \hookrightarrow \Lambda_\gamma$;*

(2) *$\det(F) \mid \det(C)$ and D_b restricts to negative identity on the orthogonal complement of the glue domain of $C \hookrightarrow \Lambda_\gamma$.*

Proof. Let γ be an (a, b)-equivariant glue map and let $h := a \oplus b \in O(\Lambda_\gamma)$. According to the proof of Proposition 2.17, D_{Λ_γ} is isometric to $\Gamma^\perp/\Gamma$ where Γ is the graph of γ in $D_F \oplus D_C$. Moreover the action of h on D_{Λ_γ} coincides with the one of $a \oplus b$ on $\Gamma^\perp/\Gamma$. Now

(1) If $\det(C) \mid \det(F)$, we write $D_F = S \oplus T$ where $S \simeq D_\Lambda$ and $T := S^\perp \simeq D_C(-1)$ is the glue domain of $F \hookrightarrow \Lambda_\gamma$. In that case, the action of $a \oplus b$ on $\Gamma^\perp/\Gamma$ is given by $(D_a)_{|S}$;

(2) If $\det(F) \mid \det(C)$, we write $D_C = S \oplus T$ where $S \simeq D_\Lambda$ and $T := S^\perp \simeq D_F(-1)$ is the glue domain of $C \hookrightarrow \Lambda_\gamma$. In that case, the action of $a \oplus b$ on $\Gamma^\perp/\Gamma$ is given by $(D_b)_{|S}$. □

From a computational point of view, Lemma 9.27 together with Theorem 9.22 allows us to decide which equivariant gluings of (F, a) and C will not give rise to spines. This is featured in Algorithms 3 and 4 to compute only the relevant equivariant primitive extensions (for our purpose).

Proposition 9.28. *For any stable symplectic sublattice $C \leq \Lambda$ and any symplectic head (F, a) of C such that $\det(C) \mid \det(F)$, Algorithm 3 returns the correct output.*

Algorithm 3: Simplied extensions I

Input: A stable symplectic sublattice $C \leq \Lambda$ and a symplectic head (F, a) of C such that $\det(C) \mid \det(F)$.

Output: Representatives for the conjugacy classes of pairs (Λ', H) where $\Lambda' \simeq \Lambda$ and $H \leq O^+(\Lambda')$ is a symplectic finite subgroup such that $H^\# \leq O^{+,\#}(\Lambda')$ is saturated, $\#\overline{H} = 2$, the heart of H is C and its head is (F, a).

Initialise the empty list $E = []$.
Let $\mathcal{H}_F$ be a complete set of representatives of classes in $\{H_F \leq D_F \mid H_F \simeq D_\Lambda\} / \overline{O(F,a)}$.
for $[S] \in \mathcal{H}_F$ **do**
- $T \leftarrow S^\perp$.
- **if** $D_a T \neq T$ **then**
  - Discard $[S]$ and continue the for loop with the next representative.
- **if** $(D_a)_{|S} \neq -\mathrm{id}_S$ **then**
  - Discard $[S]$ and continue the for loop with the next representative.
- Let $\gamma \colon T \to D_C$ be a glue map.
- $S_T^F \leftarrow \mathrm{Stab}_{\overline{O(F,a)}}(T)$.
- $S_T^T \leftarrow \mathrm{im}(S_T^F \to O(T))$.
- $S_T^\gamma \leftarrow \gamma\, S_T^T\, \gamma^{-1}$.
- **for** $[g] \in \overline{O(C)} \backslash O(D_C) / S_T^\gamma$ **do**
  - $\gamma_g \leftarrow g \circ \gamma$.
  - $\bar{b} \leftarrow \gamma_g D_a \gamma_g^{-1}$.
  - **if** $\bar{b} \notin \overline{O(C)}$ **then**
    - Discard $[g]$ and continue the for loop with the next double coset.
  - Let $b \in O(C)$ such that $D_b = \bar{b}$.
  - Let Λ' be the overlattice associated to the glue map γ_g.
  - $h \leftarrow a \oplus b \in O(\Lambda')$.
  - $H \leftarrow \langle O^\#(C), h \rangle$.
  - **if** $\Lambda'_H \cap \mathcal{W}^{pex}(\Lambda') \neq \emptyset$ **then**
    - Discard $[g]$ and continue the for loop with the next double coset.
  - Append (Λ', H) to E.

Return E.

Proof. Since $\det(C) \mid \det(F)$, Proposition 9.24 tells us that for any primitive extension $F \oplus C \leq \Lambda'$ with $\Lambda' \simeq \Lambda$, then the glue domain of $C \hookrightarrow \Lambda'$ is the discriminant group D_C of C. Let $(\Lambda', H) \in E$ be in the output of the algorithm. Since C and F_a are negative definite, the condition in Line 22 ensures that H is symplectic (Theorem 6.9). Since H is generated by $O^{\#}(C)$ and h where h lies in $O^{+}(\Lambda') \setminus O^{\#}(\Lambda')$, we have that $H^{\#} = O^{\#}(C)$ is saturated in $O^{+,\#}(\Lambda')$. Note that here, we view $O^{\#}(C)$ as a saturated subgroup of $O^{+,\#}(\Lambda')$ after extending with the identity on F. Moreover Line 7, together with Lemma 9.27, ensures that D_h acts by negative identity on $D_{\Lambda'} \simeq S$. Therefore, together with the conditions in Lines 5 and 16, we know that γ is a spine between C and (F, a). Moreover, Lemma 9.20 tells us that the definition of H does not depend on the choice of b in Line 18. Finally, we know $\overline{H} \leq D_{\Lambda'}$ is nontrivial by definition of h in Line 20. The rest of the proof follows from Theorem 9.23: note that the set of double cosets defined in Line 13 corresponds to the ones defined in the aforementioned theorem. $\square$

Algorithm 4: Simplied extensions II

Input: A stable symplectic sublattice $C \leq \Lambda$ and a symplectic head (F, a) of C such that $\det(F) \mid \det(C)$.

Output: Representatives of conjugacy classes of pairs (Λ', H) where $\Lambda' \simeq \Lambda$ and $H \leq O^{+}(\Lambda')$ is a symplectic finite subgroup such that $H^{\#} \leq O^{+,\#}(\Lambda')$ is saturated, $\#\overline{H} = 2$, the heart of H is C and its head is (F, a).

1 Initialise the empty list $E = []$.
2 Let $\mathcal{H}_C$ be a complete set of representatives of classes in $\overline{O(C)} \backslash \{H_C \leq D_C \mid H_C \simeq D_\Lambda\}$.
3 **for** $[S] \in \mathcal{H}_C$ **do**
4   $T \leftarrow S^{\perp}$.
5   $S_T^C \leftarrow \mathrm{Stab}_{\overline{O(C)}}(T)$.
6   $S_T^T \leftarrow \mathrm{im}(S_T^C \to O(T))$.
7   Let $\gamma \colon D_F \to T$ be a glue map.
8   $S_F^\gamma \leftarrow \gamma \overline{O(F, a)} \gamma^{-1}$.
9   **for** $[g] \in S_T^T \backslash O(T) / S_F^\gamma$ **do**
10     $\gamma_g \leftarrow g \circ \gamma$.
11     $\tilde{b} \leftarrow \gamma_g D_a \gamma_g^{-1}$.
12   **if** $\tilde{b} \notin S_T^T$ **then**
13   Discard $[g]$ and continue the for loop with the next double coset.
14     $\widehat{b} \leftarrow \tilde{b} \oplus (-\mathrm{id}_S) \in O(D_C)$.
15   **if** $\widehat{b} \notin S_T^C$ **then**
16   Discard $[g]$ and continue the for loop with the next double coset.
17   Let $b \in O(C)$ such that $D_b = \widehat{b}$.
18   Let Λ' be the overlattice of the glue map γ_g.
19     $h \leftarrow a \oplus b \in O(\Lambda')$.
20     $H \leftarrow \langle O^{\#}(C), h \rangle$.
21   **if** $a \neq \mathrm{id}_F$ **and** $\Lambda'_H \cap \mathcal{W}^{pex}(\Lambda') \neq \emptyset$ **then**
22   Discard $[g]$ and continue the for loop with the next double coset.
23   Append (Λ', H) to E.
24 Return E.

Proposition 9.29. *For any stable symplectic sublattice $C \leq \Lambda$, and any symplectic head (F, a) of C such that $\det(F) \mid \det(C)$, Algorithm 4 returns the correct output.*

Proof. The proof is similar to the proof of Proposition 9.28. Note that the main difference is that we do not start with a fixed isometry of C, so the translation of the double cosets from Theorem 9.23 to this context has to be adapted accordingly.

Let us note that if $a = \mathrm{id}_F$, it follows that H is the saturation of $O^{\#}(C)$ in $O^+(\Lambda')$ as described in Proposition 7.13. By definition of the saturation, $\Lambda'_H = C$ and the latter implies that $\Lambda'_H \cap \mathcal{W}^{pex}(\Lambda')$ is necessarily empty. $\square$

Note that according to the computational comments made in Section 9.1, both of the previous algorithms can effectively be implemented in any computer algebra system which supports the computational infrastructure described in the preliminaries of this thesis. This is the case of OSCAR [OSC25], together with its dependency Hecke [FHHJ17]. Therefore, we can effectively apply these algorithms to our list of stable symplectic sublattices of Λ.

Remark 9.30. Each entry of Table 16 determines a pair (C, F) where $C \leq \Lambda$ is stable symplectic and F is its orthogonal complement in Λ. In the cases where $\det(C) \mid \det(F)$ and C has rank 21, then F is positive definite and it admits no nontrivial isometries with negative definite coinvariant sublattice. All the other cases are uniquely determined by C and F, up to isometry, except for the pair of cases 47a and 47b. In those cases, $F \in \mathrm{II}_{(3,3)}2^{-2}3^2 9^1$ and we have that $\det(F) \mid \det(C)$. However, one can actually show that in this situation the set $\mathcal{H}_C$, as defined in Algorithm 4, has actually cardinality 2 and which is why we obtain these two nonisomorphic primitive sublattices of Λ. The upshot is the following. In our particular setting, for each pair (Λ', H) in output of Algorithms 3 and 4, it is effectively possible to determine to which entry of Table 16 the stable sublattice $\Lambda'_{H\#} \leq \Lambda' \simeq \Lambda$ is isomorphic.

9.3.2. Results and comments

Theorem 9.31. *Let $C \leq \Lambda$ be a stable symplectic sublattice as described in Table 16 and let $\mathcal{H}$ be an $O(\Lambda)$-conjugacy class of finite symplectic subgroups $H \leq O^+(\Lambda)$ which are stably saturated and whose heart is isomorphic to $C \leq \Lambda$, as primitive sublattice. Then a representative of $\mathcal{H}$ is computable. Moreover, the folder "dataset" of [MM25c] contains representatives for each such conjugacy class.*

Proof. This follows from Theorems 9.22 and 9.23, together with Algorithms 3 and 4. We apply the previous algorithms to the set of stable symplectic sublattices $C \leq \Lambda$ determined in Section 7.5.2. We obtain 921 such conjugacy classes $\mathcal{H}$: we refer to [MM25c, Table 4] for details about each entry of the dataset. $\square$

Remark 9.32. Recall that for $\mathcal{T} = \mathrm{OG10}$, we have that $\mathrm{Mon}^2(\Lambda) = O^+(\Lambda)$ is maximal and therefore, since $O(\Lambda)/O^+(\Lambda)$ is generated by the coset represented by the central involution $-\mathrm{id}_\Lambda$, $O(\Lambda)$-conjugacy classes are $\mathrm{Mon}^2(\Lambda)$-conjugacy classes.

Theorem 9.33. *Let X be an IHS manifold of OG10-type and let $G \leq \mathrm{Bir}_s(X)$ be finite. Then there exists a marking $\eta\colon H^2(X, \mathbb{Z}) \to \Lambda$ such that $\eta\rho_X(G)\eta^{-1} \leq O^+(\Lambda)$ is a subgroup of one of the 375 saturated groups appearing in [MM25c]. Moreover, all the finite groups $H \leq O^+(\Lambda)$ appearing in [MM25c] are symplectic. Numerical data about each saturated groups of [MM25c] are given in Appendix E, Table 19.*

Proof. The fact that all the groups in [MM25c] are symplectic follows by conctruction (Algorithms 3 and 4), Theorem 9.23 and Theorem 6.9. For the first part of the statement, we apply Proposition 7.16 to each entry in [MM25c, "dataset"] to determine which groups among the complete set of 921 representatives of conjugacy classes we have computed, thanks to Theorem 9.31, are saturated. $\square$

9.4. Comments on actions of maximal order

Following the results in [MM25c], we obtain the largest cohomological actions (i.e. groups with nontrivial action on second cohomology) for IHS manifolds of known deformation type. Recall that a finite group $G \leq \mathrm{Bir}(X)$ is *mixed* if it contains both nontrivial symplectic and nonsymplectic birational automorphisms.

Theorem 9.34. *The largest finite cohomological actions for IHS manifolds of all known deformation types have order **6531840** in the symplectic case, and **39191040** in the mixed case.*

Proof. According to Table 19, the largest symplectic finite subgroup $H \leq O^+(\Lambda)$ we have determined in the OG10-case is isomorphic to a semidirect product $\mathrm{PSU}(4,3) \rtimes C_2$ (entry 163a.1), and its order is $\#H = 6531840$. Note that this group is saturated in $O^+(\Lambda)$. We prove that this group is maximal (in size) among all the finite groups of symplectic isometries of $H^2(X,\mathbb{Z})$ for X an IHS manifold of known deformation type.

According to Theorem 7.24 and [HM16, Table 2] (and also Table 4), we see that the largest finite group acting faithfully by symplectic automorphisms on an IHS manifold of $\mathrm{K3}^{[n]}$-type ($n \geq 1$) has order less than 245760. In particular, according to the discussion in Section 6.3, the largest finite birational symplectic action has order bounded above by $2 \cdot 245760 = 491520$. For an IHS manifold X of Kum_n-type ($n \geq 2$) or OG6-type, the negative signature of $H^2(X,\mathbb{Z})$ is $k = 4$ or $k = 5$ respectively. By the known bounds on the order of finite subgroups of $\mathrm{GL}_k(\mathbb{Z})$ for $k = 4, 5$, we know that the maximal orders in those cases is smaller than 200000.

Let $H = \mathrm{PSU}(4,3) \rtimes C_2 \leq O^+(\Lambda)$. The invariant sublattice $F := \Lambda^H$ has Gram matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ in a given basis. This $\mathbb{Z}$-lattice admits an order 6 isometry $a \in O(F)$ given by

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with characteristic polynomial $\Phi_1\Phi_6$. Applying the techniques from the BBH-procedure, described in Section 9.1, there exists an isometry $b \in O(\Lambda_H)$ such that we have an equivariant primitive extension

$$(F, a) \oplus (\Lambda_H, b) \leq (\Lambda', c)$$

where $\Lambda' \simeq \Lambda$. Since Λ_H is negative definite, such an isometry b is effectively computable using an algorithm of Plesken–Souvignier [PS97]. The invariant sublattice $(\Lambda')^c$ has signatures $(1, *)$ and the isometry $c \in O^+(\Lambda')$. Moreover, $N := (\ker \Phi_1(c)\Phi_6(c))^\perp_{\Lambda'} \subseteq F^\perp_{\Lambda'} = \Lambda_H$ satisfies that $N \cap \mathcal{W}^{pex}(\Lambda') = \emptyset$: in particular, according to Theorem 6.12, we can conclude that $c \in O^+(\Lambda')$ is effective and $\langle H, c\rangle$ is an effective finite subgroup of $O^+(\Lambda')$. In particular, we have that $\langle H, c\rangle$ has a faithful action by birational automorphisms on an IHS manifold of OG10 type (see the [MM25c, Notebook "Maximal"] for the computational details). As before, we claim that the action of this group on cohomology has maximal order for all the known deformation types. This is clear for the Kum_n types ($n \geq 2$) and the deformation type OG6. For the $\mathrm{K3}^{[n]}$ types ($n \geq 1$), we use again Theorem 7.24 and [HM16, Table 2], together with the arguments relative to the BBH-procedure (Section 9.1) to show that it is indeed the case. □

Remark 9.35. The statement of Theorem 9.34 has to be restricted to finite order actions, for obvious reasons, but also to the order of the actions on cohomology because of the Kum_n types.

Indeed, by looking at Table 4, we see that for an IHS manifold X of Kum_n-type, $n \geq 2$, the kernel of ρ_X is nontrivial and has order $2(n+1)^4$. Hence, by choosing n large enough, we see that the order for faithful finite group actions on the known IHS manifolds is unbounded (from above).

Remark 9.36. The group $\mathrm{PSU}(4,3)$ is a well-known group, and it appears to act faithfully on some other special objects. For instance, the authors in [OS24] show that $\mathrm{PSU}(4,3)$ is the largest finite group acting symplectically on a supersingular K3 surface of Artin invariant 1. Another recent occurrence of this group is in [YYZ25] where it was shown that the automorphism group of the most symmetric sextic fourfold is a degree 2 extension of $\mathrm{PSU}(4,3)$. Such sextic fourfold has been known since Todd [Tod50], and it is intrinsically related to the Coxeter–Todd lattice K_{12}, see [CS93].

Part III.
Geometric applications

10. Symmetric K3 surfaces

Recall that in [BH21], Brandhorst and Hashimoto classify 42 isomorphism classes of pairs (S, G) where S is a projective K3 surface and $G \leq \mathrm{Aut}(S)$ is such that the normal subgroup $G_s \trianglelefteq G$ of symplectic automorphisms is a proper subgroup, and it is isomorphic to one of the 11 maximal groups determined by Mukai (see Theorem 9.1). Such pairs (S, G) come equipped with a canonical G-invariant polarization L. For each case, such a surface S has maximal Picard rank. The authors moreover exhibit 25 cases for which an explicit projective model of S, by means of equations, is known. In particular, all of the triples (S, G, G_s) of degree at most 10 have been treated, except one where the associated symplectic action has order 192 (case 77b). Following the notation in [BH21] (except the polarization that we denote "L" here and not "l", and the transcendental lattice which we write "$\mathrm{T}(S)$"), we display in Table 11 some information about this isomorphism class of K3 surfaces.

Table 11: Specification for the triple (S, G, G_s) corresponding to [BH21, case 77b]

case	G_s	$\Lambda_{K3}^{G_s}$	$\mathrm{SO}(\Lambda_{K3}^{G_s})$	G/G_s	$c_1(L)^2$	$\mathrm{T}(S)$	G
77b	T_{192}	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{pmatrix}$	D_6	μ_2	8	$\begin{pmatrix} 4 & 0 \\ 0 & 24 \end{pmatrix}$	$T_{192} \rtimes \mu_2$ GAP Id $[384, 5602]$

The K3 surface S described by the data in Table 11 has already been studied, and it has few known properties.

Firstly, since the $\mathbb{Z}$-lattice $\mathrm{T}(S) = L(2)$ where $L := \langle 2, 12 \rangle$ is even, [Mor84, Corollary 4.4] tells us that S is a Kummer surface. In fact, the $\mathbb{Z}$-lattice L admits a primitive embedding into $U^{\oplus 3}$, meaning that $\mathrm{T}(S)$ embeds primitively into $U(2)^{\oplus 3}$ and therefore $\mathrm{NS}(S)$ contains a primitive copy of the so-called *Kummer lattice* (see [Nik75, §3, Definition 1]). The latter is equivalent to S being a Kummer surface [Nik75, §3, Theorem 3].

Secondly, S is the unique K3 surface of degree 8 with a faithful symplectic action of T_{192} [Deg24, Example 4.3]: this follows from the fact that the $\mathbb{Z}$-lattice $F := \begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{pmatrix}$ has only one $O^+(F)$-orbit of vectors of norm 8.

Finally, the surface S is the *Barth–Bauer octic* with the second largest number (160) of conics [Deg24, Theorem 4.2].

However no equations describing S were known before the author's work [Mul24]. This section is adapted from the latter work. In what follows, we prove the following.

Theorem 10.1. *The polarized K3 surface (S, L) corresponding to the case 77b in [BH21] admits a projective model in $\mathbb{P}^5_{\mathbb{C}}$ given by*

$$S\colon \begin{cases} ix_0x_1 + x_0x_2 + x_1x_3 + ix_2x_3 + x_5^2 = 0 \\ ix_0x_1 - x_0x_2 - x_1x_3 + ix_2x_3 + x_4^2 = 0 \\ -x_0x_3 - x_1x_2 - x_4x_5 = 0 \end{cases}.$$

It admits a maximal symplectic action of T_{192} and is invariant under the linear action of

$G := T_{192} \rtimes \mu_2$ on $\mathbb{P}^5_{\mathbb{C}}$ given by

$$\sigma_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^6-1 & -z^6-1 \\ 0 & 0 & 0 & 0 & 1-z^6 & -z^6-1 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z^6 \end{pmatrix}$$

$$\sigma_4 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -z^5+z^3+z & -z^5-z^3+z \\ 0 & 0 & 0 & 0 & z^5+z^3-z & z^5-z^3-z \end{pmatrix} \quad \sigma_5 = \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

where T_{192} is one of the maximal subgroups of symplectic automorphisms classified by Mukai (see Theorem 7.1) and z is a primitive 24th root of unity.

Remark 10.2. From these equations and the description of the group action, one could possibly compute explicitly representatives for the G-orbits of 160 conics on S, similarly to [BS21, Nas22] for the 3 orbits of 800 conics on the M_{20}-quartic. Nonetheless, performing such computations appears to be expensive, in terms of computing Gröbner basis.

The goal of this section is to prove Theorem 10.1. We recall the theoretical background on projective representations of finite groups, and we describe a procedure to parametrize candidate ideals for symmetric intersections of hypersurfaces of the same degree. We conclude by giving further geometric comments about the symmetric K3 surfaces (S, G) from Theorem 10.1.

10.1. Preliminaries on representation theory

In this section we work over algebraically closed fields of characteristic zero, mainly $\mathbb{C}$, all groups are supposed to be finite and all vector spaces are of finite dimension.

10.1.1. Linear representations and group algebra modules

Let K be an algebraically closed field of characteristic zero and let E be a finite group. By Maschke's theorem [EGH+11, Theorem 3.1], the group algebra KE is *semisimple*, that is, all of its modules are semisimple and therefore can be decomposed as the direct sum of simple submodules. Throughout, we describe KE-modules as pairs (V, ρ), where V is a finite-dimensional K-vector space and ρ is a *K-linear representation of E on V*, that is ρ is a homomorphism

$$\rho \colon E \to \mathrm{GL}(V).$$

Given two KE-modules $M = (V, \rho)$ and $M' = (V', \rho')$, we say M and M' are *equivalent*, and we write $M \simeq M'$, if there exists an invertible K-linear map $\mathcal{L} \colon V \to V'$ such that for all $e \in E$,

$$\rho'(e) = \mathcal{L} \circ \rho(e) \circ \mathcal{L}^{-1}.$$

If $V = V'$, we also say that ρ and ρ' are themselves *equivalent*.

Remark 10.3. By the Krull–Schmidt theorem [EGH+11, Theorem 2.19], if a KE-module is semisimple then its decomposition into a direct sum of simple submodules is unique up to the order of the summands. Moreover, according to [Isa76, Corollary (2.5)], the equivalence classes of simple KE-modules correspond bijectively to the conjugacy classes of E.

If $M = (V, \rho)$ is a KE-module, then by Maschke's theorem, one can write

$$M \simeq \bigoplus_{i=1}^{l} W_i^{\oplus f_i}$$

where the W_i's are pairwise nonequivalent simple KE-modules. We call this decomposition an *isotypical decomposition* of M. For all $1 \leq i \leq l$, we call the summand $W_i^{f_i}$ an *isotypical component* of M. It is itself a KE-module which we say to be *isotypical of weight* $\dim_K(W_i)$ (to be understood, the K-dimension of the underlying vector space). Although the decomposition of M into the sum of its isotypical components is unique, the decomposition of each $W_i^{f_i}$ into a sum of simple modules is unique only up to equivalence.

Theorem 10.4 (Schur's lemma; [EGH+11, Proposition 1.16 & Corollary 1.17]). *Let $M = W^{\oplus t}$ and $M' = W'^{\oplus t'}$ be two isotypical KE-modules, where W and W' are simple. Then, under the assumption that K is algebraically closed, one has*

$$\mathrm{Hom}_{KE}(M, M') \cong \begin{cases} \mathrm{Mat}_{t,t'}(K) & \text{if} \quad W \simeq W' \\ 0 & \text{else} \end{cases}$$

where $\mathrm{Mat}_{t,t'}(K)$ denotes the set of t-by-t' matrices with entries in K. In particular, the KE-automorphism group of a simple KE-module can be identified with $K^\times$.

10.1.2. Characters of representations

Let again K be an algebraically closed field of characteristic 0, let E be a finite group, and let $M = (V, \rho)$ be a KE-module. We define the *K-character* χ_M of M to be the mapping

$$\chi_M \colon E \to K,\ e \mapsto \mathrm{Tr}(\rho(e)).$$

We say also that M *affords* χ_M and that χ_M is *afforded by* M. One notes that $\chi_M(1_E) = \dim_K(V)$ and χ_M is constant on each conjugacy class of E. More generally, K-characters of E are a special case of what we call *class functions* on E. Moreover, for any K-character χ of E, there is a KE-module M such that $\chi = \chi_M$. We define sum and product of K-characters of E as pointwise sum and product of their respective images in K. So for instance, if χ and χ' are two K-characters of E afforded by M and M' respectively, then $\chi + \chi'$ is afforded by $M \oplus M'$ and vice-versa. A K-character χ of E is said to be *simple*, or *irreducible*, if χ cannot be written nontrivially as a sum of other K-characters of E.

Proposition 10.5 ([Isa76, Corollary (2.5)]). *The number of simple K-characters of E is equal to the number of conjugacy classes of E (recall that E is a finite group here). In particular, simple K-characters of E are afforded by simple KE-modules.*

Proposition 10.6 ([Isa76, Corollary (2.9)]). *Two KE-modules M and M' are equivalent if and only if they afford the same K-character of E.*

We define the *degree* of a K-character χ of E as $\chi(1_E)$. For all $n \geq 1$, we denote by $\mathrm{Irr}_K^n(E)$ the set of all simple K-characters of E of degree n, and we define $\mathrm{Irr}_K(E) := \bigsqcup_{n \geq 1} \mathrm{Irr}_K^n(E)$. According to [Isa76, Theorem (2.8)], any K-character χ of E admits a unique decomposition

$$\chi = \sum_{\mu \in \mathrm{Irr}_K(E)} e_\mu \mu$$

where $e_\mu \in \mathbb{Z}_{\geq 0}$ is called the *multiplicity* of the irreducible character μ in χ. Given two K-characters $\chi = \sum_{\mu \in \mathrm{Irr}_K(E)} e_\mu \mu$ and $\chi' = \sum_{\mu \in \mathrm{Irr}_K(E)} e'_\mu \mu$ of E, we define their *scalar product* by

$$\langle \chi, \chi' \rangle := \sum_{\mu \in \mathrm{Irr}_K(E)} e_\mu e'_\mu.$$

In particular, for $\mu \in \mathrm{Irr}_K(E)$ we have that $\langle \mu, \mu \rangle = 1$, and $\langle \chi, \mu \rangle$ is equal to the multiplicity of μ in χ. If $\chi = \sum_{\mu \in \mathrm{Irr}_K(E)} e_\mu \mu$ and $\chi' = \sum_{\mu \in \mathrm{Irr}_K(E)} e'_\mu \mu$ are two K-characters of E such that $0 \leq e_\mu \leq e'_\mu$ for all $\mu \in \mathrm{Irr}_K(E)$, then we say that χ is a *constituent* of χ'.

We see that the decomposition of the K-character afforded by a KE-module depends only on its isotypical decomposition. We say that a K-character χ of E is *isotypical* if χ is afforded by an isotypical KE-module, i.e. it is a positive multiple of an irreducible K-character of E.

10.1.3. Group actions and projective representations

Let K be algebraically closed of characteristic zero. Given a finite group G and a finite-dimensional K-vector space V, we call a *projective representation* of G on V any homomorphism

$$\overline{\rho} \colon G \to \mathrm{PGL}(V).$$

Such a representation is called *faithful* if it is injective. For any group G, there exists a finite abelian group $M(G)$ called the *Schur multiplier* of G [Isa76, Definition (11.12)], which can be identified with $H^2(G, K^\times)$, the second cohomology group of G with coefficients in $K^\times$. In [Sch04], Schur proves that for any finite group G, there exists a group E and an exact sequence

$$1 \to H \xrightarrow{i} E \xrightarrow{p} G \to 1$$

such that $H \cong M(G)$, and $i(H) \leq [E, E] \cap Z(E)$ where $[E, E]$ is the derived subgroup of E and $Z(E)$ is its center. This exact sequence satisfies the following: for any projective representation $\overline{\rho} \colon G \to \mathrm{PGL}(V)$ of G on a finite-dimensional K-vector space V, there exists a linear representation $\rho \colon E \to \mathrm{GL}(V)$ making the following diagram with exact rows commute

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "i"] \arrow[d, "\beta"'] & E \arrow[r, "p"] \arrow[d, "\exists \rho"'] & G \arrow[r] \arrow[d, "\overline{\rho}"'] & 1 \\ 1 \arrow[r] & K^\times \arrow[r, "\cdot \mathrm{id}_V"'] & \mathrm{GL}(V) \arrow[r, "\pi"'] & \mathrm{PGL}(V) \arrow[r] & 1 \end{tikzcd} . \tag{38}$$

Here β is induced by the restriction of ρ to $i(H)$. This result is known as Schur's theorem [Isa76, Theorem (11.17)] and we call $p \colon E \twoheadrightarrow G$ (or just E) a *Schur cover* of G. We moreover refer to ρ as a *p-lift* of $\overline{\rho}$ and the latter as the *p-reduction* of the former. Schur's theorem allows us to use the results from the theory of linear representations of finite groups to work with projective representations of finite groups. In particular, given a finite group G, a Schur cover $p \colon E \twoheadrightarrow G$ and a finite-dimensional K-vector space V, one can relate a classification of projective representations of G on V to a classification of linear representations of E on V.

Remark 10.7. In general, a Schur cover E of a finite group G is not unique. If $G = [G, G]$ is perfect, then E is unique up to isomorphism. Otherwise, an upper bound on the number of nonisomorphic Schur covers of G can be found in [Kar87, Theorem 2.5.14].

Definition 10.8 ([Isa76, Definition(1.18) & Page 177])**.** Let G be a finite group and let V be a finite-dimensional K-vector space. Two projective representations $\overline{\rho}, \overline{\rho}' \colon G \to \mathrm{PGL}(V)$ are called *similar* if there exists an automorphism $\mathcal{L}\colon V \to V$ such that, for all $g \in G$,

$$\overline{\mathcal{L}} \circ \overline{\rho}(g) = \overline{\rho}'(g) \circ \overline{\mathcal{L}}$$

where $\overline{\mathcal{L}}\colon \mathbb{P}(V) \to \mathbb{P}(V)$ is induced by $\mathcal{L}$.

The next result can be found in [Isa76, Page 178], we give it a proof for completeness.

Lemma 10.9. *Let G be a finite group, let $p\colon E \twoheadrightarrow G$ be a Schur cover of G and let V be a finite-dimensional K-vector space. Assume that there are two projective representations $\overline{\rho}, \overline{\rho}'\colon G \to$* $\mathrm{PGL}(V)$ *with respective p-lifts $\rho, \rho'\colon E \to \mathrm{GL}(V)$ as in Equation (38). Then $\overline{\rho}$ and $\overline{\rho}'$ are similar if and only if there exists a homomorphism $\epsilon\colon E \to K^\times$ such that ρ and $\epsilon\rho'$ are equivalent.*

Proof. Suppose that $\overline{\rho}$ and $\overline{\rho}'$ are similar and let $\mathcal{L} \in \mathrm{GL}(V)$ be such that, for all $g \in G$,

$$\overline{\mathcal{L}} \circ \overline{\rho}(g) \circ \overline{\mathcal{L}}^{-1} = \overline{\rho}'(g).$$

By commutativity of the diagram in Equation (38), for all $e \in E$, one obtains that

$$\pi(\mathcal{L} \circ \rho(e) \circ \mathcal{L}^{-1}) = \overline{\mathcal{L}} \circ \underbrace{(\pi(\rho(e)))}_{=\overline{\rho}(p(e))} \circ \overline{\mathcal{L}}^{-1} = \overline{\rho}'(p(e)) = \pi(\rho'(e)).$$

Hence, for all $e \in E$, there exists a unique scalar $\epsilon(e) \in K^\times$ such that $\mathcal{L} \circ \rho(e) \circ \mathcal{L}^{-1} = \epsilon(e)\rho'(e)$. By straightforward computations, one can show that the previous assignment $\epsilon\colon E \to K^\times$ is a group homomorphism, and ρ and $\epsilon\rho'$ are equivalent. Now suppose that there exist a homomorphism $\epsilon\colon E \to K^\times$ and $\mathcal{L} \in \mathrm{GL}(V)$ such that, for all $e \in E$,

$$\mathcal{L} \circ \rho(e) \circ \mathcal{L}^{-1} = \epsilon(e)\rho'(e).$$

By commutativity of Equation (38) and surjectivity of p, it clear that $\overline{\rho}$ and $\overline{\rho}'$ are similar $\square$

Note that, in the context of Lemma 10.9, given a linear representation ρ of E on V whose restriction to $M(G) = \ker(p)$ maps to $K^\times \mathrm{id}_V$, one can always define a projective representation of G on V which makes the diagram in Equation (38) commute. Indeed, we can define a p-reduction of ρ as $\overline{\rho} := \pi \circ \rho \circ s$ where s is any section of p that maps 1_G to 1_E (it can be easily shown that this definition does not depend on the choice of s). In this case, we say that ρ is *p-projective*.

10.2. Parametrizing submodules of a given group algebra module

Let K be an algebraically closed field of characteristic zero, let E a finite group and let $M = (V, \rho)$ a KE-module. In this section we show that the set parametrizing the KE-submodules of M of a given dimension is a projective variety, and that its irreducible components are rational.

10.2.1. Invariant Grassmannians

Let $M = (V, \rho)$ be a KE-module, and let $n := \dim_K(V)$. For all $1 \leq t \leq n$, we define $\mathrm{Gr}(t, M)$ to be the set of t-dimensional KE-submodules of M. Using iteratively an argument from [MWY22, Theorem 5.32], we have that $\mathrm{Gr}(t, M)$ is a closed subvariety of the ordinary Grassmannian variety $\mathrm{Gr}(t, V) \subseteq \mathbb{P}(\bigwedge^t V)$. In general, $\mathrm{Gr}(t, M)$ is not irreducible, and we give two ways to decompose it: we use the first one computationally to parametrize all t-dimensional submodules of M, and the second one can be used to compute the defining ideal of $\mathrm{Gr}(t, M)$ in the Plücker space $\mathbb{P}(\bigwedge^t V)$. Let χ be the K-character of E afforded by M.

Definition 10.10. For all $1 \leq t \leq \chi(1_E)$, we define $C_\chi(t)$ to be the set of all K-characters of E of degree t which are a consistuent of χ.

For $1 \leq t \leq \chi(1_E)$, each $\eta \in C_\chi(t)$ defines an equivalence class of t-dimensional submodules of M. For $\eta \in C_\chi(t)$, let N_η be a KE-module affording η and define

$$\mathrm{Gr}(\eta, M) := \mathrm{Hom}^0_{KE}(N_\eta, M)/\mathrm{Aut}_{KE}(N_\eta)$$

where $\mathrm{Hom}^0_{KE}(N_\eta, M)$ denotes the set of all injective KE-module homomorphisms from N_η to M. By Proposition 10.6, this definition does not depend on the choice of N_η, and one can see that $\mathrm{Gr}(\eta, M)$ corresponds to the set of t-dimensional submodules of M affording η. Therefore, as a set,

$$\mathrm{Gr}(t, M) = \bigsqcup_{\eta \in C_\chi(t)} \mathrm{Gr}(\eta, M).$$

Now, let $M = \bigoplus_{\mu \in \mathrm{Irr}_K(E)} W_\mu^{\oplus f_\mu}$ be an isotypical decomposition of M where for all $\mu \in \mathrm{Irr}_K(E)$, W_μ affords μ, and let $\eta := \sum_{\mu \in \mathrm{Irr}_K(E)} e_\mu \mu \in C_\chi(t)$. Then, the KE-module $\bigoplus_{\mu \in \mathrm{Irr}_K(E)} W_\mu^{\oplus e_\mu}$ affords η, and we have that

$$\mathrm{Gr}(\eta, M) = \mathrm{Hom}^0_{KE}\left(\bigoplus_{\mu \in \mathrm{Irr}_K(E)} W_\mu^{\oplus e_\mu}, \bigoplus_{\mu \in \mathrm{Irr}_K(E)} W_\mu^{\oplus f_\mu}\right) \Big/ \mathrm{Aut}_{KE}\left(\bigoplus_{\mu \in \mathrm{Irr}_K(E)} W_\mu^{\oplus e_\mu}\right).$$

Using Schur's Lemma (Theorem 10.4), we compute

$$\mathrm{Gr}(\eta, M) \cong \prod_{\mu \in \mathrm{Irr}_K(E),\, e_\mu \neq 0} \mathrm{Hom}^0_{KE}(W_\mu^{e_\mu}, W_\mu^{f_\mu})/\mathrm{Aut}_{KE}(W_\mu^{e_\mu}) = \prod_{\mu \in \mathrm{Irr}_K(E),\, e_\mu \neq 0} \mathrm{Gr}(e_\mu \mu, W_\mu^{f_\mu}). \tag{39}$$

We are therefore reduced to understanding KE-submodules of isotypical KE-modules.

10.2.2. Isotypical modules and Gauss elimination

The goal of this subsection is to bring a constructive approach to the proof of the existence of a Gauss elimination theorem for isotypical KE-modules. Thanks to Schur's lemma, this immediatly follows from the fact that Grassmannians of isotypical submodules are the same as ordinary Grassmannians.

Let $M = W^{\oplus t}$ be isotypical of weight n and K-dimension tn with W simple. We want to study the set $\mathrm{Gr}(k, M)$ of equivalence classes of k-dimensional KE-submodules of M, for some $k \geq 1$. As a first remark, since the character χ afforded by M is isotypical, it can be written as $\chi = t\mu$ where $\mu \in \mathrm{Irr}_K(E)$ is afforded by W. So, in particular, since μ is of degree n,

$$\mathrm{Gr}(k, M) = \begin{cases} \mathrm{Gr}(r\mu, M) & \text{if } k = rn \text{ for some } 1 \leq r \leq t \\ \emptyset & \text{otherwise} \end{cases}.$$

Theorem 10.11 (Gauss elimination)**.** *Let $M = W^{\oplus t}$ be an isotypical KE-module of weight n and K-dimension tn. Then for all $1 \leq r \leq t$, the set $\mathrm{Gr}(rn, M)$ of KE-submodules of M of dimension rn can be identified with the ordinary Grassmannian $\mathrm{Gr}(r, t)$.*

Proof. Let $1 \leq r \leq t$ and let $\chi = t\mu$, with $\mu \in \mathrm{Irr}_K(E)$, be the character afforded by M. Using the fact that $\mathrm{Gr}(rn, M) = \mathrm{Gr}(r\mu, M)$, since M is isotypical, we have that

$$\mathrm{Gr}(rn, M) = \mathrm{Hom}^0_{KE}(W^{\oplus r}, W^{\oplus t})/\mathrm{Aut}_{KE}(W^{\oplus r}).$$

Fixing a basis of the underlying K-vector space of W, Schur's Lemma (Theorem 10.4) gives us then that

$$\mathrm{Gr}(rn, M) \simeq \mathrm{Mat}^0_{r,t}(K)/\mathrm{GL}_r(K) = \mathrm{Gr}(r, t)$$

where $\mathrm{Mat}^0_{r,t}(K)$ denotes the set of full rank r-by-t matrices with entries in K. □

The important point of the proof of Theorem 10.11 is the following: given a KE-module $M = \bigoplus_{\mu\in\mathrm{Irr}_K(E)} W_\mu^{\oplus f_\mu}$, one can algorithmically compute a basis of $\mathrm{Hom}_{KE}(W_\mu, M)$ for all μ such that $f_\mu \neq 0$. In fact, denoting $M = (V, \rho)$ and $W_\mu = (V', \rho')$, and fixing respective K-bases B and B' of V and V', $\mathrm{Hom}_{KE}(W_\mu, M)$ corresponds to the set $\mathrm{M}_{\rho,\rho'}(B, B')$ of $\dim_K(V) \times \dim_K(V')$ matrices P such that, for all $e \in E$,

$$\rho(e)P = P\rho'(e).$$

(Here we identify $\mathrm{GL}(V)$ and $\mathrm{GL}(V')$ with the respective groups of invertible matrices using the fixed bases B and B') This is a K-vector space of finite dimension f_μ and any nonzero matrix in $\mathrm{M}_{\rho,\rho'}(B, B')$ is of full rank (since W_μ is simple). A basis of this vector space can be computed using for instance an algorithm of [CIK97, Theorem 2]. In particular, for all $1 \leq e \leq f_\mu$, one has that any embedding $W_\mu^{\oplus e} \hookrightarrow M$ corresponds to the choice of an e-space in $\mathrm{M}_{\rho,\rho'}(B, B')$.

10.2.3. Rationality and irreducible components

In this subsection we show that the space $\mathrm{Gr}(\eta, M)$ of KE-submodules with a given character η of any KE-module M is a rational projective variety. Once again, thanks to Schur's lemma, this is a direct consequence of the fact that $\mathrm{Gr}(\eta, M)$ is a finite product of ordinary Grassmannians.

Let M be a KE-module and let us denote χ the character afforded by M.

Theorem 10.12. *For all $1 \leq t \leq \chi(1_E)$ and for all $\eta \in C_\chi(t)$, the space $\mathrm{Gr}(\eta, M)$ is a rational subvariety of $\mathrm{Gr}(t, M)$ of dimension $\langle \eta, \chi - \eta\rangle$. In particular, $\{\mathrm{Gr}(\eta, M)\}_{\eta\in C_\chi(t)}$ is the set of irreducible components of $\mathrm{Gr}(t, M)$.*

Proof. Using Equation (39) and Theorem 10.11, one can endow $\mathrm{Gr}(\eta, M)$ with a scheme structure as a direct product of rational varieties, allowing us to see it as rational subvariety of $\mathrm{Gr}(t, M)$. More precisely, if $M = \bigoplus_{\mu\in\mathrm{Irr}_K(E)} W_\mu^{\oplus f_\mu}$, we set that

$$\mathrm{Gr}(\eta, M) \simeq \prod_{\mu\in\mathrm{Irr}_K(E),\ e_\mu\neq 0} \mathrm{Gr}(e_\mu, f_\mu)$$

as a projective variety. It is known that the ordinary Grassmannian $\mathrm{Gr}(r, t)$ has dimension $r(t-r)$, as a complex projective variety. Thus, one deduces that

$$\dim(\mathrm{Gr}(\eta, M)) = \sum_{\mu\in\mathrm{Irr}_K(E),\ e_\mu\neq 0} e_\mu(f_\mu - e_\mu) = \sum_{\mu\in\mathrm{Irr}_K(E),\ e_\mu\neq 0} \langle e_\mu\mu, f_\mu\mu - e_\mu\mu\rangle = \langle \eta, \chi - \eta\rangle$$

by the orthogonality relations between simple characters with respect to the scalar product. □

Theorem 10.12 offers a feasible way to parametrizing t-dimensional submodules of a given KE-module M. Indeed, for all t-dimensional constituent η of the character χ of M, one may use Theorem 10.11 and Theorem 10.12 to construct a concrete parametrization of $\mathrm{Gr}(\eta, M)$.

10.2.4. Determinant character

Let $1 \leq t \leq \chi(1_E)$, where we recall that χ is the character afforded by a fixed KE-module $M = (V, \rho)$. Any element of the t-th exterior power $\bigwedge^t V$ of V over K is called a *t-multivector* of V and those of the form $v_1 \wedge \ldots \wedge v_t$ are called *pure*. Any element of $\bigwedge^t V$ can be written as a finite sum of pure t-multivectors. There is moreover an induced action of E on $\bigwedge^t V$ given by, for all $e \in E$ and for any pure t-multivector $v_1 \wedge \ldots \wedge v_t$ of V,

$$e \cdot (v_1 \wedge \ldots \wedge v_t) := (\rho(e)v_1) \wedge \ldots \wedge (\rho(e)v_t). \tag{40}$$

We denote by $\bigwedge^t \rho$ the previous representation of E on $\bigwedge^t V$ and $\bigwedge^t M := (\bigwedge^t V, \bigwedge^t \rho)$ the corresponding KE-module. We call it the *t-th exterior power* of M.

Remark 10.13. The action defined in Equation (40) is so that the *Plücker embedding* $\mathrm{Gr}(t, V) \hookrightarrow \mathbb{P}(\bigwedge^t V)$ is equivariant with respect to ρ and $\bigwedge^t \rho$. The image of $\mathrm{Gr}(t, V)$ under this closed embedding consists of all lines spanned by a pure t-multivector in $\bigwedge^t V$. Hence one deduces that $\mathrm{Gr}(t, M)$ is nonempty if and only if $\bigwedge^t M$ admits a 1-dimensional KE-submodule whose underlying K-vector space is spanned by a pure t-multivector. If we let $\bigwedge^t M = \bigoplus_{\mu \in \mathrm{Irr}_K(E)} U_\mu^{\oplus g_\mu}$ be an isotypical decomposition of $\bigwedge^t M$, then the latter holds if and only if there exists $\mu \in \mathrm{Irr}_K^1(E)$ such that $g_\mu \neq 0$ and $\mathbb{P}(U_\mu^{\oplus g_\mu}) \cap \mathrm{Gr}(t, V) \neq \emptyset$. Indeed, any E-invariant line of $\bigwedge^t V$ lies in one of the isotypical components of weight 1 of $\bigwedge^t M$.

For any $\eta \in C_\chi(t)$, we call the *determinant character* of η, denoted $\det(\eta)$, the character afforded by the t-th exterior power of any KE-module affording η. This is a 1-dimensional character, constituent of the character $\bigwedge^t \chi$ afforded by $\bigwedge^t M$. Note that two distinct constituents $\eta, \eta' \in C_\chi(t)$ of χ can have the same determinant character. For any $\mu \in \mathrm{Irr}_K^1(E)$, we denote by $\mathrm{Gr}_\mu(t, M) \subseteq \mathrm{Gr}(t, M)$ the subset of t-dimensional KE-submodules of M having determinant character equal to μ. Then we have the decompositions, as sets,

$$\mathrm{Gr}(t, M) = \bigsqcup_{\mu \in \mathrm{Irr}_K^1(E)} \mathrm{Gr}_\mu(t, M)$$

and for all $\mu \in \mathrm{Irr}_K^1(E)$

$$\mathrm{Gr}_\mu(t, M) = \bigsqcup_{\eta \in C_\chi(t),\, \det(\eta) = \mu} \mathrm{Gr}(\eta, M).$$

By identifying the Grassmannian variety $\mathrm{Gr}(t, V)$ with its image via the Plücker embedding, our Remark 10.13 tells us that

$$\mathrm{Gr}_\mu(t, M) \simeq \mathbb{P}(U_\mu^{\oplus g_\mu}) \cap \mathrm{Gr}(t, V).$$

In this way, we can define a scheme structure on $\mathrm{Gr}_\mu(t, M)$ turning it into a closed subvariety of $\mathrm{Gr}(t, M)$, allowing us to effictively compute the defining ideal of $\mathrm{Gr}(t, M)$ in the Plücker space.

10.3. Finding intersections with prescribed symmetry

Let X be a complex projective variety in $\mathbb{P}^n_{\mathbb{C}}$ given as an intersection of t hypersurfaces of the same degree d. The ideal I defining X is homogeneous and generated by t homogeneous polynomials $f_1, \ldots, f_t \in \mathbb{C}[x_0, \ldots, x_n]$ of common total degree d. Let G be a finite group and let $p\colon E \twoheadrightarrow G$ be a Schur cover of G. Suppose that there exists a faithful linear action of G on $\mathbb{P}^n_{\mathbb{C}}$ preserving X, which is not necessarily given (we assume existence without any explicit description). Our strategy to find I consists in classifying all faithful linear actions of G on $\mathbb{P}^n_{\mathbb{C}}$ and in determining, for each class of them, a parametrization of all ideals defining intersections of the same type of X, and which are preserved under the provided action of G. Any linear action of G on $\mathbb{P}^n_{\mathbb{C}}$ is given by a projective representation $\overline{\rho}\colon G \to \mathrm{PGL}(\mathbb{C}^{n+1})$, which can be lifted along p to $\rho\colon E \to \mathrm{GL}(\mathbb{C}^{n+1})$ making the following commutative diagram with exact rows commute

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "i"] \arrow[d, "\beta"'] & E \arrow[r, "p"] \arrow[d, "\rho"'] & G \arrow[r] \arrow[d, "\overline{\rho}"'] & 1 \\ 1 \arrow[r] & \mathbb{C}^\times \arrow[r, "\cdot \mathrm{id}_{\mathbb{C}^{n+1}}"'] & \mathrm{GL}(\mathbb{C}^{n+1}) \arrow[r, "\pi"'] & \mathrm{PGL}(\mathbb{C}^{n+1}) \arrow[r] & 1 \end{tikzcd} . \tag{41}$$

10.3.1. From invariant ideals to group algebra modules

Fix a linear action of the finite group G on $\mathbb{P}^n_{\mathbb{C}}$ and we assume that X is preserved under this action. We denote by $R_\bullet := \bigoplus_{h\geq 0} \mathbb{C}[x_0,\dots,x_n]_h$ the $\mathbb{Z}$-graded $\mathbb{C}$-algebra of polynomials in $n+1$ variables. Considering Equation (41), the action of E on $\mathbb{C}^{n+1}$ defined by $\rho\colon E \to \mathrm{GL}(\mathbb{C}^{n+1})$ naturally induces, for all $h \geq 0$, a linear action on R_h. It is given as follows: for any $h \geq 0$, any $P \in R_h$, any $e \in E$ and any $x \in \mathbb{C}^{n+1}$,

$$(e \cdot P)(x) := P(\rho(e)^{-1}(x)).$$

It is a well-defined left action, because the action of E on $\mathbb{C}^{n+1}$ is linear, which we denote by ρ_h. Collecting these actions for all $h \geq 0$ gives $(R_\bullet, \rho_\bullet)$ the structure of a $\mathbb{C}E$-algebra — $R_\bullet$ is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra and all of its homogeneous components R_h ($h \geq 0$), equipped with the action ρ_h, are $\mathbb{C}E$-modules. The following key result shows how to simplify the search for the ideal I defining X by restricting ourselves into determining a basis for a finite-dimensional vector space.

Proposition 10.14. *Let K be a field, let E be a group and let $(R_\bullet, \rho_\bullet)$ be a $\mathbb{Z}$-graded KE-algebra. Let I be a homogeneous ideal of $R_\bullet$ being finitely generated by t homogeneous elements $r_1,\dots,r_t \in R_\bullet$ of respective degrees $d_1,\dots,d_t$. We denote by $I_h := I \cap R_h$ the h-homogeneous part of I. Then, I is invariant for the given action of E on $R_\bullet$ if and only if (I_{d_i}, ρ_{d_i}) is a KE-submodule of (R_{d_i}, ρ_{d_i}) for all $i = 1,\dots,t$ (here we use the same notation for the restriction of ρ_h to I_h, $h \geq 0$).*

Proof. First remark that $I = \bigoplus_{h\in\mathbb{Z}} I_h = \sum_{i=1}^t I_{h_i}$ as R_0-modules since I is generated by the t homogeneous elements $r_1,\dots,r_t$. Therefore, if $E \cdot I = I$ (i.e. I is E-invariant) then for all $i = 1,\dots,t$, we see that $E \cdot I_{h_i} = E \cdot (I \cap R_{h_i}) \subseteq I_{h_i}$ because E acts on R_{h_i}. Therefore, (I_{h_i}, ρ_{h_i}) is a $\mathbb{C}E$-submodule of (R_{h_i}, ρ_{h_i}), for all $i = 1,\dots,t$.
Now suppose that for all $i = 1,\dots,t$, the h_i- homogeneous part I_{h_i} of I is E-invariant. Since I is generated by $\bigcup_{i=1}^t I_{h_i}$ as a $R_\bullet$-module and $(R_\bullet, \rho_\bullet)$ is a KE-module, then I is E-invariant. □

Considering Equation (41) with R_d instead of $\mathbb{C}^{n+1}$, we know that ρ_d reduces to a unique projective representation of G on R_d. By commutativity of the diagram in Equation (41), and Proposition 10.14, one sees that X is preserved under the action of G on $\mathbb{P}^n_{\mathbb{C}}$ if and only if (I_d, ρ_d) is a t-dimensional $\mathbb{C}E$-submodule of (R_d, ρ_d).

Remark 10.15. This is where the notion of *invariant Grassmannian* previously defined has importance. An E-invariant homogeneous ideal I_d in R_d might not be generated by semi-invariant polynomials, i.e. homogeneous polynomials whose $\mathbb{C}$-spans in R_d are themselves $\mathbb{C}E$-modules. Indeed, if the underlying $\mathbb{C}E$-module of I_d is simple of dimension greater than one, then the ideal cannot be generated by such semi-invariant polynomials. This is also why we cannot use the known algorithms from invariant theory for finite groups, as described in [DK15] for instance.

10.3.2. Classification of projectively faithful representations

Recall that we are given a Schur cover $p\colon E \twoheadrightarrow G$ of G. In Definition 10.8 we define an equivalence relation on the set of linear representations of E. For geometric purposes, we coarsen this equivalence relation.

Definition 10.16. Let V be a finite-dimensional $\mathbb{C}$-vector space. A p-projective linear representation $\rho\colon E \to \mathrm{GL}(V)$ is said to be *p-projectively faithful* if its p-reduction $\bar{\rho}\colon G \to \mathrm{PGL}(V)$ is faithful.

By commutativity of the diagram in Equation (41), and by surjectivity of p, we see that any p-projective representation ρ of E on V is p-projectively faithful if and only

$$\ker(\pi \circ \rho) = \ker(p).$$

In order to find a $\mathbb{C}E$-module W whose underlying vector space generates the defining ideal I of X (see Section 10.3.1), we start by classifying the p-projectively faithful representations of E on $\mathbb{C}^{n+1}$.

Definition 10.17 ([Isa76, Definition (2.26)])**.** Let χ be the $\mathbb{C}$-character of E afforded by a $\mathbb{C}E$-module (V, ρ). Then, the *center of the character* χ is defined to be

$$Z(\chi) := \left\{ e \in E \ : \ \frac{\chi(e)}{\chi(1_E)} \text{ is a root of unity} \right\}.$$

Proposition 10.18. *With the notations of Definition 10.17 and Equation (41), $Z(\chi) = \ker(\pi \circ \rho)$.*

Proof. According to [Isa76, Lemma (2.27)],

$$Z(\chi) = \left\{ e \in E \ : \ \rho(e) \in \mathbb{C}^{\times}\mathrm{id}_V \right\} \tag{42}$$

so the first inclusion $Z(\chi) \leq \ker(\pi \circ \rho)$ holds. Now, let $e \in E$ be such that $\pi(\rho(e)) = 1_{\mathrm{PGL}(V)}$. In particular $\rho(e) \in \ker(\pi) = \mathbb{C}^{\times}\mathrm{id}_V$. Therefore according to Equation (42), we have that $e \in Z(\chi)$. □

Corollary 10.19. *A projective representation $\overline{\rho}\colon G \to \mathrm{PGL}(V)$ is faithful if and only if $Z(\chi) = \ker(p)$ where χ is afforded by any p-lift $\rho\colon E \to \mathrm{GL}(V)$ of $\overline{\rho}$.*

Using Corollary 10.19, we are now able to decide whether a linear representation of E is p-projectively faithful or not. In fact, checking p-projectivity and the condition of Corollary 10.19 are both possible using only character theory. Now, let us note that if $\overline{\rho}$ and $\overline{\rho}'$ are two similar projective representations of G on V, then there exists an automorphism $\mathcal{L} \in \mathrm{Aut}(V)$ such that, for all $g \in G$,

$$\overline{\mathcal{L}} \circ \overline{\rho}(g) \circ \overline{\mathcal{L}}^{-1} = \overline{\rho}'(g).$$

This means that if a projective variety in $\mathbb{P}(V)$ is preserved under $\overline{\rho}$, then after a projective base change, it will also be preserved under $\overline{\rho}'$, and vice-versa. We can therefore consider only one representatives of each similarity class, and thus classify p-projectively faithful representations of E up to similarity of their respective p-reductions. Using Lemma 10.9, this is equivalent to classifying p-projectively faithful representations of E on $\mathbb{C}^{n+1}$ by equivalence modulo $\mathrm{Irr}^1_{\mathbb{C}}(E)$.

Notation. We denote by $\mathrm{PFR}(E, G, \mathbb{C}^{n+1})$ a set of representatives of classes in

$$\left\{ p\text{-projectively faithful representations of } E \text{ on } \mathbb{C}^{n+1} \right\} / \left\{ \rho \sim \rho' \text{ iff } \exists \epsilon \in \mathrm{Irr}^1_{\mathbb{C}}(E) \text{ s.t. } \chi_\rho = \epsilon \chi_{\rho'} \right\}.$$

The set $\mathrm{PFR}(E, G, \mathbb{C}^{n+1})$ is finite since the number of equivalence classes of linear representations of E on $\mathbb{C}^{n+1}$ is actually finite [Isa76, Corollary (2.5)].

Note that using character theory one can efficiently compute a set of characters afforded by the representatives of classes in $\mathrm{PFR}(E, G, \mathbb{C}^{n+1})$. However, given this set of characters, the computation of the actual representations (given in matrix form) can be much more challenging (see [DA05] for an algorithm and complexity discussion).

10.3.3. Application to the case of K3 surfaces

Let (S, G, G_s) be the triple given in Table 11. From [BH21, 77b], we know that the canonical G-invariant polarization L of the pair (S, G) has degree $c_1(L)^2 = 8$. According to a remark in [SD74, Page 615], either the polarization L is *hyperelliptic* and the linear system $|L|$ defines a degree 2 map onto a surface of degree 4 in $\mathbb{P}^5_{\mathbb{C}}$, or it is not hyperelliptic and S is birational to a surface of degree 8 in $\mathbb{P}^5_{\mathbb{C}}$. By [SD74, Theorem 5.2.], there are numerical conditions to decide whether such a polarization is hyperelliptic, and such conditions can be checked in practice using an algorithm of Shimada [Shi15, Algorithm 2.2]. In the database of Brandhorst and Hofmann [BH23, 77.2.1.3], one can recover workable lattice data about such a triple (S, G, G_s) and show that the G-invariant polarization does not satisfy any of the conditions of [SD74, Theorem 5.2.]. Therefore L is not hyperelliptic and the linear system $|L|$ defines a birational map $\varphi_{|L|}$ from S onto a surface of degree 8 in $\mathbb{P}^5_{\mathbb{C}}$. We moreover know that $c_1(L)$ lies in $2\mathrm{NS}(S)^\vee$: according to [SD74, Theorem 7.2.], this implies that $\varphi_{|L|}(S) \subseteq \mathbb{P}^5_{\mathbb{C}}$ is given as a complete intersection of 3 quadrics. Finally, the orthogonal complement of $c_1(L)$ in $\mathrm{NS}(S)$ contains no vector of norm -2, so the previous complete intersection is actually smooth. The surface S can therefore be described as a smooth complete intersection of 3 quadrics in $\mathbb{P}^5_{\mathbb{C}}$ (see also [Deg24, Theorem 1.3]).

Let G be the group with Id $[384, 5602]$ (in the Small Group Library [BEOH24]). Using GAP [GAP21], one can show that this group has Schur multiplier $M(G)$ isomorphic to C_2^3, and therefore any Schur cover of G has order 3072. We compute such a Schur cover $p\colon E \twoheadrightarrow G$, using for instance the GAP method EpimorphismSchurCover: the following steps may differ depending on the choice of the Schur cover, yet the final result shall remain true. The Schur cover E chosen for these computations has 10 classes of p-projectively faithful representations on F^6, where $F := \mathbb{Q}(\zeta_{24})$ is the 24th cyclotomic field with 24 being the exponent of E. Here we work over F instead of $\mathbb{C}$ for computational reasons: according to [Isa76, Corollary (9.15)], the field F is a splitting field for E, so we are allowed to restrict to F (the results remain true over $\mathbb{C}$). In what follows, we denote $z := \zeta_{24}$ and $i := z^6$.

Let M be the FE-module (F^6, ρ), where ρ is given by the σ_i's in Theorem 10.1. The representation ρ is p-projectively faithful, and $N := (R_2, \rho_2)$ is a 21-dimensional FE-module (where $(R_\bullet, \rho_\bullet)$ is defined as in Section 10.3.1). Let χ be the F-character of E afforded by N. One has that $C_\chi(3) = \{\mu\}$ where $\mu \in \mathrm{Irr}^3_F(E)$ and $\langle \chi, \mu \rangle = 2$. Therefore, we have that $\mathrm{Gr}(3, N)$ is irreducible of dimension 1, equal to $\mathrm{Gr}(\mu, N)$. Let W be the isotypical component of N affording 2μ. The FE-module W consists of the sum of two equivalent simple FE-modules, affording μ, with respective F-bases

$$\left\{ w_1 := \begin{pmatrix} ix_0x_1 + x_0x_2 + x_1x_3 + ix_2x_3 \\ ix_0x_1 - x_0x_2 - x_1x_3 + ix_2x_3 \\ -x_0x_3 - x_1x_2 \end{pmatrix}, w_2 := \begin{pmatrix} x_5^2 \\ x_4^2 \\ -x_4x_5 \end{pmatrix} \right\}$$

where $\{x_0, \ldots, x_5\}$ is a basis for the dual space of F^6. Note that these bases are chosen in such a way that the actions of E on each of them have the same matrix representations. We know that any 3-dimensional submodule of N is then generated by a linear combination of w_1 and w_2 (Theorem 10.11). However, it is easy to see that the ideals respectively generated by w_1 and w_2 do not define smooth varieties. Now let $\lambda \in \mathbb{C}^\times$. The ideal generated by $w_1 + \lambda w_2$ defines a variety S_λ which is by construction a K3 surface. Moreover, for distinct nonzero $\lambda_1 \neq \lambda_2$, the projective change of coordinates

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto \left[x_0 : x_1 : x_2 : x_3 : \sqrt{\frac{\lambda_2}{\lambda_1}}x_4 : \sqrt{\frac{\lambda_2}{\lambda_1}}x_5\right]$$

commutes with the action of G on $\mathbb{P}^5_{\mathbb{C}}$ and it maps S_{λ_2} to S_{λ_1}, which are therefore G-equivariantly isomorphic K3 surfaces. In this case, we say that $\{S_\lambda\}_{\lambda\in\mathbb{C}^\times}$ is a 1-*dimensional* G-*isotrivial family*. As expected, up to equivariance, this K3 surface is unique and given by $S := S_1$ in Theorem 10.1. Finally, to ensure that S corresponds to the case 77b in [BH21], we need that the subgroup G_s of automorphisms of G acting symplectically on S is isomorphic to the group T_{192}. We use the following lemma, which is a direct consequence of [Muk88a, Lemma (2.1)]:

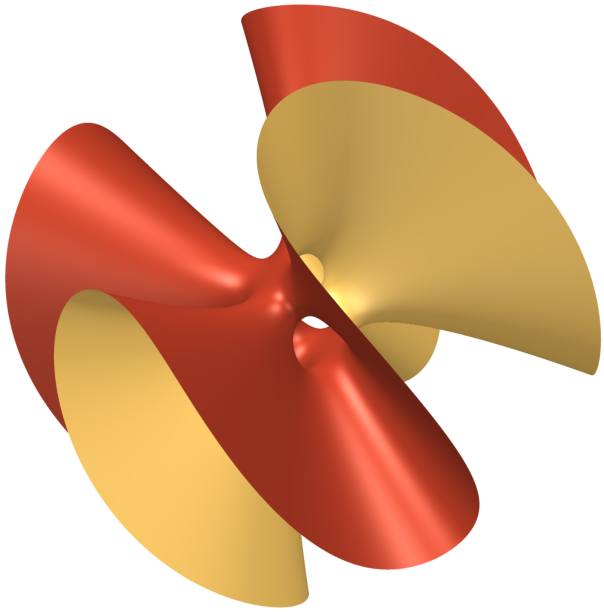
Lemma 10.20. *Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a smooth complete intersection of t hypersurfaces of degree d such that $td = n + 1$. Suppose that X is preserved under a faithful linear action of a finite group G on $\mathbb{P}^n_{\mathbb{C}}$ and let $p\colon E \twoheadrightarrow G$ be a Schur cover. Let $\rho\colon E \to \mathrm{GL}_{n+1}(\mathbb{C})$ be p-projectively faithful, such that there exists a t-dimensional $\mathbb{C}E$-submodule M of (R_d, ρ_d) generating the ideal defining X. Let us denote by χ the $\mathbb{C}$-character of E afforded by $(\mathbb{C}^{n+1}, \rho)$. Then, the normal subgroup G_s of G consisting of elements whose induced action on $H^{2,0}(X)$ is trivial is given by*

$$G_s = p\left(\{e \in E \ : \ \det(\chi)(e) = \det(\chi_M)(e)\}\right).$$

Lemma 10.20 offers a practical, and computationally feasible, way to compute G_s. Here, we can apply it to the group generated by the σ_i's on S_λ. One finds that $G_s \cong T_{192}$ with, in particular, σ_i acting symplectically on S_λ for $i = 1, 2, 3, 4$ and σ_5 being a nonsymplectic involution. This concludes the proof of Theorem 10.1.

We refer to Table 20 for more equations of symmetric polarized K3 surfaces of degree 8. In Figure 7, we give a representation of the set of real points of a $(D_8 \times C_2)$-symmetric K3-quartic, in a given affine chart. The projective equation of such a K3 surface have been obtained using the methods explained in this section, but for a homogeneous ideal generated by a homogeneous polynomial of degree 4 in 4 variables.

Figure 7: $(D_8 \times C_2)$-symmetric K3 surface: $y + x^3 z + \alpha(y^3 + xz^3) = 0$



10.4. Further geometric comments

In this subsection, we make some geometric comments about the pair (S, G) (Section 10.3.3). In particular, starting from the K3 surface S, we compute projective models of a new K3 surface and of an IHS manifold of higher dimension. For each of them, we inspect to which extent the group G, acting on the K3 surface S, acts on the new varieties constructed.

10.4.1. Symplectic quotient

The following observations were made with the help of Benedetta Piroddi, who notably pointed out the work in [vGS07].

As already mentioned, any two elements in the family of K3 surfaces $\{S_\lambda\}_{\lambda \in \mathbb{C}^\times}$ obtained in Section 10.3.3 are isomorphic to each other, and such an isomorphism can be made equivariant with respect to the prescribed G-actions. Therefore, in the associated moduli space, this isotrivial family is just a point.

Remark 10.21. This can already be seen by the rank of the Picard lattice: for $\lambda \neq 0$, the K3 surface S_λ has Picard rank 20. Moreover, we have already commented on the fact that there is a unique K3-octic with a faithful symplectic action of the group T_{192}.

Therefore, we do not have any degeneration of this family inside the associated moduli space. What would then happen at the limit points of this family ?
In the case where $\lambda = 0$, we obtain the union of 8 copies $P_1, \ldots, P_8$ of $\mathbb{P}^2$, pairwise meeting at the same rational line $l := V(x_0, x_1, x_2, x_3) \subseteq \mathbb{P}^5$. On the other side, when $\lambda = \infty$, we obtain a nonreduced variety $Z = V(x_4^2, x_5^2, x_4 x_5)$ with reduced structure $Z_{red} \simeq \mathbb{P}^3$. Now, both of these extremal cases are pointwise fixed under the involution $\sigma := \sigma_3^2(\sigma_1\sigma_3)^2$ (see Theorem 10.1). Therefore, the 8 points $p_1, \ldots, p_8 \in \mathbb{P}^5$ obtained by intersecting each of the P_i's with Z_{red} are fixed under σ too. Each of these points lies on S_λ for all $\lambda \in \mathbb{C}^\times$. Since σ acts symplectically on any of the S_λ's, it only fixes 8 points and for all $\lambda \in \mathbb{C}^\times$, the σ-fixed points of S_λ are exactly $p_1, \ldots, p_8$.

Remark 10.22. Note that the choice of σ here is not arbitrary: the class $[\sigma] \in \mathrm{PGL}_6(\mathbb{C})$ generates the unique order 2 normal subgroup of G_s (defined as in Lemma 10.20).

Via this observation, and the shape of the equations defining S_λ, one may notice that we fit in the case $\mathcal{M}_{\widetilde{8}}$ of [vGS07, §3.7]. According to the authors, each of the quotient varieties S_λ/σ maps, by projection from the σ-invariant line l onto Z_{red}, to the quartic surface $Y \subseteq Z_{red}$ defined by the equation

$$(z_0z_1 + z_2z_3)^2 + (z_0z_2 + z_1z_3)^2 + (z_0z_3 + z_1z_2)^2 = 0.$$

This is a nodal surface with 8 singularities, which are respectively the images of the 8 σ-fixed points p_i under the projection. The resolution of these singularities gives rise to a degree 4 quasipolarized K3 surface $\widetilde{Y} \to Y$. Since the subgroup generated by σ is normal both in G_s and G, the K3 surface $\widetilde{Y}$ carries an action of the finite group $G/\sigma \cong C_2^4 \rtimes D_6$ with normal symplectic subaction given by the subgroup $C_2^2 \rtimes S_4$.

Remark 10.23. The finite group $H := G/\sigma$ of order 192 is abstractly isomorphic to the group H_{192} in Mukai's list of maximal symplectic actions on K3 surfaces [Muk88a]. However, its normal subgroup H_s consisting of symplectic automorphisms has order 96: it is not maximal (in the sense of Mukai), and it corresponds to the group #65 in Xiao's list [Xia96, Table 2].

10.4.2. A symmetric IHS fourfold

This second part of comments follows from suggestions of Enrico Fatighenti during a poster presentation of this project. Starting from a construction of quadric bundles [Bea77], we use an already known construction to obtain an IHS fourfold from our K3 surface S. We then show that the induced action of G on this new variety coincide with its natural induced action as described in [Boi12]

Let V_6 be a 6-dimensional $\mathbb{C}$-vector space such that $S \subseteq \mathbb{P}(V_6) \simeq \mathbb{P}^5$ and let V_3 be a 3-dimensional complex vector space. We see the space Q_S of global quadric sections on S, generated by

$$\begin{aligned} q_1 &:= ix_0x_1 + x_0x_2 + x_1x_3 + ix_2x_3 + x_5^2, \\ q_2 &:= ix_0x_1 - x_0x_2 - x_1x_3 + ix_2x_3 + x_4^2, \text{ and} \\ q_3 &:= -x_0x_3 - x_1x_2 - x_4x_5, \end{aligned}$$

as the image of an injective linear map $q \colon V_3 \hookrightarrow \mathrm{Sym}^2 V_6^\vee$. The map q induces an isomorphism $\mathbb{P}(Q_S) \simeq \mathbb{P}(V_3)$ which sends any class of nonzero quadrics on S to the class of coordinates of one representative in the basis $\{q_1, q_2, q_3\}$. We see that through this description, we have a quadric bundle [Bea77, Définition 1.1] $f \colon X \to \mathbb{P}(V_3)$ whose fiber X_v over $[v] \in \mathbb{P}(V_3)$ has projective model $V([q(v)]) \subseteq \mathbb{P}(V_6)$. According to [Bea77, Proposition 1.2], f is a flat morphism whose general fiber is a smooth quadric fourfold.

Remark 10.24. The bundle f has singular fibers over a curve $C \subseteq \mathbb{P}(V_3)$ defined by the zero locus of the discriminant form

$$\Delta([v]) = \mathrm{disc}([q(v)])$$

where we see $q(v)$ as a complex quadratic form on V_6. This curve C is of degree 6 with at most nodal singularities [Bea77, Proposition 1.2]. In our case, for a system of coordinates $\{y_1, y_2, y_3\}$ on $\mathbb{P}(V_3)$, one can check that C is defined by

$$4y_1^5y_2 - 4y_1^4y_3^2 + 8y_1^3y_2^3 + 4y_1y_2^5 - 7y_1y_2y_3^4 - 4y_2^4y_3^2 - y_3^6 = 0$$

and has 14 nodal points. The double cover of $\mathbb{P}(V_3)$ branched over C is a nodal K3 surface, whose resolution $\widetilde{S}$ is therefore a quasipolarized K3 surface of genus 2, and Picard rank at least 15.

The map $q\colon V_3 \hookrightarrow \mathrm{Sym}^2 V_6^\vee$ also corresponds to a global section

$$s := y_1 q_1 + y_2 q_2 + y_3 q_3 \in V_3^\vee \otimes \mathrm{Sym}^2 V_6^\vee \simeq H^0\left(\mathbb{P}(V_3) \times \mathbb{P}(V_6), \mathcal{O}_{\mathbb{P}(V_3)\times\mathbb{P}(V_6)}(1,2)\right).$$

In this setting, one can view the quadric bundle f previously defined as the projection from the Fano sixfold Y defined by s,

$$Y := V(y_1 q_1 + y_2 q_2 + y_3 q_3) \subseteq \mathbb{P}(V_3) \times \mathbb{P}(V_6), \tag{43}$$

onto the first factor. Recall that a *Fano variety* is a smooth projective variety Y whose anticanonical bundle $-K_Y$ is ample. The construction in Equation (43) is known to the experts as a systematic way of producing examples of Fano varieties of *K3-type* given a general polarized K3 surface of degree 8 (see for instance [Fati22, §4.1], and see [Fati22, §2.2] for a definition of K3-type).

Remark 10.25 (due to Fatighenti)**.** The fact that Y in Equation (43) is of K3-type can been proved independently for the Hodge-theoretical sense and for the derived-categorical sense [BFM21, Propositions 48 & 49]. An evidence of such a fact is the relation between Y and the nodal K3 surface constructed in Remark 10.24.

The study of Fano varieties of K3-type have known growing interest regarding their relation with the geometry of IHS manifolds (see [FM21] and the references therein). The relation between Fano varieties and IHS manifolds is already known at the level of K3 surfaces. Indeed, by definition, we have that smooth anticanonical divisors in Fano threefolds are K3 surfaces. Some of the known general descriptions of projective models of K3 surfaces of small degree can be recovered in such a way (see [Muk88b] or the survey [Deb22, §2.3]). In the preliminaries to this thesis, we have also described one of the early examples of IHS fourfolds which can be constructed from a Fano variety of K3-type. Indeed, cubic fourfolds are Fano varieties of K3-type and we have seen that their variety of lines, in the smooth case, is an IHS fourfold. A useful trick we have used to prove this last fact is to invoke Borel–Weil theorem (Theorem 5.11) to see the equation defining a cubic fourfold as a global section of some other bundle over a Grassmannian manifold. Such a trick can also be used to construct another less trivial example: the so-called *Debarre–Voisin fourfold* [DV10]. In that situation, one starts with a Fano twentyfold defined as a general hypersurface in the Grassmannian $\mathrm{Gr}(3,10)$. By changing perspective, one describes its defining equation as a global section of the bundle $\bigwedge^3 \mathcal{U}^\vee_{\mathrm{Gr}(6,10)}$ (which is globally generated). The latter bundle being of rank 20 on the manifold $\mathrm{Gr}(6,10)$ which is 24-dimensional, we obtain a fourfold which was proved to be IHS of $\mathrm{K3}^{[2]}$-type [DV10, Theorem 1.2].

Remark 10.26. Debarre–Voisin fourfolds actually form one of the few known locally complete families of IHS manifolds, and they are equipped with a canonical polarization of BBF norm 22.

We have seen now that starting from our K3 surface S, we have obtain a Fano variety of K3-type Y. We may therefore wonder whether we can construct an IHS manifold from Y, by describing its defining section s as a global section of another vector bundle. The answer is yes, and it has been shown by Benedetti in his PhD thesis [Ben18, Proposition 3.1.3]. Let us review the construction.

Using Borel–Weil theorem (Theorem 5.11), we have the following isomorphisms of $\mathbb{C}$-vector spaces

$$V_3^\vee \cong H^0(\mathrm{Gr}(2,V_3),\ \mathcal{U}^\vee_{\mathrm{Gr}(2,V_3)})$$

and

$$\mathrm{Sym}^2 V_6^\vee \cong H^0(\mathrm{Gr}(2,V_6),\ \mathrm{Sym}^2\mathcal{U}^\vee_{\mathrm{Gr}(2,V_6)})$$

where once again, we denote by $\mathcal{U}$ the tautological bundle. We define on the product variety $\mathcal{G} := \mathrm{Gr}(2,V_3)\times\mathrm{Gr}(2,V_6)$ the homogeneous bundle $\mathcal{F} := \mathcal{U}^\vee_{\mathrm{Gr}(2,V_3)} \boxtimes \mathrm{Sym}^2\mathcal{U}^\vee_{\mathrm{Gr}(2,V_6)}$. The notation "$\boxtimes$" refers to the *external tensor product*: $\mathcal{F}$ is defined as the (internal) tensor product of the pullbacks of $\mathcal{U}^\vee_{\mathrm{Gr}(2,V_3)}$ and $\mathrm{Sym}^2\mathcal{U}^\vee_{\mathrm{Gr}(2,V_6)}$ along the respective projections. Then, by the application of Borel–Weil theorem above, we observe that

$$H^0(\mathcal{G},\mathcal{F}) \cong V_3^\vee \otimes \mathrm{Sym}^2 V_6^\vee$$

and we can therefore see s as a global section of $\mathcal{F}$. To make a clear distinction, we denote by $\widetilde{s} \in V_3^\vee \otimes \mathrm{Sym}^2 V_6^\vee$ to be the same as s but seen as a global section of $\mathcal{F}$.

Proposition 10.27 ([Ben18, Proposition 3.1.3])**.** *The vanishing locus $\widetilde{Y} := V(\widetilde{s}) \subseteq \mathcal{G}$ is isomorphic to $S^{[2]}$, where S is the original K3-octic we started with.*

Proof. Let us briefly review the proof of Benedetti: in particular, we define the isomorphism between $\widetilde{Y}$ and $S^{[2]}$ which we need for later use. We refer to the proof of [Ben18, Proposition 3.1.3] for omitted details.

Let $(A,B) \in \widetilde{Y} \subseteq \mathrm{Gr}(2,V_3)\times\mathrm{Gr}(2,V_6)$. We see A and B as $\mathbb{C}$-subvector spaces of V_3 and V_6 respectively. Recall that we have defined an injective linear map $q\colon V_3 \to \mathrm{Sym}^2 V_6^\vee$ which satisfies that $q(V_3) = Q_S$ the space of global quadric sections on the K3 surface $S \subseteq \mathbb{P}(V_6)$. This linear map is actually defined from $s = \widetilde{s} \in V_3^\vee \otimes \mathrm{Sym}^2 V_6^\vee$. In particular, we can define a linear map

$$\Phi_B\colon V_3 \xrightarrow{q} \mathrm{Sym}^2 V_6^\vee \to \mathrm{Sym}^2 B^\vee,$$

where the second arrow corresponds to restriction to B, whose kernel contains A by definition of $(A,B) \in V(\widetilde{s})$. Note that the kernel of Φ_B cannot by V_3 because S is a complete intersection. It follows that Φ_B has rank 1, and its image is generated by some quadric $Q \in \mathrm{Sym}^2 B^\vee \simeq H^0(\mathbb{P}(B),\mathcal{O}_{\mathbb{P}(B)}(2))$. Since $\mathbb{P}(B)\simeq\mathbb{P}^1$ has dimension 1, we obtain that the vanishing locus of Q in $\mathbb{P}(B)$ is 0-dimensional of degree 2, so it defines a length-2 subscheme $Z_{A,B}$ of $\mathbb{P}(B)\subseteq\mathbb{P}(V_6)$. By definition of $\widetilde{s}$, we infer that $Z_{A,B}\subseteq S$, and we obtain a map $\varphi\colon \widetilde{Y} \to S^{[2]}$, $(A,B)\mapsto Z_{A,B}$ which turns out to be an isomorphism. □

Remark 10.28. Set-theoretically, we observe that $\widetilde{Y}$ is the set of pairs $(A,B)\in\mathcal{G}$ such that $\ker\Phi_B = A$.

Hence, as claimed, we obtain that $\widetilde{Y}$ is an IHS fourfold of $\mathrm{K3}^{[2]}$-type. In comparison to the Fano variety of lines on cubic fourfolds, or Debarre–Voisin fourfolds, we obtain a priori a less interesting example here. In fact, $\widetilde{Y}$ is not only deformation equivalent to the Hilbert scheme of points on a K3 surface, but it is isomorphic to one. Moreover, we show now that this isomorphism is natural, in the sense that it is compatible with induced actions.

Proposition 10.29. *The finite group G of automorphisms of S acts faithfully on $\widetilde{Y}$ with symplectic subaction also given by G_s.*

Proof. The surface S is preserved under the faithful action of G on $\mathbb{P}(V_6)$ because the set of global quadric sections Q_S on S (given as in the beginning of Section 10.3.1) has a structure of $\mathbb{C}E$-module, where E is a Schur cover of G (see Section 10.3.3). Recall that the section $\widetilde{s}$ defining $\widetilde{Y}$ actually describes a linear map

$$q\colon V_3 \to \mathrm{Sym}^2 V_6^\vee$$

whose image is Q_S. We therefore get an left-action of G on V_3 defined by

$$G \times V_3 \to V_3,\ (g, v) \mapsto q^{-1}(g \cdot q(v)).$$

Since the action of G on $\mathbb{P}(V_6)$ is faithful, we obtain a well-defined faithful left-action of G on $\mathbb{P}(V_3)\times\mathbb{P}(V_6)$. Seeing G as a subgroup of $\mathrm{PGL}(V_3)\times\mathrm{PGL}(V_6) \cong \mathrm{Aut}(\mathrm{Gr}(2, V_3))\times\mathrm{Aut}(\mathrm{Gr}(2, V_6)) \subseteq \mathrm{PGL}(\bigwedge^2 V_3) \times \mathrm{PGL}(\bigwedge^2 V_6)$, we see that the latter action induces a faithful left-action of G on the product variety $\mathrm{Gr}(2, V_3) \times \mathrm{Gr}(2, V_6)$. We aim to prove that for all $g \in G$, the action of g on $\mathcal{G} = \mathrm{Gr}(2, V_3) \times \mathrm{Gr}(2, V_6)$ preserves $\widetilde{Y} = V(\widetilde{s}) \subseteq \mathcal{G}$ and its induced action via φ on $S^{[2]}$ coincides with the natural induced action $g^{[2]}$ of g on $S^{[2]}$ (where $\varphi\colon \widetilde{Y} \to S^{[2]}$ is the isomoprhism defined in the proof of Proposition 10.27).

First of all if $(A, B) \in V(\widetilde{s})$, we recall that the linear map

$$\Phi_B \colon V_3 \xrightarrow{q} \mathrm{Sym}^2 V_6^\vee \to \mathrm{Sym}^2 B^\vee$$

satisfies that $\ker \Phi_B = A$, seeing again A and B as $\mathbb{C}$-subvector spaces of V_3 and V_6 respectively. In particular, for all $a \in A$ and for all $b \in B$, we have that $q(a)(b) = 0$. Therefore for all $g \in G$, $a \in A$ and $b \in B$, since G acts faithfully on $\mathbb{P}(V_6)$,

$$q(g \cdot a)(g \cdot b) = (g \cdot q(a))(g \cdot b) = q(a)(g^{-1} \cdot (g \cdot (b))) = q(a)(b) = 0.$$

Hence $(g \cdot A, g \cdot B) \in V(\widetilde{s})$ (Remark 10.28) and $V(\widetilde{s})$ is preserved by the faithful action of G on $\mathcal{G}$.

Now, let $S \supseteq Z_{A,B} = \varphi((A, B))$ be a length 2 subscheme, defined as $V(\Phi_B(v_0))$ for some $v_0 \in V_3 \setminus A$. Then we observe that for all $g \in G$

$$g^{[2]} Z_{A,B} \subseteq \mathbb{P}(B)$$

by definition of the natural action $g^{[2]}$, and we have that

$$q(g \cdot v_0)(g^{[2]} Z_{A,B}) = g^{-1} \cdot q(g \cdot v_0)(Z_{A,B}) = q(v_0)(Z_{A,B}) = 0$$

by definition. From this, and Remark 10.28, we deduce that $g^{[2]} Z_{A,B} = V(\Phi_{g \cdot B}(g \cdot v_0))$ and so $g^{[2]} Z_{A,B} = \varphi((g \cdot A, g \cdot B))$, which proves our claim. We conclude thanks to [Boi12, Theorem 1] which tells us that symplectic automorphisms on S induce naturally symplectic automorphisms on $S^{[2]}$. $\square$

So the variety $\widetilde{Y}$ we have constructed is isomorphic in a natural way to the Hilbert scheme of two points on S. We end this section by highlighting why such a variety is still interesting: the following has been shared to the author of the thesis by Fatighenti in a series of private communications.

Let us consider the first projection $\widetilde{f}\colon \widetilde{Y} \twoheadrightarrow \mathrm{Gr}(2, V_3) \simeq \mathbb{P}(V_3^\vee)$. We have already seen that the ideal defining our original K3 surface S is generated by quadrics, and moreover we know from [Deg24, Table 1] that S does not contain lines. Therefore, according to [Deb22, Remark 3.12], the pullback $\mathcal{L} := \widetilde{f}^* \mathcal{O}_{\mathrm{Gr}(2,V_3)}(1)$ is a primitive line bundle on $\widetilde{Y} \simeq S^{[2]}$ and $c_1(\mathcal{L}) \in \mathrm{NS}(\widetilde{Y})$ is isotropic. Therefore, [DHMV24, Theorem 1.3] tells us that under these conditions $\widetilde{f}$ defines a Lagrangian fibration. Thus $\widetilde{Y}$ comes equipped with a (noncanonical) Lagrangian fibration.

Remark 10.30. Not all IHS manifolds can be equipped with a Lagrangian fibrations, which makes $\widetilde{Y}$ special on its own. The SYZ conjecture for IHS manifolds [DHMV24, Conjecture 1.2] predicts that any isotropic nef line bundle on an IHS manifold defines a Lagrangian fibration. By deformation theory, solving the previous conjecture would imply that any IHS manifold can be deformed into a another IHS manifold equipped with a Lagrangian fibration [DHMV24, Conjecture 1.1]. At the time this thesis is written, both of the previous conjectures have only been proved to be true for the known deformation types of IHS manifolds [DHMV24, Theorem 1.3].

We show now that the fibers of the Lagrangian fibration $\widetilde{f}\colon \widetilde{Y} \twoheadrightarrow \mathrm{Gr}(2, V_3)$ can be described geometrically, which turns the construction of this section into a very practical way to construct explicit examples of Lagrangian fibrations. Let $A \in \mathrm{Gr}(2, V_3)$ be general. The fiber over A consists of all 2-spaces B in V_6 such that Φ_B vanishes exactly at A (Remark 10.28). Denoting by Q and Q' two quadratic forms spanning $q(A)$ over $\mathbb{C}$, we have that $\widetilde{f}^{-1}(A)$ consists of all the 2-spaces in V_6 which are maximal isotropic with respect to both Q and Q' simultaneously. Therefore, seeing both Q and Q' as sections of the bundle $\mathrm{Sym}^2\mathcal{U}^\vee_{\mathrm{Gr}(2,V_6)}$ over $\mathrm{Gr}(2, V_6)$, we can identity $\widetilde{f}^{-1}(A)$ with the intersection $\widetilde{A} := V(Q) \cap V(Q') \subseteq \mathrm{Gr}(2, V_6)$.

Remark 10.31 (due to Fatighenti)**.** The variety $\widetilde{A}$ is often called a *doubly orthogonal Grassmannian* and is denoted $\mathrm{OGr}_2(2, V_6)$ (both $V(Q)$ and $V(Q')$ define copies of the *orthogonal Grassmannian* $\mathrm{OGr}(2, V_6) \subseteq \mathrm{Gr}(2, V_6)$).

This variety $\widetilde{A}$ is actually an abelian surface: indeed, considering now Q and Q' as quadric sections on $\mathbb{P}(V_6)$, via Borel–Weil theorem (Theorem 5.11), the variety $\widetilde{A}$ can be seen as the variety of lines on the complete intersection $V(Q, Q') \subseteq \mathbb{P}(V_6)$. As shown in [Rei72, §3], $\widetilde{A}$ is thus isomorphic to the Jacobian variety of the genus 2 curve obtained by taking the double cover of $\mathbb{P}(q(A))$ branched along the 6 classes of singular quadrics in $q(A)$.

Remark 10.32. The fact that there are only 6 classes of singular quadrics in $q(A)$ follows from the fact that A is chosen to be general and the discriminant form on $\mathrm{Sym}^2V_6^\vee$ is homogeneous of degree 6. Therefore, the divisor in $\mathbb{P}(\mathrm{Sym}^2V_6^\vee)$ parametrising singular complex quadratic forms on V_6 cuts $\mathbb{P}(q(A))$ in exactly 6 points.

11. Birational automorphisms of double EPW-cubes

The content of this section is adapted from a joint work with Billi and Wawak [BMW25]. As we have already mentioned in this thesis, constructing explicit geometric examples of IHS manifolds is a hard problem. This is why one sometimes uses the Torelli-type theorems (Proposition 5.33, Theorems 5.35 and 5.48) to infer the existence of an IHS manifold with a finite group action, and a possible invariant ample line bundle (see Section 10 for an example in the case of K3 surfaces). In this section, we apply the classification of finite symplectic actions on $\mathrm{K3}^{[3]}$-type IHS manifolds we have obtained (see Theorem 7.49 and Table 15) to the varieties known as *double EPW-cubes*. Such double EPW-cubes were constructed in [IKKR19], and the construction is similar to the one of *double EPW-sextics* by O'Grady [O'G12]. Both examples are constructed as natural double covers of certain degeneracy loci associated to certain maps of vector bundles. For the reader's convenience, we will review both constructions as they are similar in various aspects. The aforementioned degeneracy loci are sometimes referred to as *EPW loci* [EPW01, DK20], which explains the naming "double EPW-...". However, the "cube" and "sextic" refer to two different things; we will comment on that later.

In what follows we review the constructions of double EPW-sextics and double EPW-cubes. We then give a description of the automorphism group for double EPW-cubes, in a general setting (to be made precise later). From that we present then how to reconstruct explicit symmetric examples of double EPW-cubes. We finally comment on the description of the full birational groups for general double EPW-cubes, in the smooth and singular cases. This latter part shows interesting geometric phenomena, and allows us to see an explicit example of the birational Kähler cone of an IHS manifold (Definition 5.36).

11.1. Definitions and notation

Throughout this section, we denote by W a fixed 6-dimensional complex vector space. We equip the third exterior power $\bigwedge^3 W$ with a symplectic form

$$\eta\colon \bigwedge^3 W \times \bigwedge^3 W \to \mathbb{C}$$

determined by a volume form on $\bigwedge^6 W$. Let $\mathrm{LG}_\eta(10, \bigwedge^3 W)$ be the Grassmannian of η-Lagrangian subspaces $A \leq \bigwedge^3 W$. Recall that a linear subspace $A \leq \bigwedge^3 W$ is called *η-isotropic* if $\eta(A, A) = 0$, and *η-Lagrangian* if moreover $\dim_{\mathbb{C}}(A) = 10$. The manifold $\mathrm{LG}_\eta(10, \bigwedge^3 W)$ is 55-dimensional, and the set

$$\Sigma := \left\{ [A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \ : \ \mathbb{P}(A) \cap \mathrm{Gr}(3, W) \neq \emptyset \right\}$$

parametrizing η-Lagrangian subspaces of $\bigwedge^3 W$ containing a pure 3-multivector, is a prime divisor in $\mathrm{LG}_\eta(10, \bigwedge^3 W)$ [O'G12, Proposition 3.1]. For a class $[v] \in \mathbb{P}(W)$, one can define an η-Lagrangian subspace

$$F_v := v \wedge \bigwedge^2 W \leq \bigwedge^3 W.$$

We define $\mathcal{F} \leq \bigwedge^3 W \otimes \mathcal{O}_{\mathbb{P}(W)}$ to be the subbundle whose fiber over $[v] \in \mathbb{P}(W)$ is given by F_v. Given an η-Lagrangian subspace $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W)$, one can define a map of vector bundles

$$\lambda_A\colon \mathcal{F} \hookrightarrow \bigwedge^3 W \otimes \mathcal{O}_{\mathbb{P}(W)} \twoheadrightarrow \left(\left(\bigwedge^3 W \right) \Big/ A \right) \otimes \mathcal{O}_{\mathbb{P}(W)}$$

where the second map is induced by the quotient map $\bigwedge^3 W \twoheadrightarrow \left(\bigwedge^3 W\right)/A$. For $[A] \notin \Sigma$, we have that the degeneracy locus $Y_A := V(\det(\lambda_A)) \subseteq \mathbb{P}(W)$ is a normal sextic hypersurface, which is singular along a surface of degree 40 [DK20, Theorem 5.1]. The variety Y_A is called an *EPW-sextic*, the "sextic" referring to its degree as a hypersurface. Alternatively, another elementary description of Y_A is the following. Let again $[A] \notin \Sigma$, and define

$$Y_A[k] := \{[v] \in \mathbb{P}(W) \ : \ \dim_{\mathbb{C}}(A \cap F_v) \geq k\} \subseteq \mathbb{P}(W).$$

Then according to O'Grady (see for instance [O'G13]), we have that $Y_A = Y_A[1]$ and for all $k \geq 1$, the following holds $\mathrm{Sing}(Y_A[k]) = Y_A[k+1]$. Moreover, since $A \notin \Sigma$ does not contain pure 3-multivectors, we have that $Y_A[4] = \emptyset$ and $Y_A[3]$ is a finite set of points [DK20, Theorem 5.1]. The equivalence between both definitions can be recovered by working with local coordinates around $[v] \in \mathbb{P}(W)$, and seeing the dimension of $A \cap F_v$ as the corank of some quadratic form defined on A (see [O'G13, §1.3]). We define

$$\Delta := \left\{ [A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \ : \ Y_A[3] \neq \emptyset \right\}$$

which is a prime divisor on $\mathrm{LG}_\eta(10, \bigwedge^3 W)$ which shares no common component with Σ. [O'G13, Proposition 2.2].

Theorem 11.1 ([O'G06, Theorem 1.1], [DK20, Theorem 5.2])**.** *Let $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus \Sigma$. Then there exists a double cover*

$$\pi_A^Y \colon \widetilde{Y_A} \to Y_A$$

branched over $Y_A[2] = \mathrm{Sing}(Y_A)$, and $\mathrm{Sing}(\widetilde{Y_A}) = (\pi_A^Y)^{-1}(Y_A[3])$ is finite. Moreover, one of the following two holds:

1. *either $[A] \notin \Delta$, and $\widetilde{Y_A}$ is smooth, $h_A := (\pi_A^Y)^{-1}\mathcal{O}_{\mathbb{P}(W)}(1)$ is ample and the pair $(\widetilde{Y_A}, h_A)$ is a $(2,1)$-polarized IHS manifold of $\mathrm{K3}^{[2]}$-type;*

2. *or $[A] \in \Delta \setminus \Sigma$, and there exists a projective resolution $\widehat{Y_A} \to \widetilde{Y_A}$, given by contractions of $\mathbb{P}^2$'s, where $\widehat{Y_A}$ is a $(2,1)$-quasipolarized IHS manifold of $\mathrm{K3}^{[2]}$-type (we refer back to Definition 5.25 for a definition of quasipolarized IHS manifolds).*

The variety $\widetilde{Y_A}$ is referred to as double EPW-sextic.

O'Grady shows in [O'G06, Theorem 1.1] that, for $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Delta)$, the local deformation family of $(\widetilde{Y_A}, h_A)$ in the moduli space $\mathcal{M}_{\mathrm{K3}^{[2]}}^{(2,1)}$ of $(2,1)$-polarized IHS manifolds of $\mathrm{K3}^{[2]}$-type has maximal dimension. In particular, double EPW-sextics form a locally complete family of polarized IHS manifolds, and for a very general choice of $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Delta)$, the IHS manifold $\widetilde{Y_A}$ has Picard rank 1.

Let us now introduce double EPW-cubes and compare results. For a point $[U] \in \mathrm{Gr}(3, W)$, we define another η-Lagrangian subspace of $\bigwedge^3 W$ by

$$T_U := \bigwedge^2 U \wedge W \leq \bigwedge^3 W.$$

Following the second description we have given in the case of EPW-sextics, for an η-Lagrangian subspace $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus \Sigma$ and for any integer $k \geq 0$, one defines

$$Z_A[k] := \{[U] \in \mathrm{Gr}(3, W) \ : \ \dim_{\mathbb{C}}(A \cap T_U) \geq k\} \subseteq \mathrm{Gr}(3, W).$$

We have that $Z_A[1]$ is a normal quartic hypersurface in $\mathrm{Gr}(3, W)$, and for all $k \geq 1$, we have again that $\mathrm{Sing}(Z_A[k]) = Z_A[k+1]$ with $Z_A[5] = \emptyset$ (see [IKKR19, Corollary 2.10] for the original statement, [DK20, Theorem 5.6] for a proof using degeneracy loci as before and [Riz24, §2] for the an actual proof of the smoothness of $Z_A[4]$). We denote $Z_A := Z_A[2]$ which we call an *EPW-cube*.

Remark 11.2. The variety Z_A is 6-dimensional and has degree 480. The naming "cube" has nothing to do with the degree of any variety involved. In what follows, we recall that for suitable choices of $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus \Sigma$, there is a smooth double cover of Z_A which is an IHS manifold of $\mathrm{K3}^{[3]}$-type, hence the cube.

Similarly to the case of EPW-sextics, let us denote

$$\Gamma := \left\{ [A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \ : \ Z_A[4] \neq \emptyset \right\}$$

which is a divisor and does not share any connected component with Σ [IKKR19, Lemma 3.6].

Proposition 11.3 ([IKKR19, Theorem 1.1], [DK20, Theorem 5.2], [Riz24, Proposition 5.1])**.** *Let $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus \Sigma$. Then there exists a natural double cover*

$$\pi_A^Z \colon \widetilde{Z_A} \to Z_A$$

branched over $Z_A[3] = \mathrm{Sing}(Z_A)$, and $\mathrm{Sing}(\widetilde{Z_A}) = (\pi_A^Z)^{-1}(Z_A[4])$ is finite. Moreover, one of the following two holds:

1. *either $[A] \notin \Gamma$, and $\widetilde{Z_A}$ is smooth, $H_A := (\pi_A^Z)^{-1}\mathcal{O}_{\mathrm{Gr}(3,W)}(1)$ is ample and the pair $(\widetilde{Z_A}, H_A)$ is a $(4,2)$-polarized IHS manifold of $\mathrm{K3}^{[3]}$-type;*
2. *or $[A] \in \Gamma \setminus \Sigma$, and there exists a projective resolution $\widehat{Z_A} \to \widetilde{Z_A}$, given by contractions of $\mathbb{P}^3$'s, where $\widehat{Z_A}$ is a $(4,2)$-quasipolarized IHS manifold of deformation type $\mathrm{K3}^{[3]}$-type.*

The variety $\widetilde{Z_A}$ is referred to as double EPW-cube.

Similarly to O'Grady's result, the authors in [IKKR19] show that, for $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Gamma)$, the local deformation family of $(\widetilde{Z_A}, H_A)$ in the moduli space $\mathcal{M}^{(4,2)}_{\mathrm{K3}^{[3]}}$ of $(4,2)$-polarized IHS manifolds of $\mathrm{K3}^{[3]}$-type has maximal dimension 20. Hence, we obtain that for a very general choice of $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Gamma)$, the IHS manifold $\widetilde{Z_A}$ has Picard rank 1.

11.2. Moduli of double EPW-sextics and double EPW-cubes

Let us review some relations between the moduli spaces of double EPW-sextics and double EPW-cubes, as studied in [KKM24].

Let us consider the even $\mathbb{Z}$-lattice $\Lambda_0 := U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_1^{\oplus 2}$.

Proposition 11.4. *Let $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W)$ be very general, in particular $[A] \notin \Sigma \cap \Delta \cup \Gamma$, and let $(\widetilde{Y_A}, h_A)$ and $(\widetilde{Z_A}, H_A)$ be the associated smooth double EPW-sextics and double EPW-cubes, respectively equipped with their canonical polarization. Then, as $\mathbb{Z}$-lattices, we have*

$$c_1(h_A)^\perp \simeq c_1(H_A)^\perp \simeq \Lambda_0.$$

Proof. Let us recall that $H^2(\widetilde{Y_A}, \mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_1 =: \Lambda_{\mathrm{K3}^{[2]}}$ and $H^2(\widetilde{Z_A}, \mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_1(2) =: \Lambda_{\mathrm{K3}^{[3]}}$. Now, according to a result of Eichler [Eic52, Satz 10.4], since both $\Lambda_{\mathrm{K3}^{[2]}}$ and $\Lambda_{\mathrm{K3}^{[3]}}$ contain two copies of the hyperbolic plane U, and their respective discriminant groups have only element of order 2, we have that

(1) $\Lambda_{\mathrm{K3}^{[2]}}$ has only one $O(\Lambda_{\mathrm{K3}^{[2]}})$-orbit of vectors of type $(2,1)$; and

(2) $\Lambda_{\mathrm{K3}^{[3]}}$ has only one $O(\Lambda_{\mathrm{K3}^{[3]}})$-orbit of vectors of type $(4,2)$.

We have seen in Example 2.7, Item (2), that the glue map $D_{A_1} \to D_{A_1}(-1)$ defines a primitive extension $A_1 \oplus A_1(-1) \leq U$, which implies the existence of a primitive extension $\Lambda_0 \oplus \langle 2\rangle \leq \Lambda_{\mathrm{K3}^{[2]}}$: by (1) above we therefore get that $c_1(h_A)^\perp \leq H^2(\widetilde{Y_A}, \mathbb{Z})$ is isometric to Λ_0. Moreover, according to Table 18 there exists a primitive extension $\Lambda_0 \oplus \langle 4\rangle \leq \Lambda_{\mathrm{K3}^{[3]}}$: hence, by (2) above we conclude similarly that $c_1(H_A)^\perp \leq H^2(\widetilde{Z_A}, \mathbb{Z})$ is also isometric to Λ_0. □

Throughout, let us fix $h \in \Lambda_{\mathrm{K3}^{[2]}}$ and $H \in \Lambda_{\mathrm{K3}^{[3]}}$ primitive of respective type $(2,1)$ and $(4,2)$.

Lemma 11.5. *The following hold*

(1) *The image $O_h \leq O(\Lambda_0)$ of $O^+(\Lambda_{\mathrm{K3}^{[2]}}, h) \to O(\Lambda_0)$ is equal to $O^{+,\#}(\Lambda_0)$*

(2) *The image $O_H \leq O(\Lambda_0)$ of $O^+(\Lambda_{\mathrm{K3}^{[2]}}, H) \to O(\Lambda_0)$ is equal to $O^+(\Lambda_0)$.*

In particular, $O_h \leq O_H$ is an index two subgroup.

Proof.

(1) Note that $D_{\Lambda_{\mathrm{K3}^{[2]}}} \cong \mathbb{Z}/2\mathbb{Z}$ has no nontrivial automorphisms, meaning that $O^+(\Lambda_{\mathrm{K3}^{[2]}}) = O^{+,\#}(\Lambda_{\mathrm{K3}^{[2]}})$. Since any isometry of $O^+(\Lambda_{\mathrm{K3}^{[2]}}, h)$ is the identity on $\mathbb{Z}h$, we have that they restrict to $O^+(\Lambda_0)$ along the primitive extension $\Lambda_0 \oplus \mathbb{Z}h \leq \Lambda_{\mathrm{K3}^{[2]}}$. Hence the result follows from Corollary 2.21 after remarking that $O^+(\Lambda_{\mathrm{K3}^{[2]}}, h)$ is the pointwise stabilizer of $\mathbb{Z}h$ in $O^{+,\#}(\Lambda_{\mathrm{K3}^{[2]}})$.

(2) The proof follows almost similarly: this time however $[O^+(\Lambda_{\mathrm{K3}^{[3]}}) : O^{+,\#}(\Lambda_{\mathrm{K3}^{[3]}})] = 2$ and any nonstable isometry in $O^+(\Lambda_{\mathrm{K3}^{[3]}}, H)$ restricts to a nonstable isometry of $O^+(\Lambda_0)$, whose discriminant group $D_{\Lambda_0} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ has exactly two isometries. □

Let us denote by

$$\Omega_Y := O_h \backslash \left\{\mathbb{C}\omega \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) \;:\; \omega^2 = 0,\, \omega.\overline{\omega} > 0\right\}^+$$
$$\text{and } \Omega_Z := O_H \backslash \left\{\mathbb{C}\omega \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) \;:\; \omega^2 = 0,\, \omega.\overline{\omega} > 0\right\}^+$$

the respective period spaces for the moduli spaces of $(2,1)$-polarized IHS manifolds of deformation type $\mathrm{K3}^{[2]}$, respectively of $(4,2)$-polarized IHS manifolds of deformation type $\mathrm{K3}^{[3]}$ (Section 5.6). According to Lemma 11.5 there exists a degree 2 covering map

$$\rho\colon \Omega_Y \to \Omega_Z$$

given by the inclusion $O_h \leq O_H$ (see [LO19, Proposition 1.2.3]).

Now, to every $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Delta)$, we can associate the period of $(\widetilde{Y_A}, h_A)$ in Ω_Y, giving rise to a rational period map

$$\mathcal{P}_Y\colon \mathrm{LG}_\eta(10, \bigwedge^3 W) \dashrightarrow \Omega_Y$$

which is well-defined on the open locus consisting of η-Lagrangian subspaces of $\bigwedge^3 W$ not contained in $\Sigma \cup \Delta$. In a similar way, we obtain a rational period map

$$\mathcal{P}_Z\colon \operatorname{LG}_\eta(10, \bigwedge^3 W) \dashrightarrow \Omega_Z$$

which is well-defined on the open locus consisting of η-Lagrangian subspaces of $\bigwedge^3 W$ not contained in $\Sigma \cup \Gamma$. Let $\mathcal{M}_Y$ be the GIT quotient

$$\mathcal{M}_Y := \operatorname{LG}_\eta(10, \bigwedge^3 W) /\!/ \operatorname{PGL}(W) :$$

according to [O'G16, §1.3] and reference therein, the dense open subspace

$$\left(\operatorname{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Delta)\right) /\!/ \operatorname{PGL}(W)$$

parametrizing smooth double EPW-sextics lies in the stable locus of $\mathcal{M}_Y$. The space $\mathcal{M}_Y$ can therefore be seen as a compactification of the moduli space of smooth double EPW-sextics, and we have a rational map

$$\mathcal{M}_Y \dashrightarrow \Omega_Y^{\mathrm{BB}}$$

to the Baily–Borel compactification of Ω_Y [LO19, Section 2.3]. Now the automorphism group of $\operatorname{Gr}(3, W)$ can be identified with $\operatorname{PGL}(W) \times \langle \delta \rangle$ where δ sends any 3-space $U \leq W$ to its dual for the symplectic form η. Note that since $\operatorname{Gr}(3, W)$ spans the Plücker space $\mathbb{P}(\bigwedge^3 W)$, we have an action of δ on $\mathbb{P}(\bigwedge^3 W)$. By definition of δ, the previous induces an action on $\operatorname{LG}_\eta(10, \bigwedge^3 W)$.

Remark 11.6. The symplectic form η is unique up to scalar, so δ is uniquely determined as projective linear transformation of $\bigwedge^3 W$.

From this we obtain a quotient map

$$p\colon \mathcal{M}_Y \to \operatorname{LG}_\eta(10, \bigwedge^3 W) /\!/ \operatorname{Aut}(\operatorname{Gr}(3, W))$$

which is generically 2-to-1, except over the locus of η-Lagrangian subspaces A which are in the same orbit as $\delta(A)$ for the action of $\operatorname{PGL}(W)$ on $\operatorname{LG}_\eta(10, \bigwedge^3 W)$. Similarly as before, one can see $\operatorname{LG}_\eta(10, \bigwedge^3 W) /\!/ \operatorname{Aut}(\operatorname{Gr}(3, W))$ as a compactification of the moduli space of double EPW-cubes.

We therefore see that they are "twice as many" double EPW-sextics then cubes. In fact, it is known that in general, the double EPW-sextics associated to an η-Lagrangian and its dual are not isomorphic [O'G08, Theorem 1.1]. The proof of the following corollary is due to Billi [BMW25, Lemma 3.2] (see also [Bil23])

Lemma 11.7. *Let* $[A_1], [A_2] \in \operatorname{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Gamma)$. *Then the EPW-cubes* Z_{A_1} *and* Z_{A_2} *are* $\operatorname{PGL}(\bigwedge^3 W)$*-isomorphic if and only if* $[A_1]$ *and* $[A_2]$ *are in the same orbit under the action of* $\operatorname{Aut}(\operatorname{Gr}(3, W))$ *on* $\operatorname{LG}_\eta(10, \bigwedge^3 W)$.

Proof. One direction is clear: if $[A_1]$ and $[A_2]$ are in the same orbit, and since any automorphism of the Grassmannian $\operatorname{Gr}(3, W)$ is projectively linear, we obtain that Z_{A_1} and Z_{A_2} are linearly equivalent.

Now suppose that there exists $f \in \operatorname{PGL}(\bigwedge^3 W)$ such that $f(Z_{A_1}) = Z_{A_2}$. Since the double covers $\pi^Z_{A_1}$ and $\pi^Z_{A_2}$ are natural, we obtain that $\widetilde{Z_{A_1}}$ and $\widetilde{Z_{A_2}}$ are isomorphic, as polarized IHS manifolds. We conclude by invoking the proof of [IKKR19, Proposition 5.1] which tells us that the latter implies that there exists $g \in \operatorname{Aut}(\operatorname{Gr}(3, W))$ such that $g([A_1]) = [A_2]$. ☐

11.3. Polarized automorphisms of smooth double EPW-cubes

In [BMW25], the authors study birational automorphisms of double EPW-cubes. In particular, given a Lagrangian $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W)\setminus(\Sigma\cup\Gamma)$ so that A and $\delta(A)$ are not $\mathrm{PGL}(W)$-isomorphic, they give the structure of the group of automorphisms on $\widetilde{Z_A}$ preserving the double cover

$$\widetilde{Z_A} \to Z_A := Z_A[2] \subseteq \mathrm{Gr}(3, W).$$

Such automorphisms are said to be *polarized*, since they preserve the primitive ample line bundle $L = \pi^*\mathcal{O}_{Z_A}(1)$ on $\widetilde{Z_A}$.

We denote by $\mathrm{Aut}(\widetilde{Z_A}, L)$ the group of such polarized automorphisms: we know thanks to Proposition 6.4 that it is finite. In order to describe $\mathrm{Aut}(\widetilde{Z_A}, L)$, we adapt the proof of [DM22, Proposition A.2]. First let us prove the following:

Lemma 11.8. *For $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Gamma)$ we have an isomorphism*

$$\mathrm{Stab}_{\mathrm{PGL}(\bigwedge^3 W)}(Z_A) \cong \mathrm{Aut}(\mathrm{Gr}(3, W))_A$$

where the righthand side denotes the stabilizer of $[A]$ under the action of $\mathrm{Aut}(\mathrm{Gr}(3, W))$ on $\mathrm{LG}_\eta(10, \bigwedge^3 W)$

Proof. Note that according to [IKKR19, Lemma 5.2], if $g \in \mathrm{Stab}_{\mathrm{PGL}(\bigwedge^3 W)}(Z_A)$, then g is already an automorphism of the Grassmannian $\mathrm{Gr}(3, W)$. Hence the results follows from Lemma 11.7 □

Corollary 11.9. *Suppose $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Gamma)$ does not lie in the same orbit as $\delta(A)$ for the action of $\mathrm{PGL}(W)$ on $\mathrm{LG}_\eta(10, \bigwedge^3 W)$. Then there is a group homomorphism*

$$\mathrm{Aut}(\widetilde{Z_A}, L) \to \mathrm{PGL}(W)_A$$

where the righthand side is the stabilizer of A under the action of $\mathrm{PGL}(W)$ on $\mathrm{LG}(10, \bigwedge^3 W)$.

Proof. By definition, any automorphism $f \in \mathrm{Aut}(\widetilde{Z_A}, L)$ preserves the ample line bundle L, which defines the morphism

$$\widetilde{Z_A} \xrightarrow{2:1} Z_A \subseteq \mathbb{P}\left(\bigwedge^3 W\right).$$

In particular, f induces a regular action on $\mathbb{P}(\bigwedge^3 W) = |L|^\vee$ which preserves Z_A. This gives rise to a group homomorphism

$$\mathrm{Aut}(\widetilde{Z_A}, L) \to \mathrm{Stab}_{\mathrm{PGL}(\bigwedge^3 W)}(Z_A).$$

Now since A and $\delta(A)$ are not $\mathrm{PGL}(W)$-isomorphic, we deduce that $\mathrm{Aut}(\mathrm{Gr}(3, W))_A \cong \mathrm{PGL}(W)_A$ by the description

$$\mathrm{Aut}(\mathrm{Gr}(3, W)) \cong \mathrm{PGL}(W) \times \langle\delta\rangle.$$

This gives rise to the wanted group homomorphism. □

Remark 11.10. Note that according to [O'G16, Corollary 2.5.1], any point $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W)\setminus \Sigma$ lies in the stable locus of the GIT quotient $\mathrm{LG}_\eta(10, \bigwedge^3 W)/\!/ \mathrm{PGL}(W)$. In particular, $\mathrm{PGL}(W)$ acts on $\mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus \Sigma$ with finite stabilizers, and $\mathrm{PGL}(W)_A$ is finite for any Lagrangian subspace A containing no pure 3-multivectors.

We now prove the following proposition, which appeared first in the PhD thesis of Billi [Bil23, Proposition 2.1.17]. The statement is different from the previous result of Billi since it requires stronger restrictions of the Lagrangian subspaces we consider. We comment about this a bit later.

Proposition 11.11. *Let* $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cap \Gamma)$ *be a Lagrangian so that* A *and* $\delta(A)$ *are not in the same* $\mathrm{PGL}(W)$*-orbit (for the action of* $\mathrm{PGL}(W)$ *on* $\mathrm{LG}_\eta(10, \bigwedge^3 W)$*). Consider* $\pi\colon \widetilde{Z_A} \to Z_A$ *the associated double EPW-cube with its polarization* $L := \pi^*\mathcal{O}_{Z_A}(1)$*. Then*

$$\mathrm{Aut}(\widetilde{Z_A}, L) \cong \mathrm{PGL}(W)_A \times \langle \iota \rangle$$

where ι *is the covering involution. Moreover the group* $\mathrm{PGL}(W)_A$ *corresponds to the subgroup* $\mathrm{Aut}_s(\widetilde{Z_A}, L)$ *of polarized symplectic automorphisms.*

Proof. Since $[A] \notin \Sigma$, by [DK20, Theorem 4.2, Theorem 5.7] we have that

$$\widetilde{Z_A} = \mathrm{Spec}_{\mathcal{O}_{Z_A}}(\mathcal{O}_{Z_A} \oplus \mathcal{R}_2(-2))$$

where $\mathcal{R}_2$ is a reflexive sheaf of rank 1 on Z_A. Note that $\mathcal{R}_2(-2) \simeq \omega_{Z_A}$ is the canonincal sheaf of Z_A. Since π is a double cover, and A and $\delta(A)$ are not $\mathrm{PGL}(W)$-isomorphic, Corollary 11.9 tells us that there is an exact sequence

$$1 \to \langle \iota \rangle \to \mathrm{Aut}(\widetilde{Z_A}, L) \to \mathrm{PGL}(W)_A.$$

Let $\widetilde{G}$ be the preimage of $\mathrm{PGL}(W)_A$ via the map $\mathrm{SL}(W) \to \mathrm{PGL}(W)$: there is a central extension

$$1 \to \mu_6 \to \widetilde{G} \to \mathrm{PGL}(W)_A \to 1.$$

Following the ideas of [DM22, Appendix A.1], we have that the induced action of $\widetilde{G}$ on $\bigwedge^3 W$ factors through the quotient $\widetilde{G}/\mu_3$, and according to [DM22, Lemma A.1] we have that the latter embeds into $\mathrm{GL}(A)$. By similar arguments as in the proof of [DM22, Lemma A.2], and according to the description of $\mathcal{R}_2$ given in [DK20, Theorem 4.2], we observe that $\widetilde{G}/\mu_3$ acts on $\widetilde{Z_A}$ and preserves L. Actually, the subgroup $\mu_6/\mu_3 \leq \widetilde{G}/\mu_3$ acts trivially on $\mathcal{R}_2(-2)$ via this action. Hence we get an injective morphism

$$\psi\colon \mathrm{PGL}(W)_A \hookrightarrow \mathrm{Aut}(\widetilde{Z_A}, L)$$

which is a section of the group homomorphism determined in Corollary 11.9.

Now, the action of $\mathrm{Aut}(\widetilde{Z_A}, L)$ on $H^2(\widetilde{Z_A}, \mathcal{O}_{\widetilde{Z_A}}) \simeq \mathbb{C}\sigma_{\widetilde{Z_A}}$ determines a character $\mathrm{Aut}(\widetilde{Z_A}, L) \to \mathbb{C}^\times$, with finite (cyclic) image of order $r \geq 2$, that sends ι to -1 (Proposition 6.1). Note moreover that $\mathrm{PGL}(W)_A$ acts trivially on $H^2(\widetilde{Z_A}, \mathcal{O}_{\widetilde{Z_A}})$ since $\mathrm{PGL}(W)$ has no nontrivial characters. Hence $\psi(\mathrm{PGL}(W)_A) \leq \mathrm{Aut}_s(\widetilde{Z_A}, L)$ and $\mathrm{Aut}(\widetilde{Z_A}, L) \cong \langle \iota \rangle \times \mathrm{PGL}(W)_A$. □

Remark 11.12. With the notation of Proposition 11.11, if A and $\delta(A)$ are $\mathrm{PGL}(W)$-isomorphic, then there exists $\varphi \in \mathrm{PGL}(W)$ such that $f := \bigwedge^3 \varphi \circ \delta \in \mathrm{Aut}(\mathrm{Gr}(3, W))_A$. Following similar arguments as in the previous proof, we can show that in some cases, such automorphism f admits a lift in $\mathrm{Aut}(\widetilde{Z_A}, L)$ which is purely nonsymplectic of order 4 (Remark 11.17). However, it does not seem that this is systematically true.

Example 11.13. By investigating Table 18, we have that there exists a family of IHS manifolds of $\mathrm{K3}^{[3]}$-type, equipped with a purely nonsymplectic automorphism g of order 4 which is nonstable. For any such manifold X, with associated automorphism g, we have that

$$H^2(X, \mathbb{Z})^{\rho_X(g)} \simeq \langle 4 \rangle \text{ and } H^2(X, \mathbb{Z})_{\rho_X(g)} \simeq U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_1^{\oplus 2}$$

and X comes equipped with a canonical polarization of numerical type $(4, 2)$. This family of $(4, 2)$-polarized IHS fourfolds of $\mathrm{K3}^{[3]}$-type determines a 9-dimensional subspace of Ω_Z, the period space defined in Section 11.2. It then follows that the general element in this family is a double EPW-cube, and the isomorphism g^2 is conjugate to the associated covering involution.

11.4. Geometric examples

Similarly to the techniques used in [DM22, BW24], Billi and Wawak construct in [BMW25] explicit examples of symmetric double EPW-cubes based on the results of the previous section. In order to determine if such examples can exist, the authors in [BMW25] show the following.

Lemma 11.14. *Let (X, L) be a $(4,2)$-quasipolarized IHS manifold of $\mathrm{K3}^{[3]}$-type, and let $\mathrm{Bir}_s(X, L)$ be the finite subgroup of symplectic birational automorphisms of X preserving L. Suppose that there exists a finite stable subgroup $G \leq \mathrm{Bir}_s(X, L)$ such that $H := \rho_X(G) \leq O^{+,\#}(H^2(X, \mathbb{Z}))$ is saturated with $\mathrm{rank}_{\mathbb{Z}}(H^2(X, \mathbb{Z})_H) = 20$. Then G is given in Table 12, together with the genus of $\mathrm{NS}(X)$ and a representative for the isometry class of $\mathrm{T}(X)$.*

Proof. According to Proposition 6.5, Item (1), we know in this situation that $\mathrm{NS}(X)$, which contains $c_1(L)$ and $H^2(X, \mathbb{Z})_H$, have rank 21 and thus be of signatures $(1, 20)$. This also implies that $\mathrm{T}(X)$ is positive definite of rank 2. We can investigate now Table 15 in Appendix B to determine which conjugacy classes of finite stable symplectic subgroups $G \leq O^+(\Lambda_{\mathrm{K3}^{[3]}})$ are such that the associated coinvariant sublattice has rank 20. For each such group, the associated invariant sublattice has rank 3, and by enumerating short vectors, we can determine whether they contain a vector of type $(4, 2)$. For each orbit of such vectors $v \in \Lambda^G_{\mathrm{K3}^{[3]}}$, we know from Theorem 7.49 that there exists an IHS manifold X of $\mathrm{K3}^{[3]}$-type such that $\mathrm{T}(X) \simeq v^\perp \cap \Lambda^G_{\mathrm{K3}^{[3]}}$. For each entry in Table 12, we refer the Id of the corresponding conjugacy class as given in Table 15. □

Table 12: Stable and stably saturated $(4, 2)$-polarized actions with maximal coinvariant sublattice

Id	G	Regular	$\mathrm{T}(X)$	$\mathrm{NS}(X)$	Id	G	Regular	$\mathrm{T}(X)$	$\mathrm{NS}(X)$
102d	$L_3(4)$	No	$\begin{pmatrix} 12 & 0 \\ 0 & 28 \end{pmatrix}$	$\mathrm{II}_{(1,21)}4_5^{-1}3^{-1}7^{-1}$	124	$[384, 18134]$	No	$\begin{pmatrix} 2 & 0 \\ 0 & 16 \end{pmatrix}$	$\mathrm{II}_{(1,21)}2_7^1 4_3^{-1}16_3^{-1}$
106b	$C_2^4 \rtimes A_6$	No	$\begin{pmatrix} 4 & 0 \\ 0 & 24 \end{pmatrix}$	$\mathrm{II}_{(1,21)}2^2 8_3^{-1}3^1$	131	$[192, 1494]$	No	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$	$\mathrm{II}_{(1,21)}4_5^{-1}8_4^{-2}$
108a	A_7	Yes	$\begin{pmatrix} 6 & 0 \\ 0 & 70 \end{pmatrix}$	$\mathrm{II}_{(1,21)}4_1^1 3^1 5^1 7^1$	133b	$C_2 \times M_9$	No	$\begin{pmatrix} 4 & -2 \\ -2 & 10 \end{pmatrix}$	$\mathrm{II}_{(1,21)}2_4^{-3}4_5^{-1}9^{-1}$
119a	M_{10}	Yes	$\begin{pmatrix} 4 & 0 \\ 0 & 30 \end{pmatrix}$	$\mathrm{II}_{(1,21)}2_3^{-1}4_0^2 3^{-1}5^1$	133b	$C_2 \times M_9$	No	$\begin{pmatrix} 4 & 0 \\ 0 & 36 \end{pmatrix}$	$\mathrm{II}_{(1,21)}2_4^{-3}4_5^{-1}9^{-1}$
119e	M_{10}	No	$\begin{pmatrix} 4 & 0 \\ 0 & 30 \end{pmatrix}$	$\mathrm{II}_{(1,21)}2_7^3 3^{-1}5^1$	137a	S_5	No	$\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$	$\mathrm{II}_{(1,21)}2_2^{-2}4_1^1 3^1 5^{-1}$
120a	$L_2(11)$	Yes	$\begin{pmatrix} 22 & 0 \\ 0 & 22 \end{pmatrix}$	$\mathrm{II}_{(1,21)}4_1^1 11^2$	149b	$C_2 \times F_5$	No	$\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$	$\mathrm{II}_{(1,21)}2_2^{-2}4_7^1 5^2$

From this, Billi and Wawak show the following. Recall from Proposition 11.3 that given a Lagrangian $[A] \in \mathrm{LG}_\eta(3, W) \setminus \Sigma$, either $[A] \notin \Gamma$ and the associated double EPW-cube $\widetilde{Z_A}$ is smooth, or there exists a smoth projective resolution $\widehat{Z_A} \to \widetilde{Z_A}$ which is a quasipolarized IHS manifold of $\mathrm{K3}^{[3]}$-type.

Theorem 11.15 ([BMW25, Theorem 0.2])**.** *For any group G as in Table 13, there exists a faithful projective representation $G \to \mathrm{PGL}(W)$ and an associated G-invariant Lagrangian subspace $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus \Sigma$ so that:*

(1) *if $G = L_3(4)$ then $[A] \in \Gamma$, and the double EPW-cube $\widetilde{Z_A}$ is singular. In this case G embeds into $\mathrm{Bir}_s(\widehat{Z_A}, L)$, where $\widehat{Z_A}$ is a projective IHS resolution of $\widetilde{Z_A}$ and L is the associated quasipolarization.*

(2) *if $G = A_7, M_{10}, L_2(11)$ then $[A] \notin \Gamma$, and the double EPW-cube $\widetilde{Z_A}$ is smooth. Moreover we have that $\mathrm{Aut}_s(\widetilde{Z_A}, L) \cong G$, where L is the canonical polarization on $\widetilde{Z_A}$.*

Table 13: Groups acting symplectically on double EPW-cubes of Picard rank 21

G	$L_3(4)$	A_7	M_{10}	$L_2(11)$
Regular	False	True	True	True
$\mathrm{T}(X)$	$\begin{pmatrix} 10 & 4 \\ 4 & 10 \end{pmatrix}$	$\begin{pmatrix} 6 & 0 \\ 0 & 70 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 \\ 0 & 30 \end{pmatrix}$	$\begin{pmatrix} 22 & 0 \\ 0 & 22 \end{pmatrix}$

Remark 11.16. Note that Proposition 11.11 only holds for Lagrangian subspaces for which the associated double EPW-cube is smooth. However, the proof can be adapted to show that even in the singular case, we have that $\mathrm{PGL}(W)_A$ embeds into the symplectic subgroup of $\mathrm{Bir}(\widehat{Z_A})$.

Remark 11.17. By investigating the data from Table 13, one can conclude that the A_7-symmetric and the M_{10}-symmetric smooth double EPW-cubes $\widetilde{Z_A}$ determined in Theorem 11.15, with polarization L, satisfy that $[\mathrm{Aut}(\widetilde{Z_A}, L) : \mathrm{Aut}_s(\widetilde{Z_A}, L)] = 2$. Indeed, the associated transcendental lattices only admit isometries of order at most 2. Observe that in these cases, the associated invariant Lagrangian subspaces are not $\mathrm{PGL}(W)$-isomorphic to their dual via the map δ (Corollary 11.9).

For the $L_2(11)$-symmetric one, which we denote again by $\widetilde{Z_A}$ with polarization L, we have that $a := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is an isometry of $\mathrm{T}(\widetilde{Z_A})$ and $(\mathrm{T}(\widetilde{Z_A}), a)$ is a Φ_4-lattice. There actually exists an isometry g of $H^2(\widetilde{Z_A}, \mathbb{Z})$ fixing $c_1(L)$, preserving $\mathrm{NS}(\widetilde{Z_A})$ and restricting to a on $\mathrm{T}(\widetilde{Z_A})$. Since $c_1(L)^{\perp}_{\mathrm{NS}(\widetilde{Z_A})} \cap \mathcal{W}(\widetilde{Z_A}) = \emptyset$, we have that g preserves the Kähler cone of $\widetilde{Z_A}$, and it has positive spinor norm. Hence by Theorem 5.48 we have that g is induced by a purely nonsymplectic automorphism of $\widetilde{Z_A}$. Note that g^2 acts as negative identity on $c_1(L)^{\perp}$, hence it agrees with the covering involution. In this case, note that the invariant $L_2(11)$-invariant Lagrangian is $\mathrm{PGL}(W)$-isomorphic to its dual (see [DM22]). The observations made in this remark support what was explained previously in Remark 11.12.

11.5. Birational automorphisms of general double EPW-cubes

The content of this section is adapted from [BMW25, §4]. We compute the group of birational automorphisms of the general smooth double EPW-cubes, and of the desingularization of the general singular double EPW-cube (Proposition 11.3).

For a very general $[A] \notin \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Gamma \cup \Sigma)$, the associated double EPW-cube $\pi\colon \widetilde{Z_A} \to Z_A[2]$ has Picard rank 1. In fact, its Néron–Severi lattice $\mathrm{NS}(\widetilde{Z_A})$ is spanned by the first Chern class of the polarization $L := \pi^{-1}\mathcal{O}_{Z_A[2]}(1)$, of numerical type $(4,2)$. In particular, one has that

$$\mathrm{T}(\widetilde{Z_A}) \simeq U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_1^{\oplus 2}$$

(Section 11.2). We show the following, which is a particular case of a more general result [Deb22, Proposition 4.3].

Proposition 11.18. *Let $[A] \in \mathrm{LG}_\eta(10, \bigwedge^3 W) \setminus (\Sigma \cup \Gamma)$ be very general. Then*

$$\mathrm{Bir}(\widetilde{Z_A}) = \mathrm{Aut}(\widetilde{Z_A}, L) = \langle \iota \rangle$$

where ι is the covering involution. In particular

$$\mathrm{Bir}_s(\widetilde{Z_A}) = \{\mathrm{id}\}.$$

Proof. By Theorem 5.48, any birational automorphism of $\widetilde{Z_A}$ preserves $\mathrm{NS}(\widetilde{Z_A})$ and respects the orientation fixed by the positive cone. In particular, it induces an isometry in $O^+(\mathrm{NS}(\widetilde{Z_A}))$: since the Néron–Severi lattice of $\widetilde{Z_A}$ is positive definite of rank 1, we have that any birational automorphism of $\widetilde{Z_A}$ acts trivially on $\mathrm{NS}(\widetilde{Z_A})$. In particular, we already observe that $\mathrm{Bir}(\widetilde{Z_A}) = \mathrm{Aut}(\widetilde{Z_A}, L)$ and that $\mathrm{Bir}_s(\widetilde{Z_A})$ is trivial, because $\rho_{\widetilde{Z_A}}$ is injective (Table 4). Together, we infer that $\mathrm{Bir}(\widetilde{Z_A})$ is cyclic, generated by a regular purely nonsymplectic automorphism. We conclude using the description of $\mathrm{Aut}(\widetilde{Z_A}, L)$ given in Proposition 11.11 and the fact that, for A very general, A and $\delta(A)$ are not $\mathrm{PGL}(W)$-isomorphic. □

We show in the rest of this section that we observe a similar result for the desingularization of the double EPW cube for a general $[A] \in \Gamma \subseteq \mathrm{LG}_\eta(10, \bigwedge^3 W)$.

According to [Riz24, Lemma 3.2], if $[A] \in \Gamma \setminus \Sigma \subseteq \mathrm{LG}_\eta(10, \bigwedge^3 W)$ is general, then $Z_A[4]$ consists of a unique point, which is also the only singular point of $\widetilde{Z_A}$. Hence, following [Riz24, Theorem 5.3] we have that $\widetilde{Z_A}$ admits two smooth projective small resolutions

$$\begin{tikzcd}
\widetilde{Z_A}^{\epsilon_1} \arrow[rr, dashed, "f"] \arrow[dr, "q_1"'] & & \widetilde{Z_A}^{\epsilon_2} \arrow[dl, "q_2"] \\
& \widetilde{Z_A} \arrow[d, "\pi"] & \\
& Z_A &
\end{tikzcd}.$$

Here by small resolution we mean that the exceptional loci of q_1 and q_2, lying above the singular point of $\widetilde{Z_A}$, have codimension at least 2. Indeed, in our situation, their respective exceptional loci are just a copy of $\mathbb{P}^3$. Note that the projectivity statement here is important: in general, if $Z_A[4]$ consist of more than only point, the variety $Z_A = Z_A[2]$ could admit some small resolutions which are not Kähler. According to [Riz24, Proposition 5.1] the projective resolutions $\widetilde{Z_A}^{\epsilon_1}$ and $\widetilde{Z_A}^{\epsilon_2}$ are IHS manifolds of $\mathrm{K3}^{[3]}$-type. They are moreover $(4,2)$-quasipolarized [Riz24, Lemma 5.2]. The birational map f relating these two resolutions is a so-called *flop*: it is well-defined outside of the contracted $\mathbb{P}^3$'s.

Remark 11.19. By [IKKR19, Lemma 3.7], the family of resolutions of singular double EPW-cubes parametrized by $\Gamma \setminus \Sigma$ is 19-dimensional. Therefore, given a very general $[A] \in \Gamma \subseteq \mathrm{LG}_\eta(10, \bigwedge^3 W)$, the Picard rank of $\widetilde{Z_A}^{\epsilon_1}$ is at most 2.

Lemma 11.20. *Let $[A] \in \Gamma$ be general. We have*

$$\mathrm{NS}(\widetilde{Z_A}^{\epsilon_1}) \simeq \begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix}$$

and

$$\mathrm{T}(\widetilde{Z_A}^{\epsilon_1}) \simeq U^{\oplus 2} \oplus A_2 \oplus D_7 \oplus E_8.$$

Proof. The quasipolarization of numerical type $(4,2)$ of $\widetilde{Z_A}^{\epsilon_1}$ gives a class $L \in \mathrm{NS}(\widetilde{Z_A}^{\epsilon_1})$. Now, by [Riz24, Theorem 5.3], we know that the map $q_1 \colon \widetilde{Z_A}^{\epsilon_1} \to \widetilde{Z_A}$ is a small resolution that contracts a $\mathbb{P}^3$ to the singular point of $\widetilde{Z_A}$. According to [HT10, Table H3], this corresponds to the existence of a vector $D \in \mathrm{NS}(\widetilde{Z_A}^{\epsilon_1})$ of type $(-12,2)$ in $H^2(\widetilde{Z_A}^{\epsilon_1}, \mathbb{Z})$ and which is orthogonal to L. This implies that $\mathrm{NS}(\widetilde{Z_A}^{\epsilon_1})$ has rank 2 (Remark 11.19) and we have embeddings $\mathbb{Z}L + \mathbb{Z}D \leq \mathrm{NS}(\widetilde{Z_A}^{\epsilon}) < \Lambda_{\mathrm{K3}^{[3]}}$, where the first is of finite index and the second is primitive. Observe that since $\mathrm{div}(L, \Lambda_{\mathrm{K3}^{[3]}}) = \mathrm{div}(D, \Lambda_{\mathrm{K3}^{[3]}}) = 2$ and the discriminant group of $\Lambda_{\mathrm{K3}^{[3]}}$ has only one element of order 2 (see Table 4), we have that $\mathbb{Z}L + \mathbb{Z}\frac{L+D}{2}$ is an even overlattice of $\mathbb{Z}L + \mathbb{Z}D$ of index 2. The $\mathbb{Z}$-lattice $\mathbb{Z}L + \mathbb{Z}\frac{L+D}{2}$ has determinant -12, and none of its overlattices are even (see Proposition 2.4): it is actually primitive in $\Lambda_{\mathrm{K3}^{[3]}}$. In conclusion, we have $\mathrm{NS}(\widetilde{Z_A}^{\epsilon_1}) = \mathbb{Z}L + \mathbb{Z}\frac{L+D}{2}$ which has the wanted Gram matrix. Finally, one can check (using for instance Algorithm 2) that the primitive sublattice $\mathrm{NS}(\widetilde{Z_A}^{\epsilon_1}) < \Lambda_{\mathrm{K3}^{[3]}}$ is unique up to the action of $O^+(\Lambda_{\mathrm{K3}^{[3]}})$ and that its orthogonal complement is isometric to $U^{\oplus 2} \oplus A_2 \oplus D_7 \oplus E_8$. □

We give an explicit description of the movable cone $\mathrm{Mov}(\widetilde{Z_A}^{\epsilon_1})$ for a general Lagrangian space $[A] \in \Gamma$ (see Remark 5.37 for a definition of the movable cone). We recall that such a cone is given as the closure of the birational Kähler cone $\mathcal{BK}_{\widetilde{Z_A}^{\epsilon_1}}$ in $H^{1,1}(\widetilde{Z_A}^{\epsilon_1}, \mathbb{R})$. The MBM classes whose orthogonal complements describe the walls of the closure of the Kähler cone of $\widetilde{Z_A}^{\epsilon_1}$ are numerically characterized in Example 5.43. In particular, the possible norms of such classes are $-2, -4, -12$ or -36. Using the description from Lemma 11.20, one can easily check that $\mathrm{NS}(\widetilde{Z_A}^{\epsilon_1})$ has no isotropic vectors nor primitive vectors of norm -4 or -36. The only possibility left for the walls of $\mathrm{Mov}(\widetilde{Z_A}^{\epsilon_1})$ are vectors of type $(-2,1)$ or $(-12,2)$.

Recall that the birational Kähler cone of $\widetilde{Z_A}^{\epsilon_1}$ is the fundamental domain for the action of $W_{pex}(\widetilde{Z_A}^{\epsilon_1})$ on the decomposition of the positive cone of $\widetilde{Z_A}^{\epsilon_1}$, which contains the vector L of type $(4,2)$ (see Proposition 5.47). The group $W_{pex}(\widetilde{Z_A}^{\epsilon_1})$ is generated by the reflections in the vectors of $\mathcal{W}^{pex}(\widetilde{Z_A}^{\epsilon_1})$. Since the latter are necessarily of numerical type $(-2,1)$ (Example 5.43), we can apply an algorithm of Vinberg [Vin75] to find such a set of vectors whose orthogonal span the walls of $\mathrm{Mov}(\widetilde{Z_A}^{\epsilon_1})$ over $\mathbb{R}_{\geq 0}$.

Remark 11.21. Vinberg's algorithm has been implemented on the computer algebra system OSCAR [DEF+25]. See the notebook "NSgeneral" of [BMW24] to see how it has been used in our case.

We determine that $\mathcal{W}^{pex}(\widetilde{Z_A}^{\epsilon_1})$ consists of the two primitive vectors

$$(L-D)/2 \quad \text{and} \quad (L+D)/2$$

and their respective orthogonal complements are spanned by

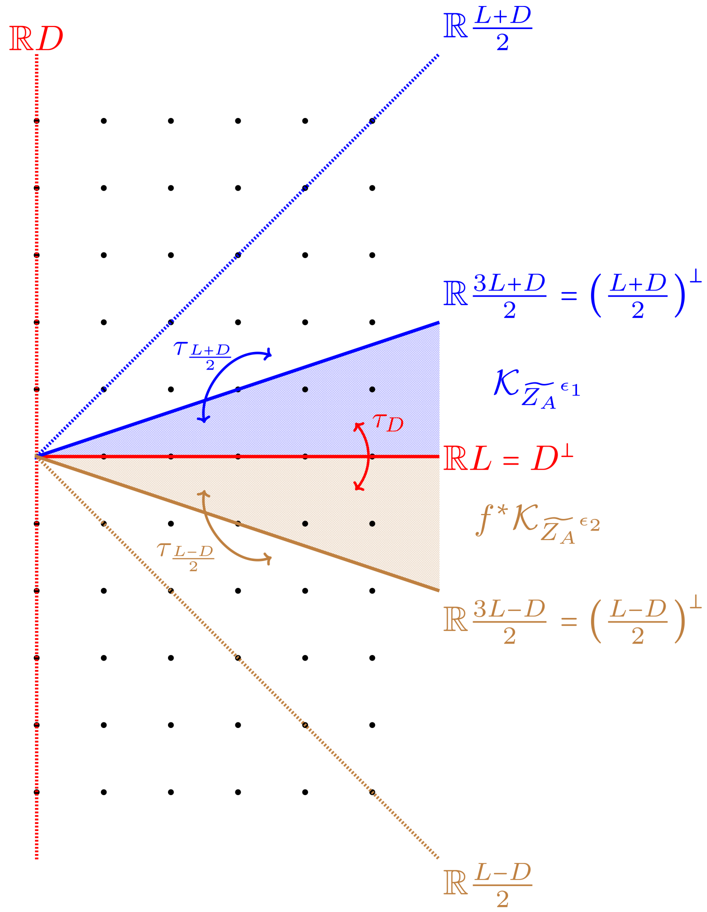
$$(3L-D)/2 \quad \text{and} \quad (3L+D)/2$$

which have both type $(6,1)$. Similarly, we find that the Kähler cone of $\widetilde{Z_A}^{\epsilon_1}$ is one of the two chambers obtained by cutting $\mathcal{BK}_{\widetilde{Z_A}^{\epsilon_1}}$ with $\mathbb{R}L = D^{\perp}$. The situation being symmetric for $\widetilde{Z_A}^{\epsilon_1}$

and $\widetilde{Z_A}^{\epsilon_2}$, related by a flop which is birational but nonregular, we obtain that the other chamber is the pullback by f of $\mathcal{K}_{\widetilde{Z_A}^{\epsilon_2}}$.

We draw in Figure 8 the birational Kähler cone of $\widetilde{Z_A}^{\epsilon_1}$ which contains two Kähler-type chambers being the Kähler cone $\mathcal{K}_{\widetilde{Z_A}^{\epsilon_1}}$ of $\widetilde{Z_A}^{\epsilon_1}$ and the pullback $f^*\mathcal{K}_{\widetilde{Z_A}^{\epsilon_2}}$ of the Kähler cone of $\widetilde{Z_A}^{\epsilon_2}$ by the flop $f\colon \widetilde{Z_A}^{\epsilon_1} \dashrightarrow \widetilde{Z_A}^{\epsilon_2}$.

Figure 8: Birational Kähler cone of $\widetilde{Z_A}^{\epsilon_1}$



Lemma 11.22. *The two resolutions $\widetilde{Z_A}^{\epsilon_1}$ and $\widetilde{Z_A}^{\epsilon_2}$ are isomorphic, as projective manifolds (but not as $\widetilde{Z_A}$-schemes).*

Proof. The covering involution ι on $\widetilde{Z_A}$ induces a birational automorphism $\hat{\iota}$ of $\widetilde{Z_A}^{\epsilon_1}$ whose action $\hat{\iota}^*$ on cohomology fixes L and is negative identity on the orthogonal complement. This is a nonsymplectic involution which is nonregular, as the action on $\mathrm{NS}(\widetilde{Z_A}^{\epsilon_1})$ given by the reflection τ_D in Figure 8 does not preserve $\mathcal{K}_{\widetilde{Z_A}^{\epsilon_1}}$. But, if we denote by $f^*\colon H^2(\widetilde{Z_A}^{\epsilon_2}, \mathbb{Z}) \to H^2(\widetilde{Z_A}^{\epsilon_1}, \mathbb{Z})$ the Hodge parallel transport operator induced by f, then the isometry $\hat{\iota}^* \circ f^*$ sends $\mathcal{K}_{\widetilde{Z_A}^{\epsilon_2}}$ to $\mathcal{K}_{\widetilde{Z_A}^{\epsilon_1}}$. By Theorem 5.35, this implies that the two resolutions are isomorphic. $\square$

Notation. For $[A] \in \Gamma \setminus \Sigma \subseteq \mathrm{LG}_\eta(10, \bigwedge^3 W)$ general, we write $\widetilde{Z_A}^{\epsilon} := \widetilde{Z_A}^{\epsilon_1} \simeq \widetilde{Z_A}^{\epsilon_2}$ for the smooth projective desingularization of $\widetilde{Z_A}$.

Proposition 11.23. *Let $[A] \in \Gamma \setminus \Sigma \subseteq \mathrm{LG}_\eta(10, \bigwedge^3 W)$ be general. Then*

$$\mathrm{Bir}(\widetilde{Z_A}^{\epsilon}) = \mathrm{Bir}(\widetilde{Z_A}^{\epsilon}, L) = \langle \hat{\iota} \rangle.$$

In particular,

$$\mathrm{Bir}_s(\widetilde{Z_A}^{\epsilon}) = \mathrm{Aut}(\widetilde{Z_A}^{\epsilon}) = \{\mathrm{id}\}.$$

Proof. Since the transcendental lattice $\mathrm{T}(\widetilde{Z_A}^{\epsilon})$ of the projective manifold $\widetilde{Z_A}^{\epsilon}$ has odd rank, any birational automorphism of $\widetilde{Z_A}^{\epsilon}$ is either symplectic or antisymplectic, i.e. acts as $\pm\,\mathrm{id}$ on $\mathrm{T}(\widetilde{Z_A}^{\epsilon})$ (Proposition 6.1). Moreover, any birational automorphism of $\widetilde{Z_A}^{\epsilon}$ restricts to an isometry in $O^+(\mathrm{NS}(\widetilde{Z_A}^{\epsilon}))$, because they preserve the positive cone of $\widetilde{Z_A}^{\epsilon}$. According to [BH23, Remark 4.27], we have that $O^+(\mathrm{NS}(\widetilde{Z_A}^{\epsilon}))$ is generated by the three reflections τ_D, $\tau_{(L+D)/2}$ and $\tau_{(L-D)/2}$. We refer to the notebook "NSgeneral" in [BMW24] where we perform the actual computations for this part of the proof.

Now any birational automorphim of $\widetilde{Z_A}^{\epsilon}$ preserves the birational Kähler cone of $\widetilde{Z_A}^{\epsilon}$. To support the rest of the proof, we represent in Figure 8 the previously mentioned reflections. Since $\tau_{(L+D)/2}\tau_D = \tau_D\tau_{(L-D)/2}$, it follows that any element of $O^+(\mathrm{NS}(\widetilde{Z_A}^{\epsilon}))$ is of the form $\tau_D^i\alpha$ where $i = 0, 1$ and α is a finite word in $\tau_{(L+D)/2}$ and $\tau_{(L-D)/2}$. While τ_D clearly preserves $\mathcal{BK}_{\widetilde{Z_A}^{\epsilon}}$, any nontrivial word α of the previous form maps L outside of $\mathcal{BK}_{\widetilde{Z_A}^{\epsilon}}$ (see Figure 8). Therefore, it follows that the action of any birational automorphism coincides with τ_D or id on $\mathrm{NS}(\widetilde{Z_A}^{\epsilon})$. On the one hand, since τ_D is not stable, the equivariant gluing condition (EGC) tells us we can only extend it with negative identity on $\mathrm{T}(\widetilde{Z_A}^{\epsilon})$: the resulting isometry coincides with $\rho_{\widetilde{Z_A}^{\epsilon}}(\hat{\iota})$. On the other hand, the identity on $\mathrm{NS}(\widetilde{Z_A}^{\epsilon})$ can only be extended with the identity on $\mathrm{T}(\widetilde{Z_A}^{\epsilon})$, giving rise to the identity of $H^2(\widetilde{Z_A}^{\epsilon}, \mathbb{Z})$. By Theorem 5.35, this implies that $\widetilde{Z_A}^{\epsilon}$ has no nontrivial symplectic birational automorphisms, and $\mathrm{Bir}(\widetilde{Z_A}^{\epsilon})$ has order 2 and it is generated by $\hat{\iota}$. $\square$

12. LSV manifolds and twisted analogs

We conclude by discussing the geometric counterpart of the data computed in [MM25c] (see also Table 19 in Appendix E). This section is adapted from the work [MM25b] and its content is mostly due to Marquand.

12.1. Generalities about Laza–Saccà–Voisin manifolds

Let $V \subseteq \mathbb{P}^5$ be a smooth cubic fourfold. We write the space of hyperplanes in $\mathbb{P}^5$ as $(\mathbb{P}^5)^\vee$, and for each $H \in (\mathbb{P}^5)^\vee$, we denote by $V_H := V \cap H$ the associated cubic threefold. There is a dense open subset $\mathcal{U} \subseteq (\mathbb{P}^5)^\vee$ parametrizing hyperplanes H for which V_H is smooth. For any $H \in \mathcal{U}$, we define

$$J_H := H^{2,1}(V_H)^\vee / H_3(V_H, \mathbb{Z})$$

to be the *(Griffiths) intermediate Jacobian* of the smooth cubic threefold V_H. According to [CG72, Theorem 11.19], the variety J_H is a principally polarized smooth abelian fivefold, which is isomorphic to the Albenese variety of the Fano variety of lines $F(V_H)$. According to [DM96a, §2] the family $\mathcal{V} \to \mathcal{U}$, whose fiber over $H \in \mathcal{U}$ is V_H, gives rise to a holomorphic family

$$\pi_{\mathcal{U}} \colon J \to \mathcal{U}$$

where for any hyperplane $H \in \mathcal{U}$, we define the fiber over H to be J_H. In [DM96b] (see for instance §8.5.2, Example 8.22 of this reference), Donagi and Markman showed that the total space J is equipped with a symplectic form σ_J and for all $H \in \mathcal{U}$, $J_H \subseteq J$ is Lagrangian with respect to this form; see for instance [LSV17, Theorem 1.2] for a description of such a form. In fact, they show that the map $\pi_{\mathcal{U}}$ is actually a Lagrangian fibration: we usually referred to $\pi_{\mathcal{U}}$ as the *Donagi–Markman fibration*, or *intermediate Jacobian fibration*, of V. The following result was first due to Laza, Saccà and Voisin in the case of V general in the moduli space of cubic fourfolds, and later generalized by Saccà in the case of any smooth cubic fourfold.

Theorem 12.1 ([LSV17, Main Theorem])**.** *There exists a smooth projective compactification $\mathcal{J}$ of J and a fibration*

$$\pi \colon \mathcal{J} \to (\mathbb{P}^5)^\vee$$

extending $\pi_{\mathcal{U}}$ which is Lagrangian with respect to a holomorphic symplectic 2-form on $\mathcal{J}$. Moreover, the smooth fibers of π are irreducible and the total space $\mathcal{J}$ is an IHS tenfold of OG10-type.

Note that the compactification $\mathcal{J}$ is not unique: we refer to it as an *LSV manifold* associated to V. Let us fix such an LSV manifold $\mathcal{J}$, with its Lagrangian fibration $\pi \colon \mathcal{J} \to (\mathbb{P}^5)^\vee$.

According to [Sac23, Lemma 3.5] the tenfold $\mathcal{J}$ has Picard rank at least 2. Indeed, there are two canonical algebraic classes associated to $\mathcal{J}$. The first one is the pullback $L := \pi^{-1}\mathcal{O}_{(\mathbb{P}^5)^\vee}(1)$ which is the class defining the fibration π. In particular, $c_1(L)$ is isotropic with respect to the BBF form on $H^2(\mathcal{J}, \mathbb{Z})$. The second one is the so-called *relative theta divisor* θ, which is an effective divisor on $\mathcal{J}$. For all $H \in \mathcal{U} \subseteq (\mathbb{P}^5)^\vee$, the intermediate Jacobian J_H admits a canonical (-1)-invariant theta divisor [LSV17, Lemma 5.4]: θ is defined as the closure of the (disjoint) union of the previous theta divisors in $\mathcal{J}$. According to [Sac23, Lemma 3.5, Proposition 3.6], we have that $c_1(\theta)^2 = -2$ and $c_1(L).c_1(\theta) = 1$. In particular, $\mathbb{Z}c_1(L) + \mathbb{Z}c_1(\theta) \simeq U$ defines an hyperbolic plane in $\mathrm{NS}(\mathcal{J})$: we say the tenfold $\mathcal{J}$ is *U-polarized*.

Remark 12.2. For V very general in the moduli space of cubic fourfolds, we have that $\mathrm{NS}(\mathcal{J}) \simeq U$ for any LSV $\mathcal{J}$ associated to V.

Let us finally talk about symmetries of LSV manifolds. Note that according to [LSV17, Lemma 4.7], the fibration π admits a zero section $s\colon (\mathbb{P}^5)^\vee \to \mathcal{J}$. The mapping by (-1) on the smooth fibers of π actually gives rise to a birational involution

$$\tau\colon \mathcal{J} \dashrightarrow \mathcal{J}$$

which we refer to as the *LSV involution*. Note that τ is antisymplectic. Let now $f \in \mathrm{Aut}(V)$ be an automorphism of the cubic fourfold V. Then f acts on the family $\mathcal{V} \to \mathcal{U}$ of smooth hyperplane sections on V, and such an action induces an action of f on the Donagi–Markman fibration $\pi_{\mathcal{U}}$ (where we see J_H as $\mathrm{Alb}(F(V_H))$). Therefore, we obtain an induced birational automorphism

$$\widetilde{f}\colon \mathcal{J} \dashrightarrow \mathcal{J}$$

and such a birational automorphism preserves both L and θ [Sac23, §3.1]. Thus, the action on cohomology preserves a copy of U in $\mathrm{NS}(\mathcal{J})$.

Remark 12.3.

(1) According to the proof of [BG25, Proposition 5.6] (see also [MO22, Lemma 7.1]), there is a **rational** isometry of Hodge structures

$$H^4_{prim}(V,\mathbb{Q})(-1) \to (U^\perp) \subseteq H^2(\mathcal{J},\mathbb{Q})$$

such that for all $f \in \mathrm{Aut}(V)$, the action of f on $H^4_{prim}(V,\mathbb{Q})(-1)$ coincides with the one of $\widetilde{f}$ on $(U^\perp) \subseteq H^2(\mathcal{J},\mathbb{Q})$ along this isometry. In particular, automorphisms of V induce birational automorphisms of $\mathcal{J}$ of the same transcendental value.

(2) The involution $\tau \in \mathrm{Aut}(\mathcal{J})$ commutes with all birational automorphisms induced from $\mathrm{Aut}(V)$. Indeed, τ acts trivially on the base $(\mathbb{P}^5)^\vee$ of the fibration π, whereas any automorphism induced from $f \in \mathrm{Aut}(V)$ will either permute fibers or induce an automorphism of an invariant smooth fiber that will commute with τ.

Example 12.4. In [GAL21, Theorem 3.8], the authors show the existence of a 10-dimensional family $\mathcal{C}$ of smooth cubic fourfolds V, all equipped with an automorphism $g \in \mathrm{Aut}(V)$ of order 3. For any such pair (V,g), the induced action of g on $H^4(V,\mathbb{Z})$ only fixes the square h^2 of the hyperplane class (see [BCS16, Example 6.4] or [BG25, Table 4]). For a very general $V \in \mathcal{C}$, the primitive algebraic lattice $A(V)_{prim} := (h^2)^\perp \leq H^4(V,\mathbb{Z}) \cap H^{2,2}(V)$ of V is trivial. Consequently, any LSV tenfold $\mathcal{J}$ associated to a very general V in $\mathcal{C}$, of deformation type OG10, has Picard rank 2 with $\mathrm{NS}(\mathcal{J}) \simeq U$. The automorphism g induces a nonsymplectic automorphism $\widetilde{g} \in \mathrm{Bir}(\mathcal{J})$ of order 3, since in our context its action on cohomology fixes $\mathrm{NS}(\mathcal{J})$ pointwise. Therefore $\widetilde{g}$ is algebraically trivial and stable (Section 8.2.3). Note that the LSV involution τ on $\mathcal{J}$ is also regular since it is algebraically trivial. Therefore, since τ and $\widetilde{g}$ commute, we obtain an algebraically trivial purely nonsymplectic automorphism $f := \widetilde{g} \circ \tau$ of order 6: according to Theorem 8.67 and Table 18 we obtain that f is nonstable (and so is τ).

In Section 12.2, we investigate the possible groups from [MM25c] that can act on a cubic fourfold, and we comment on the relation with actions on LSV manifolds.

In Section 12.3, we use similar techniques of [BG25] to investigate when a group of symplectic birational transformations of an IHS manifold of OG10-type is induced from a cubic fourfold via the twisted LSV construction of Voisin [Voi18].

12.2. Actions on cubic fourfolds

We prove the following.

Proposition 12.5. *Let X be an IHS manifold of OG10-type, let $G \leq \mathrm{Bir}_s(X)$ be a finite subgroup, and let us denote $\Lambda := H^2(X,\mathbb{Z})$. Suppose that $\Lambda^G = U \oplus \Gamma$ for some even $\mathbb{Z}$-lattice Γ. Then there exists a smooth cubic fourfold V and an embedding $j\colon G \hookrightarrow \mathrm{Aut}(V)$ such that $j(G^\#) \leq \mathrm{Aut}_s(V)$ and*

(1) *either $G = G^\#$ is stable;*

(2) *or G is nonstable and $\mathrm{Aut}(V)$ contains an antisymplectic automorphism.*

Proof. Let $U_1 := U$ be a distinguished hyperbolic plane in $\Lambda^G = U_1 \oplus \Gamma \leq \Lambda$, and let us define $L := (U_1)^\perp_\Lambda(-1)$. Then L is an even $\mathbb{Z}$-lattice of signatures $(20,2)$; indeed

$$L \simeq U^2 \oplus E_8^2(-1) \oplus A_2(-1).$$

By abuse of notation, since $\rho_X\colon \mathrm{Bir}(X) \to O(H^2(X,\mathbb{Z}))$ is injective, we denote by $G := \rho_X(G)$ the image of G. Since G fixes U_1, it restricts to a subgroup $H \leq O(L)$ with

$$L^H \simeq (U_1)^\perp_{\Lambda^G}(-1) \simeq \Gamma(-1) \text{ and } L_H \simeq \Lambda_G(-1).$$

We choose a pure Hodge structure H on $L \otimes_\mathbb{Z} \mathbb{C}$ of weight 4 and of type $(0,1,20,1,0)$ such that $H^{2,2} \cap L = \Lambda_G(-1)$. In particular, $H^{3,1} \cap L \leq L^H$, and since $\Lambda_G \cap \mathcal{W}^{pex}(X) = \emptyset$ we can apply the Global Torelli Theorem for cubic fourfolds ([Voi86], [Zhe19, Prop 1.3]). We obtain a smooth cubic fourfold V with $H^4(V,\mathbb{Z})_{prim} \simeq L$ Hodge isometric.

First, we assume that G acts trivially on D_Λ, and hence so does H on D_L. In order to conclude that $G \cong H$ embeds into $\mathrm{Aut}_s(V)$, we need to extend G to a group of isometries of $H^4(V,\mathbb{Z})$ fixing the square of the hyperplane class $h^2 \in H^4(V,\mathbb{Z})$. Note that $D_L \simeq D_\Lambda \simeq D_{\mathbb{Z}h^2}(-1)$ so we have that $L \oplus \mathbb{Z}h^2 \leq H^4(V,\mathbb{Z})$ is an odd unimodular primitive extension. Since H acts trivially on D_L, we can apply Corollary 2.21 to get that G extends with $\mathrm{id}_{\mathbb{Z}h^2}$ to a group of isometries of $H^4(V,\mathbb{Z})$. By the Torelli theorem for cubic fourfolds, we conclude that H acts faithfully on V. Further, in this case it follows that $G \cong H$ is identified to a group of symplectic automorphisms of V.

Next, we assume that G does not act trivially on D_Λ. In other words, there exists a short exact sequence

$$1 \to G^\# \to G \to \mu_2 \to 1.$$

In particular there exists a nonstable isometry $a \in O(\Lambda)$ such that $G = \langle G^\#, a\rangle$. The isometry a has even order and $\Lambda^G \leq \Lambda^a$ contains U_1 as well. From the previous part, we already know we can embed $G^\#$ into $\mathrm{Aut}_s(V)$. It remains to extend the isometry a.

We restrict a to L to obtain an isometry of L: since D_Λ and D_L are equivariantly isometric, via $\Lambda = U_1 \oplus L$, we have that the restriction $a' := a_{|L}$ to L is nonstable. Since $O(D_L) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the image of $-\mathrm{id}_L$, we can instead extend the isometry $-a' \in O^\#(L)$. Following the idea of the previous paragraphs, we obtain an isometry b of $H^4(V,\mathbb{Z})$, acting trivially on $\mathbb{Z}h^2$, and by the Torelli theorem for cubic fourfolds we conclude that b corresponds to an automorphism of V, which we still denote by $b \in \mathrm{Aut}(V)$. This time however, the automorphism b is antisymplectic: the isometry a acts trivially on $H^{3,1}$, hence b acts by $-\mathrm{id}_{|H^{3,1}}$. If we let $G^\# \leq \mathrm{Aut}_s(V)$ denote the image of $G^\#$, by abuse of notation, we have that G embeds into $\mathrm{Aut}(V)$ and its image is exactly the subgroup $\langle G^\#, b\rangle$ with $b^2 \in G^\#$. □

The converse of the previous theorem is not immediate, for 3 main reasons:

(1) it is not clear that automorphisms of cubic fourfolds induce stable birational automorphisms on their associated LSV's;

(2) it is conjectured but not yet proven that the rational Hodge isometry mentioned in Remark 12.3 is in fact **integral**;

(3) the LSV involution is known to be nonstable in certain cases (see Example 12.4) but this has not, to the author's knowledge, been proven in general.

Proposition 12.5 and the classification of prime order nonsymplectic automorphisms [BG25] give some evidence that the three facts in Items (1)–(3) above should actually be true. Note that showing that the conjecture in (2) is true would imply the fact that induced actions are stable in (1). Currently, the best we can do as a converse is the following, which is a direct consequence of [Sac23, §3.1] and Remark 12.3.

Proposition 12.6. *Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold, and $G \leq \mathrm{Aut}_s(V)$ be a finite subgroup of symplectic automorphisms of V. Then the group of symplectic birational transformations of any LSV manifold $\mathcal{J}$ associated to V contains a finite subgroup isomorphic to G.*

If one can prove moreover Items (2) and (3) above, then we would get an actual converse of Proposition 12.5, namely:

Conjecture 12.7. *Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold, and $G \leq \mathrm{Aut}(V)$ be a finite subgroup of automorphisms whose symplectic subgroup $G_s := G \cap \mathrm{Aut}_s(V)$ has index at most 2 in G. Then the group of symplectic birational automorphisms of any LSV manifold associated to V contains a finite subgroup isomorphic to G with stable subgroup isomorphic to G_s.*

Remark 12.8. By comparing further Table 19 and [LZ22, Theorem 1.8], we see a correspondence of lattice data between entry 101a.1 and case (1), entry 108a.1 and case (2), entry 109a.4 and case (3), entry 119 and case (4), entry 120 and case (5), and entry 128a.1 and case (6). Conjecturally, this numerical correspondence actually follows from the rational Hodge isometry in Remark 12.3 being integral, and the LSV involution τ being nonstable.

In Table 19, we mark with "$\times$" all entries for which the associated invariant sublattice contains a copy of U. Conjecturally, all these entries can be realized as induced by a group of symplectic automorphisms of a cubic fourfold on an LSV manifold associated to it.

12.3. Twisted LSV manifolds

Let $V \subseteq \mathbb{P}^5$ be a smooth cubic fourfold. There exists another IHS manifold $\mathcal{J}^t$ of OG10-type associated to V. The manifold $\mathcal{J}^t$ is equipped with a Lagrangian fibration $\pi^t : \mathcal{J}^t \to (\mathbb{P}^5)^\vee$ whose fiber J^t_H over a smooth hyperplane section $H \in \mathcal{U}$ is the torsor $\mathrm{Jac}^1(Y_H)$ parametrising 1-cycles of degree 1 on V_H, up to rational equivalence [Voi18]. We call such a manifold $\mathcal{J}^t$ a *twisted LSV manifold.* Similarly to LSV manifolds, we have that $\mathrm{NS}(\mathcal{J}^t)$ has rank at least 2, since it contains two algebraic classes which span a copy of $U(3)$ (see for instance [MO22, Lemma 7.1]). Moreover, any automorphism of V induces a birational automorphism on $\mathcal{J}^t$ [BG25, Remark 5.4]. This time however, we observe the following.

Proposition 12.9 ([BG25, Proposition 5.1])**.** *Let V be a cubic fourfold and let $\mathcal{J}^t$ be an associated twisted LSV manifold. Then there is an **integral** isometry of Hodge structures*

$$H^4_{prim}(V, \mathbb{Z}) \to U(3)^\perp \leq H^2(\mathcal{J}^t, \mathbb{Z})$$

where $U(3)$ is the algebraic twisted copy of U in $\mathrm{NS}(\mathcal{J}^t)$.

This differs from our current knowledge about LSV manifolds. In particular, it solves a problem we observed earlier: every automorphism $f \in \mathrm{Aut}(V)$ induces a **stable** birational automorphism $\widetilde{f}^t$ of $\mathcal{J}^t$, of the same transcendental value. We may use this to provide a geometric realization for some of the groups given in [MM25c].

Remark 12.10. Note that given a smooth cubic fourfold, the associated LSV manifold $\mathcal{J}$ and twisted LSV manifold $\mathcal{J}^t$ can be non-birational (see for instance [LPZ22, Theorem 1.3] or [BG25, Remark 5.3]).

In [BG25], the authors investigate when an IHS manifold of OG10-type is birational to a twisted LSV manifold, and use their criterion to determine when a nonsymplectic automorphism is induced from a cubic fourfold. By a small adaption of the arguments in [BG25, Proposition 5.2], one obtains the following proposition.

Proposition 12.11. *Let X be an IHS manifold of OG10-type, and let us denote $\Lambda := H^2(X, \mathbb{Z})$. Let $G \leq \mathrm{Bir}_s(X)$ be a finite subgroup such that $H := \rho_X(G)$ acts trivially D_Λ. Suppose that there is a primitive embedding $U(3) \hookrightarrow \Lambda^H$ such that the composition $U(3) \hookrightarrow \Lambda$ has glue domain $\mathbb{Z}/3\mathbb{Z}$. Then there exists a smooth cubic fourfold V with an embedding $G \hookrightarrow \mathrm{Aut}_s(V)$.*

Proof. Consider the $\mathbb{Z}$-lattice $L := U(3)^\perp \leq \Lambda$, which is isometric to $U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$ by the assumption on the glue domain of $U(3) \hookrightarrow \Lambda$. Note that H acts on L with $L_H = \Lambda_H$. We follow the proof of Proposition 12.5 to obtain a smooth cubic fourfold V with $H^4_{prim}(V, \mathbb{Z})$ Hodge isometric to the $\mathbb{Z}$-lattice $L(-1)$, for an appropriate choice of a Hodge structure on $L(-1) \otimes_\mathbb{Z} \mathbb{C}$. Since H is stable, we can extend H to a group of isometries of $H^4_{prim}(V, \mathbb{Z})$ as in Proposition 12.5 and conclude that $G \cong H$ embeds into $\mathrm{Aut}_s(V)$. □

Recall that the abstract $\mathbb{Z}$-lattice associated to the deformation type OG10 is

$$\Lambda_{\mathrm{OG10}} \simeq U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2.$$

In what follows, we say a finite stable symplectic subgroup $H \leq O^+(\Lambda_{\mathrm{OG10}})$ is *twisted LSV induced* if there exists a cubic fourfold V, a finite subgroup $H \cong G_V \leq \mathrm{Aut}_s(V)$ and a marking $\eta\colon H^2(\mathcal{J}^t, \mathbb{Z}) \to \Lambda_{\mathrm{OG10}}$ of a twisted LSV manifold $\mathcal{J}^t$ associated to V such that

$$H = \eta\rho_{\mathcal{J}^t}(\widetilde{G_V}^t)\eta^{-1},$$

where $\widetilde{G_V}^t \leq \mathrm{Bir}_s(\mathcal{J}^t)$ is induced by G_V.

Corollary 12.12. *A stably saturated finite symplectic subgroup $H \leq O^+(\Lambda_{\mathrm{OG10}})$ is twisted LSV induced if and only if H is stable, and the* Id *number, as listed in Table 16, is one of the following:*

$$\begin{aligned}\mathrm{Id} \in \{&1, 2, 3, 4b, 7b, 9, 13, 15b, 18b, 19b, 20, 29b, 31, 35b, 39b, 44b, 46b, 47c,\\ &52, 53, 55, 68b, 72b, 77, 82b, 84, 85b, 87, 101b, 108b, 109b, 119, 120, 128b\}.\end{aligned}$$

Proof. If $H \leq O^+(\Lambda)$ is twisted LSV induced, then we know it is stable and its invariant sublattice contains a copy of $U(3)$ that satisfies the assumptions of Proposition 12.11. The only entries in [MM25c] that satisfy the assumptions of Proposition 12.11 are those with Ids as listed above (verified by direct computation, see the notebook [MM25c, Realisations]).

Let (X, η) be a marked IHS manifold of OG10-type with $G \leq \mathrm{Bir}_s(X)$ such that $H := \eta\rho_X(G)\eta$ is one of the entries of [MM25c] corresponding to one of the Ids above. In particular, for those cases, we have that H is stable and $\Lambda^H_{\mathrm{OG10}}$ contains a copy of $U(3)$ whose primitive embedding into Λ_{OG10} has glue domain $\mathbb{Z}/3\mathbb{Z}$. According to Proposition 12.11, this implies that there exists a cubic fourfold V with an embedding $G \hookrightarrow \mathrm{Aut}_s(V)$, such that the action of G on $H^4_{prim}(V,\mathbb{Z})$ is compatible with the action of G on $U(3)^{\perp}_{H^2(X,\mathbb{Z})}$ (in the sense of the proof of Proposition 12.11). One then applies Proposition 12.9 to obtain a twisted LSV manifold $\mathcal{J}^t$ associated to V, such that $G \cong G^t \leq \mathrm{Bir}_s(\mathcal{J}^t)$ and a marking η^t of $\mathcal{J}^t$ such that

$$H = \eta^t \rho_{\mathcal{J}^t}(G^t)\eta^{t,-1}.$$

$\square$

In Table 19, we mark with "$\times^t$" all entries with the Id given as in Corollary 12.12.

Remark 12.13. Let $V \subset \mathbb{P}^5$ be the Fermat cubic, and let $G = \mathrm{Aut}_s(V) \cong C_3^4 \rtimes A_6$. The associated LSV manifold $\mathcal{J}$ and twisted LSV manifold $\mathcal{J}^t$ inherit a group of symplectic birational automorphisms isomorphic to G, but the action on the second cohomology is different. Namely, if the action on $H^2(\mathcal{J},\mathbb{Z})$ is nonstable then it is given by the entry 101a in Table 16, whereas the action on $H^2(\mathcal{J}^t,\mathbb{Z})$ is given by 101b. We point out however that $\mathcal{J}$ and $\mathcal{J}^t$ are, in that case, birational [BG25, Remark 5.3]. However, the pairs $(\mathcal{J},\widetilde{G})$ and $(\mathcal{J}^t,\widetilde{G}^t)$ are not birational conjugate.

Example 12.14. Let us remark that in [LZ22, Theorem 1.8 (2)] the authors show the existence of a smooth cubic fourfold V such that $\mathrm{Aut}(V) = \mathrm{Aut}_s(V) \cong A_7$. By the twisted LSV construction, we obtain an OG10-type IHS manifold $\mathcal{J}^t$ with a symplectic birational action of A_7 fixing a copy of $U(3) \leq \mathrm{NS}(\mathcal{J}^t)$, and $T(V) \simeq T(\mathcal{J}^t)$. By investigating Table 19, we observe the latter action corresponds to entry 108b from [MM25c]. Conjecturally, this case could be also realized by considering the action of $\mathrm{Aut}(V)$ on an LSV manifold $\mathcal{J}$ associated to V.

Remark 12.15. Given a K3 surface S, any finite subgroup of symplectic automorphisms $G \leq \mathrm{Aut}_s(S)$ acts faithfully on the (desingularizations of the) moduli spaces of semistable sheaves on the K3 surface S [FGG24, Proposition 6.7]. Following the arguments of the proof of [FGG24, Theorem 6.8], we can detect groups realized in that way by comparing $\mathbb{Z}$-lattices data with [Has12, Table 10.2]. In Table 19 for each conjugacy classes of groups which can be realized in this way, we indicate the number of the entry in Hashimoto's list.

Part IV.
Tables

A. Exceptional finite groups of isometries of the Weyl chamber

Each entry in Table 14 corresponds to an abstract isometry class of primitive sublattice $C \leq \mathbb{B}$ without (-2)-roots, of rank at most 21, and such that $O^\#(C) \leq \mathrm{Aut}(\mathcal{D})$ is exceptional (see Theorem 7.46). For each entry we give:

(1) the rank of the $\mathbb{Z}$-lattice C;

(2) the length $l(D_C)$ of the discriminant group D_C of C;

(3) a symbol $g(C)$ for the genus of the $\mathbb{Z}$-lattice C;

(4) the order of the stable subgroup $O^\#(C)$ of isometries of C.

Note that some of the $\mathbb{Z}$-lattices represented in Table 14 are in the same genus and have isomorphic stable subgroup of isometries. However, the entries represent pairwise nonisometric $\mathbb{Z}$-lattices.

Table 14: Exceptional primitive sublattices of $\mathrm{II}_{(1,25)}$ without (-2)-roots

rk(C)	$l(D_C)$	$g(C)$	$\#O^\#(C)$	$O^\#(C)$	rk(C)	$l(D_C)$	$g(C)$	$\#O^\#(C)$	$O^\#(C)$
16	10	$\mathrm{II}_{(0,16)}2_0^{10}$	2	C_2	20	6	$\mathrm{II}_{(0,20)}2^2 4_4^4$	128	$C_2 \times D_8^2$
16	10	$\mathrm{II}_{(0,16)}2^8 4^2$	8	C_2^3	20	6	$\mathrm{II}_{(0,20)}2^2 4_4^4$	128	[128, 1135]
16	10	$\mathrm{II}_{(0,16)}2^{10}$	32	C_2^5	20	6	$\mathrm{II}_{(0,20)}2^2 4_4^4$	128	[128, 2216]
17	9	$\mathrm{II}_{(0,17)}2^8 8_7^1$	16	C_2^4	20	6	$\mathrm{II}_{(0,20)}2^6 3^2$	192	$C_2^3 \times S_4$
18	7	$\mathrm{II}_{(0,18)}3^{-7}$	9	C_3^2	20	6	$\mathrm{II}_{(0,20)}2^{-2} 4_0^{-4}$	256	[256, 29598]
18	8	$\mathrm{II}_{(0,18)}2_6^4 4^4$	4	C_2^2	20	6	$\mathrm{II}_{(0,20)}2^2 4_4^4$	256	[256, 56089]
18	8	$\mathrm{II}_{(0,18)}2_6^4 4^4$	8	C_2^3	20	6	$\mathrm{II}_{(0,20)}2^4 4_3^{-1} 8_5^{-1}$	256	[256, 53380]
18	8	$\mathrm{II}_{(0,18)}2_6^4 4^4$	8	$C_2 \times C_4$	20	6	$\mathrm{II}_{(0,20)}2^{-4} 4^{-2}$	768	[768, 1090235]
18	8	$\mathrm{II}_{(0,18)}2^{-4} 4_2^{-4}$	16	C_2^4	20	6	$\mathrm{II}_{(0,20)}2^{-4} 4^{-2}$	4096	—
18	8	$\mathrm{II}_{(0,18)}2^{-4} 4_2^{-4}$	16	$C_2 \times D_8$	21	4	$\mathrm{II}_{(0,21)}2_7^{-3} 3^4$	18	$C_3 \times S_3$
18	8	$\mathrm{II}_{(0,18)}2^{-6} 4_2^{-2}$	32	C_2^5	21	5	$\mathrm{II}_{(0,21)}2_2^2 4_1^1 8_0^2$	16	$C_2 \times D_8$
18	8	$\mathrm{II}_{(0,18)}2^6 4_6^2$	32	C_2^5	21	5	$\mathrm{II}_{(0,21)}2_5^3 3^5$	18	$C_3 \rtimes S_3$
18	8	$\mathrm{II}_{(0,18)}2^{-6} 4_2^{-2}$	64	$C_2^3 \times D_8$	21	5	$\mathrm{II}_{(0,21)}2_7^3 8^{-2}$	24	$C_2 \times A_4$
19	7	$\mathrm{II}_{(0,19)}2_5^5 8^2$	8	D_8	21	5	$\mathrm{II}_{(0,21)}2_2^2 4_1^{-3}$	32	[32, 44]
19	7	$\mathrm{II}_{(0,19)}2^2 4_5^5$	8	D_8	21	5	$\mathrm{II}_{(0,21)}2^2 4_5^{-1} 8_2^{-2}$	32	$C_2^2 \wr C_2$
19	7	$\mathrm{II}_{(0,19)}2^2 4_5^5$	16	C_2^4	21	5	$\mathrm{II}_{(0,21)}2^2 4_3^{-1} 8_4^{-2}$	32	$C_8 \rtimes C_2^2$
19	7	$\mathrm{II}_{(0,19)}2_6^4 4^{-2} 8_3^{-1}$	16	C_2^4	21	5	$\mathrm{II}_{(0,21)}2_3^{-3} 4_1^1 16_3^{-1}$	32	$C_8 \rtimes C_2^2$
19	7	$\mathrm{II}_{(0,19)}2_2^{-4} 4^{-2} 8_7^1$	16	$C_2 \times D_8$	21	5	$\mathrm{II}_{(0,21)}2_4^4 8_7^1$	32	2_-^{1+4}
19	7	$\mathrm{II}_{(0,19)}2^{-2} 4_1^{-5}$	16	$C_2 \times D_8$	21	5	$\mathrm{II}_{(0,21)}2^2 4_1^1 8_2^2$	64	D_8^2
19	7	$\mathrm{II}_{(0,19)}2^{-4} 4^{-2} 8_1^1$	32	$C_2^2 \times D_8$	21	5	$\mathrm{II}_{(0,21)}2_2^{-2} 4_6^2 8_7^1$	64	D_8^2
19	7	$\mathrm{II}_{(0,19)}2^{-4} 4_2^{-2} 8_7^1$	32	$C_2^2 \times D_8$	21	5	$\mathrm{II}_{(0,21)}4_3^5$	64	$C_2^3 \times D_8$
19	7	$\mathrm{II}_{(0,19)}2^4 4_6^{-2} 8_3^{-1}$	32	$C_2^2 \wr C_2$	21	5	$\mathrm{II}_{(0,21)}2^2 4_3^{-1} 8_4^{-2}$	64	$C_2^3 \times D_8$
19	7	$\mathrm{II}_{(0,19)}2^{-6} 8_5^{-1}$	64	C_2^6	21	5	$\mathrm{II}_{(0,21)}2_6^4 16_5^{-1}$	64	$C_2 \times C_8 \rtimes C_2^2$
19	7	$\mathrm{II}_{(0,19)}2^4 4_5^3$	64	$C_2 \times C_2^2 \wr C_2$	21	5	$\mathrm{II}_{(0,21)}2_2^{-4} 16_5^{-1}$	64	[64, 124]
19	7	$\mathrm{II}_{(0,19)}2^{-6} 8_5^{-1}$	128	$C_2^2 \times C_2^2 \wr C_2$	21	5	$\mathrm{II}_{(0.21)}2_4^{-2} 4_3^{-1} 8_4^{-2}$	64	[64, 134]
19	7	$\mathrm{II}_{(0,19)}2^4 4_5^3$	128	$C_2^2 \times 2_+^{1+4}$	21	5	$\mathrm{II}_{(0,21)}2_7^3 8^{-2}$	64	[64, 134]

Table 14: Exceptional primitive sublattices of $\mathrm{II}_{(1,25)}$ without (-2)-roots (continued)

rank(C)	$l(D_C)$	$g(C)$	$\#O^{\#}(C)$	$O^{\#}(C)$	rank(C)	$l(D_C)$	$g(C)$	$\#O^{\#}(C)$	$O^{\#}(C)$
20	5	$\mathrm{II}_{(0,20)}5^5$	5	C_5	21	5	$\mathrm{II}_{(0,21)}2_7^3 8^{-2}$	64	[64, 211]
20	5	$\mathrm{II}_{(0,20)}3^{-4}9^1$	108	[108, 40]	21	5	$\mathrm{II}_{(0,21)}2_2^{-4}8_7^1 3^1$	96	$C_2^2 \times S_4$
20	6	$\mathrm{II}_{(0,20)}2^{-2}4_0^{-4}$	4	C_4	21	5	$\mathrm{II}_{(0,21)}2^4 8_7^1 3^2$	96	$C_2^2 \times S_4$
20	6	$\mathrm{II}_{(0,20)}2_6^2 4_6^{-4}$	4	C_2^2	21	5	$\mathrm{II}_{(0,21)}2^{-2}4_7^1 8_4^{-2}$	128	$D_8 \wr C_2$
20	6	$\mathrm{II}_{0,20)}2_4^{-6}3^2$	6	C_6	21	5	$\mathrm{II}_{(0,21)}2_7^3 8^{-2}$	128	[128, 2317]
20	6	$\mathrm{II}_{(0,20)}2_0^{-4}4^2$	16	$C_2 \times Q_8$	21	5	$\mathrm{II}_{(0,21)}2^2 4_1^{-3}3^{-1}$	192	$C_2 \times C_2^2 \rtimes S_4$
20	6	$\mathrm{II}_{(0,20)}2_2^{-4}8_6^2$	16	$C_2 \times D_8$	21	5	$\mathrm{II}_{(0,21)}2^2 4_4^{-2}8_3^{-1}$	256	[256, 16883]
20	6	$\mathrm{II}_{(0,20)}2_6^4 8_6^2$	16	$C_4 \bigcirc D_8$	21	5	$\mathrm{II}_{(0,21)}2^2 4_2^{-2}8_5^{-1}$	256	[256, 16888]
20	6	$\mathrm{II}_{(0,20)}2_1^{-5}16_7^1$	16	$C_4 \bigcirc D_8$	21	5	$\mathrm{II}_{(0,21)}4_3^5$	256	[256, 25886]
20	6	$\mathrm{II}_{(0,20)}2_5^3 4_7^1 8^2$	16	D_{16}	21	5	$\mathrm{II}_{(0,21)}2^2 4_2^2 8_1^1$	256	[256, 51978]
20	6	$\mathrm{II}_{(0,20)}2^{-2}3^{-6}$	18	$C_3 \times S_3$	21	5	$\mathrm{II}_{(0,21)}2^2 4_2^2 8_1^1$	384	[384, 18235]
20	6	$\mathrm{II}_{(0,20)}2^2 4_5^3 8_7^1$	32	C_2^5	21	5	$\mathrm{II}_{(0,21)}2^2 4_3^3$	384	[384, 20089]
20	6	$\mathrm{II}_{(0,20)}2^{-2}4_7^3 8_5^{-1}$	32	$C_2^2 \times D_8$	21	5	$\mathrm{II}_{(0,21)}2^{-2}4_7^{-3}$	384	[384, 20089]
20	6	$\mathrm{II}_{(0,20)}2_6^{-2}4_2^{-4}$	32	$C_2^2 \times D_8$	21	5	$\mathrm{II}_{(0,21)}2^4 8_5^{-1}3^{-1}$	384	[384, 20100]
20	6	$\mathrm{II}_{(0,20)}2_6^4 8_2^{-2}$	32	$C_2^2 \times D_8$	21	5	$\mathrm{II}_{(0,21)}2^{-2}4_6^{-2}8_1^1$	512	—
20	6	$\mathrm{II}_{(0,20)}2_6^4 8_6^2$	32	[32, 34]	21	5	$\mathrm{II}_{(0,21)}2^2 4_3^3$	512	—
20	6	$\mathrm{II}_{(0,20)}2^2 4_4^4$	48	$C_2 \times S_4$	21	5	$\mathrm{II}_{(0,21)}2^{-4}8_1^1 3^1$	576	S_4^2
20	6	$\mathrm{II}_{(0,20)}2^2 4_4^4$	64	$C_2 \times C_2^2 \wr C_2$	21	5	$\mathrm{II}_{(0,21)}2^4 8_5^{-1}3^{-1}$	768	[768, 1089108]
20	6	$\mathrm{II}_{(0,20)}2^2 4_4^4$	64	$C_2 \times C_2^2 \wr C_2$	21	5	$\mathrm{II}_{(0,21)}2^{-4}4_1^1 3^1$	768	[768, 1090213]
20	6	$\mathrm{II}_{(0,20)}2^2 4_4^4$	64	$C_2 \wr C_2^2$	21	5	$\mathrm{II}_{(0,21)}2^{-2}4_7^{-3}$	1536	—
20	6	$\mathrm{II}_{(0,20)}2^2 4_3^{-3}8_5^{-1}$	64	$C_2 \wr C_2^2$	21	5	$\mathrm{II}_{(0,21)}2^2 4_3^3$	1536	—
20	6	$\mathrm{II}_{(0,20)}2^{-4}8_4^{-2}$	64	D_8^2	21	5	$\mathrm{II}_{(0,21)}2^{-4}8_3^{-1}$	4608	—
20	6	$\mathrm{II}_{(0,20)}2^{-4}4_2^2 3^1$	96	$C_2^2 \times S_4$	21	5	$\mathrm{II}_{(0,21)}2^{-2}4_7^{-3}$	8192	—
20	6	$\mathrm{II}_{(0,20)}2^{-4}4_6^2 3^{-1}$	96	$C_2 \times C_2^2 \rtimes A_4$	21	5	$\mathrm{II}_{(0,21)}2^{-4}8_3^{-1}$	24576	—
20	6	$\mathrm{II}_{(0,20)}2^{-4}4_7^1 8_5^{-1}$	128	$C_2^3 \wr C_2$					

A.1. Isometry classes of three exceptional primitive sublattices of $\mathrm{II}_{(1,25)}$

We display the Gram matrices for the exceptional primitive sublattices of $\mathbb{B}$ without (-2)-roots which could potentially embed primitively into the $\mathbb{Z}$-lattice $\Lambda_{\mathrm{OG10}} := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$. These are the only $\mathbb{Z}$-lattices represented in Table 14 for which $\mathrm{rank}(-) + l(D_-) = 25$.

$$E_{18} := \begin{pmatrix} -4 & 2 & 2 & -2 & -2 & 2 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -1 & 1 \\ 2 & -4 & -2 & 0 & 1 & -1 & 2 & -2 & -2 & -1 & -1 & 1 & 1 & -1 & -1 & -2 & -1 & -2 \\ 2 & -2 & -4 & 1 & 0 & -2 & 2 & -2 & -2 & -1 & -1 & -1 & 1 & -1 & 0 & -2 & 0 & -1 \\ -2 & 0 & 1 & -4 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 2 & 0 & -1 \\ -2 & 1 & 0 & -2 & -4 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 2 & 2 & -1 & 1 \\ 2 & -1 & -2 & 0 & 1 & -4 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 1 & -1 \\ -2 & 2 & 2 & 0 & 0 & 2 & -4 & 2 & 2 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 & 2 \\ 2 & -2 & -2 & 0 & 0 & -2 & 2 & -4 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 0 & -1 \\ 2 & -2 & -2 & 0 & 0 & -2 & 2 & -1 & -4 & 0 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 1 & 1 & 0 & -1 & 1 & 0 & -4 & -1 & -1 & 2 & -2 & 0 & -2 & -1 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 & 1 & -1 & 1 & -1 & -4 & 2 & 2 & -1 & 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 0 & -1 & 1 & -1 & -1 & 2 & -4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & 2 & 2 & 0 & -4 & 2 & -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & -2 & -1 & 0 & 2 & -4 & 1 & -1 & 0 & -1 \\ 2 & -1 & 0 & 0 & 2 & -2 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & -4 & -1 & 1 & -1 \\ 2 & -2 & -2 & 2 & 2 & -1 & 0 & -1 & -1 & -2 & -1 & 0 & 1 & -1 & -1 & -4 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 \\ 1 & -2 & -1 & -1 & 1 & -1 & 2 & -1 & -1 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -4 \end{pmatrix}$$

$$E_{20} := \begin{pmatrix} -4 & -2 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -2 & -2 & -1 & -2 & 1 & 1 & -1 & 0 \\ -2 & -4 & 1 & 2 & 1 & -2 & 1 & 1 & 1 & 1 & -1 & -2 & -1 & -2 & -1 & 0 & -1 & -1 & 0 & -1 \\ -1 & 1 & -4 & -2 & -2 & 2 & 0 & -2 & -2 & -2 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 2 & -2 & -1 \\ 1 & 2 & -2 & -4 & -2 & 2 & -1 & 0 & -2 & -2 & 1 & 2 & 0 & 0 & 0 & 1 & -1 & 2 & -1 & -1 \\ -1 & 1 & -2 & -2 & -4 & 0 & 0 & 0 & -1 & -1 & 2 & 2 & 0 & 0 & -1 & -1 & 0 & 2 & -2 & 0 \\ -1 & -2 & 2 & 2 & 0 & -4 & 1 & 2 & 2 & 2 & 1 & -1 & -1 & -1 & -2 & 0 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 0 & 1 & -4 & 0 & -1 & -1 & 1 & 1 & -2 & -1 & -1 & 0 & 1 & 1 & 0 & 1 \\ -1 & 1 & -2 & 0 & 0 & 2 & 0 & -4 & -2 & -2 & 0 & -1 & 1 & 1 & 2 & -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & -2 & -2 & -1 & 2 & -1 & -2 & -4 & -2 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & -2 & -2 & -1 & 2 & -1 & -2 & -2 & -4 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 2 & 0 & -1 \\ 1 & -1 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & -4 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -2 & 1 & 2 & 2 & -1 & 1 & -1 & 0 & 1 & 0 & -4 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & -1 & -2 & 1 & 0 & -1 & 0 & 0 & -4 & -2 & -2 & 0 & 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & -1 & -2 & -4 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & -1 & -2 & -1 & 2 & 1 & 1 & 1 & 0 & -2 & -2 & -4 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 1 & -1 & 0 & 0 & -2 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 1 & -1 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & -1 & -1 \\ 1 & -1 & 2 & 2 & 2 & -1 & 1 & 1 & 1 & 2 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & -4 & 2 & 1 \\ -1 & 0 & -2 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 2 & -4 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -4 \end{pmatrix}$$

$$
E_{21} := \begin{pmatrix}
-4 & 2 & 2 & -1 & 2 & -2 & 2 & 1 & -1 & -2 & -2 & -2 & 1 & 0 & -2 & -2 & -1 & -2 & 2 & -2 & 2 \\
2 & -4 & -2 & 2 & -1 & 0 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & -1 & 1 & 2 & 2 & 2 & -2 & 1 & -1 \\
2 & -2 & -4 & 2 & -2 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & 2 & 0 \\
-1 & 2 & 2 & -4 & 0 & -1 & -1 & -1 & -1 & -1 & -2 & -1 & -1 & 1 & -1 & -2 & -1 & -1 & 1 & -2 & -1 \\
2 & -1 & -2 & 0 & -4 & 2 & -2 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -2 & 1 & -2 \\
-2 & 0 & 1 & -1 & 2 & -4 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 1 \\
2 & 0 & 0 & -1 & -2 & 1 & -4 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 2 & 1 & -1 & 1 & -1 & 0 & -2 \\
1 & 1 & 1 & -1 & 1 & 1 & 0 & -4 & 0 & 2 & 1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
-1 & 2 & 0 & -1 & -1 & 1 & -1 & 0 & -4 & -2 & -1 & -1 & 0 & 1 & 0 & -1 & -2 & -2 & 1 & -1 & 0 \\
-2 & 1 & 0 & -1 & 0 & -1 & 0 & 2 & -2 & -4 & -2 & -1 & 0 & 0 & -1 & -1 & -2 & -2 & 2 & -2 & 1 \\
-2 & 2 & 2 & -2 & 0 & -1 & 0 & 1 & -1 & -2 & -4 & -2 & 0 & 0 & -1 & -1 & -2 & -1 & 1 & -2 & 0 \\
-2 & 2 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & -1 & -2 & -4 & -1 & 1 & -1 & -2 & -2 & -1 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & -4 & 2 & -1 & -1 & -1 & -1 & 0 & 1 & 0 \\
0 & -1 & 1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 2 & -4 & 2 & 1 & 0 & 1 & -1 & 0 & -1 \\
-2 & 1 & 0 & -1 & 1 & -1 & 2 & 0 & 0 & -1 & -1 & -1 & -1 & 2 & -4 & -1 & 0 & -2 & 1 & -1 & 2 \\
-2 & 2 & 1 & -2 & 1 & -1 & 1 & 0 & -1 & -1 & -1 & -2 & -1 & 1 & -1 & -4 & -1 & -1 & 2 & -1 & 1 \\
-1 & 2 & 0 & -1 & -1 & 0 & -1 & 1 & -2 & -2 & -2 & -2 & -1 & 0 & 0 & -1 & -4 & -1 & 1 & 0 & 0 \\
-2 & 2 & 0 & -1 & 0 & 0 & 1 & 0 & -2 & -2 & -1 & -1 & -1 & 1 & -2 & -1 & -1 & -4 & 1 & -1 & 1 \\
2 & -2 & -1 & 1 & -2 & 1 & -1 & 0 & 1 & 2 & 1 & 1 & 0 & -1 & 1 & 2 & 1 & 1 & -4 & 2 & -2 \\
-2 & 1 & 2 & -2 & 1 & -1 & 0 & 0 & -1 & -2 & -2 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 2 & -4 & 0 \\
2 & -1 & 0 & -1 & -2 & 1 & -2 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 2 & 1 & 0 & 1 & -2 & 0 & -4
\end{pmatrix}
$$

B. Finite stable symplectic subgroups for the deformation type $\mathrm{K3}^{[3]}$

The lattice-theoretic counterpart of Theorem 7.49 is presented in Table 15 (see [BMW24] for explicit data in terms of matrix representations). Each entry in this latter table corresponds to an isomorphism class of stable symplectic sublattice $C \leq \Lambda_{\mathrm{K3}^{[3]}}$. For each entry, we give:

(1) the label of the associated $\mathbb{Z}$-lattice C. Since C embeds primitively into the Leech lattice $\mathbb{L}$, the Id corresponds to the one of the associated stable symplectic sublattice of $\mathbb{L}$ as given in [HM19, Table 2]. In case where $\Lambda_{\mathrm{K3}^{[3]}}$ has several classes of primitive sublattices abstractly isometric to C (see Theorem 7.10), we add letters to distinguish each class;

(2) a description of the group $O^{\#}(C)$, or its Id in the Small Group Library [BEOH24], or its order;

(3) a symbol for the genus of the invariant sublattice $\Lambda_{\mathrm{K3}^{[3]}}^{O^{\#}(C)}$ following Conway–Sloane's convention (Section 1.3).

Moreover, in each case, the finite stable subgroup $O^{\#}(C) \leq O^{+,\#}(\Lambda_{\mathrm{K3}^{[3]}})$ is symplectic, but it may not be regular symplectic. To be able to distinguish between the regular and the nonregular cases, we also give the number of vectors of type $(-12, 2)$ and type $(-36, 4)$ in C (column "Walls"), up to sign (see Example 5.43).

Table 15: Finite stable symplectic subgroups for $\mathrm{K3}^{[3]}$

Id	$O^{\#}(C)$	Walls	$g(\Lambda_{\mathrm{K3}^{[3]}}^{O^{\#}(C)})$	Id	$O^{\#}(C)$	Walls	$g(\Lambda_{\mathrm{K3}^{[3]}}^{O^{\#}(C)})$
1	C_1	$(0,0)$	$\mathrm{II}_{(3,20)}4_7^1$	71b	$[384, 20164]$	$(312,64)$	$\mathrm{II}_{(3,1)}2^{-2}4_6^{-2}$
2	C_2	$(0,0)$	$\mathrm{II}_{(3,12)}2^{-8}4_3^{-1}$	72a	A_6	$(0,0)$	$\mathrm{II}_{(3,1)}4_2^{-2}3^25^1$
3a	C_2^2	$(0,0)$	$\mathrm{II}_{(3,8)}2^{-6}4_7^{-3}$	72b	A_6	$(380,0)$	$\mathrm{II}_{(3,1)}2_6^{-2}3^25^1$
3b	C_2^2	$(64,0)$	$\mathrm{II}_{(3,8)}2^{-4}4_7^{-3}$	73a	$A_{4,4}$	$(0,0)$	$\mathrm{II}_{(3,1)}2^24_7^18_7^13^2$
4	C_3	$(0,0)$	$\mathrm{II}_{(3,8)}4_7^13^6$	73b	$A_{4,4}$	$(304,0)$	$\mathrm{II}_{(3,1)}4_7^18_7^13^2$
5	C_2	$(32,0)$	$\mathrm{II}_{(3,8)}2_4^{10}4_7^1$	74a	H_{192}	$(0,0)$	$\mathrm{II}_{(3,1)}4_5^{-3}8_7^13^1$
6a	C_2^3	$(0,0)$	$\mathrm{II}_{(3,6)}2^64_5^3$	74b	H_{192}	$(286,0)$	$\mathrm{II}_{(3,1)}2_2^24_1^18_5^{-1}3^1$
6b	C_2^3	$(96,0)$	$\mathrm{II}_{(3,6)}2^{-4}4_1^{-3}$	75a	$[192, 1538]$	$(280,0)$	$\mathrm{II}_{(3,1)}2^24_3^{-1}8_1^13^1$
6c	C_2^3	$(112,0)$	$\mathrm{II}_{(3,6)}2_6^{-8}4_3^{-1}$	75b	$[192, 1538]$	$(240,0)$	$\mathrm{II}_{(3,1)}2^24_3^{-1}8_1^13^1$
7	S_3	$(0,0)$	$\mathrm{II}_{(3,6)}2^24_3^{-1}3^{-5}$	76a	T_{192}	$(0,0)$	$\mathrm{II}_{(3,1)}4_0^{-4}3^{-1}$
8	C_2^2	$(48,0)$	$\mathrm{II}_{(3,6)}2^64_5^3$	76b	T_{192}	$(370,0)$	$\mathrm{II}_{(3,1)}2_4^{-2}4_0^23^{-1}$
9a	C_4	$(0,0)$	$\mathrm{II}_{(3,6)}2_2^{-2}4_7^5$	76c	T_{192}	$(360,0)$	$\mathrm{II}_{(3,1)}2^24^{-2}3^{-1}$
9b	C_4	$(104,0)$	$\mathrm{II}_{(3,6)}4_5^{-5}$	76d	T_{192}	$(360,64)$	$\mathrm{II}_{(3,1)}4^{-2}3^{-1}$
10a	C_2^4	$(0,0)$	$\mathrm{II}_{(3,5)}2^64_7^18_7^1$	77a	$L_2(7)$	$(0,0)$	$\mathrm{II}_{(3,1)}4_6^27^2$
10b	C_2^4	$(160,0)$	$\mathrm{II}_{(3,5)}2^{-4}4_3^{-1}8_7^1$	77b	$L_2(7)$	$(364,0)$	$\mathrm{II}_{(3,1)}2_6^27^2$
11	C_2^4	$(112,0)$	$\mathrm{II}_{(3,5)}2^64_6^2$	79a	$[128, 1758]$	$(192,0)$	$\mathrm{II}_{(3,1)}4_3^{-3}8_3^{-1}$
12	C_2^3	$(80,0)$	$\mathrm{II}_{(3,5)}2^64_5^{-1}8_5^{-1}$	79b	$[128, 1758]$	$(262,0)$	$\mathrm{II}_{(3,1)}4_3^{-3}8_3^{-1}$
13a	D_4	$(0,0)$	$\mathrm{II}_{(3,5)}4_6^6$	81a	$[128, 1755]$	$(224,0)$	$\mathrm{II}_{(3,1)}4_1^38_1^1$

Table 15: Finite stable symplectic subgroups for $\mathrm{K3}^{[3]}$ (continued)

Id	$O^{\#}(C)$	Walls	$g(\Lambda_{\mathrm{K3}^{[3]}}^{O^{\#}(C)})$	Id	$O^{\#}(C)$	Walls	$g(\Lambda_{\mathrm{K3}^{[3]}}^{O^{\#}(C)})$
13b	D_4	$(124,0)$	$\mathrm{II}_{(3,5)}2_0^2 4_6^4$	81b	$[128,1755]$	$(260,0)$	$\mathrm{II}_{(3,1)}4_3^{-3}8_3^{-1}$
14	2_+^{1+8}	$(256,0)$	$\mathrm{II}_{(3,4)}2^{-6}4_3^{-1}$	82	S_5	$(0,0)$	$\mathrm{II}_{(3,1)}4_4^{-2}3^{-1}5^{-2}$
15	$A_{3,3}$	$(0,0)$	$\mathrm{II}_{(3,4)}4_7^1 3^4 9^1$	83	$C_2^2 \times S_4$	$(208,0)$	$\mathrm{II}_{(3,1)}2^2 4_6^2 3^2$
16a	C_2^4	$(128,0)$	$\mathrm{II}_{(3,4)}2^4 4_7^3$	84a	M_9	$(0,0)$	$\mathrm{II}_{(3,1)}2_7^3 4_5^{-1}3^1 9^1$
16b	C_2^4	$(144,0)$	$\mathrm{II}_{(3,4)}2^4 4_7^3$	84b	M_9	$(344,0)$	$\mathrm{II}_{(3,1)}2_7^1 4_5^{-1}3^1 9^1$
17a	$\Gamma_2 a_1$	$(0,0)$	$\mathrm{II}_{(3,4)}2^2 4_7^5$	85	N_{72}	$(0,0)$	$\mathrm{II}_{(3,1)}4_6^2 3^2 9^1$
17b	$\Gamma_2 a_1$	$(144,0)$	$\mathrm{II}_{(3,4)}4_7^5$	86	D_4^2	$(188,32)$	$\mathrm{II}_{(3,1)}4_2^4$
17c	$\Gamma_2 a_1$	$(128,0)$	$\mathrm{II}_{(3,4)}4_7^5$	87	T_{48}	$(0,0)$	$\mathrm{II}_{(3,1)}2_7^1 4_1^1 8^{-2}3^1$
17d	$\Gamma_2 a_1$	$(148,0)$	$\mathrm{II}_{(3,4)}2_2^4 4_5^3$	88	$C_2 \times S_4$	$(180,0)$	$\mathrm{II}_{(3,1)}4_0^{-4}3^{-1}$
18	D_6	$(0,0)$	$\mathrm{II}_{(3,4)}2^4 4_7^1 3^4$	89a	$[32,43]$	$(208,0)$	$\mathrm{II}_{(3,1)}2_3^{-1}4_3^{-1}8^2$
19a	A_4	$(0,0)$	$\mathrm{II}_{(3,4)}2^{-2}4_7^{-3}3^2$	89b	$[32,43]$	$(244,0)$	$\mathrm{II}_{(3,1)}2_3^{-1}4_7^1 8^{-2}$
19b	A_4	$(172,0)$	$\mathrm{II}_{(3,4)}4_3^3 3^2$	90	S_4	$(188,0)$	$\mathrm{II}_{(3,1)}2^{-2}4_3^{-1}8_7^1 3^2$
20	D_5	$(0,0)$	$\mathrm{II}_{(3,4)}4_7^1 5^4$	91	S_4	$(156,24)$	$\mathrm{II}_{(3,1)}4_2^4$
21	D_4	$(72,0)$	$\mathrm{II}_{(3,4)}2^{-2}4_3^{-5}$	92	S_4	$(270,0)$	$\mathrm{II}_{(3,1)}2_6^{-2}4_5^{-1}8_5^{-1}3^{-1}$
23	C_2^2	$(72,0)$	$\mathrm{II}_{(3,4)}2_6^{-2}4_5^{-5}$	94	$C_2 \times Q_8$	$(316,0)$	$\mathrm{II}_{(3,1)}2_4^{-2}4_7^1 16_3^{-1}$
24a	$C_2^2 \rtimes A_4$	$(0,0)$	$\mathrm{II}_{(3,3)}2^4 4_3^{-1}8_7^1 3^1$	95	$C_2 \times D_4$	$(196,8)$	$\mathrm{II}_{(3,1)}2_4^{-2}8_6^{-2}$
24b	$C_2^2 \rtimes A_4$	$(232,0)$	$\mathrm{II}_{(3,3)}2^2 4_1^1 8_5^{-1}3^1$	97	D_6	$(230,0)$	$\mathrm{II}_{(3,1)}2_2^2 4_0^2 3^{-2}$
25a	C_2^5	$(208,0)$	$\mathrm{II}_{(3,3)}2^4 4_1^1 8_7^1$	99	#245760	$(640,0)$	$\mathrm{II}_{(3,0)}4_3^3$
25b	C_2^5	$(192,0)$	$\mathrm{II}_{(3,3)}2^4 4_1^1 8_7^1$	100	#30720	$(576,0)$	$\mathrm{II}_{(3,0)}2^2 4_3^{-1}5^{-1}$
26a	$\Gamma_4 a_1$	$(0,0)$	$\mathrm{II}_{(3,3)}2^2 4_1^3 8_7^1$	102a	$L_3(4)$	$(0,0)$	$\mathrm{II}_{(3,0)}2^2 4_3^{-1}3^1 7^1$
26b	$\Gamma_4 a_1$	$(208,0)$	$\mathrm{II}_{(3,3)}4_1^3 8_7^1$	102b	$L_3(4)$	$(0,0)$	$\mathrm{II}_{(3,0)}2^{-2}4_7^1 3^1 7^1$
26c	$\Gamma_4 a_1$	$(214,0)$	$\mathrm{II}_{(3,3)}2_2^{-4}4_1^1 8_5^{-1}$	102c	$L_3(4)$	$(560,0)$	$\mathrm{II}_{(3,0)}4_3^{-1}3^1 7^1$
27a	$C_2 \times \Gamma_2 a_1$	$(144,0)$	$\mathrm{II}_{(3,3)}2^2 4_0^4$	102d	$L_3(4)$	$(560,0)$	$\mathrm{II}_{(3,0)}4_3^{-1}3^1 7^1$
27b	$C_2 \times \Gamma_2 a_1$	$(168,0)$	$\mathrm{II}_{(3,3)}2^{-2}4_4^{-4}$	103	#12288	$(480,0)$	$\mathrm{II}_{(3,0)}4_4^{-2}8_3^{-1}$
28a	2_+^{1+4}	$(0,0)$	$\mathrm{II}_{(3,3)}4_0^6$	104	#9216	$(448,0)$	$\mathrm{II}_{(3,0)}2^2 4_7^1 3^2$
28b	2_+^{1+4}	$(172,0)$	$\mathrm{II}_{(3,3)}2_4^{-2}4_0^{-4}$	105	#6144	$(416,0)$	$\mathrm{II}_{(3,0)}4_5^{-3}3^1$
28c	2_+^{1+4}	$(168,0)$	$\mathrm{II}_{(3,3)}2^2 4^4$	106a	$C_2^4 \rtimes A_6$	$(0,0)$	$\mathrm{II}_{(3,0)}4_0^2 8_5^{-1}3^{-1}$
28d	2_+^{1+4}	$(168,16)$	$\mathrm{II}_{(3,3)}4^4$	106b	$C_2^4 \rtimes A_6$	$(520,0)$	$\mathrm{II}_{(3,0)}2^2 8_5^{-1}3^{-1}$
29a	S_4	$(0,0)$	$\mathrm{II}_{(3,3)}4_4^4 3^2$	106c	$C_2^4 \rtimes A_6$	$(530,0)$	$\mathrm{II}_{(3,0)}2_2^2 8_3^{-1}3^{-1}$
29b	S_4	$(196,0)$	$\mathrm{II}_{(3,3)}2_4^{-2}4_4^{-2}3^2$	106d	$C_2^4 \rtimes A_6$	$(520,80)$	$\mathrm{II}_{(3,0)}8_5^{-1}3^{-1}$
30	D_4	$(104,0)$	$\mathrm{II}_{(3,3)}2_0^2 4_1^3 8_7^1$	108a	A_7	$(0,0)$	$\mathrm{II}_{(3,0)}4_7^1 3^{-1}5^1 7^{-1}$
31a	Q_8	$(0,0)$	$\mathrm{II}_{(3,3)}2_1^{-3}4_7^1 8^{-2}$	108b	A_7	$(0,0)$	$\mathrm{II}_{(3,0)}4_7^1 3^{-1}5^1 7^{-1}$
31b	Q_8	$(200,0)$	$\mathrm{II}_{(3,3)}2_7^1 4_5^{-1}8^{-2}$	108c	A_7	$(0,0)$	$\mathrm{II}_{(3,0)}4_7^1 3^{-1}5^1 7^{-1}$
32	C_2^3	$(84,0)$	$\mathrm{II}_{(3,3)}4_0^6$	110a	$[1920,240993]$	$(0,0)$	$\mathrm{II}_{(3,0)}4_2^{-2}8_1^1 5^{-1}$

Table 15: Finite stable symplectic subgroups for $\mathrm{K3}^{[3]}$ (continued)

Id	$O^{\#}(C)$	Walls	$g(\Lambda_{\mathrm{K3}^{[3]}}^{O^{\#}(C)})$	Id	$O^{\#}(C)$	Walls	$g(\Lambda_{\mathrm{K3}^{[3]}}^{O^{\#}(C)})$
33	#1536	$(352,0)$	$\mathrm{II}_{(3,2)}2^4 4_3^{-1}3^1$	110b	$[1920,240993]$	$(0,0)$	$\mathrm{II}_{(3,0)}4_2^{-2}8_1^1 5^{-1}$
34	#1024	$(320,0)$	$\mathrm{II}_{(3,2)}2^2 4_1^3$	110c	$[1920,240993]$	$(0,0)$	$\mathrm{II}_{(3,0)}4_2^{-2}8_1^1 5^{-1}$
36a	$[192,1541]$	$(304,0)$	$\mathrm{II}_{(3,2)}2^2 4_1^3$	110d	$[1920,240993]$	$(430,0)$	$\mathrm{II}_{(3,0)}2_6^{-2}8_1^1 5^{-1}$
36b	$[192,1541]$	$(288,64)$	$\mathrm{II}_{(3,2)}2^4 4_1^1$	110e	$[1920,240993]$	$(430,0)$	$\mathrm{II}_{(3,0)}2_6^{-2}8_1^1 5^{-1}$
37a	$C_4^2 \rtimes A_4$	$(0,0)$	$\mathrm{II}_{(3,2)}2^2 4_3^{-1}8_2^{-2}$	111a	$[1344,11686]$	$(0,0)$	$\mathrm{II}_{(3,0)}4_5^3 7^{-1}$
37b	$C_4^2 \rtimes A_4$	$(304,0)$	$\mathrm{II}_{(3,2)}4_1^1 8_0^2$	111b	$[1344,11686]$	$(490,0)$	$\mathrm{II}_{(3,0)}2_6^{-2}4_3^{-1}7^{-1}$
38a	$C_2^2 \rtimes S_4$	$(0,0)$	$\mathrm{II}_{(3,2)}2^2 4_4^{-2}8_7^1 3^1$	112	$[1152,155478]$	$(0,0)$	$\mathrm{II}_{(3,0)}4_7^1 8_2^{-2}3^1$
38b	$C_2^2 \rtimes S_4$	$(256,0)$	$\mathrm{II}_{(3,2)}4_4^{-2}8_7^1 3^1$	113a	$[1152,157862]$	$(376,0)$	$\mathrm{II}_{(3,0)}4_5^{-3}3^1$
38c	$C_2^2 \rtimes S_4$	$(262,0)$	$\mathrm{II}_{(3,2)}2_2^4 8_5^{-1}3^1$	113b	$[1152,157862]$	$(360,64)$	$\mathrm{II}_{(3,0)}2^2 4_5^{-1}3^1$
39	$A_{4,3}$	$(0,0)$	$\mathrm{II}_{(3,2)}4_7^{-3}3^3$	115a	$[768,1090134]$	$(288,0)$	$\mathrm{II}_{(3,0)}4_3^{-1}8_4^{-2}$
40a	$C_2 \times 2_+^{1+4}$	$(160,0)$	$\mathrm{II}_{(3,2)}4_1^5$	115b	$[768,1090134]$	$(358,0)$	$\mathrm{II}_{(3,0)}4_3^{-1}8_4^{-2}$
40b	$C_2 \times 2_+^{1+4}$	$(196,0)$	$\mathrm{II}_{(3,2)}4_1^5$	117	$[768,1090070]$	$(336,64)$	$\mathrm{II}_{(3,0)}4_3^3$
42a	$[64,202]$	$(208,0)$	$\mathrm{II}_{(3,2)}2^2 4_2^{-2}8_3^{-1}$	118a	S_6	$(0,0)$	$\mathrm{II}_{(3,0)}2^2 4_3^{-1}3^2 5^1$
42b	$[64,202]$	$(232,0)$	$\mathrm{II}_{(3,2)}2^2 4_2^{-2}8_3^{-1}$	118b	S_6	$(0,0)$	$\mathrm{II}_{(3,0)}2^2 4_3^{-1}3^2 5^1$
43a	$\Gamma_{25}a_1$	$(0,0)$	$\mathrm{II}_{(3,2)}4_0^4 8_1^1$	118c	S_6	$(360,0)$	$\mathrm{II}_{(3,0)}4_3^{-1}3^2 5^1$
43b	$\Gamma_{25}a_1$	$(232,0)$	$\mathrm{II}_{(3,2)}2^2 4^2 8_1^1$	118d	S_6	$(360,0)$	$\mathrm{II}_{(3,0)}4_3^{-1}3^2 5^1$
43c	$\Gamma_{25}a_1$	$(236,0)$	$\mathrm{II}_{(3,2)}2_2^2 4^2 8_7^1$	119a	M_{10}	$(0,0)$	$\mathrm{II}_{(3,0)}2_1^1 4_0^2 3^1 5^1$
43d	$\Gamma_{25}a_1$	$(238,0)$	$\mathrm{II}_{(3,2)}2_2^{-2}4_4^{-2}8_7^1$	119b	M_{10}	$(0,0)$	$\mathrm{II}_{(3,0)}2_7^1 4_2^2 3^1 5^1$
43e	$\Gamma_{25}a_1$	$(232,32)$	$\mathrm{II}_{(3,2)}4^2 8_1^1$	119c	M_{10}	$(0,0)$	$\mathrm{II}_{(3,0)}2_1^1 4_0^2 3^1 5^1$
44a	A_5	$(0,0)$	$\mathrm{II}_{(3,2)}2^{-2}4_7^1 3^{-1}5^{-2}$	119d	M_{10}	$(470,0)$	$\mathrm{II}_{(3,0)}2_1^3 3^1 5^1$
44b	A_5	$(280,0)$	$\mathrm{II}_{(3,2)}4_3^{-1}3^{-1}5^{-2}$	119e	M_{10}	$(470,0)$	$\mathrm{II}_{(3,0)}2_1^3 3^1 5^1$
45a	$C_2 \times S_4$	$(0,0)$	$\mathrm{II}_{(3,2)}2^2 4_5^3 3^2$	119f	M_{10}	$(470,72)$	$\mathrm{II}_{(3,0)}2_1^1 3^1 5^1$
45b	$C_2 \times S_4$	$(192,0)$	$\mathrm{II}_{(3,2)}4_5^3 3^2$	120a	$L_2(11)$	$(0,0)$	$\mathrm{II}_{(3,0)}4_7^1 11^2$
45c	$C_2 \times S_4$	$(220,0)$	$\mathrm{II}_{(3,2)}2_4^4 4_1^1 3^2$	120b	$L_2(11)$	$(0,0)$	$\mathrm{II}_{(3,0)}4_7^1 11^2$
46a	$[36,9]$	$(0,0)$	$\mathrm{II}_{(3,2)}2_6^{-2}4_3^{-1}3^2 9^1$	121a	$[576,8654]$	$(0,0)$	$\mathrm{II}_{(3,0)}4_0^2 8_7^1 3^2$
46b	$[36,9]$	$(272,0)$	$\mathrm{II}_{(3,2)}4_5^{-1}3^2 9^1$	121b	$[576,8654]$	$(0,0)$	$\mathrm{II}_{(3,0)}4_0^2 8_7^1 3^2$
47	S_3^2	$(0,0)$	$\mathrm{II}_{(3,2)}2^2 4_3^{-1}3^{-3}9^1$	121c	$[576,8654]$	$(334,0)$	$\mathrm{II}_{(3,0)}2_2^{-2}8_5^{-1}3^2$
48	$\Gamma_5 a_2$	$(168,0)$	$\mathrm{II}_{(3,2)}4_1^5$	122	$[500,23]$	$(0,0)$	$\mathrm{II}_{(3,0)}4_7^1 5^3$
49a	$[32,27]$	$(144,0)$	$\mathrm{II}_{(3,2)}4_1^5$	124	$[384,18134]$	$(456,0)$	$\mathrm{II}_{(3,0)}2_1^1 4_5^{-1}16_5^{-1}$
49b	$[32,27]$	$(180,0)$	$\mathrm{II}_{(3,2)}4_1^5$	125a	$[384,17948]$	$(256,0)$	$\mathrm{II}_{(3,0)}4_4^{-2}8_1^1 3^1$
49c	$[32,27]$	$(168,32)$	$\mathrm{II}_{(3,2)}2^{-2}4_5^{-3}$	125b	$[384,17948]$	$(256,0)$	$\mathrm{II}_{(3,0)}4_4^{-2}8_1^1 3^1$
50	$C_2^2 \times S_3$	$(112,0)$	$\mathrm{II}_{(3,2)}2^4 4_3^{-1}3^{-3}$	126a	$[384,20089]$	$(352,0)$	$\mathrm{II}_{(3,0)}4_1^{-3}3^{-1}$
51a	S_4	$(168,0)$	$\mathrm{II}_{(3,2)}2^{-2}4_3^3 3^{-1}$	126b	$[384,20089]$	$(388,0)$	$\mathrm{II}_{(3,0)}4_1^{-3}3^{-1}$
51b	S_4	$(190,0)$	$\mathrm{II}_{(3,2)}2^2 4_7^{-3}3^{-1}$	127	A_6	$(430,0)$	$\mathrm{II}_{(3,0)}2^2 4_7^1 3^2$

Table 15: Finite stable symplectic subgroups for $K3^{[3]}$ (continued)

Id	$O^{\#}(C)$	Walls	$g(\Lambda_{K3^{[3]}}^{O^{\#}(C)})$	Id	$O^{\#}(C)$	Walls	$g(\Lambda_{K3^{[3]}}^{O^{\#}(C)})$
52	F_{21}	$(0,0)$	$II_{(3,2)}4_7^1 7^{-3}$	128a	$\Gamma L_2(\mathbb{F}_4)$	$(0,0)$	$II_{(3,0)}4_7^1 3^{-2}5^{-2}$
53	F_5	$(0,0)$	$II_{(3,2)}2_4^{-2}4_5^{-1}5^3$	128b	$\Gamma L_2(\mathbb{F}_4)$	$(0,0)$	$II_{(3,0)}4_7^1 3^{-2}5^{-2}$
54	$C_2 \times D_4$	$(116,0)$	$II_{(3,2)}4_2^{-4}8_3^{-1}$	129a	$C_2 \times L_2(7)$	$(0,0)$	$II_{(3,0)}2^2 4_7^1 7^2$
55a	QD_{16}	$(0,0)$	$II_{(3,2)}2_1^1 4_4^{-2}8^{-2}$	129b	$C_2 \times L_2(7)$	$(0,0)$	$II_{(3,0)}2^2 4_7^1 7^2$
55b	QD_{16}	$(222,0)$	$II_{(3,2)}2_1^3 8^2$	131	$[192, 1494]$	$(312,0)$	$II_{(3,0)}4_3^{-1}8_4^{-2}$
55c	QD_{16}	$(222,40)$	$II_{(3,2)}2_1^1 8^2$	132	T_{192}	$(312,48)$	$II_{(3,0)}4_3^3$
56	A_4	$(144,0)$	$II_{(3,2)}4_1^5$	133a	$C_2 \times M_9$	$(428,0)$	$II_{(3,0)}2_2^2 4_1^1 9^1$
57	C_2^3	$(108,16)$	$II_{(3,2)}4_1^5$	133b	$C_2 \times M_9$	$(428,0)$	$II_{(3,0)}2_4^{-2}4_3^{-1}9^1$
58	C_2^3	$(146,0)$	$II_{(3,2)}2_0^2 4^2 8_1^1$	134a	$A\Gamma L_1(\mathbb{F}_9)$	$(0,0)$	$II_{(3,0)}2_1^1 4_4^{-2}3^1 9^1$
59a	$C_2 \times C_4$	$(172,0)$	$II_{(3,2)}2_2^2 4_5^{-1}8_6^{-2}$	134b	$A\Gamma L_1(\mathbb{F}_9)$	$(0,0)$	$II_{(3,0)}2_1^1 4_4^{-2}3^1 9^1$
59b	$C_2 \times C_4$	$(136,0)$	$II_{(3,2)}2_6^{-2}4_7^1 8_4^{-2}$	135a	$S_3 \times S_4$	$(204,0)$	$II_{(3,0)}4_3^3 3^{-2}$
60	S_3	$(210,0)$	$II_{(3,2)}2_2^4 4_7^1 3^{-2}$	135b	$S_3 \times S_4$	$(204,0)$	$II_{(3,0)}4_3^3 3^{-2}$
63	C_6	$(140,0)$	$II_{(3,2)}2_2^{-4}4_1^1 3^3$	136	S_5	$(298,0)$	$II_{(3,0)}2^2 4_7^1 5^{-2}$
64	C_4	$(84,16)$	$II_{(3,2)}4_1^5$	137a	S_5	$(350,0)$	$II_{(3,0)}2_0^2 4_1^1 3^{-1}5^{-1}$
65	#6144	$(448,0)$	$II_{(3,1)}2^2 4_3^{-1}8_3^{-1}$	137b	S_5	$(350,0)$	$II_{(3,0)}2_0^2 4_1^1 3^{-1}5^{-1}$
66	#3072	$(384,0)$	$II_{(3,1)}2^2 4_4^{-2}3^1$	139	$[96, 195]$	$(252,0)$	$II_{(3,0)}4_7^3 3^2$
67	#2048	$(352,0)$	$II_{(3,1)}4_2^4$	143	$[64, 257]$	$(270,16)$	$II_{(3,0)}2_3^{-1}8^{-2}$
69a	M_{20}	$(0,0)$	$II_{(3,1)}2^2 4_3^{-1}8_7^1 5^{-1}$	146	$C_2 \times S_4$	$(290,0)$	$II_{(3,0)}4_0^2 8_5^{-1}3^{-1}$
69b	M_{20}	$(400,0)$	$II_{(3,1)}4_1^1 8_5^{-1}5^{-1}$	148	$C_2 \times S_4$	$(296,24)$	$II_{(3,0)}2_4^{-2}8_1^1 3^{-1}$
70a	F_{384}	$(0,0)$	$II_{(3,1)}4_4^{-2}8_2^{-2}$	149a	$C_2 \times F_5$	$(232,0)$	$II_{(3,0)}2_4^{-2}4_3^{-1}5^2$
70b	F_{384}	$(334,0)$	$II_{(3,1)}2_2^2 8_0^2$	149b	$C_2 \times F_5$	$(232,0)$	$II_{(3,0)}2_4^{-2}4_3^{-1}5^2$
70c	F_{384}	$(328,0)$	$II_{(3,1)}2^2 8_2^2$	152	$C_2 \times QD_{16}$	$(340,40)$	$II_{(3,0)}2_4^{-2}16_3^{-1}$
70d	F_{384}	$(328,48)$	$II_{(3,1)}8_2^2$	160	$C_2^2 \rtimes C_4$	$(174,24)$	$II_{(3,0)}4_3^{-1}8_4^{-2}$
71a	$[384, 20164]$	$(328,0)$	$II_{(3,1)}4_2^4$				

C. Finite stable symplectic subgroups for the deformation type OG10

The lattice-theoretic counterpart of Theorem 7.54 is presented in Table 16 (see [MM25c] for explicit data in terms of matrix representations). Each entry in this table corresponds to an isomorphism class of stable symplectic sublattice $C \leq \Lambda_{\mathrm{OG10}}$. For each entry we give

(1) the label of the associated $\mathbb{Z}$-lattice C in Höhn–Mason database [HM19, Table 2]. If C embeds primitively into the Leech lattice $\mathbb{L}$, the Id corresponds to the one of the associated primitive sublattice of $\mathbb{L}$ as given in [HM19, Table 2]. Otherwise, the Id corresponds to the name of the associated $\mathbb{Z}$-lattice in Appendix A.1. In case where Λ_{OG10} has several classes of primitive sublattices abstractly isometric to C (see Theorem 7.10), we add letters to distinguish each class;

(2) a description of the group $O^{\#}(C)$, or its Id in the Small Group Library [BEOH24], or its order;

(3) a symbol for the genus of the invariant sublattice $\Lambda_{\mathrm{OG10}}^{O^{\#}(C)}$ following Conway–Sloane's convention (Section 1.3).

Remark C.1. For an IHS manifold $X \sim \mathrm{OG10}$, the wall divisors of X which are not stably prime exceptional correspond to vectors in $\mathrm{NS}(X)$ which are of numerical type $(-4, 1)$ or $(-24, 3)$ (Example 5.46). Excluding the *ambiguous* pairs

$$\{(194\mathrm{a}, 194\mathrm{b}), (200\mathrm{b}, 200\mathrm{c}), (203\mathrm{a}, 203\mathrm{b}), (208\mathrm{b}, 208\mathrm{c})\}$$

all entries in [MM25c] are uniquely determined by

(1) the isometry class of the stable symplectic sublattice $C \leq \Lambda_{\mathrm{OG10}}$;

(2) the isometry class of the orthogonal complement of C in Λ_{OG10};

(3) the number of vectors of norm -4 in C, up to sign;

(4) the number of vectors of norm -24 in C which have divisibility 3 in Λ_{OG10}, up to sign.

Table 16: Finite stable symplectic subgroups for OG10

Id	$O^{\#}(C)$	$g(\Lambda_{\mathrm{OG10}}^{O^{\#}(C)})$	Saturated	Id	$O^{\#}(C)$	$g(\Lambda_{\mathrm{OG10}}^{O^{\#}(C)})$	Saturated
1	C_1	$\mathrm{II}_{(3,21)}3^1$	true	108b	A_7	$\mathrm{II}_{(3,1)}3^{-2}5^1 7^{-1}$	true
2	C_2	$\mathrm{II}_{(3,13)}2^8 3^1$	true	109a	$[1944, 3559]$	$\mathrm{II}_{(3,1)}2_6^2 3^2$	false
3	C_2^2	$\mathrm{II}_{(3,9)}2^{-6}4^{-2}3^1$	true	109b	$[1944, 3559]$	$\mathrm{II}_{(3,1)}2_6^2 3^{-4}$	true
4a	C_3	$\mathrm{II}_{(3,9)}3^{-5}$	false	110	$[1920, 240993]$	$\mathrm{II}_{(3,1)}4_3^{-1}8_1^1 3^1 5^{-1}$	true
4b	C_3	$\mathrm{II}_{(3,9)}3^7$	true	111	$[1344, 11686]$	$\mathrm{II}_{(3,1)}4_6^2 3^1 7^{-1}$	true
5	C_2	$\mathrm{II}_{(3,9)}2_0^{-12}3^1$	true	112	$[1152, 155478]$	$\mathrm{II}_{(3,1)}8_2^{-2}3^2$	true
6	C_2^3	$\mathrm{II}_{(3,7)}2^6 4_6^2 3^1$	true	114	$[972, 812]$	$\mathrm{II}_{(3,1)}2^{-2}3^{-3}$	false
7a	S_3	$\mathrm{II}_{(3,7)}2^{-2}3^4$	false	116	$[768, 1090135]$	$\mathrm{II}_{(3,1)}2_2^2 8_6^{-2}3^1$	true
7b	S_3	$\mathrm{II}_{(3,7)}2^{-2}3^{-6}$	true	118a	S_6	$\mathrm{II}_{(3,1)}2^{-2}3^{-1}5^1$	false

Table 16: Finite stable symplectic subgroups for OG10 (continued)

Id	$O^{\#}(C)$	$g(\Lambda_{\mathrm{OG10}}^{O^{\#}(C)})$	Saturated	Id	$O^{\#}(C)$	$g(\Lambda_{\mathrm{OG10}}^{O^{\#}(C)})$	Saturated
9	$[4,1]$	$\mathrm{II}_{(3,7)}2_2^{-2}4^4 3^1$	true	118b	S_6	$\mathrm{II}_{(3,1)}2^{-2}3^3 5^1$	true
10	C_2^4	$\mathrm{II}_{(3,6)}2^6 8_7^1 3^1$	true	119	M_{10}	$\mathrm{II}_{(3,1)}2_5^{-1}4_1^1 3^2 5^1$	true
13	D_8	$\mathrm{II}_{(3,6)}4_7^5 3^1$	true	120	$L_2(11)$	$\mathrm{II}_{(3,1)}3^1 11^2$	true
15a	$A_{3,3}$	$\mathrm{II}_{(3,5)}3^{-3}9^1$	false	121a	$[576,8654]$	$\mathrm{II}_{(3,1)}4_1^1 8_7^1 3^{-1}$	false
15b	$A_{3,3}$	$\mathrm{II}_{(3,5)}3^5 9^1$	true	121b	$[576,8654]$	$\mathrm{II}_{(3,1)}4_1^1 8_7^1 3^3$	true
17	$C_2 \times D_8$	$\mathrm{II}_{(3,5)}2^2 4_0^4 3^1$	true	122	$[500,23]$	$\mathrm{II}_{(3,1)}3^1 5^3$	true
18a	D_{12}	$\mathrm{II}_{(3,5)}2^4 3^{-3}$	false	123a	$[384,20097]$	$\mathrm{II}_{(3,1)}2_6^{-2}4^{-2}$	false
18b	D_{12}	$\mathrm{II}_{(3,5)}2^4 3^5$	true	123b	$[384,20097]$	$\mathrm{II}_{(3,1)}2_6^{-2}4^{-2}3^{-2}$	true
19a	A_4	$\mathrm{II}_{(3,5)}2^{-2}4^{-2}3^{-1}$	false	124	$[384,18134]$	$\mathrm{II}_{(3,1)}2_7^{-3}16_1^1 3^1$	true
19b	A_4	$\mathrm{II}_{(3,5)}2^{-2}4^{-2}3^3$	true	128a	$\Gamma\mathrm{L}_2(\mathbb{F}_4)$	$\mathrm{II}_{(3,1)}3^1 5^{-2}$	false
20	D_{10}	$\mathrm{II}_{(3,5)}3^1 5^4$	true	128b	$\Gamma\mathrm{L}_2(\mathbb{F}_4)$	$\mathrm{II}_{(3,1)}3^{-3}5^{-2}$	true
22	S_3	$\mathrm{II}_{(3,5)}3^{-7}$	true	129	$C_2 \times L_3(2)$	$\mathrm{II}_{(3,1)}2^2 3^1 7^2$	true
23	C_2^2	$\mathrm{II}_{(3,5)}2_6^{-4}4_6^4 3^1$	true	131	$[192,1494]$	$\mathrm{II}_{(3,1)}2_2^2 8_6^{-2}3^1$	true
24	$C_2^2 \rtimes A_4$	$\mathrm{II}_{(3,4)}2^{-4}8_7^1 3^2$	true	133	$C_2 \times M_9$	$\mathrm{II}_{(3,1)}2_0^{-4}3^1 9^1$	true
26	$C_2^2 \wr C_2$	$\mathrm{II}_{(3,4)}2^2 4_2^2 8_7^1 3^1$	true	134	$\Gamma\mathrm{L}_1(\mathbb{F}_9)$	$\mathrm{II}_{(3,1)}2_7^1 4_7^1 3^2 9^1$	true
28	2_+^{1+4}	$\mathrm{II}_{(3,4)}4_1^5 3^1$	true	137a	S_5	$\mathrm{II}_{(3,1)}2_6^{-4}5^{-1}$	false
29a	S_4	$\mathrm{II}_{(3,4)}4_5^3 3^{-1}$	false	137b	S_5	$\mathrm{II}_{(3,1)}2_6^{-4}3^{-2}5^{-1}$	true
29b	S_4	$\mathrm{II}_{(3,4)}4_5^3 3^3$	true	138	$[108,17]$	$\mathrm{II}_{(3,1)}3^{-1}9^2$	true
30	D_8	$\mathrm{II}_{(3,4)}2_2^4 4^{-2}8_7^1 3^1$	true	141a	N_{72}	$\mathrm{II}_{(3,1)}2_2^4 9^1$	false
31	Q_8	$\mathrm{II}_{(3,4)}2_5^3 8^{-2}3^1$	true	141b	N_{72}	$\mathrm{II}_{(3,1)}2_2^4 3^{-2}9^1$	true
35a	$[486,249]$	$\mathrm{II}_{(3,3)}3^4$	false	143	$[64,257]$	$\mathrm{II}_{(3,1)}2_1^1 4_3^{-1}8^2 3^1$	true
35b	$[486,249]$	$\mathrm{II}_{(3,3)}3^{-6}$	true	144	A_5	$\mathrm{II}_{(3,1)}2^{-2}3^{-3}$	true
37	$C_4^2 \rtimes A_4$	$\mathrm{II}_{(3,3)}2^{-2}8_2^{-2}3^1$	true	148a	$C_2 \times S_4$	$\mathrm{II}_{(3,1)}2_2^{-2}4_3^{-1}8_1^1$	false
38	$C_2^2 \rtimes S_4$	$\mathrm{II}_{(3,3)}2^2 4_5^{-1}8_7^1 3^2$	true	148b	$C_2 \times S_4$	$\mathrm{II}_{(3,1)}2_0^2 4_1^1 8_1^1 3^{-2}$	true
39a	$A_{4,3}$	$\mathrm{II}_{(3,3)}4^{-2}3^{-2}$	false	149	$C_2 \times F_5$	$\mathrm{II}_{(3,1)}2_0^{-4}3^1 5^2$	true
39b	$A_{4,3}$	$\mathrm{II}_{(3,3)}4^{-2}3^4$	true	150	$[36,13]$	$\mathrm{II}_{(3,1)}2_0^{-4}3^{-3}$	false
41	$[64,266]$	$\mathrm{II}_{(3,3)}2_6^{-2}4^4 3^1$	true	151a	S_3^2	$\mathrm{II}_{(3,1)}2_6^4 3^2$	false
43	$\Gamma_{25}a_1$	$\mathrm{II}_{(3,3)}4_1^3 8_1^1 3^1$	true	151b	S_3^2	$\mathrm{II}_{(3,1)}2_6^4 3^2$	false
44a	A_5	$\mathrm{II}_{(3,3)}2^{-2}5^{-2}$	false	151c	S_3^2	$\mathrm{II}_{(3,1)}2_6^4 3^{-4}$	true
44b	A_5	$\mathrm{II}_{(3,3)}2^{-2}3^{-2}5^{-2}$	true	152	$C_2 \times QD_{16}$	$\mathrm{II}_{(3,1)}2_0^2 4_5^{-1}16_7^1 3^1$	true
45a	$C_2 \times S_4$	$\mathrm{II}_{(3,3)}2^{-2}4_2^{-2}3^{-1}$	false	154a	$C_2^2 \times S_3$	$\mathrm{II}_{(3,1)}2_0^2 4_4^{-2}3^1$	false
45b	$C_2 \times S_4$	$\mathrm{II}_{(3,3)}2^{-2}4_2^{-2}3^3$	true	154b	$C_2^2 \times S_3$	$\mathrm{II}_{(3,1)}2_0^2 4_4^{-2}3^{-3}$	true
46a	$C_3^2 \rtimes C_4$	$\mathrm{II}_{(3,3)}2_2^{-2}3^{-1}9^1$	false	157a	D_{24}	$\mathrm{II}_{(3,1)}2_2^{-2}4^{-2}3^2$	false
46b	$C_3^2 \rtimes C_4$	$\mathrm{II}_{(3,3)}2_2^{-2}3^3 9^1$	true	157b	D_{24}	$\mathrm{II}_{(3,1)}2_2^{-2}4^{-2}3^{-4}$	true

Table 16: Finite stable symplectic subgroups for OG10 (continued)

Id	$O^{\#}(C)$	$g(\Lambda_{\mathrm{OG10}}^{O^{\#}(C)})$	Saturated	Id	$O^{\#}(C)$	$g(\Lambda_{\mathrm{OG10}}^{O^{\#}(C)})$	Saturated
47a	S_3^2	$\mathrm{II}_{(3,3)}2^{-2}3^29^1$	false	161	D_{12}	$\mathrm{II}_{(3,1)}2_0^{-4}3^{-3}$	true
47b	S_3^2	$\mathrm{II}_{(3,3)}2^{-2}3^29^1$	false	163a	$PSU(4,3)$	$\mathrm{II}_{(3,0)}4_1^13^{-1}$	false
47c	S_3^2	$\mathrm{II}_{(3,3)}2^{-2}3^{-4}9^1$	true	163b	$PSU(4,3)$	$\mathrm{II}_{(3,0)}4_1^13^3$	true
48	$[32,34]$	$\mathrm{II}_{(3,3)}2_6^{-2}4^43^1$	true	165a	M_{22}	$\mathrm{II}_{(3,0)}4_3^{-1}3^111^{-1}$	true
52	$C_7 \rtimes C_3$	$\mathrm{II}_{(3,3)}3^17^{-3}$	true	165b	M_{22}	$\mathrm{II}_{(3,0)}4_3^{-1}3^111^{-1}$	true
53	F_5	$\mathrm{II}_{(3,3)}2_6^23^15^3$	true	167a	$PSU(3,5)$	$\mathrm{II}_{(3,0)}2_5^{-1}3^15^{-2}$	true
55	QD_{16}	$\mathrm{II}_{(3,3)}2_5^{-1}4_5^{-1}8^{-2}3^1$	true	167b	$PSU(3,5)$	$\mathrm{II}_{(3,0)}2_5^{-1}3^15^{-2}$	true
56	A_4	$\mathrm{II}_{(3,3)}2_2^24^{-4}3^1$	true	169	#58320	$\mathrm{II}_{(3,0)}2_7^13^29^{-1}$	true
58	C_2^3	$\mathrm{II}_{(3,3)}2_4^{-2}4_3^{-3}8_3^{-1}3^1$	true	170	#40320	$\mathrm{II}_{(3,0)}4_5^{-1}3^27^1$	true
59	$C_2 \times C_4$	$\mathrm{II}_{(3,3)}2_0^{-4}8_6^23^1$	true	171a	#40320	$\mathrm{II}_{(3,0)}2_7^{-3}3^17^1$	true
60a	S_3	$\mathrm{II}_{(3,3)}2_6^{-6}3^1$	false	171b	#40320	$\mathrm{II}_{(3,0)}2_7^{-3}3^17^1$	true
60b	S_3	$\mathrm{II}_{(3,3)}2_6^{-6}3^{-3}$	true	172	#40320	$\mathrm{II}_{(3,0)}8_7^13^17^{-1}$	true
61a	S_3	$\mathrm{II}_{(3,3)}2_2^{-6}3^3$	false	175a	A_8	$\mathrm{II}_{(3,0)}4_7^15^1$	false
61b	S_3	$\mathrm{II}_{(3,3)}2_2^{-6}3^{-5}$	true	175b	A_8	$\mathrm{II}_{(3,0)}4_7^13^{-2}5^1$	true
63a	C_6	$\mathrm{II}_{(3,3)}2_0^63^{-2}$	false	178a	#11520	$\mathrm{II}_{(3,0)}2_2^28_1^1$	false
63b	C_6	$\mathrm{II}_{(3,3)}2_0^63^4$	true	178b	#11520	$\mathrm{II}_{(3,0)}2_6^{-2}8_5^{-1}3^{-2}$	true
68a	$[972,776]$	$\mathrm{II}_{(3,2)}2_7^13^3$	false	180	#10752	$\mathrm{II}_{(3,0)}2_2^216_3^{-1}3^1$	true
68b	$[972,776]$	$\mathrm{II}_{(3,2)}2_7^13^{-5}$	true	182	M_{11}	$\mathrm{II}_{(3,0)}2_5^{-1}3^211^1$	true
69	M_{20}	$\mathrm{II}_{(3,2)}2^{-2}8_7^13^15^{-1}$	true	183	#5760	$\mathrm{II}_{(3,0)}8_7^13^25^{-1}$	true
70	F_{384}	$\mathrm{II}_{(3,2)}4_5^{-1}8_2^{-2}3^1$	true	184	#4608	$\mathrm{II}_{(3,0)}2_1^18^{-2}3^1$	true
72a	A_6	$\mathrm{II}_{(3,2)}4_3^{-1}3^{-1}5^1$	false	186	#3888	$\mathrm{II}_{(3,0)}4_7^13^2$	false
72b	A_6	$\mathrm{II}_{(3,2)}4_3^{-1}3^35^1$	true	187a	#3888	$\mathrm{II}_{(3,0)}2_1^{-3}3^1$	false
73a	$A_{4,4}$	$\mathrm{II}_{(3,2)}2^28_7^13^{-1}$	false	187b	#3888	$\mathrm{II}_{(3,0)}2_1^{-3}3^{-3}$	true
73b	$A_{4,4}$	$\mathrm{II}_{(3,2)}2^28_7^13^3$	true	191	$[1944,3536]$	$\mathrm{II}_{(3,0)}2_3^33^{-2}$	false
74	H_{192}	$\mathrm{II}_{(3,2)}4_6^{-2}8_7^13^2$	true	193a	$[1440,5844]$	$\mathrm{II}_{(3,0)}2_0^24_1^13^15^1$	true
76a	T_{192}	$\mathrm{II}_{(3,2)}4_1^{-3}$	false	193b	$[1440,5844]$	$\mathrm{II}_{(3,0)}2_0^24_1^13^15^1$	true
76b	T_{192}	$\mathrm{II}_{(3,2)}4_1^{-3}3^{-2}$	true	194a	$[1440,5841]$	$\mathrm{II}_{(3,0)}2_7^{-3}3^25^1$	true
77	$L_2(7)$	$\mathrm{II}_{(3,2)}4_7^13^17^2$	true	194b	$[1440,5841]$	$\mathrm{II}_{(3,0)}2_7^{-3}3^25^1$	true
80	$[128,1759]$	$\mathrm{II}_{(3,2)}2_2^24^{-2}8_1^13^1$	true	197a	$[768,1086051]$	$\mathrm{II}_{(3,0)}2_7^14_1^116_5^{-1}3^1$	true
82a	S_5	$\mathrm{II}_{(3,2)}4_5^{-1}5^{-2}$	false	197b	$[768,1086051]$	$\mathrm{II}_{(3,0)}2_7^14_1^116_5^{-1}3^1$	true
82b	S_5	$\mathrm{II}_{(3,2)}4_5^{-1}3^{-2}5^{-2}$	true	200a	S_6	$\mathrm{II}_{(3,0)}2_6^24_3^{-1}3^{-1}$	false
84	M_9	$\mathrm{II}_{(3,2)}2_5^33^29^1$	true	200b	S_6	$\mathrm{II}_{(3,0)}2_6^24_3^{-1}3^3$	true
85a	N_{72}	$\mathrm{II}_{(3,2)}4_7^13^{-1}9^1$	false	200c	S_6	$\mathrm{II}_{(3,0)}2_6^24_3^{-1}3^3$	true
85b	N_{72}	$\mathrm{II}_{(3,2)}4_7^13^39^1$	true	201	$\mathrm{AGL}_2(\mathbb{F}_3)$	$\mathrm{II}_{(3,0)}2_5^{-1}3^{-1}9^1$	true

Table 16: Finite stable symplectic subgroups for OG10 (continued)

Id	$O^{\#}(C)$	$g(\Lambda_{\mathrm{OG10}}^{O^{\#}(C)})$	Saturated	Id	$O^{\#}(C)$	$g(\Lambda_{\mathrm{OG10}}^{O^{\#}(C)})$	Saturated
87	T_{48}	$\mathrm{II}_{(3,2)}2_1^1 8^{-2} 3^2$	true	203a	$C_2 \times \mathrm{A\Gamma L}_1(\mathbb{F}_9)$	$\mathrm{II}_{(3,0)}2_2^{-2} 4_7^1 3^1 9^1$	true
89	$\mathrm{Aut}(D_8)$	$\mathrm{II}_{(3,2)}2_7^{-3} 8^2 3^1$	true	203b	$C_2 \times \mathrm{A\Gamma L}_1(\mathbb{F}_9)$	$\mathrm{II}_{(3,0)}2_2^{-2} 4_7^1 3^1 9^1$	true
92a	S_4	$\mathrm{II}_{(3,2)}2_2^4 8_7^1$	false	205a	$C_2 \times S_5$	$\mathrm{II}_{(3,0)}2_0^2 4_3^{-1} 5^{-1}$	false
92b	S_4	$\mathrm{II}_{(3,2)}2_2^4 8_7^1 3^{-2}$	true	205b	$C_2 \times S_5$	$\mathrm{II}_{(3,0)}2_0^2 4_3^{-1} 3^{-2} 5^{-1}$	true
94	$C_2 \times Q_8$	$\mathrm{II}_{(3,2)}2_0^{-4} 16_7^1 3^1$	true	205c	$C_2 \times S_5$	$\mathrm{II}_{(3,0)}2_0^2 4_3^{-1} 3^{-2} 5^{-1}$	true
95	$C_2 \times D_8$	$\mathrm{II}_{(3,2)}2_0^2 4_3^{-1} 8_0^2 3^1$	true	207	$\mathrm{A\Gamma L}_1(\mathbb{F}_8)$	$\mathrm{II}_{(3,0)}2_1^1 8^{-2} 3^1$	true
97a	D_{12}	$\mathrm{II}_{(3,2)}2_0^4 4_3^{-1} 3^1$	false	208a	$S_3 \times S_4$	$\mathrm{II}_{(3,0)}2_6^2 8_3^{-1} 3^1$	false
97b	D_{12}	$\mathrm{II}_{(3,2)}2_0^4 4_3^{-1} 3^{-3}$	true	208b	$S_3 \times S_4$	$\mathrm{II}_{(3,0)}2_6^2 8_3^{-1} 3^{-3}$	true
98a	D_{12}	$\mathrm{II}_{(3,2)}2_2^{-4} 4_3^{-1} 3^{-2}$	false	208c	$S_3 \times S_4$	$\mathrm{II}_{(3,0)}2_6^2 8_3^{-1} 3^{-3}$	true
98b	D_{12}	$\mathrm{II}_{(3,2)}2_2^{-4} 4_3^{-1} 3^4$	true	211	S_5	$\mathrm{II}_{(3,0)}2_3^3 3^{-2}$	true
101a	$C_3^4 \rtimes A_6$	$\mathrm{II}_{(3,1)}3^{-1} 9^{-1}$	false	212	$C_2 \times T_{48}$	$\mathrm{II}_{(3,0)}2_0^2 16_7^1 3^2$	true
101b	$C_3^4 \rtimes A_6$	$\mathrm{II}_{(3,1)}3^3 9^{-1}$	true	214a	$C_2 \times S_3^2$	$\mathrm{II}_{(3,0)}2_6^{-2} 4_5^{-1} 3^2$	false
102	$L_3(4)$	$\mathrm{II}_{(3,1)}2^{-2} 3^2 7^1$	true	214b	$C_2 \times S_3^2$	$\mathrm{II}_{(3,0)}2_6^{-2} 4_5^{-1} 3^2$	false
106a	$C_2^4 \rtimes A_6$	$\mathrm{II}_{(3,1)}4_1^1 8_5^{-1}$	false	220a	F_7	$\mathrm{II}_{(3,0)}2_1^{-3} 3^1 7^{-2}$	true
106b	$C_2^4 \rtimes A_6$	$\mathrm{II}_{(3,1)}4_1^1 8_5^{-1} 3^{-2}$	true	220b	F_7	$\mathrm{II}_{(3,0)}2_1^{-3} 3^1 7^{-2}$	true
108a	A_7	$\mathrm{II}_{(3,1)}5^1 7^{-1}$	false				
E18a	C_3^2	$\mathrm{II}_{(3,3)}3^{-6}$	false	E20c	[108, 40]	$\mathrm{II}_{(3,1)}3^3 9^{-1}$	false
E18b	C_3^2	$\mathrm{II}_{(3,3)}3^{-6}$	true	E21a	$C_3 \times S_3$	$\mathrm{II}_{(3,0)}2_1^{-3} 3^{-3}$	true
E20a	[108, 40]	$\mathrm{II}_{(3,1)}3^3 9^{-1}$	false	E21b	$C_3 \times S_3$	$\mathrm{II}_{(3,0)}2_1^{-3} 3^{-3}$	false
E20b	[108, 40]	$\mathrm{II}_{(3,1)}3^3 9^{-1}$	true				

D. Algebraically trivial cohomological actions

D.1. Tables of results from Theorem 8.67

Table 17: Stable isometries — nonprime orders

(a) $\mathcal{T} = \mathrm{Kum}_n$ ($n \geq 2$); $M = U^{\oplus 4}$

m	M^g	M_g	Induced
$4, 6, 12$	$\mathrm{II}_{(2,2)}$	$\mathrm{II}_{(2,2)}$	1
4	$\mathrm{II}_{(2,2)}2^2$	$\mathrm{II}_{(2,2)}2^2$	1
9	$\mathrm{II}_{(2,0)}3^{-1}$	$\mathrm{II}_{(2,4)}3^1$	0

(b) $\mathcal{T} = \mathrm{OG6}$; $M = U^{\oplus 5}$

m	M^g	M_g
$4, 6, 12$	$\mathrm{II}_{(3,3)}$	$\mathrm{II}_{(2,2)}$
4	$\mathrm{II}_{(3,3)}2^2$	$\mathrm{II}_{(2,2)}2^2$
9	$\mathrm{II}_{(3,1)}3^{-1}$	$\mathrm{II}_{(2,4)}3^1$

(c) $\mathcal{T} = \mathrm{K3}^{[n]}$ ($n \geq 2$); $M = U^{\oplus 4} \oplus E_8^{\oplus 2}$

m	M^g	M_g	Induced
4,6,12,22 33,44,66	$\mathrm{II}_{(2,2)}$	$\mathrm{II}_{(2,18)}$	1
4	$\mathrm{II}_{(2,2)}2^2$	$\mathrm{II}_{(2,18)}2^2$	1
4	$\mathrm{II}_{(2,2)}2^4$	$\mathrm{II}_{(2,18)}2^4$	2
$4, 8, 16$	$\mathrm{II}_{(2,6)}2^{-2}$	$\mathrm{II}_{(2,14)}2^{-2}$	1
$4, 8$	$\mathrm{II}_{(2,6)}2^{-4}$	$\mathrm{II}_{(2,14)}2^{-4}$	1
4	$\mathrm{II}_{(2,6)}2^{-6}$	$\mathrm{II}_{(2,14)}2^{-6}$	2
4,6,9,12 14,18,21 28,36,42	$\mathrm{II}_{(2,10)}$	$\mathrm{II}_{(2,10)}$	1
4	$\mathrm{II}_{(2,10)}2^2$	$\mathrm{II}_{(2,10)}2^2$	1
4	$\mathrm{II}_{(2,10)}2^4$	$\mathrm{II}_{(2,10)}2^4$	1
4	$\mathrm{II}_{(2,10)}2^6$	$\mathrm{II}_{(2,10)}2^6$	1
$4, 8$	$\mathrm{II}_{(2,14)}2^{-2}$	$\mathrm{II}_{(2,6)}2^{-2}$	1
4	$\mathrm{II}_{(2,14)}2^{-4}$	$\mathrm{II}_{(2,6)}2^{-4}$	1
$4, 6, 12$	$\mathrm{II}_{(2,18)}$	$\mathrm{II}_{(2,2)}$	1
4	$\mathrm{II}_{(2,18)}2^2$	$\mathrm{II}_{(2,2)}2^2$	1
$9, 27$	$\mathrm{II}_{(2,4)}3^1$	$\mathrm{II}_{(2,16)}3^{-1}$	1
9	$\mathrm{II}_{(2,4)}3^{-3}$	$\mathrm{II}_{(2,16)}3^3$	1
9	$\mathrm{II}_{(2,10)}3^{-2}$	$\mathrm{II}_{(2,10)}3^{-2}$	1
9	$\mathrm{II}_{(2,16)}3^{-1}$	$\mathrm{II}_{(2,4)}3^1$	1
25	$\mathrm{II}_{(2,2)}5^{-1}$	$\mathrm{II}_{(2,18)}5^{-1}$	1

(d) $\mathcal{T} = \mathrm{OG10}$; $M = U^{\oplus 5} \oplus E_8^{\oplus 2}$

m	M^g	M_g
4,6,12,22 33,44,66	$\mathrm{II}_{(3,3)}$	$\mathrm{II}_{(2,18)}$
4	$\mathrm{II}_{(3,3)}2^2$	$\mathrm{II}_{(2,18)}2^2$
4	$\mathrm{II}_{(3,3)}2^4$	$\mathrm{II}_{(2,18)}2^4$
4	$\mathrm{II}_{(3,3)}2^6$	$\mathrm{II}_{(2,18)}2^6$
$4, 8, 16$	$\mathrm{II}_{(3,7)}2^{-2}$	$\mathrm{II}_{(2,14)}2^{-2}$
$4, 8$	$\mathrm{II}_{(3,7)}2^{-4}$	$\mathrm{II}_{(2,14)}2^{-4}$
4	$\mathrm{II}_{(3,7)}2^{-6}$	$\mathrm{II}_{(2,14)}2^{-6}$
4	$\mathrm{II}_{(3,7)}2^{-8}$	$\mathrm{II}_{(2,14)}2^{-8}$
4,6,9,12,14,18 21,28,36,42	$\mathrm{II}_{(3,11)}$	$\mathrm{II}_{(2,10)}$
4	$\mathrm{II}_{(3,11)}2^2$	$\mathrm{II}_{(2,10)}2^2$
4	$\mathrm{II}_{(3,11)}2^4$	$\mathrm{II}_{(2,10)}2^4$
4	$\mathrm{II}_{(3,11)}2^6$	$\mathrm{II}_{(2,10)}2^6$
$4, 8$	$\mathrm{II}_{(3,15)}2^{-2}$	$\mathrm{II}_{(2,6)}2^{-2}$
4	$\mathrm{II}_{(3,15)}2^{-4}$	$\mathrm{II}_{(2,6)}2^{-4}$
$4, 6, 12$	$\mathrm{II}_{(3,19)}$	$\mathrm{II}_{(2,2)}$
4	$\mathrm{II}_{(3,19)}2^2$	$\mathrm{II}_{(2,2)}2^2$
$9, 27$	$\mathrm{II}_{(3,5)}3^1$	$\mathrm{II}_{(2,16)}3^{-1}$
9	$\mathrm{II}_{(3,5)}3^{-3}$	$\mathrm{II}_{(2,16)}3^3$
9	$\mathrm{II}_{(3,11)}3^{-2}$	$\mathrm{II}_{(2,10)}3^{-2}$
9	$\mathrm{II}_{(3,17)}3^{-1}$	$\mathrm{II}_{(2,4)}3^1$
25	$\mathrm{II}_{(3,3)}5^{-1}$	$\mathrm{II}_{(2,18)}5^{-1}$

D.2. Tables of results from Theorem 8.40

Table 18: Nonstable isometries — twice prime powers

$\mathcal{T}$	m	M^g	Λ^h	Λ_h
OG6	4	$\mathrm{II}_{(3,1)}2_6^{-2}$	U	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(3,1)}2_6^{-2}$	$U(2)$	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(3,1)}2_2^{4}$	$U(2)$	$U \oplus U(2) \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(3,5)}2_6^{2}$	$U \oplus D_4$	$\langle 2\rangle^{\oplus 2}$
	8	$\mathrm{II}_{(3,3)}2^{2}$	$U \oplus \langle -2\rangle^{\oplus 2}$	$U \oplus U(2)$
K3[3]	4	$\mathrm{II}_{(2,0)}2_2^{2}$	$\langle 4\rangle$	$U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(2,4)}2_6^{2}$	$U \oplus A_3$	$U^{\oplus 2} \oplus E_7^{\oplus 2}$
	4	$\mathrm{II}_{(2,4)}2_2^{-4}$	$U(2) \oplus A_3$	$U \oplus U(2) \oplus E_7^{\oplus 2}$
	4	$\mathrm{II}_{(2,8)}2_2^{2}$	$U \oplus D_7$	$U^{\oplus 2} \oplus E_8 \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(2,8)}2_2^{4}$	$U(2) \oplus D_7$	$U^{\oplus 2} \oplus D_8 \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(2,8)}2_2^{6}$	$U(2) \oplus A_3 \oplus D_4$	$U^{\oplus 2} \oplus D_4^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(2,8)}2_6^{-8}$	$U \oplus E_7(2)$	$U \oplus U(2) \oplus D_4^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(2,12)}2_6^{2}$	$U \oplus A_3 \oplus E_8$	$U^{\oplus 2} \oplus D_6$
	4	$\mathrm{II}_{(2,12)}2_6^{4}$	$U \oplus A_3 \oplus D_8$	$U^{\oplus 2} \oplus D_4 \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(2,12)}2_6^{6}$	$U(2) \oplus A_3 \oplus D_8$	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 6}$
	4	$\mathrm{II}_{(2,16)}2_2^{2}$	$U \oplus D_7 \oplus E_8$	$U^{\oplus 2} \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(2,16)}2_2^{4}$	$U \oplus A_3 \oplus D_4 \oplus E_8$	$U \oplus U(2) \oplus \langle -2\rangle^{\oplus 2}$
	4	$\mathrm{II}_{(2,20)}2_6^{2}$	$U \oplus A_3 \oplus E_8^{\oplus 2}$	$\langle 2\rangle^{\oplus 2}$
	8	$\mathrm{II}_{(2,2)}2^{2}$	$U \oplus \langle -4\rangle$	$U^{\oplus 2} \oplus D_8 \oplus E_8$
	8	$\mathrm{II}_{(2,2)}2^{4}$	$U(2) \oplus \langle -4\rangle$	$U^{\oplus 2} \oplus D_4^{\oplus 2} \oplus E_8$
	8	$\mathrm{II}_{(2,10)}2^{2}$	$U \oplus E_8 \oplus \langle -4\rangle$	$U^{\oplus 2} \oplus D_8$
	8	$\mathrm{II}_{(2,10)}2^{4}$	$U \oplus D_8 \oplus \langle -4\rangle$	$U^{\oplus 2} \oplus D_4^{\oplus 2}$
	8	$\mathrm{II}_{(2,18)}2^{2}$	$U \oplus E_8^{\oplus 2} \oplus \langle -4\rangle$	$U \oplus U(2)$
	16	$\mathrm{II}_{(2,14)}2^{-2}$	$U \oplus D_5 \oplus E_8$	$U^{\oplus 2} \oplus D_4$
	32	$\mathrm{II}_{(2,6)}2^{-2}$	$U \oplus D_5$	$U^{\oplus 2} \oplus D_4 \oplus E_8$
OG10	6	$\mathrm{II}_{(3,1)}3^{-1}$	U	$U^{\oplus 2} \oplus A_2 \oplus E_8^{\oplus 2}$
	6	$\mathrm{II}_{(3,9)}3^{-1}$	$U \oplus E_8$	$U^{\oplus 2} \oplus A_2 \oplus E_8$
	6, 18	$\mathrm{II}_{(3,17)}3^{-1}$	$U \oplus E_8^{\oplus 2}$	$U^{\oplus 2} \oplus A_2$
K3[4]	6	$\mathrm{II}_{(2,0)}3^{-1}$	$\langle 2\rangle$	$U^{\oplus 2} \oplus A_2 \oplus E_8^{\oplus 2}$
	6	$\mathrm{II}_{(2,8)}3^{-1}$	$E_8 \oplus \langle 2\rangle$	$U^{\oplus 2} \oplus A_2 \oplus E_8$
	6, 18	$\mathrm{II}_{(2,16)}3^{-1}$	$E_8^{\oplus 2} \oplus \langle 2\rangle$	$U^{\oplus 2} \oplus A_2$
K3[6]	10, 50	$\mathrm{II}_{(2,2)}5^{-1}$	$U \oplus \langle -2\rangle$	$U^{\oplus 2} \oplus E_8 \oplus L_8^5$
	10	$\mathrm{II}_{(2,10)}5^{-1}$	$U \oplus E_8 \oplus \langle -2\rangle$	$U^{\oplus 2} \oplus L_8^5$
	10	$\mathrm{II}_{(2,18)}5^{-1}$	$U \oplus E_8^{\oplus 2} \oplus \langle -2\rangle$	$U \oplus H_5$
K3[8]	14	$\mathrm{II}_{(2,16)}7^{1}$	$U \oplus E_7 \oplus E_8$	$U^{\oplus 2} \oplus K_7$
K3[14]	26	$\mathrm{II}_{(2,10)}13^{-1}$	$U \oplus E_8 \oplus \langle -2\rangle$	$U^{\oplus 2} \oplus L_8^{13}$
K3[24]	46	$\mathrm{II}_{(2,0)}23^{1}$	$\langle 2\rangle$	$U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus K_{23}$

E. Saturated finite symplectic subgroups for the deformation type OG10

The lattice-theoretic counterpart of Theorem 9.33 is presented in Table 19 (see [MM25c] for explicit data in terms of matrix representations). For simplicity, let us denote $\Lambda := \Lambda_{\mathrm{OG10}} := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$ the OG10 BBF form. Each entry of Table 19 records information about a certain $O^+(\Lambda)$-conjugacy class of saturated finite symplectic subgroup $H \leq O^+(\Lambda)$. Let $\mathcal{H}$ be such a conjugacy class and let $H \leq O^+(\Lambda)$ be a representative. The entry corresponding to $\mathcal{H}$ in the table records:

Id: an ID for the class $\mathcal{H}$, matching the ones given in [MM25b, Table 4].

ρ: the rank of the coinvariant sublattice Λ_H of H.

$\#H$: the order of H.

H: a description of H as an abstract finite group. If there exists a short expression of H, in terms of standard operations on known groups, then it is given. Otherwise, if it is available, we refer to the identification numbers of the group in the GAP Small Groups Library [BEOH24]. If none of the two latter are possible, we leave the entry blank (—).

$g(\Lambda_H)$: a symbol for the genus of the coinvariant sublattice of H, following Conway–Sloane convention (Section 1.3).

$g(\Lambda^H)$: a symbol for the genus of the invariant sublattice of H, following Conway–Sloane convention (Section 1.3).

K3: if the entry in the database can be realized as induced from a K3 surface S on a desingularization of a moduli space of semistable sheaves on S, we enter the number of the corresponding group in [Has12, Table 10.2].

Cubic: $\times$ if the entry in the database can be realized on an LSV manifold (conjecturally, see Proposition 12.5), $\times^t$ if it can be realized on a twisted LSV manifold (Corollary 12.12), and blank otherwise.

$\#(-4,1)$: the number of vectors of norm -4 in Λ_H, up to sign.

$\#(-24,3)$: the number of classes of norm -24 in Λ_H which have divisibility 3 in Λ, up to sign.

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$

H	ρ	$\#H$	H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	Cubic	$\#(-4,1)$	$\#(-24,3)$
1	0	1	C_1	$\mathrm{II}_{(0,0)}$	$\mathrm{II}_{(3,21)}3^1$	(0)	$\times\times^t$	0	0
1.4	6	2	C_2	$\mathrm{II}_{(0,6)}2^{-6}3^1$	$\mathrm{II}_{(3,15)}2^{-6}$	—	$\times$	36	27
2	8	2	C_2	$\mathrm{II}_{(0,8)}2^8$	$\mathrm{II}_{(3,13)}2^83^1$	(1)	$\times\times^t$	120	0
1.3	10	2	C_2	$\mathrm{II}_{(0,10)}2_4^{-10}3^1$	$\mathrm{II}_{(3,11)}2_0^{10}$	—	$\times$	72	19
2.12	10	4	C_2^2	$\mathrm{II}_{(0,10)}2^44^{-2}3^1$	$\mathrm{II}_{(3,11)}2^44^{-2}$	—	$\times$	168	51

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
1.2	12	2	C_2	$\mathrm{II}_{(0,12)}2_2^{-12}3^1$	$\mathrm{II}_{(3,9)}2_2^{12}$	—	—	78	15
5	12	2	C_2	$\mathrm{II}_{(0,12)}2_4^{12}$	$\mathrm{II}_{(3,9)}2_0^{-12}3^1$	—	—	132	0
4b	12	3	C_3	$\mathrm{II}_{(0,12)}3^6$	$\mathrm{II}_{(3,9)}3^7$	(2)	$\times$	378	0
2.11	12	4	C_2^2	$\mathrm{II}_{(0,12)}2^{-6}4_2^2 3^1$	$\mathrm{II}_{(3,9)}2^6 4_2^{-2}$	—	$\times$	190	39
3	12	4	C_2^2	$\mathrm{II}_{(0,12)}2^{-6}4^{-2}$	$\mathrm{II}_{(3,9)}2^{-6}4^{-2}3^1$	(3)	$\times\times^t$	324	0
4a.6	12	6	S_3	$\mathrm{II}_{(0,12)}3^6$	$\mathrm{II}_{(3,9)}3^{-5}$	—	$\times$	378	81
3.33	13	8	C_2^3	$\mathrm{II}_{(0,13)}2^{-6}8_1^1 3^1$	$\mathrm{II}_{(3,8)}2^{-6}8_7^1$	—	$\times$	372	73
9	14	4	C_4	$\mathrm{II}_{(0,14)}2_2^2 4^4$	$\mathrm{II}_{(3,7)}2_2^{-2}4^4 3^1$	(4)	$\times\times^t$	606	0
2.8	14	4	C_2^2	$\mathrm{II}_{(0,14)}2_0^2 4_4^{-4}3^1$	$\mathrm{II}_{(3,7)}2_4^{-2}4_4^{-4}$	—	$\times$	364	51
7b	14	6	S_3	$\mathrm{II}_{(0,14)}2^{-2}3^5$	$\mathrm{II}_{(3,7)}2^{-2}3^{-6}$	(6)	$\times$	624	0
3.32	14	8	D_8	$\mathrm{II}_{(0,14)}2^{-2}4^4 3^1$	$\mathrm{II}_{(3,7)}2^{-2}4^4$	—	$\times$	372	51
3.31	14	8	C_2^3	$\mathrm{II}_{(0,14)}2^{-6}4^2 3^1$	$\mathrm{II}_{(3,7)}2^{-6}4^2$	—	—	380	59
6	14	8	C_2^3	$\mathrm{II}_{(0,14)}2^6 4_2^2$	$\mathrm{II}_{(3,7)}2^6 4_6^2 3^1$	(9)	—	630	0
7.4	14	12	D_{12}	$\mathrm{II}_{(0,14)}2^{-2}3^5$	$\mathrm{II}_{(3,7)}2^{-2}3^4$	—	$\times$	624	99
5.9	15	4	C_2^2	$\mathrm{II}_{(0,15)}2_6^{-6}4_1^3 3^1$	$\mathrm{II}_{(3,6)}2_4^{-6}4_5^{-3}$	—	—	233	39
2.7	15	4	C_2^2	$\mathrm{II}_{(0,15)}2_6^6 4_5^{-3}3^1$	$\mathrm{II}_{(3,6)}2_4^{-6}4_5^{-3}$	—	—	269	27
9.43	15	8	D_8	$\mathrm{II}_{(0,15)}2_2^2 4^2 8_1^1 3^1$	$\mathrm{II}_{(3,6)}2_2^{-2}4^2 8_3^{-1}$	—	$\times$	670	85
13	15	8	D_8	$\mathrm{II}_{(0,15)}4_1^5$	$\mathrm{II}_{(3,6)}4_7^5 3^1$	(10)	$\times\times^t$	815	0
3.29	15	8	C_2^3	$\mathrm{II}_{(0,15)}4_3^{-5}3^1$	$\mathrm{II}_{(3,6)}4_5^{-5}$	—	$\times$	513	63
6.31	15	16	C_2^4	$\mathrm{II}_{(0,15)}2^{-6}8_7^1 3^1$	$\mathrm{II}_{(3,6)}2^{-6}8_1^1$	—	—	694	93
10	15	16	C_2^4	$\mathrm{II}_{(0,15)}2^6 8_1^1$	$\mathrm{II}_{(3,6)}2^6 8_7^1 3^1$	(21)	—	1170	0
1.1	16	2	C_2	$\mathrm{II}_{(0,16)}2_6^{-8}3^1$	$\mathrm{II}_{(3,5)}2_6^8$	—	—	1186	135
2.5	16	4	C_4	$\mathrm{II}_{(0,16)}2_2^4 4^{-4}3^1$	$\mathrm{II}_{(3,5)}2_6^4 4^4$	—	—	370	23
5.8	16	4	C_2^2	$\mathrm{II}_{(0,16)}2^4 4_2^4 3^1$	$\mathrm{II}_{(3,5)}2^4 4_6^4$	—	—	318	43
2.6	16	4	C_2^2	$\mathrm{II}_{(0,16)}2_4^4 4_6^{-4}3^1$	$\mathrm{II}_{(3,5)}2_0^4 4_6^4$	—	—	326	35
23	16	4	C_2^2	$\mathrm{II}_{(0,16)}2_0^{-4}4_4^{-4}$	$\mathrm{II}_{(3,5)}2_6^{-4}4_6^4 3^1$	—	—	520	0
22	16	6	S_3	$\mathrm{II}_{(0,16)}3^8$	$\mathrm{II}_{(3,5)}3^{-7}$	—	—	360	45
4b.8	16	6	S_3	$\mathrm{II}_{(0,16)}2_6^4 3^{-5}$	$\mathrm{II}_{(3,5)}2_2^4 3^{-4}$	—	$\times$	579	51
4b.7	16	6	C_6	$\mathrm{II}_{(0,16)}2^4 3^{-6}$	$\mathrm{II}_{(3,5)}2^4 3^5$	—	$\times$	411	27
4b.9	16	6	C_6	$\mathrm{II}_{(0,16)}2_4^4 3^{-4}$	$\mathrm{II}_{(3,5)}2_0^{-4}3^3$	—	$\times$	945	91
9.40	16	8	$C_2 \times C_4$	$\mathrm{II}_{(0,16)}2_6^{-2}4^4 3^1$	$\mathrm{II}_{(3,5)}2_6^2 4^4$	—	$\times$	666	63
9.41	16	8	D_8	$\mathrm{II}_{(0,16)}2_6^{-6}4^2 3^1$	$\mathrm{II}_{(3,5)}2_6^6 4^2$	—	—	666	63
20	16	10	D_{10}	$\mathrm{II}_{(0,16)}5^4$	$\mathrm{II}_{(3,5)}3^1 5^4$	(16)	$\times\times^t$	1320	0
19b	16	12	A_4	$\mathrm{II}_{(0,16)}2^{-2}4^{-2}3^2$	$\mathrm{II}_{(3,5)}2^{-2}4^{-2}3^3$	(17)	$\times\times^t$	1386	0
18b	16	12	D_{12}	$\mathrm{II}_{(0,16)}2^4 3^4$	$\mathrm{II}_{(3,5)}2^4 3^5$	(18)	$\times$	924	0
6.29	16	16	$C_2 \times D_8$	$\mathrm{II}_{(0,16)}2^{-2}4_6^4 3^1$	$\mathrm{II}_{(3,5)}2^2 4_6^{-4}$	—	—	686	59
13.35	16	16	$C_2 \times D_8$	$\mathrm{II}_{(0,16)}4_1^3 8_5^{-1}3^1$	$\mathrm{II}_{(3,5)}4_7^3 8_3^{-1}$	—	$\times$	887	97

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
17	16	16	$C_2 \times D_8$	$\mathrm{II}_{(0,16)}2^2 4_0^4$	$\mathrm{II}_{(3,5)}2^2 4_0^4 3^1$	(22)	—	1064	0
6.28	16	16	C_2^4	$\mathrm{II}_{(0,16)}2^2 4_2^{-4} 3^1$	$\mathrm{II}_{(3,5)}2^2 4_6^{-4}$	—	—	702	75
15b	16	18	$A_{3,3}$	$\mathrm{II}_{(0,16)}3^4 9^{-1}$	$\mathrm{II}_{(3,5)}3^5 9^1$	(30)	$\times$	1224	0
19a.6	16	24	S_4	$\mathrm{II}_{(0,16)}2^{-2} 4^{-2} 3^2$	$\mathrm{II}_{(3,5)}2^{-2} 4^{-2} 3^{-1}$	—	$\times$	1386	153
18a.5	16	24	$C_2^2 \times S_3$	$\mathrm{II}_{(0,16)}2^4 3^4$	$\mathrm{II}_{(3,5)}2^4 3^{-3}$	—	$\times$	924	117
10.8	16	32	C_2^5	$\mathrm{II}_{(0,16)}2^{-4} 4_6^2 3^1$	$\mathrm{II}_{(3,5)}2^4 4_6^{-2}$	—	—	1266	127
15a.8	16	36	S_3^2	$\mathrm{II}_{(0,16)}3^4 9^{-1}$	$\mathrm{II}_{(3,5)}3^{-3} 9^1$	—	$\times$	1224	135
5.7	17	4	C_2^2	$\mathrm{II}_{(0,17)}2_2^{-2} 4_3^{-5} 3^1$	$\mathrm{II}_{(3,4)}2_0^2 4_7^5$	—	—	419	39
5.6	17	4	C_2^2	$\mathrm{II}_{(0,17)}2^{-4} 4_5^{-3} 3^1$	$\mathrm{II}_{(3,4)}2^4 4_7^3$	—	—	775	75
5.5	17	4	C_2^2	$\mathrm{II}_{(0,17)}2_6^6 4_3^{-1} 3^1$	$\mathrm{II}_{(3,4)}2_0^6 4_7^1$	—	—	1487	147
9.35	17	8	$C_2 \times C_4$	$\mathrm{II}_{(0,17)}2_6^{-4} 4^2 8_7^1 3^1$	$\mathrm{II}_{(3,4)}2_6^4 4^2 8_1^1$	—	—	664	41
3.19	17	8	D_8	$\mathrm{II}_{(0,17)}2_6^{-4} 4^2 8_7^1 3^1$	$\mathrm{II}_{(3,4)}2_6^4 4^2 8_1^1$	—	—	552	57
23.32	17	8	D_8	$\mathrm{II}_{(0,17)}2_6^{-4} 4^2 8_7^1 3^1$	$\mathrm{II}_{(3,4)}2_6^4 4^2 8_1^1$	—	—	552	57
30	17	8	D_8	$\mathrm{II}_{(0,17)}2_6^4 4^{-2} 8_1^1$	$\mathrm{II}_{(3,4)}2_2^4 4^{-2} 8_7^1 3^1$	—	—	928	0
9.34	17	8	D_8	$\mathrm{II}_{(0,17)}2_1^3 8^2 3^1$	$\mathrm{II}_{(3,4)}2_7^3 8^2$	—	$\times$	1094	85
31	17	8	Q_8	$\mathrm{II}_{(0,17)}2_7^{-3} 8^{-2}$	$\mathrm{II}_{(3,4)}2_5^3 8^{-2} 3^1$	(12)	$\times\times^t$	1818	0
23.31	17	8	C_2^3	$\mathrm{II}_{(0,17)}2_6^4 4_2^{-2} 8_1^1 3^1$	$\mathrm{II}_{(3,4)}2_0^{-4} 4_4^{-2} 8_7^1$	—	—	568	69
3.18	17	8	C_2^3	$\mathrm{II}_{(0,17)}2_2^4 4_0^2 8_7^1 3^1$	$\mathrm{II}_{(3,4)}2_0^{-4} 4_4^{-2} 8_7^1$	—	—	622	51
7b.7	17	12	D_{12}	$\mathrm{II}_{(0,17)}2^{-2} 4_1^1 3^{-5}$	$\mathrm{II}_{(3,4)}2^{-2} 4_7^1 3^{-4}$	—	$\times$	764	63
7b.8	17	12	D_{12}	$\mathrm{II}_{(0,17)}2_4^{-2} 4_3^{-1} 3^{-4}$	$\mathrm{II}_{(3,4)}2_0^2 4_5^{-1} 3^3$	—	$\times$	1202	99
13.33	17	16	$C_2 \times D_8$	$\mathrm{II}_{(0,17)}2_4^{-4} 4_1^3 3^1$	$\mathrm{II}_{(3,4)}2_0^4 4_7^3$	—	—	879	67
13.31	17	16	$C_2 \times D_8$	$\mathrm{II}_{(0,17)}4_1^{-5} 3^1$	$\mathrm{II}_{(3,4)}4_7^{-5}$	—	—	883	71
13.32	17	16	$C_4 \bigcirc D_8$	$\mathrm{II}_{(0,17)}4_1^5 3^1$	$\mathrm{II}_{(3,4)}4_7^5$	—	$\times$	875	63
29b	17	24	S_4	$\mathrm{II}_{(0,17)}4_3^3 3^2$	$\mathrm{II}_{(3,4)}4_5^3 3^3$	(34)	$\times\times^t$	1731	0
19b.15	17	24	$C_2 \times A_4$	$\mathrm{II}_{(0,17)}2^{-2} 8_1^1 3^3$	$\mathrm{II}_{(3,4)}2^{-2} 8_7^1 3^2$	—	$\times$	1482	121
17.45	17	32	$C_2^2 \wr C_2$	$\mathrm{II}_{(0,17)}2^{-2} 4_4^{-2} 8_5^{-1} 3^1$	$\mathrm{II}_{(3,4)}2^{-2} 4_4^{-2} 8_3^{-1}$	—	—	1128	93
26	17	32	$C_2^2 \wr C_2$	$\mathrm{II}_{(0,17)}2^2 4_6^2 8_1^1$	$\mathrm{II}_{(3,4)}2^2 4_2^2 8_7^1 3^1$	(39)	—	1860	0
10.7	17	32	$C_2^2 \times D_8$	$\mathrm{II}_{(0,17)}2^2 4^2 8_5^{-1} 3^1$	$\mathrm{II}_{(3,4)}2^2 4^2 8_3^{-1}$	—	—	1242	75
17.46	17	32	$C_2^2 \times D_8$	$\mathrm{II}_{(0,17)}2^{-2} 4_6^{-2} 8_3^{-1} 3^1$	$\mathrm{II}_{(3,4)}2^{-2} 4_4^{-2} 8_3^{-1}$	—	—	1144	109
28	17	32	2_+^{1+4}	$\mathrm{II}_{(0,17)}4_7^5$	$\mathrm{II}_{(3,4)}4_1^5 3^1$	(40)	—	1353	0
29a.4	17	48	$C_2 \times S_4$	$\mathrm{II}_{(0,17)}4_3^3 3^2$	$\mathrm{II}_{(3,4)}4_5^3 3^{-1}$	—	$\times$	1731	165
24	17	48	$C_2^2 \rtimes A_4$	$\mathrm{II}_{(0,17)}2^{-4} 8_1^1 3^{-1}$	$\mathrm{II}_{(3,4)}2^{-4} 8_7^1 3^2$	(49)	—	2133	0
5.4	18	4	C_4	$\mathrm{II}_{(0,18)}4_0^6 3^1$	$\mathrm{II}_{(3,3)}4_0^6$	—	—	512	27
5.3	18	4	C_2^2	$\mathrm{II}_{(0,18)}4_0^6 3^1$	$\mathrm{II}_{(3,3)}4_0^6$	—	—	544	43
5.2	18	4	C_2^2	$\mathrm{II}_{(0,18)}2^4 4_0^2 3^1$	$\mathrm{II}_{(3,3)}2^4 4_0^2$	—	—	1772	135
4b.2	18	6	S_3	$\mathrm{II}_{(0,18)}2_2^6 3^6$	$\mathrm{II}_{(3,3)}2_2^{-6} 3^{-5}$	—	—	474	15
61b	18	6	S_3	$\mathrm{II}_{(0,18)}2_6^{-6} 3^{-4}$	$\mathrm{II}_{(3,3)}2_2^{-6} 3^{-5}$	—	—	735	0

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
4b.5	18	6	S_3	$\mathrm{II}_{(0,18)}2_6^6 3^4$	$\mathrm{II}_{(3,3)}2_6^{-6}3^{-3}$	—	—	820	39
60b	18	6	S_3	$\mathrm{II}_{(0,18)}2_2^{-6}3^{-2}$	$\mathrm{II}_{(3,3)}2_6^{-6}3^{-3}$	—	—	2089	0
4b.3	18	6	C_6	$\mathrm{II}_{(0,18)}2_4^{-6}3^5$	$\mathrm{II}_{(3,3)}2_0^6 3^4$	—	—	588	19
63b	18	6	C_6	$\mathrm{II}_{(0,18)}2_0^6 3^{-3}$	$\mathrm{II}_{(3,3)}2_0^6 3^4$	—	—	1221	0
9.26	18	8	$C_2 \times C_4$	$\mathrm{II}_{(0,18)}2_2^4 8_6^2 3^1$	$\mathrm{II}_{(3,3)}2_6^4 8_2^2$	—	—	1082	55
3.14	18	8	$C_2 \times C_4$	$\mathrm{II}_{(0,18)}2^{-2}4_4^{-4}3^1$	$\mathrm{II}_{(3,3)}2^2 4_0^4$	—	—	1084	55
59	18	8	$C_2 \times C_4$	$\mathrm{II}_{(0,18)}2_0^{-4}8_2^2$	$\mathrm{II}_{(3,3)}2_0^{-4}8_6^2 3^1$	—	—	1584	0
3.7	18	8	D_8	$\mathrm{II}_{(0,18)}2_6^{-2}4_5^3 8_1^1 3^1$	$\mathrm{II}_{(3,3)}2_2^{-2}4_1^3 8_5^{-1}$	—	—	779	39
9.24	18	8	D_8	$\mathrm{II}_{(0,18)}2_0^4 8_0^2 3^1$	$\mathrm{II}_{(3,3)}2_0^4 8_0^2$	—	—	972	69
23.25	18	8	C_2^3	$\mathrm{II}_{(0,18)}2_6^2 4_3^{-3}8_7^1 3^1$	$\mathrm{II}_{(3,3)}2_2^{-2}4_1^3 8_5^{-1}$	—	—	705	65
3.8	18	8	C_2^3	$\mathrm{II}_{(0,18)}2_6^2 4_3^{-3}8_7^1 3^1$	$\mathrm{II}_{(3,3)}2_2^{-2}4_1^3 8_5^{-1}$	—	—	729	57
23.29	18	8	C_2^3	$\mathrm{II}_{(0,18)}2^2 4_0^4 3^1$	$\mathrm{II}_{(3,3)}2^2 4_0^4$	—	—	988	87
3.13	18	8	C_2^3	$\mathrm{II}_{(0,18)}2^2 4^4 3^1$	$\mathrm{II}_{(3,3)}2^2 4^4$	—	—	1088	51
58	18	8	C_2^3	$\mathrm{II}_{(0,18)}2_4^{-2}4_5^{-3}8_5^{-1}$	$\mathrm{II}_{(3,3)}2_4^{-2}4_3^{-3}8_3^{-1}3^1$	—	—	1169	0
23.30	18	8	C_2^3	$\mathrm{II}_{(0,18)}2_2^{-4}4_2^2 3^1$	$\mathrm{II}_{(3,3)}2_0^4 4_0^2$	—	—	1828	159
E18b	18	9	C_3^2	$\mathrm{II}_{(0,18)}3^{-7}$	$\mathrm{II}_{(3,3)}3^{-6}$			1053	81
56	18	12	A_4	$\mathrm{II}_{(0,18)}2_6^2 4^4$	$\mathrm{II}_{(3,3)}2_2^2 4^{-4}3^1$	—	—	1530	0
61a.4	18	12	D_{12}	$\mathrm{II}_{(0,18)}2_2^6 3^{-4}$	$\mathrm{II}_{(3,3)}2_2^{-6}3^3$	—	—	735	87
7b.5	18	12	D_{12}	$\mathrm{II}_{(0,18)}2_2^6 3^{-4}$	$\mathrm{II}_{(3,3)}2_2^{-6}3^3$	—	—	843	51
63a.4	18	12	D_{12}	$\mathrm{II}_{(0,18)}2_4^{-6}3^{-3}$	$\mathrm{II}_{(3,3)}2_0^6 3^{-2}$	—	—	1221	117
7b.6	18	12	D_{12}	$\mathrm{II}_{(0,18)}2_4^{-6}3^{-3}$	$\mathrm{II}_{(3,3)}2_0^6 3^{-2}$	—	—	1299	91
60a.4	18	12	D_{12}	$\mathrm{II}_{(0,18)}2_6^6 3^{-2}$	$\mathrm{II}_{(3,3)}2_6^{-6}3^1$	—	—	2089	171
55	18	16	QD_{16}	$\mathrm{II}_{(0,18)}2_1^1 4_5^{-1}8^2$	$\mathrm{II}_{(3,3)}2_5^{-1}4_5^{-1}8^{-2}3^1$	(26)	$\times\times^t$	2219	0
13.24	18	16	D_{16}	$\mathrm{II}_{(0,18)}2_3^{-1}4_1^1 8^2 3^1$	$\mathrm{II}_{(3,3)}2_7^1 4_1^1 8^2$	—	$\times$	1367	85
30.15	18	16	$C_2 \times D_8$	$\mathrm{II}_{(0,18)}2_4^{-2}4_0^{-4}3^1$	$\mathrm{II}_{(3,3)}2_0^2 4_0^4$	—	—	984	83
30.17	18	16	$C_2 \times D_8$	$\mathrm{II}_{(0,18)}2_2^{-4}8_2^2 3^1$	$\mathrm{II}_{(3,3)}2_2^4 8_6^2$	—	—	992	85
30.16	18	16	$C_2 \times D_8$	$\mathrm{II}_{(0,18)}2^2 4_0^4 3^1$	$\mathrm{II}_{(3,3)}2^2 4_0^4$	—	—	992	91
31.18	18	16	$C_4 \bigcirc D_8$	$\mathrm{II}_{(0,18)}2_6^{-2}8_6^2 3^1$	$\mathrm{II}_{(3,3)}2_2^{-2}8_6^{-2}$	—	$\times$	1922	127
E18a.2	18	18	$C_3 \times S_3$	$\mathrm{II}_{(0,18)}3^{-7}$	$\mathrm{II}_{(3,3)}3^{-6}$			1053	81
53	18	20	F_5	$\mathrm{II}_{(0,18)}2_6^{-2}5^3$	$\mathrm{II}_{(3,3)}2_6^2 3^1 5^3$	(32)	$\times\times^t$	2246	0
20.5	18	20	D_{20}	$\mathrm{II}_{(0,18)}2^2 3^1 5^{-3}$	$\mathrm{II}_{(3,3)}2^2 5^{-3}$	—	$\times$	1466	87
52	18	21	$C_7 \rtimes C_3$	$\mathrm{II}_{(0,18)}7^3$	$\mathrm{II}_{(3,3)}3^1 7^{-3}$	(33)	$\times\times^t$	2709	0
18b.4	18	24	$C_3 \rtimes D_8$	$\mathrm{II}_{(0,18)}4^{-2}3^5$	$\mathrm{II}_{(3,3)}4^{-2}3^4$	—	$\times$	972	51
18b.5	18	24	$C_2^2 \times S_3$	$\mathrm{II}_{(0,18)}4_2^2 3^{-4}$	$\mathrm{II}_{(3,3)}4_6^2 3^3$	—	$\times$	1495	111
48	18	32	$[32, 34]$	$\mathrm{II}_{(0,18)}2_2^{-2}4^4$	$\mathrm{II}_{(3,3)}2_6^{-2}4^4 3^1$	—	—	1626	0
17.39	18	32	$[32, 38]$	$\mathrm{II}_{(0,18)}2_0^2 4_0^4 3^1$	$\mathrm{II}_{(3,3)}2_0^2 4_0^4$	—	—	1120	59
17.38	18	32	$C_2^2 \times D_8$	$\mathrm{II}_{(0,18)}2_4^{-2}4_0^4 3^1$	$\mathrm{II}_{(3,3)}2_0^2 4_0^4$	—	—	1136	75
17.41	18	32	$C_2 \times C_4 \bigcirc D_8$	$\mathrm{II}_{(0,18)}2^{-2}4_4^{-4}3^1$	$\mathrm{II}_{(3,3)}2^2 4_0^4$	—	—	1128	67

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
17.40	18	32	2_+^{1+4}	$\mathrm{II}_{(0,18)}2_4^{-2}4_0^43^1$	$\mathrm{II}_{(3,3)}2_0^24_0^4$	—	—	1132	71
46b	18	36	$C_3^2\rtimes C_4$	$\mathrm{II}_{(0,18)}2_2^23^29^{-1}$	$\mathrm{II}_{(3,3)}2_2^{-2}3^39^1$	(46)	$\times$	2790	0
47c	18	36	S_3^2	$\mathrm{II}_{(0,18)}2^{-2}3^39^{-1}$	$\mathrm{II}_{(3,3)}2^{-2}3^{-4}9^1$	(48)	—	1632	0
15b.6	18	36	$C_2\times C_3\rtimes S_3$	$\mathrm{II}_{(0,18)}2_4^{-2}3^5$	$\mathrm{II}_{(3,3)}2_0^23^4$	—	$\times$	1716	115
29b.10	18	48	$C_2\times S_4$	$\mathrm{II}_{(0,18)}4_5^{-1}8_7^13^3$	$\mathrm{II}_{(3,3)}4_5^{-1}8_7^13^2$	—	$\times$	1835	129
45b	18	48	$C_2\times S_4$	$\mathrm{II}_{(0,18)}2^{-2}4_6^{-2}3^2$	$\mathrm{II}_{(3,3)}2^{-2}4_2^{-2}3^3$	(51)	—	2124	0
44b	18	60	A_5	$\mathrm{II}_{(0,18)}2^{-2}3^15^{-2}$	$\mathrm{II}_{(3,3)}2^{-2}3^{-2}5^{-2}$	(55)	$\times\times^t$	2910	0
28.14	18	64	$\Gamma_{25}a_1$	$\mathrm{II}_{(0,18)}4_7^38_5^{-1}3^1$	$\mathrm{II}_{(3,3)}4_1^{-3}8_7^1$	—	—	1417	93
43	18	64	$\Gamma_{25}a_1$	$\mathrm{II}_{(0,18)}4_7^38_7^1$	$\mathrm{II}_{(3,3)}4_1^38_1^13^1$	(56)	—	2277	0
26.19	18	64	D_8^2	$\mathrm{II}_{(0,18)}2^28_4^{-2}3^1$	$\mathrm{II}_{(3,3)}2^28_4^{-2}$	—	—	1956	125
26.21	18	64	$C_2\times C_2^2\wr C_2$	$\mathrm{II}_{(0,18)}2_4^{-4}4_0^23^1$	$\mathrm{II}_{(3,3)}2_0^44_0^2$	—	—	1964	135
26.20	18	64	$C_2\times C_2^2\wr C_2$	$\mathrm{II}_{(0,18)}4_0^{-4}3^1$	$\mathrm{II}_{(3,3)}4_0^{-4}$	—	—	1972	143
41	18	64	$[64,266]$	$\mathrm{II}_{(0,18)}2_2^{-2}4^4$	$\mathrm{II}_{(3,3)}2_6^{-2}4^43^1$	—	—	1690	0
47b.4	18	72	N_{72}	$\mathrm{II}_{(0,18)}2^{-2}3^39^{-1}$	$\mathrm{II}_{(3,3)}2^{-2}3^29^1$	—	$\times$	1632	135
46a.4	18	72	N_{72}	$\mathrm{II}_{(0,18)}2_6^{-2}3^29^{-1}$	$\mathrm{II}_{(3,3)}2_2^{-2}3^{-1}9^1$	—	$\times$	2790	213
39b	18	72	$A_{4,3}$	$\mathrm{II}_{(0,18)}4^{-2}3^{-3}$	$\mathrm{II}_{(3,3)}4^{-2}3^4$	(61)	$\times\times^t$	2421	0
47a.3	18	72	$C_2\times S_3^2$	$\mathrm{II}_{(0,18)}2^{-2}3^39^{-1}$	$\mathrm{II}_{(3,3)}2^{-2}3^29^1$	—	$\times$	1632	153
45a.4	18	96	$C_2^2\times S_4$	$\mathrm{II}_{(0,18)}2^24_2^23^2$	$\mathrm{II}_{(3,3)}2^{-2}4_2^{-2}3^{-1}$	—	—	2124	177
38	18	96	$C_2^2\rtimes S_4$	$\mathrm{II}_{(0,18)}2^24_3^{-1}8_1^13^{-1}$	$\mathrm{II}_{(3,3)}2^24_5^{-1}8_7^13^2$	(65)	—	2598	0
24.8	18	96	$C_2\times C_2^2\rtimes A_4$	$\mathrm{II}_{(0,18)}2^24_6^23^{-2}$	$\mathrm{II}_{(3,3)}2^24_2^23^1$	—	—	2253	151
44a.4	18	120	S_5	$\mathrm{II}_{(0,18)}2^{-2}3^15^{-2}$	$\mathrm{II}_{(3,3)}2^{-2}5^{-2}$	—	$\times$	2910	225
39a.3	18	144	$S_3\times S_4$	$\mathrm{II}_{(0,18)}4^{-2}3^{-3}$	$\mathrm{II}_{(3,3)}4^{-2}3^{-2}$	—	$\times$	2421	189
37	18	192	$C_4^2\rtimes A_4$	$\mathrm{II}_{(0,18)}2^{-2}8_6^{-2}$	$\mathrm{II}_{(3,3)}2^{-2}8_2^{-2}3^1$	(75)	—	3168	0
35b	18	486	$[486,249]$	$\mathrm{II}_{(0,18)}3^5$	$\mathrm{II}_{(3,3)}3^{-6}$	—	—	3240	0
35a.6	18	972	$[972,812]$	$\mathrm{II}_{(0,18)}3^5$	$\mathrm{II}_{(3,3)}3^4$	—	$\times$	3240	243
9.9	19	8	C_8	$\mathrm{II}_{(0,19)}2^24_7^18^23^1$	$\mathrm{II}_{(3,2)}2^{-2}4_1^18^{-2}$	—	—	1329	51
23.24	19	8	C_2^3	$\mathrm{II}_{(0,19)}4_6^48_1^13^1$	$\mathrm{II}_{(3,2)}4_0^48_1^1$	—	—	980	59
23.16	19	8	C_2^3	$\mathrm{II}_{(0,19)}4_7^53^1$	$\mathrm{II}_{(3,2)}4_1^5$	—	—	1193	83
7b.2	19	12	D_{12}	$\mathrm{II}_{(0,19)}2_4^{-4}4_7^13^5$	$\mathrm{II}_{(3,2)}2_0^44_1^13^4$	—	—	773	27
7b.3	19	12	D_{12}	$\mathrm{II}_{(0,19)}2_2^44_3^{-1}3^4$	$\mathrm{II}_{(3,2)}2_0^44_3^{-1}3^{-3}$	—	—	1037	43
98b	19	12	D_{12}	$\mathrm{II}_{(0,19)}2_6^{-4}4_5^{-1}3^{-3}$	$\mathrm{II}_{(3,2)}2_2^{-4}4_3^{-1}3^4$	—	—	1502	0
97b	19	12	D_{12}	$\mathrm{II}_{(0,19)}2_2^{-4}4_7^13^{-2}$	$\mathrm{II}_{(3,2)}2_0^44_3^{-1}3^{-3}$	—	—	2510	0
6.7	19	16	$C_2^2\rtimes C_4$	$\mathrm{II}_{(0,19)}4_7^53^1$	$\mathrm{II}_{(3,2)}4_1^5$	—	—	1361	59
6.12	19	16	$C_2^2\times C_4$	$\mathrm{II}_{(0,19)}2^24_7^33^1$	$\mathrm{II}_{(3,2)}2^24_1^3$	—	—	2317	119
30.11	19	16	$C_2\times D_8$	$\mathrm{II}_{(0,19)}2_6^24_1^18_4^{-2}3^1$	$\mathrm{II}_{(3,2)}2_6^24_1^18_2^2$	—	—	1193	81
13.10	19	16	$C_2\times D_8$	$\mathrm{II}_{(0,19)}2_6^24_1^18_4^{-2}3^1$	$\mathrm{II}_{(3,2)}2_6^24_1^18_2^2$	—	—	1217	73
58.12	19	16	$C_2\times D_8$	$\mathrm{II}_{(0,19)}2_6^24_1^18_0^23^1$	$\mathrm{II}_{(3,2)}2_6^24_1^18_2^2$	—	—	1217	73
13.9	19	16	$C_2\times D_8$	$\mathrm{II}_{(0,19)}2_4^{-2}4_1^18_6^23^1$	$\mathrm{II}_{(3,2)}2_6^24_1^18_2^2$	—	—	1363	55

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
6.6	19	16	$C_2 \times D_8$	$\mathrm{II}_{(0,19)}4_7^5 3^1$	$\mathrm{II}_{(3,2)}4_1^5$	—	—	1365	55
59.13	19	16	$C_2 \times D_8$	$\mathrm{II}_{(0,19)}2_6^{-2}4_4^{-2}8_5^{-1}3^1$	$\mathrm{II}_{(3,2)}2_4^{-2}4_0^2 8_5^{-1}$	—	—	1672	111
95	19	16	$C_2 \times D_8$	$\mathrm{II}_{(0,19)}2_0^2 4_5^{-1}8_0^2$	$\mathrm{II}_{(3,2)}2_0^2 4_3^{-1}8_0^2 3^1$	—	—	1929	0
31.13	19	16	$C_2 \times Q_8$	$\mathrm{II}_{(0,19)}2_7^{-3}8^{-2}3^1$	$\mathrm{II}_{(3,2)}2_1^{-3}8^{-2}$	—	—	1884	69
94	19	16	$C_2 \times Q_8$	$\mathrm{II}_{(0,19)}2_4^4 16_1^1$	$\mathrm{II}_{(3,2)}2_0^{-4}16_7^1 3^1$	—	—	3712	0
59.14	19	16	$C_4 \bigcirc D_8$	$\mathrm{II}_{(0,19)}2_3^3 8^{-2}3^1$	$\mathrm{II}_{(3,2)}2_1^{-3}8^{-2}$	—	—	1656	97
31.16	19	16	$C_4 \bigcirc D_8$	$\mathrm{II}_{(0,19)}2_2^4 16_5^{-1}3^1$	$\mathrm{II}_{(3,2)}2_2^{-4}16_3^{-1}$	—	—	2314	119
58.13	19	16	C_2^4	$\mathrm{II}_{(0,19)}4_7^5 3^1$	$\mathrm{II}_{(3,2)}4_1^5$	—	—	1241	99
58.14	19	16	C_2^4	$\mathrm{II}_{(0,19)}2_2^{-2}4_1^3 3^1$	$\mathrm{II}_{(3,2)}2_0^2 4_1^3$	—	—	2209	171
E18b.1	19	18	$C_3^2 \rtimes C_2$	$\mathrm{II}_{(0,19)}2_5^{-1}3^6$	$\mathrm{II}_{(3,2)}2_3^{-1}3^{-5}$			1539	117
19b.9	19	24	S_4	$\mathrm{II}_{(0,19)}2_6^{-4}8_3^{-1}3^2$	$\mathrm{II}_{(3,2)}2_2^{-4}8_1^1 3^{-1}$	—	—	1908	105
92b	19	24	S_4	$\mathrm{II}_{(0,19)}2_6^4 8_1^1 3^1$	$\mathrm{II}_{(3,2)}2_2^4 8_7^1 3^{-2}$	—	—	3052	0
98a.2	19	24	$C_2^2 \times S_3$	$\mathrm{II}_{(0,19)}2_2^4 4_5^{-1}3^{-3}$	$\mathrm{II}_{(3,2)}2_2^{-4}4_3^{-1}3^{-2}$	—	—	1502	129
18b.3	19	24	$C_2^2 \times S_3$	$\mathrm{II}_{(0,19)}2_0^4 4_7^1 3^{-3}$	$\mathrm{II}_{(3,2)}2_0^4 4_1^1 3^{-2}$	—	—	1592	99
97a.2	19	24	$C_2^2 \times S_3$	$\mathrm{II}_{(0,19)}2_6^4 4_7^1 3^{-2}$	$\mathrm{II}_{(3,2)}2_0^4 4_3^{-1}3^1$	—	—	2510	183
17.26	19	32	$C_2 \times D_{16}$	$\mathrm{II}_{(0,19)}2_3^{-3}8^2 3^1$	$\mathrm{II}_{(3,2)}2_1^3 8^2$	—	—	1684	85
55.15	19	32	$C_4 \bigcirc D_{16}$	$\mathrm{II}_{(0,19)}4_7^1 8_0^2 3^1$	$\mathrm{II}_{(3,2)}4_7^1 8_2^2$	—	×	2323	127
17.25	19	32	$\mathrm{Aut}(D_{16})$	$\mathrm{II}_{(0,19)}2_3^3 8^{-2}3^1$	$\mathrm{II}_{(3,2)}2_1^{-3}8^{-2}$	—	—	1680	89
55.14	19	32	$\mathrm{Aut}(D_{16})$	$\mathrm{II}_{(0,19)}4_7^1 8_4^{-2}3^1$	$\mathrm{II}_{(3,2)}4_3^{-1}8_6^2$	—	—	2331	135
89	19	32	$\mathrm{Aut}(D_{16})$	$\mathrm{II}_{(0,19)}2_1^{-3}8^2$	$\mathrm{II}_{(3,2)}2_7^{-3}8^2 3^1$	—	—	2664	0
52.3	19	42	F_7	$\mathrm{II}_{(0,19)}2_3^{-1}3^1 7^{-2}$	$\mathrm{II}_{(3,2)}2_5^{-1}7^{-2}$	—	×	3465	217
87	19	48	T_{48}	$\mathrm{II}_{(0,19)}2_3^{-1}8^{-2}3^{-1}$	$\mathrm{II}_{(3,2)}2_1^1 8^{-2}3^2$	(54)	$\times\times^t$	3021	0
92a.4	19	48	$C_2 \times S_4$	$\mathrm{II}_{(0,19)}2_6^4 8_5^{-1}3^1$	$\mathrm{II}_{(3,2)}2_2^4 8_7^1$	—	—	3052	207
26.13	19	64	$[64, 215]$	$\mathrm{II}_{(0,19)}2_0^2 4_0^2 8_7^1 3^1$	$\mathrm{II}_{(3,2)}2_4^{-2}4_0^2 8_5^{-1}$	—	—	1932	75
48.6	19	64	D_8^2	$\mathrm{II}_{(0,19)}2_2^{-2}4^2 8_1^1 3^1$	$\mathrm{II}_{(3,2)}2_6^{-2}4^{-2}8_7^1$	—	—	1706	109
28.9	19	64	$[64, 266]$	$\mathrm{II}_{(0,19)}4_7^5 3^1$	$\mathrm{II}_{(3,2)}4_1^5$	—	—	1425	75
85b	19	72	N_{72}	$\mathrm{II}_{(0,19)}4_1^1 3^2 9^{-1}$	$\mathrm{II}_{(3,2)}4_7^1 3^3 9^1$	(62)	—	3335	0
84	19	72	M_9	$\mathrm{II}_{(0,19)}2_3^3 3^{-1}9^{-1}$	$\mathrm{II}_{(3,2)}2_5^3 3^2 9^1$	(63)	×	4050	0
45b.5	19	96	$C_2^2 \times S_4$	$\mathrm{II}_{(0,19)}2^{-2}8_7^1 3^3$	$\mathrm{II}_{(3,2)}2^{-2}8_1^1 3^2$	—	—	2236	141
82b	19	120	S_5	$\mathrm{II}_{(0,19)}4_3^{-1}3^1 5^{-2}$	$\mathrm{II}_{(3,2)}4_5^{-1}3^{-2}5^{-2}$	(70)	$\times\times^t$	3471	0
43.12	19	128	$C_2 \wr D_8$	$\mathrm{II}_{(0,19)}4_7^1 8_4^{-2}3^1$	$\mathrm{II}_{(3,2)}4_3^{-1}8_6^2$	—	—	2373	125
43.13	19	128	$C_2 \times \Gamma_{25}a_1$	$\mathrm{II}_{(0,19)}2_4^{-2}4_7^3 3^1$	$\mathrm{II}_{(3,2)}2_0^2 4_1^3$	—	—	2389	143
41.4	19	128	$[128, 1757]$	$\mathrm{II}_{(0,19)}2_6^2 4^2 8_1^1 3^1$	$\mathrm{II}_{(3,2)}2_6^{-2}4^{-2}8_7^1$	—	—	1754	93
80	19	128	$[128, 1759]$	$\mathrm{II}_{(0,19)}2_2^{-2}4^{-2}8_7^1$	$\mathrm{II}_{(3,2)}2_2^2 4^{-2}8_1^1 3^1$	—	—	2742	0
85a.1	19	144	$[144, 186]$	$\mathrm{II}_{(0,19)}4_1^1 3^2 9^{-1}$	$\mathrm{II}_{(3,2)}4_7^1 3^{-1}9^1$	—	×	3335	225
39b.2	19	144	$C_2 \times A_{4,3}$	$\mathrm{II}_{(0,19)}8_1^1 3^{-4}$	$\mathrm{II}_{(3,2)}8_7^1 3^3$	—	×	2541	145
77	19	168	$\mathrm{PSL}(2,7)$	$\mathrm{II}_{(0,19)}4_1^1 7^2$	$\mathrm{II}_{(3,2)}4_7^1 3^1 7^2$	(74)	$\times\times^t$	4263	0
74	19	192	H_{192}	$\mathrm{II}_{(0,19)}4_2^{-2}8_1^1 3^{-1}$	$\mathrm{II}_{(3,2)}4_6^{-2}8_7^1 3^2$	(76)	—	3111	0

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
76b	19	192	T_{192}	$\mathrm{II}_{(0,19)}4_7^{-3}3^1$	$\mathrm{II}_{(3,2)}4_1^{-3}3^{-2}$	(77)	—	4317	0
38.8	19	192	$C_2 \times C_2^2 \rtimes S_4$	$\mathrm{II}_{(0,19)}2_4^{-4}4_5^{-1}3^{-2}$	$\mathrm{II}_{(3,2)}2_0^4 4_3^{-1}3^1$	—	—	2718	151
38.7	19	192	$C_2 \times C_2^2 \rtimes S_4$	$\mathrm{II}_{(0,19)}4_5^3 3^{-2}$	$\mathrm{II}_{(3,2)}4_3^3 3^1$	—	—	2726	159
82a.2	19	240	$C_2 \times S_5$	$\mathrm{II}_{(0,19)}4_3^{-1}3^1 5^{-2}$	$\mathrm{II}_{(3,2)}4_5^{-1}5^{-2}$	—	$\times$	3471	237
73b	19	288	$A_{4,4}$	$\mathrm{II}_{(0,19)}2^2 8_1^1 3^2$	$\mathrm{II}_{(3,2)}2^2 8_7^1 3^3$	(78)	—	3528	0
72b	19	360	A_6	$\mathrm{II}_{(0,19)}4_5^{-1}3^2 5^1$	$\mathrm{II}_{(3,2)}4_3^{-1}3^3 5^1$	(79)	$\times\times^t$	4455	0
70	19	384	F_{384}	$\mathrm{II}_{(0,19)}4_3^{-1}8_6^{-2}$	$\mathrm{II}_{(3,2)}4_5^{-1}8_2^{-2}3^1$	(80)	—	3777	0
37.6	19	384	$[384, 18235]$	$\mathrm{II}_{(0,19)}4_0^2 8_3^{-1}3^1$	$\mathrm{II}_{(3,2)}4_0^2 8_5^{-1}$	—	—	3312	175
76a.4	19	384	$[384, 20089]$	$\mathrm{II}_{(0,19)}4_7^{-3}3^1$	$\mathrm{II}_{(3,2)}4_1^{-3}$	—	—	4317	279
73a.4	19	576	S_4^2	$\mathrm{II}_{(0,19)}2^2 8_1^1 3^2$	$\mathrm{II}_{(3,2)}2^2 8_7^1 3^{-1}$	—	—	3528	225
72a.3	19	720	S_6	$\mathrm{II}_{(0,19)}4_5^{-1}3^2 5^1$	$\mathrm{II}_{(3,2)}4_3^{-1}3^{-1}5^1$	—	$\times$	4455	285
69	19	960	M_{20}	$\mathrm{II}_{(0,19)}2^{-2}8_1^1 5^{-1}$	$\mathrm{II}_{(3,2)}2^{-2}8_7^1 3^1 5^{-1}$	(81)	—	4740	0
68b	19	972	$[972, 776]$	$\mathrm{II}_{(0,19)}2_1^1 3^{-4}$	$\mathrm{II}_{(3,2)}2_7^1 3^{-5}$	—	—	4698	0
68a.4	19	1944	$[1944, 3536]$	$\mathrm{II}_{(0,19)}2_5^{-1}3^{-4}$	$\mathrm{II}_{(3,2)}2_7^1 3^3$	—	$\times$	4698	297
23.3	20	8	$C_2 \times C_4$	$\mathrm{II}_{(0,20)}4_6^{-2}8_4^{-2}3^1$	$\mathrm{II}_{(3,1)}4_0^2 8_2^2$	—	—	1406	53
23.9	20	8	C_2^3	$\mathrm{II}_{(0,20)}4_6^4 3^1$	$\mathrm{II}_{(3,1)}4_2^4$	—	—	2526	151
61b.3	20	12	D_{12}	$\mathrm{II}_{(0,20)}2_6^{-2}4^{-2}3^{-5}$	$\mathrm{II}_{(3,1)}2_6^2 4^2 3^{-4}$	—	—	771	39
61b.5	20	12	D_{12}	$\mathrm{II}_{(0,20)}2_2^2 4_2^{-2}3^4$	$\mathrm{II}_{(3,1)}2_0^2 4_4^{-2}3^{-3}$	—	—	1090	59
63b.4	20	12	D_{12}	$\mathrm{II}_{(0,20)}2_2^{-2}4_4^{-2}3^3$	$\mathrm{II}_{(3,1)}2_0^2 4_6^2 3^2$	—	—	1832	95
61b.4	20	12	D_{12}	$\mathrm{II}_{(0,20)}2^2 4_4^{-2}3^4$	$\mathrm{II}_{(3,1)}2^{-2}4_0^2 3^{-3}$	—	—	1106	75
161	20	12	D_{12}	$\mathrm{II}_{(0,20)}2_4^4 3^4$	$\mathrm{II}_{(3,1)}2_0^{-4}3^{-3}$	—	—	1872	105
60b.2	20	12	D_{12}	$\mathrm{II}_{(0,20)}2_2^{-2}4^2 3^{-3}$	$\mathrm{II}_{(3,1)}2_6^{-2}4^{-2}3^{-2}$	—	—	2149	63
20.1	20	20	D_{20}	$\mathrm{II}_{(0,20)}2_6^4 3^1 5^2$	$\mathrm{II}_{(3,1)}2_2^4 5^2$	—	—	2078	115
154b	20	24	$C_2^2 \times S_3$	$\mathrm{II}_{(0,20)}2_4^{-2}4_4^{-2}3^{-2}$	$\mathrm{II}_{(3,1)}2_0^2 4_4^{-2}3^{-3}$	—	—	2971	0
18b.1	20	24	$C_2^2 \times S_3$	$\mathrm{II}_{(0,20)}2_0^2 4_4^{-2}3^4$	$\mathrm{II}_{(3,1)}2_0^2 4_4^{-2}3^{-3}$	—	—	1282	51
157b	20	24	D_{24}	$\mathrm{II}_{(0,20)}2_2^2 4^{-2}3^3$	$\mathrm{II}_{(3,1)}2_2^{-2}4^{-2}3^{-4}$	—	—	1716	0
18b.2	20	24	D_{24}	$\mathrm{II}_{(0,20)}2_2^{-2}4^2 3^{-3}$	$\mathrm{II}_{(3,1)}2_6^{-2}4^{-2}3^{-2}$	—	—	1861	71
19b.8	20	24	S_4	$\mathrm{II}_{(0,20)}2_4^{-2}4_5^{-1}8_7^1 3^{-2}$	$\mathrm{II}_{(3,1)}2_4^{-2}4_3^{-1}8_5^{-1}3^1$	—	—	2266	99
56.3	20	24	S_4	$\mathrm{II}_{(0,20)}4_5^{-3}8_5^{-1}3^1$	$\mathrm{II}_{(3,1)}4_1^3 8_1^1$	—	—	1899	93
152	20	32	$C_2 \times QD_{16}$	$\mathrm{II}_{(0,20)}2_4^{-2}4_3^{-1}16_1^1$	$\mathrm{II}_{(3,1)}2_0^2 4_5^{-1}16_7^1 3^1$	—	—	4345	0
17.8	20	32	$[32, 28]$	$\mathrm{II}_{(0,20)}4_6^4 3^1$	$\mathrm{II}_{(3,1)}4_2^4$	—	—	2762	123
55.11	20	32	$\mathrm{Aut}(D_{16})$	$\mathrm{II}_{(0,20)}2_0^2 4_5^{-1}16_1^1 3^1$	$\mathrm{II}_{(3,1)}2_4^{-2}4_3^{-1}16_7^1$	—	—	2755	119
55.7	20	32	$\Gamma_6 a_2$	$\mathrm{II}_{(0,20)}2_5^{-1}4_5^{-1}8^2 3^1$	$\mathrm{II}_{(3,1)}2_1^1 4_1^1 8^2$	—	—	2285	69
95.5	20	32	2_+^{1+4}	$\mathrm{II}_{(0,20)}2_5^{-1}4_5^{-1}8^2 3^1$	$\mathrm{II}_{(3,1)}2_1^1 4_1^1 8^2$	—	—	2001	97
94.2	20	32	2_-^{1+4}	$\mathrm{II}_{(0,20)}2_3^3 16_3^{-1}3^1$	$\mathrm{II}_{(3,1)}2_1^{-3}16_5^{-1}$	—	—	3852	165
95.6	20	32	$C_2^2 \times D_8$	$\mathrm{II}_{(0,20)}4_7^3 8_7^1 3^1$	$\mathrm{II}_{(3,1)}4_1^3 8_1^1$	—	—	2017	115
151c	20	36	S_3^2	$\mathrm{II}_{(0,20)}2_6^{-4}3^3$	$\mathrm{II}_{(3,1)}2_6^4 3^{-4}$	—	—	3352	0
15b.1	20	36	S_3^2	$\mathrm{II}_{(0,20)}2_6^4 3^{-3}9^{-1}$	$\mathrm{II}_{(3,1)}2_2^4 3^{-2}9^1$	—	—	1479	51

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
15b.3	20	36	$C_2 \times C_3 \rtimes S_3$	$\mathrm{II}_{(0,20)}2_0^{-4}3^{-2}9^{-1}$	$\mathrm{II}_{(3,1)}2_0^{-4}3^1 9^1$	—	—	2115	91
149	20	40	$C_2 \times F_5$	$\mathrm{II}_{(0,20)}2_4^4 5^2$	$\mathrm{II}_{(3,1)}2_0^{-4}3^1 5^2$	—	—	3512	0
53.4	20	40	$C_2 \times F_5$	$\mathrm{II}_{(0,20)}2_6^{-4}3^1 5^{-2}$	$\mathrm{II}_{(3,1)}2_6^4 5^{-2}$	—	—	2412	97
148b	20	48	$C_2 \times S_4$	$\mathrm{II}_{(0,20)}2_4^{-2}4_7^1 8_7^1 3^1$	$\mathrm{II}_{(3,1)}2_0^2 4_1^1 8_1^1 3^{-2}$	—	—	3593	0
154a.3	20	48	$C_2^3 \times S_3$	$\mathrm{II}_{(0,20)}2_2^2 4_2^2 3^{-2}$	$\mathrm{II}_{(3,1)}2_0^2 4_4^{-2}3^1$	—	—	2971	195
157a.2	20	48	$D_8 \times S_3$	$\mathrm{II}_{(0,20)}2_2^2 4^{-2}3^3$	$\mathrm{II}_{(3,1)}2_2^{-2}4^{-2}3^2$	—	—	1716	123
29b.4	20	48	$C_2 \times S_4$	$\mathrm{II}_{(0,20)}2_2^{-2}4_1^1 8_1^1 3^2$	$\mathrm{II}_{(3,1)}2_6^2 4_1^1 8_5^{-1}3^{-1}$	—	—	2289	105
92b.3	20	48	$C_2 \times S_4$	$\mathrm{II}_{(0,20)}2_0^2 4_0^2 3^2$	$\mathrm{II}_{(3,1)}2_4^{-2}4_0^2 3^{-1}$	—	—	3172	147
144	20	60	A_5	$\mathrm{II}_{(0,20)}2^{-2}3^4$	$\mathrm{II}_{(3,1)}2^{-2}3^{-3}$	—	—	3780	195
143	20	64	$D_8 \bigcirc D_{16}$	$\mathrm{II}_{(0,20)}2_3^{-1}4_5^{-1}8^2$	$\mathrm{II}_{(3,1)}2_1^1 4_3^{-1}8^2 3^1$	—	—	3153	0
28.6	20	64	$D_8 \bigcirc D_{16}$	$\mathrm{II}_{(0,20)}2_3^{-1}4_3^{-1}8^{-2}3^1$	$\mathrm{II}_{(3,1)}2_1^1 4_1^1 8^2$	—	—	2041	89
89.4	20	64	$D_8 \bigcirc D_{16}$	$\mathrm{II}_{(0,20)}2_6^2 8_4^{-2}3^1$	$\mathrm{II}_{(3,1)}2_2^2 8_0^2$	—	—	2776	135
89.5	20	64	$[64, 256]$	$\mathrm{II}_{(0,20)}2^{-2}8_6^{-2}3^1$	$\mathrm{II}_{(3,1)}2^{-2}8_2^{-2}$	—	—	2768	131
141b	20	72	N_{72}	$\mathrm{II}_{(0,20)}2_2^{-4}3^1 9^{-1}$	$\mathrm{II}_{(3,1)}2_2^4 3^{-2}9^1$	—	—	3432	0
150.3	20	72	$C_2 \times S_3^2$	$\mathrm{II}_{(0,20)}2_4^4 3^4$	$\mathrm{II}_{(3,1)}2_0^{-4}3^{-3}$	—	—	2064	153
47c.4	20	72	$C_2 \times S_3^2$	$\mathrm{II}_{(0,20)}2_0^{-4}3^4$	$\mathrm{II}_{(3,1)}2_0^{-4}3^{-3}$	—	—	2178	115
151a.3	20	72	N_{72}	$\mathrm{II}_{(0,20)}2_6^{-4}3^3$	$\mathrm{II}_{(3,1)}2_6^4 3^2$	—	—	3352	189
151b.3	20	72	$S_2 \times S_3^2$	$\mathrm{II}_{(0,20)}2_6^{-4}3^3$	$\mathrm{II}_{(3,1)}2_6^4 3^2$	—	—	3352	207
46b.7	20	72	$C_2 \times C_3^2 \rtimes C_4$	$\mathrm{II}_{(0,20)}2_2^4 3^3$	$\mathrm{II}_{(3,1)}2_6^4 3^2$	—	$\times$	3642	151
148a.2	20	96	$C_2^2 \times S_4$	$\mathrm{II}_{(0,20)}2_0^2 4_7^1 8_3^{-1}3^1$	$\mathrm{II}_{(3,1)}2_2^{-2}4_3^{-1}8_1^1$	—	—	3593	219
138	20	108	$[108, 17]$	$\mathrm{II}_{(0,20)}3^2 9^2$	$\mathrm{II}_{(3,1)}3^{-1}9^2$	—	—	2556	117
E20b	20	108	[108,40]	$\mathrm{II}_{(0,20)}3^{-4}9^1$	$\mathrm{II}_{(3,1)}3^3 9^1$			2592	108
44b.5	20	120	S_5	$\mathrm{II}_{(0,20)}2_2^4 3^1 5^1$	$\mathrm{II}_{(3,1)}2_2^{-4}5^1$	—	—	4460	225
137b	20	120	S_5	$\mathrm{II}_{(0,20)}2_6^4 3^1 5^{-1}$	$\mathrm{II}_{(3,1)}2_6^{-4}3^{-2}5^{-1}$	—	—	4496	0
134	20	144	$\mathrm{A\Gamma L}_1(\mathbb{F}_9)$	$\mathrm{II}_{(0,20)}2_7^1 4_3^{-1}3^{-1}9^{-1}$	$\mathrm{II}_{(3,1)}2_7^1 4_7^1 3^2 9^1$	—	—	4739	0
141a.1	20	144	$[144, 186]$	$\mathrm{II}_{(0,20)}2_2^{-4}3^1 9^{-1}$	$\mathrm{II}_{(3,1)}2_2^4 9^1$	—	—	3432	213
133	20	144	$C_2 \times M_9$	$\mathrm{II}_{(0,20)}2_4^4 9^{-1}$	$\mathrm{II}_{(3,1)}2_0^{-4}3^1 9^1$	—	—	5736	0
131	20	192	$[192, 1494]$	$\mathrm{II}_{(0,20)}2_2^{-2}8_2^{-2}$	$\mathrm{II}_{(3,1)}2_2^2 8_6^{-2}3^1$	—	—	4290	0
E20a.1	20	216	[216,158]	$\mathrm{II}_{(0,20)}3^{-4}9^1$	$\mathrm{II}_{(3,1)}3^3 9^1$			2592	135
E20c.1	20	216	S_3^3	$\mathrm{II}_{(0,20)}3^{-4}9^1$	$\mathrm{II}_{(3,1)}3^3 9^1$			2592	189
137a.2	20	240	$C_2 \times S_5$	$\mathrm{II}_{(0,20)}2_6^4 3^1 5^{-1}$	$\mathrm{II}_{(3,1)}2_6^{-4}5^{-1}$	—	—	4496	255
80.3	20	256	$[256, 26541]$	$\mathrm{II}_{(0,20)}2_2^{-2}8_0^2 3^1$	$\mathrm{II}_{(3,1)}2_2^2 8_0^2$	—	—	2838	125
77.3	20	336	$\mathrm{PGL}(2,7)$	$\mathrm{II}_{(0,20)}2_3^{-1}4_1^1 3^1 7^{-1}$	$\mathrm{II}_{(3,1)}2_7^1 4_1^1 7^1$	—	$\times$	5327	273
129	20	336	$C_2 \times \mathrm{PSL}(3,2)$	$\mathrm{II}_{(0,20)}2^2 7^2$	$\mathrm{II}_{(3,1)}2^2 3^1 7^2$	—	—	4992	0
128b	20	360	$\Gamma\mathrm{L}_2(\mathbb{F}_4)$	$\mathrm{II}_{(0,20)}3^{-2}5^{-2}$	$\mathrm{II}_{(3,1)}3^{-3}5^{-2}$	—	$\times^t$	4593	0
76b.3	20	384	$[384, 5602]$	$\mathrm{II}_{(0,20)}4_7^1 8_1^1 3^2$	$\mathrm{II}_{(3,1)}4_1^1 8_7^1 3^{-1}$	—	—	4477	189
74.2	20	384	$[384, 17948]$	$\mathrm{II}_{(0,20)}2_4^{-2}4_4^{-2}3^{-2}$	$\mathrm{II}_{(3,1)}2_0^2 4_4^{-2}3^1$	—	—	3239	159
37.4	20	384	$[384, 18134]$	$\mathrm{II}_{(0,20)}2_3^3 16_3^{-1}3^1$	$\mathrm{II}_{(3,1)}2_1^{-3}16_5^{-1}$	—	—	3888	153

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
124	20	384	$[384, 18134]$	$\mathrm{II}_{(0,20)}2_5^3 16_7^1$	$\mathrm{II}_{(3,1)}2_7^{-3}16_1^1 3^1$	—	—	6088	0
123b	20	384	$[384, 20097]$	$\mathrm{II}_{(0,20)}2_6^2 4^{-2}3^1$	$\mathrm{II}_{(3,1)}2_6^{-2}4^{-2}3^{-2}$	—	—	5038	0
122	20	500	$[500, 23]$	$\mathrm{II}_{(0,20)}5^3$	$\mathrm{II}_{(3,1)}3^1 5^3$	—	—	6150	0
121b	20	576	$[576, 8654]$	$\mathrm{II}_{(0,20)}4_7^1 8_1^1 3^2$	$\mathrm{II}_{(3,1)}4_1^1 8_7^1 3^3$	—	—	4137	0
73b.3	20	576	$[576, 8657]$	$\mathrm{II}_{(0,20)}4_2^{-2}3^3$	$\mathrm{II}_{(3,1)}4_6^{-2}3^2$	—	—	3672	175
120	20	660	$\mathrm{PSL}(2, 11)$	$\mathrm{II}_{(0,20)}11^2$	$\mathrm{II}_{(3,1)}3^1 11^2$	—	$\times\times^t$	6270	0
118b	20	720	S_6	$\mathrm{II}_{(0,20)}2^{-2}3^2 5^1$	$\mathrm{II}_{(3,1)}2^{-2}3^3 5^1$	—	—	5216	0
119	20	720	M_{10}	$\mathrm{II}_{(0,20)}2_3^{-1}4_7^1 3^{-1}5^1$	$\mathrm{II}_{(3,1)}2_5^{-1}4_1^1 3^2 5^1$	—	$\times$	6291	0
128a.1	20	720	$S_3 \times S_5$	$\mathrm{II}_{(0,20)}3^{-2}5^{-2}$	$\mathrm{II}_{(3,1)}3^1 5^{-2}$	—	$\times$	4593	261
70.4	20	768	$[768, 1090134]$	$\mathrm{II}_{(0,20)}2_4^{-2}4_7^1 8_7^1 3^1$	$\mathrm{II}_{(3,1)}2_4^{-2}4_1^1 8_5^{-1}$	—	—	3921	175
116	20	768	$[768, 1090135]$	$\mathrm{II}_{(0,20)}2_6^2 8_2^{-2}$	$\mathrm{II}_{(3,1)}2_2^2 8_6^{-2}3^1$	—	—	4434	0
123a.2	20	768	$[768, 1090220]$	$\mathrm{II}_{(0,20)}2_6^2 4^{-2}3^1$	$\mathrm{II}_{(3,1)}2_6^{-2}4^{-2}$	—	—	5038	279
112	20	1152	$[1152, 155478]$	$\mathrm{II}_{(0,20)}8_6^{-2}3^{-1}$	$\mathrm{II}_{(3,1)}8_2^{-2}3^2$	—	—	4995	0
121a.2	20	1152	$[1152, 157849]$	$\mathrm{II}_{(0,20)}4_7^1 8_1^1 3^2$	$\mathrm{II}_{(3,1)}4_1^1 8_7^1 3^{-1}$	—	—	4137	225
111	20	1344	$[1344, 11686]$	$\mathrm{II}_{(0,20)}4_2^2 7^1$	$\mathrm{II}_{(3,1)}4_6^2 3^1 7^{-1}$	—	—	6531	0
118a.1	20	1440	$C_2 \times S_6$	$\mathrm{II}_{(0,20)}2^{-2}3^2 5^1$	$\mathrm{II}_{(3,1)}2^{-2}3^{-1}5^1$	—	—	5216	297
110	20	1920	$[1920, 240993]$	$\mathrm{II}_{(0,20)}4_5^{-1}8_7^1 5^{-1}$	$\mathrm{II}_{(3,1)}4_3^{-1}8_1^1 3^1 5^{-1}$	—	—	5541	0
69.4	20	1920	$[1920, 240995]$	$\mathrm{II}_{(0,20)}4_6^2 3^1 5^{-1}$	$\mathrm{II}_{(3,1)}4_2^2 5^{-1}$	—	—	4916	207
114.3	20	1944	$[1944, 3537]$	$\mathrm{II}_{(0,20)}2^{-2}3^4$	$\mathrm{II}_{(3,1)}2^{-2}3^{-3}$	—	—	3972	243
109b	20	1944	$[1944, 3559]$	$\mathrm{II}_{(0,20)}2_6^{-2}3^3$	$\mathrm{II}_{(3,1)}2_6^2 3^{-4}$	—	$\times^t$	6642	0
108b	20	2520	A_7	$\mathrm{II}_{(0,20)}3^1 5^1 7^1$	$\mathrm{II}_{(3,1)}3^{-2}5^1 7^{-1}$	—	$\times\times^t$	6741	0
109a.4	20	3888	—	$\mathrm{II}_{(0,20)}2_6^{-2}3^3$	$\mathrm{II}_{(3,1)}2_6^2 3^2$	—	$\times$	6642	351
108a.1	20	5040	S_7	$\mathrm{II}_{(0,20)}3^1 5^1 7^1$	$\mathrm{II}_{(3,1)}5^1 7^{-1}$	—	$\times$	6741	357
106b	20	5760	$C_2^4 \rtimes A_6$	$\mathrm{II}_{(0,20)}4_7^1 8_3^{-1}3^1$	$\mathrm{II}_{(3,1)}4_1^1 8_5^{-1}3^{-2}$	—	—	7065	0
106a.2	20	11520	$C_2^4 \rtimes S_6$	$\mathrm{II}_{(0,20)}4_7^1 8_3^{-1}3^1$	$\mathrm{II}_{(3,1)}4_1^1 8_5^{-1}$	—	—	7065	375
102	20	20160	$\mathrm{PSL}(3, 4)$	$\mathrm{II}_{(0,20)}2^{-2}3^{-1}7^{-1}$	$\mathrm{II}_{(3,1)}2^{-2}3^2 7^1$	—	—	7560	0
101b	20	29160	$C_3^4 \times A_6$	$\mathrm{II}_{(0,20)}3^2 9^1$	$\mathrm{II}_{(3,1)}3^3 9^{-1}$	—	$\times^t$	7695	0
101a.1	20	58320	$C_3^4 \rtimes S_6$	$\mathrm{II}_{(0,20)}3^2 9^1$	$\mathrm{II}_{(3,1)}3^{-1}9^{-1}$	—	$\times$	7695	405
59.4	21	16	$C_2^2 \rtimes C_4$	$\mathrm{II}_{(0,21)}4_6^2 8_7^1 3^1$	$\mathrm{II}_{(3,0)}4_2^2 8_1^1$	—	—	3986	183
59.1	21	16	$\mathrm{M}_4(2)$	$\mathrm{II}_{(0,21)}4_2^{-2}16_3^{-1}3^1$	$\mathrm{II}_{(3,0)}4_2^2 16_1^1$	—	—	3172	117
E21a	21	18	$C_3 \times S_3$	$\mathrm{II}_{(0,21)}2_7^{-3}3^4$	$\mathrm{II}_{(3,0)}2_1^{-3}3^{-3}$			3051	135
56.1	21	24	S_4	$\mathrm{II}_{(0,21)}4_5^{-1}8_4^{-2}3^1$	$\mathrm{II}_{(3,0)}4_3^{-1}8_4^{-2}$	—	—	3127	117
56.2	21	24	S_4	$\mathrm{II}_{(0,21)}4_5^3 3^1$	$\mathrm{II}_{(3,0)}4_3^3$	—	—	5671	255
98b.1	21	24	$C_2^2 \times S_3$	$\mathrm{II}_{(0,21)}4_1^3 3^3$	$\mathrm{II}_{(3,0)}4_7^3 3^2$	—	—	2173	99
E21b.1	21	36	S_3^2	$\mathrm{II}_{(0,21)}2_7^{-3}3^4$	$\mathrm{II}_{(3,0)}2_1^{-3}3^{-3}$			3051	153
220a	21	42	F_7	$\mathrm{II}_{(0,21)}2_7^{-3}7^{-2}$	$\mathrm{II}_{(3,0)}2_1^{-3}3^1 7^{-2}$	—	—	4119	0
220b	21	42	F_7	$\mathrm{II}_{(0,21)}2_7^{-3}7^{-2}$	$\mathrm{II}_{(3,0)}2_1^{-3}3^1 7^{-2}$	—	—	4119	0
157b.1	21	48	$D_8 \times S_3$	$\mathrm{II}_{(0,21)}2_6^{-2}8_5^{-1}3^4$	$\mathrm{II}_{(3,0)}2_2^{-2}8_7^1 3^{-3}$	—	—	1780	85

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
154b.1	21	48	$D_8 \times S_3$	$\mathrm{II}_{(0,21)}2_2^{-2}8_7^1 3^{-3}$	$\mathrm{II}_{(3,0)}2_6^{-2}8_5^{-1}3^{-2}$	—	—	3051	105
48.1	21	64	$[64, 34]$	$\mathrm{II}_{(0,21)}4_5^{-1}8_4^{-2}3^1$	$\mathrm{II}_{(3,0)}4_3^{-1}8_4^{-2}$	—	—	3215	101
48.2	21	64	$[64, 241]$	$\mathrm{II}_{(0,21)}4_5^3 3^1$	$\mathrm{II}_{(3,0)}4_3^3$	—	—	5671	255
152.1	21	64	$D_8 \bigcirc QD_{16}$	$\mathrm{II}_{(0,21)}2_5^{-1}4_5^{-1}16_7^1 3^1$	$\mathrm{II}_{(3,0)}2_1^1 4_1^1 16_1^1$	—	—	4485	165
46b.1	21	72	F_9	$\mathrm{II}_{(0,21)}2^{-2}4_7^1 3^{-2}9^{-1}$	$\mathrm{II}_{(3,0)}2^2 4_5^{-1}3^1 9^1$	—	—	3009	75
46b.2	21	72	F_9	$\mathrm{II}_{(0,21)}2^{-2}4_7^1 3^{-2}9^{-1}$	$\mathrm{II}_{(3,0)}2^{-2}4_1^1 3^1 9^1$	—	—	3009	75
47c.1	21	72	$C_2 \times S_3^2$	$\mathrm{II}_{(0,21)}2_0^2 4_3^{-1}3^{-2}9^{-1}$	$\mathrm{II}_{(3,0)}2_0^2 4_5^{-1}3^1 9^1$	—	—	2480	99
45b.1	21	96	$C_4 \times S_4$	$\mathrm{II}_{(0,21)}4_3^{-3}3^{-2}$	$\mathrm{II}_{(3,0)}4_5^{-3}3^1$	—	—	3652	131
212	21	96	$C_2 \times T_{48}$	$\mathrm{II}_{(0,21)}2_4^{-2}16_1^1 3^{-1}$	$\mathrm{II}_{(3,0)}2_0^2 16_7^1 3^2$	—	—	5611	0
87.1	21	96	$[96, 190]$	$\mathrm{II}_{(0,21)}2_7^1 8^{-2}3^{-2}$	$\mathrm{II}_{(3,0)}2_1^1 8^{-2}3^1$	—	—	3087	69
87.2	21	96	$[96, 193]$	$\mathrm{II}_{(0,21)}2_2^{-2}16_5^{-1}3^{-2}$	$\mathrm{II}_{(3,0)}2_2^2 16_3^{-1}3^1$	—	—	3637	119
211	21	120	S_5	$\mathrm{II}_{(0,21)}2_1^{-3}3^{-3}$	$\mathrm{II}_{(3,0)}2_3^3 3^{-2}$	—	—	5256	225
143.1	21	128	$[128, 2317]$	$\mathrm{II}_{(0,21)}4_7^1 8_6^2 3^1$	$\mathrm{II}_{(3,0)}4_3^{-1}8_4^{-2}$	—	—	3265	139
39b.1	21	144	$S_3 \times S_4$	$\mathrm{II}_{(0,21)}2_2^{-2}8_3^{-1}3^{-3}$	$\mathrm{II}_{(3,0)}2_6^{-2}8_5^{-1}3^{-2}$	—	—	3051	105
208b	21	144	$S_3 \times S_4$	$\mathrm{II}_{(0,21)}2_6^{-2}8_5^{-1}3^{-2}$	$\mathrm{II}_{(3,0)}2_6^2 8_3^{-1}3^{-3}$	—	—	4675	0
208c	21	144	$S_3 \times S_4$	$\mathrm{II}_{(0,21)}2_6^{-2}8_5^{-1}3^{-2}$	$\mathrm{II}_{(3,0)}2_6^2 8_3^{-1}3^{-3}$	—	—	4675	0
214b.1	21	144	$[144, 186]$	$\mathrm{II}_{(0,21)}2_4^{-2}4_1^1 3^3$	$\mathrm{II}_{(3,0)}2_6^{-2}4_5^{-1}3^2$	—	—	3893	189
85b.1	21	144	$[144, 186]$	$\mathrm{II}_{(0,21)}2_4^{-2}4_1^1 3^3$	$\mathrm{II}_{(3,0)}2_6^2 4_1^1 3^2$	—	—	4259	151
214a.1	21	144	$C_2^2 \times S_3^2$	$\mathrm{II}_{(0,21)}2_4^{-2}4_1^1 3^3$	$\mathrm{II}_{(3,0)}2_6^{-2}4_5^{-1}3^2$	—	—	3893	219
207	21	168	$\mathrm{A\Gamma L}_1(\mathbb{F}_8)$	$\mathrm{II}_{(0,21)}2_3^{-1}8^{-2}$	$\mathrm{II}_{(3,0)}2_1^1 8^{-2}3^1$	—	—	6846	0
82b.1	21	240	$C_2 \times S_5$	$\mathrm{II}_{(0,21)}2_2^2 4_7^1 3^1 5^1$	$\mathrm{II}_{(3,0)}2_4^{-2}4_7^1 5^1$	—	—	5153	225
82b.2	21	240	$C_2 \times S_5$	$\mathrm{II}_{(0,21)}2_6^2 4_3^{-1}3^1 5^1$	$\mathrm{II}_{(3,0)}2_4^{-2}4_7^1 5^1$	—	—	5153	225
205b	21	240	$C_2 \times S_5$	$\mathrm{II}_{(0,21)}2_6^{-2}4_3^{-1}3^1 5^{-1}$	$\mathrm{II}_{(3,0)}2_0^2 4_3^{-1}3^{-2}5^{-1}$	—	—	5197	0
205c	21	240	$C_2 \times S_5$	$\mathrm{II}_{(0,21)}2_6^{-2}4_3^{-1}3^1 5^{-1}$	$\mathrm{II}_{(3,0)}2_0^2 4_3^{-1}3^{-2}5^{-1}$	—	—	5197	0
203a	21	288	$C_2 \times \mathrm{A\Gamma L}_1(\mathbb{F}_9)$	$\mathrm{II}_{(0,21)}2_6^{-2}4_1^1 9^{-1}$	$\mathrm{II}_{(3,0)}2_2^{-2}4_7^1 3^1 9^1$	—	—	6593	0
203b	21	288	$C_2 \times \mathrm{A\Gamma L}_1(\mathbb{F}_9)$	$\mathrm{II}_{(0,21)}2_6^{-2}4_1^1 9^{-1}$	$\mathrm{II}_{(3,0)}2_2^{-2}4_7^1 3^1 9^1$	—	—	6593	0
208a.1	21	288	$C_2 \times S_3 \times S_4$	$\mathrm{II}_{(0,21)}2_2^2 8_1^1 3^{-2}$	$\mathrm{II}_{(3,0)}2_6^2 8_3^{-1}3^1$	—	—	4675	243
201	21	432	$\mathrm{AGL}_2(\mathbb{F}_3)$	$\mathrm{II}_{(0,21)}2_7^1 3^2 9^{-1}$	$\mathrm{II}_{(3,0)}2_5^{-1}3^{-1}9^1$	—	—	6117	252
205a.1	21	480	$C_2^2 \times S_5$	$\mathrm{II}_{(0,21)}2_4^{-2}4_5^{-1}3^1 5^{-1}$	$\mathrm{II}_{(3,0)}2_0^2 4_3^{-1}5^{-1}$	—	—	5197	267
129.1	21	672	$[672, 1254]$	$\mathrm{II}_{(0,21)}2_3^{-3}3^1 7^{-1}$	$\mathrm{II}_{(3,0)}2_1^3 7^1$	—	—	6140	273
129.2	21	672	$[672, 1254]$	$\mathrm{II}_{(0,21)}2_3^{-3}3^1 7^{-1}$	$\mathrm{II}_{(3,0)}2_1^3 7^1$	—	—	6140	273
200b	21	720	S_6	$\mathrm{II}_{(0,21)}2_6^{-2}4_5^{-1}3^2$	$\mathrm{II}_{(3,0)}2_6^2 4_3^{-1}3^3$	—	—	6601	0
200c	21	720	S_6	$\mathrm{II}_{(0,21)}2_6^{-2}4_5^{-1}3^2$	$\mathrm{II}_{(3,0)}2_6^2 4_3^{-1}3^3$	—	—	6601	0
70.1	21	768	$[768, 1086051]$	$\mathrm{II}_{(0,21)}2_3^{-1}4_3^{-1}16_3^{-1}3^1$	$\mathrm{II}_{(3,0)}2_1^1 4_1^1 16_1^1$	—	—	4545	153
197a	21	768	$[768, 1086051]$	$\mathrm{II}_{(0,21)}2_5^{-1}4_3^{-1}16_7^1$	$\mathrm{II}_{(3,0)}2_7^1 4_1^1 16_5^{-1}3^1$	—	—	7001	0
197b	21	768	$[768, 1086051]$	$\mathrm{II}_{(0,21)}2_5^{-1}4_3^{-1}16_7^1$	$\mathrm{II}_{(3,0)}2_7^1 4_1^1 16_5^{-1}3^1$	—	—	7001	0
123b.1	21	768	$[768, 1088556]$	$\mathrm{II}_{(0,21)}2_6^2 8_1^1 3^2$	$\mathrm{II}_{(3,0)}2_2^2 8_3^{-1}3^{-1}$	—	—	5198	189
121b.1	21	1152	$[1152, 157852]$	$\mathrm{II}_{(0,21)}2_4^{-2}4_1^1 3^3$	$\mathrm{II}_{(3,0)}2_6^2 4_1^1 3^2$	—	—	4281	175

Table 19: Saturated finite symplectic subgroups of $O^+(\Lambda_{\mathrm{OG10}})$ (continued)

H	ρ	$\#H$	Description of H	$g(\Lambda_H)$	$g(\Lambda^H)$	K3	LSV	$\#(-4,1)$	$\#(-24,3)$
194a	21	1440	$[1440, 5841]$	$\mathrm{II}_{(0,21)}2_5^3 3^{-1} 5^1$	$\mathrm{II}_{(3,0)}2_7^{-3} 3^2 5^1$	—	—	7232	0
194b	21	1440	$[1440, 5841]$	$\mathrm{II}_{(0,21)}2_5^3 3^{-1} 5^1$	$\mathrm{II}_{(3,0)}2_7^{-3} 3^2 5^1$	—	—	7232	0
200a.1	21	1440	$C_2 \times S_6$	$\mathrm{II}_{(0,21)}2_0^2 4_7^1 3^2$	$\mathrm{II}_{(3,0)}2_6^2 4_3^{-1} 3^{-1}$	—	—	6601	315
193a	21	1440	$[1440, 5844]$	$\mathrm{II}_{(0,21)}2_4^{-2} 4_7^1 5^1$	$\mathrm{II}_{(3,0)}2_0^2 4_1^1 3^1 5^1$	—	—	8697	0
193b	21	1440	$[1440, 5844]$	$\mathrm{II}_{(0,21)}2_4^{-2} 4_7^1 5^1$	$\mathrm{II}_{(3,0)}2_0^2 4_1^1 3^1 5^1$	—	—	8697	0
110.2	21	3840	$C_2^5 \rtimes S_5$	$\mathrm{II}_{(0,21)}2_4^{-2} 4_5^{-1} 3^1 5^{-1}$	$\mathrm{II}_{(3,0)}2_0^2 4_3^{-1} 5^{-1}$	—	—	5717	207
191.1	21	3888	—	$\mathrm{II}_{(0,21)}2_1^{-3} 3^{-3}$	$\mathrm{II}_{(3,0)}2_3^3 3^{-2}$	—	—	5592	297
187b	21	3888	—	$\mathrm{II}_{(0,21)}2_3^3 3^{-2}$	$\mathrm{II}_{(3,0)}2_1^{-3} 3^{-3}$	—	—	9180	0
184	21	4608	—	$\mathrm{II}_{(0,21)}2_7^1 8^{-2}$	$\mathrm{II}_{(3,0)}2_1^1 8^{-2} 3^1$	—	—	6894	0
183	21	5760	—	$\mathrm{II}_{(0,21)}8_1^1 3^{-1} 5^{-1}$	$\mathrm{II}_{(3,0)}8_7^1 3^2 5^{-1}$	—	—	7143	0
186.1	21	7776	—	$\mathrm{II}_{(0,21)}4_1^1 3^3$	$\mathrm{II}_{(3,0)}4_7^1 3^2$	—	—	7619	351
187a.1	21	7776	—	$\mathrm{II}_{(0,21)}2_3^3 3^{-2}$	$\mathrm{II}_{(3,0)}2_1^{-3} 3^1$	—	—	9180	405
182	21	7920	M_{11}	$\mathrm{II}_{(0,21)}2_7^1 3^{-1} 11^{-1}$	$\mathrm{II}_{(3,0)}2_5^{-1} 3^2 11^1$	—	—	9581	0
180	21	10752	—	$\mathrm{II}_{(0,21)}2_2^{-2} 16_5^{-1}$	$\mathrm{II}_{(3,0)}2_2^2 16_3^{-1} 3^1$	—	—	9730	0
178b	21	11520	—	$\mathrm{II}_{(0,21)}2_2^{-2} 8_7^1 3^1$	$\mathrm{II}_{(3,0)}2_6^{-2} 8_5^{-1} 3^{-2}$	—	—	8106	0
175b	21	20160	A_8	$\mathrm{II}_{(0,21)}4_1^1 3^1 5^1$	$\mathrm{II}_{(3,0)}4_7^1 3^{-2} 5^1$	—	—	10073	0
178a.1	21	23040	—	$\mathrm{II}_{(0,21)}2_2^{-2} 8_7^1 3^1$	$\mathrm{II}_{(3,0)}2_2^2 8_1^1$	—	—	8106	375
170	21	40320	—	$\mathrm{II}_{(0,21)}4_3^{-1} 3^{-1} 7^{-1}$	$\mathrm{II}_{(3,0)}4_5^{-1} 3^2 7^1$	—	—	8681	0
102.1	21	40320	$\mathrm{PSL}(3,4) \rtimes C_2$	$\mathrm{II}_{(0,21)}2_3^3 3^{-2}$	$\mathrm{II}_{(3,0)}2_1^{-3} 3^1$	—	—	9240	385
175a.1	21	40320	S_8	$\mathrm{II}_{(0,21)}4_1^1 3^1 5^1$	$\mathrm{II}_{(3,0)}4_7^1 5^1$		—	10073	441
171a	21	40320	—	$\mathrm{II}_{(0,21)}2_5^3 7^{-1}$	$\mathrm{II}_{(3,0)}2_7^{-3} 3^1 7^1$	—	—	10416	0
171b	21	40320	—	$\mathrm{II}_{(0,21)}2_5^3 7^{-1}$	$\mathrm{II}_{(3,0)}2_7^{-3} 3^1 7^1$	—	—	10416	0
172	21	40320	—	$\mathrm{II}_{(0,21)}8_1^1 7^1$	$\mathrm{II}_{(3,0)}8_7^1 3^1 7^{-1}$	—	—	10431	0
169	21	58320	—	$\mathrm{II}_{(0,21)}2_1^1 3^{-1} 9^1$	$\mathrm{II}_{(3,0)}2_7^1 3^2 9^{-1}$	—	—	10611	0
167a	21	126000	$\mathrm{PSU}(3,5)$	$\mathrm{II}_{(0,21)}2_7^1 5^{-2}$	$\mathrm{II}_{(3,0)}2_5^{-1} 3^1 5^{-2}$	—	—	11025	0
167b	21	126000	$\mathrm{PSU}(3,5)$	$\mathrm{II}_{(0,21)}2_7^1 5^{-2}$	$\mathrm{II}_{(3,0)}2_5^{-1} 3^1 5^{-2}$	—	—	11025	0
165a	21	443520	M_{22}	$\mathrm{II}_{(0,21)}4_5^{-1} 11^1$	$\mathrm{II}_{(3,0)}4_3^{-1} 3^1 11^{-1}$	—	—	11781	0
165b	21	443520	M_{22}	$\mathrm{II}_{(0,21)}4_5^{-1} 11^1$	$\mathrm{II}_{(3,0)}4_3^{-1} 3^1 11^{-1}$	—	—	11781	0
163b	21	3265920	$\mathrm{PSU}(4,3)$	$\mathrm{II}_{(0,21)}4_7^1 3^2$	$\mathrm{II}_{(3,0)}4_1^1 3^3$	—	—	13041	0
163a.1	21	6531840	—	$\mathrm{II}_{(0,21)}4_7^1 3^2$	$\mathrm{II}_{(3,0)}4_1^1 3^{-1}$	—	—	13041	567

F. Equations of symmetric K3 surfaces of degree 8

In Table 20, we provide equations for different triples (S, G, G_s) consisting of a polarized K3 surface S of degree 8, a finite group G of automorphisms of S, with symplectic subaction given by G_s. Using the numerical data available in the database of [BH23], together with the theory of [SD74] and an algorithm from [Shi15], we know that there exist such symmetric K3 surfaces which can be described as complete intersections of 3 quadrics in $\mathbb{P}^5_{\mathbb{C}}$. Note that now, the corresponding K3 surfaces with such symmetries are not necessarily unique, and there are also several deformation families with the same group actions and degree. Hence Table 20 is not complete.

In Table 20, the groups G and G_s are identified by their ID's in the Small Group Library [BEOH24], the column # gives the entry of [Xia96, Table 2] corresponding the groups G_s, and any ζ_n denotes a primitive nth root of unity. Whenever it makes sense, we give parameters α_i's in the equations, with $1 \leq i \leq D$, arising from the output of the algorithm explained in the paper (after removing parameters describing isotrivial families). Finally, we also tell whether the generic element in the family described by the associated equations is smooth.

Table 20: Equations of symmetric K3 surfaces of degree 8

S	G	G_s	#	D	smooth
$\begin{cases} x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+\alpha_1 x_5^2 = 0 \\ -\zeta_{10}^3x_0^2-\zeta_{10}x_1^2+\zeta_{10}^4x_2^2+\zeta_{10}^2x_3^2+x_4^2 = 0 \\ -\zeta_{10}x_0^2+\zeta_{10}^2x_1^2-\zeta_{10}^3x_2^2+\zeta_{10}^4x_3^2+x_4^2 = 0 \end{cases}$	$[160, 235]$	$[16, 14]$	21	1	yes
$\begin{cases} x_0x_1+x_2x_3+x_4x_5 = 0 \\ (1-\zeta_4)x_0^2+(\zeta_4-1)x_1^2+\zeta_4x_2^2-\zeta_4x_3^2-x_4^2+x_5^2 = 0 \\ \zeta_4x_0^2-\zeta_4x_1^2+(1-\zeta_4)x_2^2+(\zeta_4-1)x_3^2-x_4^2+x_5^2 = 0 \end{cases}$	$[96, 226]$	$[48, 48]$	51	0	yes
$\begin{cases} x_0x_1+x_2^2+x_3^2+x_4^2+x_5^2 = 0 \\ \zeta_4x_0^2-\zeta_4x_1^2-\alpha_1(x_4^2-x_5^2) = 0 \\ x_0^2+x_1^2-\alpha_1(x_2^2-x_3^2) = 0 \end{cases}$	$[128, 928]$	$[64, 138]$	56	1	yes
$\begin{cases} x_0^2+\alpha_1x_1x_2-\zeta_6x_3^2+(\zeta_6-1)x_4^2+x_5^2 = 0 \\ x_1^2+\alpha_1x_0x_2+x_3^2+x_4^2+x_5^2 = 0 \\ x_2^2+\alpha_1x_0x_1+(\zeta_6-1)x_3^2-\zeta_6x_4^2+x_5^2 = 0 \end{cases}$	$[144, 189]$	$[72, 43]$	61	1	yes
$\begin{cases} x_0^2+x_1^2+x_2^2-x_3^2-x_4^2-x_5^2 = 0 \\ \zeta_3x_0^2-(1+\zeta_3)x_1^2+x_2^2-\alpha_1((1+\zeta_3)x_3^2-\zeta_3x_4^2-x_5^2) = 0 \\ \zeta_3x_3^2-(1+\zeta_3)x_4^2+x_5^2-\alpha_1((1+\zeta_3)x_0^2-\zeta_3x_1^2-x_2^2) = 0 \end{cases}$	$[192, 1538]$	$[96, 227]$	65	1	yes
$\begin{cases} x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 = 0 \\ (\zeta_6-1)x_0^2+(\zeta_6-1)x_1^2-\zeta_6x_2^2-\zeta_6x_3^2+x_4^2+x_5^2 = 0 \\ -\zeta_6x_0^2-\zeta_6x_1^2+(\zeta_6-1)x_2^2+(\zeta_6-1)x_3^2+x_4^2+x_5^2 = 0 \end{cases}$	$[384, 18235]$	$[192, 1023]$	75	0	no
$\begin{cases} -x_0^2+x_1^2-\alpha_1(x_4^2+x_5^2) = 0 \\ -x_2^2+x_3^2-\alpha_1(x_0^2+x_1^2) = 0 \\ -x_4^2+x_5^2-\alpha_1(x_2^2+x_3^2) = 0 \end{cases}$	$[384, 18235]$	$[192, 1023]$	75	1	yes
$\begin{cases} x_0^2+\zeta_3^2x_3^2+\zeta_3x_4^2+x_5^2 = 0 \\ x_1^2+\zeta_3x_3^2+\zeta_3^2x_4^2+x_5^2 = 0 \\ x_2^2+x_3^2+x_4^2+x_5^2 = 0 \end{cases}$	$[576, 8657]$	$[288, 1026]$	78	0	yes

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A

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