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Variants of Bernstein's theorem for variational integrals with linear and nearly linear growth

Michael Bildhauer[1] · Martin Fuchs[1]

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Abstract

Using a Caccioppoli-type inequality involving negative exponents for a directional weight we establish variants of Bernstein's theorem for variational integrals with linear and nearly linear growth. We give some mild conditions for entire solutions of the equation

$$\mathrm{div}\left[Df(\nabla u)\right] = 0\,,$$

under which solutions have to be affine functions. Here f is a smooth energy density satisfying $D^2 f > 0$ together with a natural growth condition for $D^2 f$.

Keywords Bernstein's theorem · Non-parametric minimal surfaces · Variational problems with (nearly) linear growth · Equations in two variables

Mathematical Subject Classification 49Q20 · 49Q05 · 53A10 · 35J20

1 Introduction

In [1] Bernstein proved that every C^2-solution $u = u(x) = u(x_1, x_2)$ of the non-parametric minimal surface equation

$$\mathrm{div}\left[\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right] = 0 \tag{1.1}$$

✉ Michael Bildhauer
bibi@math.uni-sb.de

Martin Fuchs
fuchs@math.uni-sb.de

1 Department of Mathematics, Saarland University, P.O. Box 15 11 50, 66041 Saarbrücken, Germany

over the entire plane must be an affine function, which means that with real numbers a, b, c it holds

$$u(x_1, x_2) = ax_1 + bx_2 + c\,.$$

For a detailed discussion of this classical result the interested reader is referred for instance to [2–5] and the more recent contributions [6, 7] as well as the references quoted therein.

Starting from Bernstein's result the question arises to which classes of second order equations the Bernstein-property extends. More precisely, we replace (1.1) through the equation

$$L[u] = 0 \tag{1.2}$$

for a second order elliptic operator L and assume that $u \in C^2(\mathbb{R}^2)$ is an entire solution of (1.2) asking if u is affine. To our knowledge a complete answer to this problem is open, however we have the explicit "Nitsche-criterion" established by Nitsche and Nitsche [8].

In our note we discuss Eq. (1.2) assuming that L is the Euler-operator associated to the variational integral

$$J[u, \Omega] := \int_\Omega f(\nabla u)\,\mathrm{d}x$$

with density $f\colon \mathbb{R}^2 \to \mathbb{R}$ and for domains $\Omega \subset \mathbb{R}^2$, i.e. (1.2) is replaced by

$$\operatorname{div}\Big[Df(\nabla u)\Big] = 0\,, \tag{1.3}$$

where in the minimal surface case (1.1) we have $f(p) = \sqrt{1 + |p|^2}$, $p \in \mathbb{R}^2$, being an integrand of linear growth with repect to ∇u. For this particular class of energy densities and under the additional assumption that f is of type $f(p) = g(|p|)$, $p \in \mathbb{R}^2$, for a function $g \in C^2([0, \infty))$ such that

$$0 < g''(t) \leq c(1 + t)^{-\mu}\,, \quad t \geq 0\,,$$

with exponent $\mu \geq 3$ (including the minimal surface case) we proved the Bernstein-property in Theorem 1.2 of [9] benefiting from the work [10] of Farina, Sciunzi and Valdinoci.

Bernstein-type theorems under natural additional conditions to be imposed on the entire solutions of the Euler-equations for splitting-type variational integrals of linear growth have been established in the recent paper [11].

One of the main tools used in [11] is a Caccioppoli-inequality involving negative exponents which was already exploited in different variants in the papers [12–15].

In fact we used this inequality to show that $\partial_1 \nabla u \equiv 0$ which follows by considering the bilinear form $D^2 f(\nabla u)$ with suitable weights. Since we are in two dimensions, the use of Eq. (1.3) then completes the proof of the splitting-type results in [11].

In the manuscript at hand we observe that even without splitting-structure it is possible to discuss the Caccioppoli-inequality with a directional weight obtaining $\partial_1 \nabla u \equiv 0$ and to argue similar as before. We note that considering directional weights gives much more flexibility in choosing exponents than arguing with a full gradient (see Propositions 6.1 and 6.2 of [15]). We here already note that we also include a logarithmic variant of the Caccioppli-inequality as an approach to the limit case $\alpha = -1/2$.

Before going into these details let us have a brief look at power growth energy densities as for example

$$f(p) = (1 + |p|^2)^{\frac{s}{2}}, \quad p \in \mathbb{R}^2,$$

with exponent $s > 1$. Then the Nitsche-criterion (compare [8], Satz) shows the existence of non-affine entire solutions to Eq. (1.3), and as we will shortly discuss in the Appendix the same reasoning applies to the nearly linear growth model

$$f(p) = |p| \ln(1 + |p|), \quad p \in \mathbb{R}^2, \tag{1.4}$$

which means that we do not have the Bernstein-property for Eq. (1.3) with density of the form (1.4).

However, as indicated above, we can establish a mild condition under which any entire solution is an affine function being valid for a large class of densities f including the nearly linear and even the linear case.

Let us formulate our

Assumptions The density $f : \mathbb{R}^2 \to \mathbb{R}$ is of class C^2 such that

$$D^2 f(p)(q, q) > 0 \quad \text{for all } p, q \in \mathbb{R}^2, q \neq 0. \tag{1.5}$$

For a constant $\lambda > 0$ it holds

$$D^2 f(p)(q, q) \leq \lambda \frac{\ln(2 + |p|)}{1 + |p|} |q|^2, \quad p, q \in \mathbb{R}^2. \tag{1.6}$$

Remark 1.1 Condition (1.5) implies the strict convexity of f, whereas from (1.6) we obtain

$$|f(p)| \leq c\big(|p| \ln(1 + |p|) + 1\big), \quad p \in \mathbb{R}^2,$$

with some constant $c > 0$.

We have the following result:

Theorem 1.1 *Let f satisfy (1.5) and (1.6) and consider an entire solution $u \in C^2(\mathbb{R}^2)$ of Eq. (1.3). Suppose that with numbers $0 \leq m < 1$, $K > 0$ the solution satisfies*

$$|\partial_1 u(x)| \leq K\Big(|\partial_2 u(x)|^m + 1\Big), \quad x \in \mathbb{R}^2, \tag{1.7}$$

or

$$|\partial_2 u(x)| \le K\Big(|\partial_1 u(x)|^m + 1\Big)\,, \quad x \in \mathbb{R}^2\,. \tag{1.8}$$

Then u is affine.

Remark 1.2 Since the density f from (1.4) fulfills the the conditions (1.5) and (1.6) and since in this case non-affine entire solutions exist, the requirements (1.7) and (1.8) single out a class of entire solutions of Bernstein-type.

Of course we know nothing concerning the optimality of (1.7) and (1.8). Another unsolved problem is the question, if in the case of linear growth with radial structure, i.e. $f(\nabla u) = g(|\nabla u|)$, Bernstein's theorem holds without extra conditions on the entire solution u.

Remark 1.3 The conditions (1.7) and (1.8) are in some sense related to the "balancing conditions" used in [11] in order to exclude entire solutions of the form $u(x_1, x_2) = x_1 x_2$ for densities f of splitting type.

Let us pass to the linear growth case replacing (1.6) by

$$|D^2 f(p)| \le \Lambda \frac{1}{1+|p|}\,, \quad p \in \mathbb{R}^2\,, \tag{1.9}$$

with a positive constant Λ. Here the notion of linear growth just expresses the fact that from (1.9) it follows that

$$|f(p)| \le c\big(|p| + 1\big)\,, \quad p \in \mathbb{R}^2\,,$$

with some number $c > 0$. In this situation we have

Theorem 1.2 *Let f satisfy (1.5) together with (1.9) and let $u \in C^2(\mathbb{R}^2)$ denote an entire solution of Eq. (1.3) for which we have*

$$|\partial_1 u(x)| \ln^2\big(1 + |\partial_1 u(x)|\big) \le K\big(|\partial_2 u(x)| + 1\big)\,, \quad x \in \mathbb{R}^2\,, \tag{1.10}$$

or

$$|\partial_2 u(x)| \ln^2\big(1 + |\partial_2 u(x)|\big) \le K\big(|\partial_1 u(x)| + 1\big)\,, \quad x \in \mathbb{R}^2\,, \tag{1.11}$$

with some number $K \in (0, \infty)$. Then u is an affine function.

The results of Theorems 1.1 and 1.2 are not limited to the particular coordinate directions $e_1 = (1, 0)$ and $e_2 = (0, 1)$, more precisely it holds:

Theorem 1.3 *Let f satisfy either the assumptions of Theorem 1.1 ("case 1") or of Theorem 1.2 ("case 2") and suppose that $u \in C^2(\mathbb{R}^2)$ is an entire solution of (1.3). Assume that there exist two linearly independent vectors E_1, $E_2 \in \mathbb{R}^2$ such that*

In case 1: (1.7) holds with $\partial_\alpha u$ being replaced by $\partial_{E_\alpha} u$, $\alpha = 1, 2$, *In case 2:* (1.10) *is true again with* $\partial_\alpha u$ *being replaced by* $\partial_{E_\alpha} u$, $\alpha = 1, 2$.

Then u is an affine function.

The proof of this result follows from the observation that the function u is a local minimizer of the energy $\int f(\nabla u)\, \mathrm{d}x$ combined with a suitable linear transformation. If we let

$$
\begin{aligned}
T: \quad & \mathbb{R}^2 \to \mathbb{R}^2, \quad E_\alpha = T(e_\alpha), \quad \alpha = 1, 2, \\
E_{\alpha\beta} &:= E_\alpha \cdot E_\beta, \quad \alpha, \beta = 1, 2, \\
\tilde{u}(x) &:= u\big(T(x)\big), \quad x \in \mathbb{R}^2, \\
\tilde{f}(p) &:= f\Big(T\Big(\big(E_{\alpha\beta}\big)^{-1}_{1\le\alpha,\beta\le 2}\, p\Big)\Big), \quad p \in \mathbb{R}^2,
\end{aligned}
$$

then it holds

$$\partial_\alpha \tilde{u}(x) = \partial_{E_\alpha} u\big(T(x)\big), \quad x \in \mathbb{R}^2, \quad \alpha = 1, 2,$$

and $\tilde{u}$ is an entire solution of Eq. (1.3) with f being replaced by $\tilde{f}$, which follows from the local minimality of $\tilde{u}$ with respect to the energy $\int \tilde{f}(\nabla w)\, \mathrm{d}x$. Obviously the properties of $\tilde{f}$ required in Theorems 1.1 and 1.2, respectively, are consequences of the corresponding assumptions imposed on f, thus we can apply our previous results to $\tilde{u}$ (and $\tilde{f}$).

Our paper is organized as follows: in Sect. 2 we present the proof of Theorem 1.1 based on a Caccioppoli-inequality involving negative exponents, which has been established, for instance, in [15], Proposition 6.1.

Sect. 3 is devoted to the discussion of Theorem 1.2. We will make use of some kind of a limit version of Caccioppoli's inequality, whose proof will be presented below. With the help of this result the claim of Theorem 1.2 follows along the lines of Sect. 2. We finish Sect. 3 by presenting a technical extension of Theorem 1.2, which just follows from an inspection of the arguments (compare Theorem 3.1).

For the reader's convenience we discuss in an appendix Eq. (1.3) for the nearly linear growth case (1.4) and show that the Nitsche-criterion applies yielding non-affine solutions defined on the whole plane.

2 Proof of Theorem 1.1

Let f satisfy (1.5) and (1.6), let u denote an entire solution of (1.3) and assume w.l.o.g. that (1.7) holds. We apply inequality (107) from Proposition 6.1 in [15] with the choices $l = 1$, $i = 1$ and

$$\Omega = B_{2R} = \big\{x \in \mathbb{R}^2 : |x| < 2R\big\}$$

to obtain for any $\alpha > -1/2$ and $\eta \in C_0^\infty(B_{2R})$, $0 \le \eta < 1$,

$$\begin{aligned}&\int_{B_{2R}} \eta^2 D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\partial_1 u\big)\Gamma_1^\alpha \,\mathrm{d}x\\ &\quad\le c\int_{B_{2R}} D^2 f(\nabla u)(\nabla\eta, \nabla\eta)\Gamma_1^{\alpha+1}\,\mathrm{d}x\,, \quad \Gamma_1 := 1+|\partial_1 u|^2\,,\end{aligned} \tag{2.1}$$

with a finite constant independent of R.

Letting $\eta = 1$ on B_R and assuming $|\nabla\eta| \le c/R$ we apply (1.6) to the r.h.s. of (2.1) and get

$$\begin{aligned}&\int_{B_R} D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\partial_1 u\big)\Gamma_1^\alpha\,\mathrm{d}x\\ &\quad\le cR^{-2}\int_{R<|x|<2R} \Gamma_1^{1+\alpha}\ln\big(2+|\nabla u|\big)\big(1+|\nabla u|\big)^{-1}\,\mathrm{d}x\,.\end{aligned} \tag{2.2}$$

On account of (1.7) we deduce for any $\varepsilon > 0$

$$\begin{aligned}\Gamma_1^{1+\alpha}\ln\big(2+|\nabla u|\big)\big(1+|\nabla u|\big)^{-1} &\le c(\varepsilon)\big(1+|\partial_2 u|^2\big)^{m(1+\alpha)}\big(1+|\nabla u|\big)^{\varepsilon-1}\\ &\le \tilde c(\varepsilon)\big(1+|\partial_2 u|\big)^{2m(1+\alpha)}\big(1+|\partial_2 u|\big)^{\varepsilon-1}\\ &= \tilde c(\varepsilon)\big(1+|\partial_2 u|\big)^{2m(1+\alpha)-1+\varepsilon}\,.\end{aligned}$$

Recall that $m < 1$, hence $2m(1+\alpha)-1 < 0$ for $\alpha > -1/2$ sufficiently close to $-1/2$. We fix α with this property and finally select $\varepsilon > 0$ such that $2m(1+\alpha)-1+\varepsilon \le 0$ to obtain

$$\Gamma_1^{1+\alpha}\ln\big(2+|\nabla u|\big)\big(1+|\nabla u|\big)^{-1} \le const < \infty\,. \tag{2.3}$$

Combining (2.2) with (2.3) it is shown that

$$\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\partial_1 u\big)\Gamma_1^\alpha\,\mathrm{d}x < \infty\,. \tag{2.4}$$

We quote equation (108) from [15] again with the previous choices $l = 1$, $i = 1$, $\Omega = B_{2R}$, $\eta \in C_0^\infty(B_{2R})$, $0 \le \eta \le 1$, and with α as fixed above. The same calculations as carried out after (108) then yield

$$\begin{aligned}&\int_{B_{2R}} D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\partial_1 u\big)\eta^2\Gamma_1^\alpha\,\mathrm{d}x\\ &\quad\le c\left|\int_{B_{2R}-B_R} D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\eta^2\big)\partial_1 u\Gamma_1^\alpha\,\mathrm{d}x\right|.\end{aligned} \tag{2.5}$$

On the r.h.s. of (2.5) we apply the Cauchy–Schwarz inequality to the bilinear form $D^2 f(\nabla u)$, hence

$$
\begin{aligned}
\text{l.h.s. of (2.5)} \;\le\; & \left[\int_{B_{2R}-B_R} D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\partial_1 u\big)\Gamma_1^{\alpha}\eta^2 \,\mathrm{d}x\right]^{\frac{1}{2}} \\
& \cdot\left[\int_{B_{2R}-B_R} D^2 f(\nabla u)(\nabla\eta, \nabla\eta)\Gamma_1^{1+\alpha} \,\mathrm{d}x\right]^{\frac{1}{2}} \\
=:\; & T_1(R)^{\frac{1}{2}} \cdot T_2(R)^{\frac{1}{2}} \,.
\end{aligned}
\tag{2.6}
$$

Here η has been chosen in such a way that $\eta \equiv 1$ on B_R and therefore $\mathrm{spt}(\nabla\eta) \subset B_{2R} - B_R$. By (2.4) we have

$$\lim_{R\to\infty} T_1(R) = 0\,,$$

whereas the calculations carried out after (2.2) imply the boundedness of $T_2(R)$. Thus (2.5) and (2.6) imply

$$\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\partial_1 u\big)\Gamma_1^{\alpha} \,\mathrm{d}x = 0\,,$$

hence $\nabla\partial_1 u = 0$ on account of (1.5). This shows $\partial_1 u = a$ for some number $a \in \mathbb{R}$ and since

$$u(x_1, x_2) - u(0, x_2) = \int_0^{x_1} \frac{\mathrm{d}}{\mathrm{d}t} u(t, x_2)\,\mathrm{d}t = a x_1$$

we can write

$$u(x_1, x_2) = \varphi(x_2) + a x_1\,, \quad \varphi(x_2) := u(0, x_2)\,.$$

Equation (1.3) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial p_2}\big(a, \varphi'(t)\big) = 0$$

so that

$$\frac{\partial f}{\partial p_2}\big(a, \varphi'(t)\big) = c$$

for a constant c. Finally we observe that the function

$$y \mapsto \frac{\partial f}{\partial p_2}(a, y)$$

is strictly increasing (recall (1.5)), which shows the constancy of φ' and therefore

$$u(0, x_2) = bx_2 + c$$

for some numbers $b, c \in \mathbb{R}$. Altogether we have shown that u is affine finishing the proof of Theorem 1.1. $\square$

3 Proof of Theorem 1.2

Let the assumptions of Theorem 1.2 hold and consider an entire solution $u \in C^2(\mathbb{R}^2)$ of (1.3) without requiring (1.10) or (1.11) for the moment. If we use condition (1.9) in inequality (2.1) and if we assume that the choice $\alpha = -1/2$ is admissible in (2.1), then the calculations of Sect. 2 would immediately imply that $\nabla^2 u = 0$ yielding Bernstein's theorem, i.e. the entire solution u is an affine function without adding further hypotheses on u.

However, we do not have (2.1) in the case that $\alpha = -1/2$ and hence we provide a weaker version involving conditions like (1.10) or (1.11) in order to conclude that u is affine.

To be precise, we assume the validity of (1.10), let l, i, Ω and η as stated in front of (2.1) recalling

$$\int_{B_{2R}} D^2 f(\nabla u)\big(\nabla \partial_1 u, \nabla \psi\big)\, \mathrm{d}x = 0$$

for the choice $\psi := \eta^2 \partial_1 u \Phi(\Gamma_1)$, where

$$\Phi(t) := \frac{\ln(e^2 - 1 + t)}{\sqrt{t}}\, , \quad t \geq 1\, . \tag{3.1}$$

Note that the choice $\alpha = -1/2$ is compensated by the logarithm. Obviously $\Phi(1) = 2$, $\Phi(\infty) = 0$ together with

$$\Phi'(t) = -\frac{1}{2}\frac{1}{t^{\frac{3}{2}}} \ln(e^2 - 1 + t) + \frac{1}{\sqrt{t}(e^2 - 1 + t)} < 0\, , \quad t \geq 1\, , \tag{3.2}$$

where the negative sign for $\Phi'(t)$ follows from $\ln(e^2 - 1 + t) \geq 2$ for $t \geq 1$.

With ψ from above and Φ defined according to (3.1) we obtain

$$\begin{aligned} &\int_{B_{2R}} \eta^2 D^2 f(\nabla u)\big(\nabla \partial_1 u, \nabla \partial_1 u\big)\Phi(\Gamma_1)\, \mathrm{d}x \\ &+ \int_{B_{2R}} \eta^2 D^2 f(\nabla u)\big(\nabla \partial_1 u, \partial_1 u \nabla \Phi(\Gamma_1)\big)\, \mathrm{d}x \\ &\qquad = -2 \int_{B_{2R}} \eta D^2 f(\nabla u)\big(\nabla \partial_1 u, \nabla \eta\big)\partial_1 u \Phi(\Gamma_1)\, \mathrm{d}x\, . \end{aligned} \tag{3.3}$$

Using the identity

$$\begin{aligned}\partial_1 u \nabla \Phi(\Gamma_1) &= 2\nabla \partial_1 u (\partial_1 u)^2 \Phi'(\Gamma_1)\\ &= 2\nabla \partial_1 u \Big[\Gamma_1 \Phi'(\Gamma_1) - \Phi'(\Gamma_1)\Big],\end{aligned}$$

the left-hand side of (3.3) equals

$$\int_{B_{2R}} \eta^2 D^2 f(\nabla u)\big(\nabla \partial_1 u, \nabla \partial_1 u\big)\Big[\Phi(\Gamma_1) + 2\Gamma_1 \Phi'(\Gamma_1) - 2\Phi'(\Gamma_1)\Big]\,\mathrm{d}x\,.$$

From (3.2) it follows (recalling $\Phi'(t) \le 0$ for $t \ge 1$)

$$\begin{aligned}&\Phi(\Gamma_1) + 2\Gamma_1\Phi'(\Gamma_1) - 2\Phi'(\Gamma_1)\\ &\ge \Phi(\Gamma_1) + 2\Gamma_1\Phi'(\Gamma_1)\\ &= \frac{\ln(e^2 - 1 + \Gamma_1)}{\sqrt{\Gamma_1}} + 2\Gamma_1\Big[-\frac{1}{2}\frac{\ln(e^2-1+\Gamma_1)}{\Gamma_1^{\frac{3}{2}}} + \frac{1}{\sqrt{\Gamma_1}(e^2-1+\Gamma_1)}\Big]\\ &= 2\frac{\sqrt{\Gamma_1}}{e^2-1+\Gamma_1} \ge c\frac{1}{\sqrt{\Gamma_1}}\end{aligned}$$

for some constant $c > 0$. Altogether we deduce from (3.3) the inequality of Caccioppoli-type

$$\begin{aligned}&\int_{B_{2R}} \eta^2 D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\partial_1 u\big)\Gamma_1^{-\frac{1}{2}}\,\mathrm{d}x\\ &\quad\le -2\int_{B_{2R}} \eta D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\eta\big)\partial_1 u\Gamma_1^{-\frac{1}{2}}\ln(e^2-1+\Gamma_1)\,\mathrm{d}x\\ &=: -2S\,. \end{aligned} \tag{3.4}$$

To the quantity S we apply the Cauchy–Schwarz inequality valid for the bilinear form $D^2 f(\nabla u)$ and get

$$\begin{aligned}|S| &\le \int_{B_{2R}} \Big|D^2 f(\nabla u)\big(\eta\Gamma_1^{-\frac{1}{4}}\nabla\partial_1 u, \partial_1 u\Gamma_1^{-\frac{1}{4}}\ln(e^2-1+\Gamma_1)\nabla\eta\big)\Big|\,\mathrm{d}x\\ &\le \Big[\int_{B_{2R}} D^2 f(\nabla u)\big(\nabla\partial_1 u, \nabla\partial_1 u\big)\eta^2\Gamma_1^{-\frac{1}{2}}\,\mathrm{d}x\Big]^{\frac{1}{2}}\\ &\qquad\cdot\Big[\int_{B_{2R}} D^2 f(\nabla u)(\nabla\eta, \nabla\eta)\Gamma_1^{\frac{1}{2}}\ln^2(e^2-1+\Gamma_1)\,\mathrm{d}x\Big]^{\frac{1}{2}}\,. \end{aligned} \tag{3.5}$$

On the right-hand side of (3.5) we make use of Young's inequality yielding for any $\varepsilon > 0$

$$|S| \le \varepsilon \int_{B_{2R}} \eta^2 \Gamma_1^{-\frac{1}{2}} D^2 f(\nabla u)\big(\nabla \partial_1 u, \nabla \partial_1 u\big)\,\mathrm{d}x \\ + c(\varepsilon) \int_{B_{2R}} D^2 f(\nabla u)(\nabla \eta, \nabla \eta) \Gamma_1^{\frac{1}{2}} \ln^2(e^2 - 1 + \Gamma_1)\,\mathrm{d}x\,. \tag{3.6}$$

Finally we combine (3.6) and (3.4), thus for a fixed ε being sufficiently small it holds

$$\int_{B_{2R}} D^2 f(\nabla u)\big(\nabla \partial_1 u, \nabla \partial_1 u\big) \eta^2 \Gamma_1^{-\frac{1}{2}}\,\mathrm{d}x \\ \le c \int_{B_{2R}} D^2 f(\nabla u)(\nabla \eta, \nabla \eta) \Gamma_1^{\frac{1}{2}} \ln^2(e^2 - 1 + \Gamma_1)\,\mathrm{d}x\,. \tag{3.7}$$

The properties of η as stated after (2.1) imply that the right-hand side of (3.7) is bounded by (recall (1.9))

$$cR^{-2} \int_{B_{2R}} |D^2 f(\nabla u)| \Gamma_1^{\frac{1}{2}} \ln^2(e^2 - 1 + \Gamma_1)\,\mathrm{d}x \\ \le cR^{-2} \int_{B_{2R}} \frac{1}{1 + |\nabla u|} \Gamma_1^{\frac{1}{2}} \ln^2(e^2 - 1 + \Gamma_1)\,\mathrm{d}x \\ \le cR^{-2} \int_{B_{2R}} \frac{1}{1 + |\partial_2 u|} (1 + |\partial_1 u|) \ln^2(1 + |\partial_1 u|)\,\mathrm{d}x\,.$$

Quoting (1.10) and returning to (3.7) we find that

$$\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla \partial_1 u, \nabla \partial_1 u\big) \eta^2 \Gamma_1^{-\frac{1}{2}}\,\mathrm{d}x < \infty\,. \tag{3.8}$$

With (3.8) and on account of estimate (3.5) it is immediate (recall (3.4)) that actually

$$\int_{\mathbb{R}^2} D^2 f(\nabla u)\big(\nabla \partial_1 u, \nabla \partial_1 u\big) \Gamma_1^{-\frac{1}{2}} = 0\,,$$

hence $\nabla \partial_1 u = 0$ and we can follow the lines of Sect. 2 to prove our claim. □

An inspection of our previous arguments shows that we can replace the function $\ln(e^2 - 1 + t)$ used before by any function $\rho \colon [1, \infty) \to \mathbb{R}^+$ of class C^1 such that we have for all $t \ge 1$

$$\rho'(t) > 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{\sqrt{t}} \rho(t)\Big] \le 0\,. \tag{3.9}$$

Replacing (3.1) by

$$\Phi(t) := \frac{1}{\sqrt{t}} \rho(t) \quad t \ge 1\,, \tag{3.10}$$

and letting as before $\psi := \eta^2 \partial_1 u \Phi(\Gamma_1)$ now with Φ defined in (3.10) we obtain

Theorem 3.1 *Let f satisfy (1.5) together with (1.9) and choose ρ according to (3.9). Suppose that $u \in C^2(\mathbb{R}^2)$ is an entire solution of (1.3) such that*

$$\Gamma_1^{-\frac{1}{2}} \frac{\rho^2(\Gamma_1)}{\rho'(\Gamma_1)} \leq c\Gamma_2^{\frac{1}{2}}, \quad \Gamma_i := 1 + |\partial_i u|^2, \quad i = 1, 2,$$

or

$$\sup_{R>0} R^{-2} \int_{B_R} \Gamma_1^{-1} \frac{\rho^2(\Gamma_1)}{\rho'(\Gamma_1)} \, \mathrm{d}x < \infty$$

holds with some finite constant c. Then u is an affine function.

We leave the details to the reader just adding the obvious remark that clearly we can interchange the roles of the partial derivatives $\partial_1 u$ and $\partial_2 u$ or even work with arbitrary directional derivatives as done in Theorem 1.3.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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4 Appendix

We shortly discuss the Nitsche-criterion (see [8], Satz) for the model case

$$\int |\nabla u| \ln \big(1 + |\nabla u|\big) \, \mathrm{d}x \, .$$

With a slight abuse of notation but in accordance with the terminology of [8] we let

$$g(t) := t \ln(1+t), \quad t \geq 0, \qquad f(t) := g\big(\sqrt{t}\big),$$

so that

$$J[u] := \int |\nabla u| \ln \big(1 + |\nabla u|\big) \, \mathrm{d}x = \int f\big(|\nabla u|^2\big) \, \mathrm{d}x \, .$$

Introducing the function $\lambda(t) := 2f''(t)/f'(t)$ again for $t \geq 0$ we claim

$$\int_1^\infty \frac{1+t\lambda(t)}{2+t\lambda(t)}\frac{1}{t}\,\mathrm{d}t = \infty\,. \tag{4.1}$$

From (4.1) it follows that the Euler-equation associated to the functional J admits entire non-affine solutions.

For (4.1) we observe the formula

$$\frac{1}{t}\frac{1+t\lambda(t)}{2+t\lambda(t)} = \frac{1}{1+\frac{\sqrt{t}}{g''(\sqrt{t})}g'(\sqrt{t})} =: \Theta(t)$$

and remark that for $t \gg 1$ it holds

$$c_1 t \ln\left(1+\sqrt{t}\right) \leq \frac{\sqrt{t}g'(\sqrt{t})}{g''(\sqrt{t})} \leq c_2 t \ln\left(1+\sqrt{t}\right)$$

as well as $t \leq c_3 t \ln\left(1+\sqrt{t}\right)$, hence

$$\Theta(t) \geq c_4 \frac{1}{t\ln\left(1+\sqrt{t}\right)} \geq c_5 \frac{1}{t\ln\left(1+t\right)}$$

again for $t \gg 1$. Since

$$\int_1^\infty \frac{\mathrm{d}t}{t\ln\left(1+t\right)} = \infty\,,$$

the claim (4.1) follows. □

Remark 4.1 Formally the above model case of nearly linear growth should satisfy the $C^{3,\alpha}$-condition posed in [8]. Since the integral occurring in (4.1) is not depending on the energy density for small values of t, we may easily adjust the example to obtain a $C^{3,\alpha}$-density of nearly linear growth. For example we just let $f_\varepsilon(t) := \sqrt{\varepsilon+t}\ln(1+\sqrt{\varepsilon+t})$, $t \geq 0$, with some $\varepsilon > 0$.

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