Stochastic calculus for Lévy-driven Volterra processes

Dissertation zur Erlangung des Grades des
Doktors der Naturwissenschaften
der Naturwissenschaftlich-Technischen Fakultäten
der Universität des Saarlandes

eingereicht im Mai 2015
in Saarbrücken

von

Philip Oberacker

Tag des Kolloquiums:	03.09.2015

Mitglieder des Prüfungsausschusses:

Vorsitzender:	Professor Dr. Moritz Weber
1. Berichterstatter:	Professor Dr. Christian Bender
2. Berichterstatter:	Professor Dr. Fred Espen Benth (während des Kolloquiums vertreten durch Professor Dr. Jörg Eschmeier)
Protokollführer:	Dr. Frederik Heber
Dekan:	Professor Dr. Markus Bläser

The brain is the most important muscle for writing a thesis

(modified quote from the famous climber Wolfgang Güllich, who once noted that the brain is the most important muscle for climbing)

Acknowledgements

First of all, I want to express my gratitude towards my supervisor Professor Christian Bender, who gave me the opportunity to work in the pleasant and inspiring environment of the Stochastics group at Saarland University (with financial support by the Deutsche Forschungsgemeinschaft under grant BE3933/4-1 raised by him) with all its present and former members over the last years. Without his constant support and encouragement this thesis would not have been possible.

I want to thank my co-author and friend Dr. Robert Knobloch for all the mathematical (and political) discussions. His share in our joint publications and his experience are gratefully acknowledged.

I also want to thank Professor Fred Espen Benth for being the co-referee of this thesis.

My heartfelt thanks go to Professor Ernst Albrecht, the supervisor of my bachelor and master theses, for arousing the love for mathematics.

Over the last years, I had the opportunity to work with many nice people from different parts of Europe. These include Esko Valkeila, Tommi Sottinen, Dario Gasbarra, Heikki Tikanmäki, and Arturo Valdivia. I really enjoyed our collaboration.

Some of my closest friends (and my beloved partner Anne Wald) I got to know during my studies. Particularly, I want to mention Heiko Hoffmann, Simon Krämer, and Christian Tietz for doing so much work together and having so much fun from the beginning of the very first semester.

I want to thank my family for supporting my studies, believing in me, and not having me do too much facility management and system administration jobs at home.

I would like to thank Professor Alexander Lindner for helpful comments on the proof of Lemma 2.3 as well as Lauri Viitasaari and Anne Wald for proof-reading the introduction.

Contents

Abstract

In this thesis we investigate processes which arise from the convolution of a deterministic Volterra-type kernel with a two-sided martingale as an integrator. The class of processes of this kind is of special interest, because it contains for example the fractional Brownian motion, which has been intensely studied over the last years. We first give sufficient conditions for the convolution integral to exist in a suitable sense and show that under appropriate assumptions nice path properties of the driving process are carried over to the convoluted process. A special interest lies on the situation that the driver is a centred two-sided Lévy process, in which case the resulting convoluted process is called a *Lévy-driven Volterra process*. In particular, we are able to treat the case of fractional Lévy processes (via Mandelbrot-Van Ness representation).

The focus of this thesis is on proving a generalised Itō formula for Lévy-driven Volterra processes. We emphasise that our generalised Itō formula is unifying in the sense that it covers the well-known Itō formulas for Lévy processes and fractional Brownian motions as well as the case of fractional Lévy processes, for which such a formula has not been available in the literature.

Zusammenfassung

In dieser Arbeit untersuchen wir Prozesse, die durch die Faltung eines deterministischen Kerns vom Volterra-Typ mit einem zweiseitigen Martingal als Integrator entstehen. Die Klasse von Prozessen dieser Art ist von speziellem Interesse, da sie zum Beispiel die fraktionale Brownsche Bewegung, die in den letzten Jahren intensiv studiert wurde, enthält. Wir geben zuerst hinreichende Bedingungen dafür an, dass das Faltungsintegral in einem geeigneten Sinne existiert und zeigen, dass sich unter geeigneten Bedingungen schöne Pfadeigenschaften des treibenden Prozesses auf den gefalteten Prozess übertragen. Von speziellem Interesse ist die Situation, dass der treibende Prozess ein zentrierter zweiseitiger Lévy-Prozess ist, in welchem Fall der resultierende gefaltete Prozess *Lévy-angetriebener Volterra-Prozess* genannt wird. Insbesondere sind wir in der Lage, den Fall von fraktionalen Lévy-Prozessen (in Mandelbrot-Van Ness-Darstellung) zu behandeln.

Das Hauptaugenmerk dieser Arbeit liegt auf dem Beweis einer verallgemeinerten Itō-Formel für Lévy-angetriebene Volterra-Prozesse. Wir betonen, dass unsere Formel vereinheitlichend ist in dem Sinn, dass sie die bekannten Itō-Formeln für Lévy-Prozesse und fraktionale Brownsche Bewegungen ebenso wie für fraktionale Lévy-Prozesse, für die eine solche Formel bisher nicht bekannt war, abdeckt.

Introduction

The aim of this thesis is to investigate processes which arise from the convolution of a deterministic Volterra-type kernel with a two-sided martingale as an integrator, i.e. processes of the form

$$M(t) = \int_{-\infty}^{t} f(t,s)\ X(\mathrm{d}s), \tag{1}$$

where $t \in [0,T]$ for a fixed time horizon $T > 0$. The deterministic integration kernel f satisfies several integrability, differentiability, and growth conditions, which will be specified later. The underlying driving process X is a two-sided martingale whose behaviour at $s = -\infty$ is connected with the corresponding behaviour of the function $s \mapsto f(t,s)$ in such a way that the integral $\int_{-\infty}^{t} f(t,s)\, X(\mathrm{d}s)$ exists as the limit of integrals $\int_{-n}^{t} f(t,s)\, X(\mathrm{d}s)$ in a suitable sense as n approaches infinity.

The class of processes of the form (1) is of special interest, because it contains several types of stochastic processes which have been intensely studied over the last years. First of all, when the driving process X is a two-sided Brownian motion and the integration kernel f is given by

$$f(t,s) = f_d(t,s) = \frac{1}{\Gamma(d+1)} \left((t-s)_+^d - (-s)_+^d \right) : \tag{2}$$

the process M is a so-called *fractional Brownian motion* (via *Mandelbrot-Van Ness representation*, see for example [34]). Here, Γ denotes the Gamma function and the parameter $d \in (-1/2, 1/2)$ is the *fractional integration parameter*. It is connected to the well-known *Hurst index* $H \in (0,1)$ via $d = H - 1/2$.

Note that a fractional Brownian motion B_H as above can be characterised as a Gaussian process on $[0,T]$ with $B_H(0) = 0$ a.s., $\mathbb{E}\left(B_H(t)\right) = 0$ for all $t \in [0,T]$ and covariance function

$$\operatorname{cov}\left(B_H(t), B_H(s)\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \tag{3}$$

for all $t, s \in [0,T]$. Its features such as self-similarity, stationarity of the increments, Hölder continuity of the paths, and long-range dependence in the case $H \in (1/2, 1)$ (or equivalently $d \in (0, 1/2)$, the situation we will restrict ourselves to later on) make the fractional Brownian motion an interesting candidate for modelling.

Starting with Kolmogorov in 1940 (who used the fractional Brownian motion in the case $0 < H < 1/2$ as a model for turbulence, see [40], p. 221ff.), the fractional Brownian motion has served as an important tool both in physics and finance, see for example [14], Section 3, for a list of applications. For the purpose of applying the fractional Brownian motion to various problems, a stochastic calculus for the fractional Brownian motion has been developed via different techniques such as Malliavin calculus (e.g. [1]), white noise calculus (e.g. [23]), and the S-transform approach as in [8], to name but a few references. Note that classical results, such as the Itō formula for semimartingales in general do not apply to the fractional Brownian motion, since the only case in which the fractional Brownian motion is a semimartingale is the case $H = 1/2$, in which it reduces to the classical Brownian motion.

However, since the fractional Brownian motion is a Gaussian process it is not an appropriate tool to handle modelling beyond normal distributions. This is where a new class of stochastic processes comes into play.

Motivated by the above-mentioned Mandelbrot-Van Ness representation of the fractional Brownian motion, [6] and [33] replaced the Brownian motion as the driving process in the convolution integral (1) by a Lévy process. In this way, the richness of the class of Lévy measures is used to obtain stochastic processes which admit more flexibility from the modelling point of view, for example heavier tails than a normal distribution. At the same time, these *fractional Lévy processes* have similar properties as the fractional Brownian motion such as Hölder continuous paths and stationary increments. Furthermore these processes exhibit *long memory* in the case $0 < d < 1/2$ (in the sense of [33]) and have up to a constant the same second-order structure as a fractional Brownian motion. That is defining

$$M_d(t) = \int_{-\infty}^{t} f_d(t,s)\ L(\mathrm{d}s)$$

with a centred square-integrable pure jump Lévy process L, [33] obtains

$$\operatorname{cov}\left(M_d(t), M_d(s)\right) = \frac{\mathbb{E}\left(L(1)^2\right)}{2\,\Gamma\left(2d+2\right)\sin\left(\pi\left(d+\frac{1}{2}\right)\right)}\left(|t|^{2d+1} + |s|^{2d+1} - |t-s|^{2d+1}\right).$$

Nevertheless, unlike the fractional Brownian motion, the fractional Lévy processes as introduced above are in general not self-similar (see [33] for more details).

In recent years, fractional Lévy processes and the more general *Lévy-driven Volterra processes* have been used in diverse problems in finance (see [5, 15, 20, 24]) as well as in modelling the workload of network devices (see [43]) and in signal processing (see [42]).

Let us mention that one has to be a bit careful with the expression *fractional Lévy process*: There are different ways to construct fractional Brownian motions as convolution integrals. Besides the aforementioned Mandelbrot-Van Ness representation there exists for example also the *Molchan-Golosov representation*

$$B_H(t) = \int_{0}^{t} z_H(t,s)\ B(\mathrm{d}s),$$

where B is a Brownian motion and z_H is the so-called *Molchan-Golosov kernel*. Note that in this definition only a one-sided Brownian motion is needed since we integrate over the interval $[0,T]$. In the case of a Brownian motion as a driving process, these two representations essentially lead to the same process. This is no longer true if we replace the Brownian motion by a Lévy process, as e.g. the fractional Lévy processes obtained via Mandelbrot-Van Ness representation have stationary increments (as mentioned above), whereas the fractional Lévy processes obtained via Molchan-Golosov representation do not have stationary increments in general (see [41] for more details on the construction and the comparison of the different approaches). In the following, we will only consider the case of fractional Lévy processes obtained via Mandelbrot-Van Ness representation.

In order to go in for stochastic calculus for processes of the form (1), this thesis is subdivided in two parts:

In the first part we will investigate path and moment properties of processes of the form (1). To be more precise, we will show in Theorem 1.3 that under appropriate assumptions on the kernel function f and the driving càdlàg p-integrable martingale X the limit

$$\lim_{n\to\infty} \int_{-n}^{t} f(t,s)\ X(\mathrm{d}s), \quad 0 \le t \le T,$$

exists $\mathbb{P}$-a.s. as well as in $\mathscr{L}^p(\mathbb{P})$ and has a modification which is given by an integration by parts representation. Furthermore, we will show that the convoluted process M inherits the càdlàg paths from X and that the jumps of M and the jumps of X are related via the formula

$$\Delta M(t) = f(t,t)\Delta X(t)$$

for all $t \in [0,T]$. Further we will show that the supremum

$$M^*(T) := \sup_{t\in[0,T]} |M(t)|$$

is p-integrable by proving an $\mathscr{L}^p(\mathbb{P})$-inequality for M^* which nicely separates the influences from the kernel f and the driving process X. In the special case that $M = M_d$ is a fractional Lévy process with a centred Lévy process with finite second moment as a driver, this $\mathscr{L}^p(\mathbb{P})$-inequality reads as follows: Given a fractional integration parameter $d \in (0, 1/2)$, $p \ge 2$, and $\delta > 0$ such that $d + \delta < 1/2$ there is a constant $C_{d,p,\delta}$ which is independent of the underlying Lévy process L, such that for every $T \ge 1$ the equation

$$\left\| \sup_{t\in[0,T]} |M(t)| \right\|_p \le C_{d,p,\delta}\ \left\|L(1)\right\|_p T^{d+1/2+\delta}$$

holds. Since

$$\left\|M(T)\right\|_2 = \left(\Gamma\left(2d+2\right)\sin\left(\pi\left(d+1/2\right)\right)\right)^{-\frac{1}{2}}\ \left\|L(1)\right\|_2 T^{d+1/2}$$

by the aforementioned covariance of fractional Lévy processes obtained in [33], we emphasise that, by choosing $\delta > 0$ small, we get arbitrarily close to this expected optimal rate.

Subsequently, we particularise the above results to the case that the driving martingale is a two-sided Lévy process. This special case of *Lévy-driven Volterra processes* will be studied more extensively in the second part of the thesis.

In the second part our main goal is to prove a generalised Itō formula for Lévy-driven Volterra processes, which contain as a special case the above-mentioned fractional Lévy processes.

We start by introducing the *Segal-Bargmann transform*, an important tool from white noise analysis and provide an injectivity result for the Segal-Bargmann transform in Proposition 4.3 which can be used to identify random variables. This result roughly states that if the Segal-Bargmann transforms $S\varphi$ and $S\psi$ of two $\mathscr{L}^2(\mathbb{P})$-random variables φ and ψ coincide on a certain set Ξ of test functions, that is, if

$$S\varphi(g) = S\psi(g)$$

for all $g \in \Xi$, then the two random variables φ and ψ coincide $\mathbb{P}$-almost surely.

Having this injectivity result at hand, motivated by [8] and [11], we will use the Segal-Bargmann transform to define *Hitsuda-Skorokhod integrals*. For a better understanding of such integrals, we will briefly discuss situations in which they reduce to the well-known integrals with respect to Brownian motion and Poisson jump measures, respectively. These Hitsuda-Skorokhod integrals will appear later on in our generalised Itō formula.

The starting point for proving our generalised Itō formula will be the equation

$$S(G(M(t)))(g) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}G)(u) S\big(e^{iuM(t)}\big)(g) \; \mathrm{d}u, \tag{4}$$

which follows from the Fourier inversion theorem, as we will show. Here $g \in \Xi$, $G \in C^2(\mathbb{R})$ fulfils some additional growth conditions which will be specified later on, and $\mathcal{F}G$ denotes the Fourier transform of the function G. By deriving an explicit expression for $S\big(e^{iuM(t)}\big)(g)$, which is the Segal-Bargmann transform of the characteristic function of the Lévy-driven Volterra process M, we can show that the expression in equation (4) is continuously differentiable on $[0,T]$. Combining this with the fundamental theorem of calculus will lead to

$$S(G(M(T)))(g) = G(0) + \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} S(G(M(t)))(g) \; \mathrm{d}t. \tag{5}$$

Plugging the explicit expression for $S(G(M(t)))(g)$ obtained via equation (4) into equation (5) and using the injectivity property of the Segal-Bargmann transform will eventually result in the generalised Itō formula (see Theorem 5.1)

$$\begin{aligned} G(M(T)) = G(0) &+ \frac{\sigma^2}{2} \int_0^T G''(M(t)) \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^t f(t,s)^2 \mathrm{d}s \right) \mathrm{d}t \\ &+ \sum_{0<t\leq T} \big[G(M(t)) - G(M(t-)) - G'(M(t-)) \Delta M(t) \big] \\ &+ \int_{-\infty}^T \int_{\mathbb{R}} \int_{0\vee s}^T G'\left(M(t) + x f(t,s)\right) \frac{\partial}{\partial t} f(t,s) \; \mathrm{d}t \; \Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}s) \\ &+ \int_{-\infty}^T \int_{\mathbb{R}_0} \int_{0\vee s}^T \big(G'\left(M(t) + x f(t,s)\right) - G'(M(t)) \big) x \frac{\partial}{\partial t} f(t,s) \; \mathrm{d}t \; \nu(\mathrm{d}x) \; \mathrm{d}s \\ &+ \int_0^T G'(M(t-)) f(t,t) \; L(\mathrm{d}t). \end{aligned} \tag{6}$$

Here $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, σ is the standard deviation of the Gaussian part of the underlying Lévy process L, and ν denotes the Lévy measure of L. The diamond symbol in the $\Lambda^{\diamond}$-integral indicates that the integration is understood in the Hitsuda-Skorokhod sense. This integral combines influences from the Gaussian part and the compensated jump measure of L (see Definition 4.9). In some special cases the integration with respect to $\Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}s)$ reduces to the classical integration with respect to a Gaussian process or a compensated jump measure, respectively (cf. Remark 4.10).

We emphasise that our result particularly yields that all the terms occurring in equation (6) exist as elements of $\mathscr{L}^2(\mathbb{P})$.

Under the additional assumption that all the terms occurring in the next formula exist as elements of $\mathscr{L}^2(\mathbb{P})$, we will furthermore be able to prove the following variant of the generalised Itō formula:

$$\begin{aligned} G(M(T)) = G(0) &+ \frac{\sigma^2}{2}\int_0^T G''(M(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t f(t,s)^2\mathrm{d}s\right)\mathrm{d}t \\ &+ \sum_{0<t\leq T}\left[G(M(t)) - G(M(t-)) - G'(M(t-))\Delta M(t)\right] \\ &+ \int_{-\infty}^T\int_{\mathbb{R}_0}\int_{0\vee s}^T \left(G'\left(M(t)+xf(t,s)\right) - G'(M(t))\right) x\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ N^\diamond(\mathrm{d}x,\mathrm{d}s) \\ &+ \int_0^T G'(M(t))\ M^\diamond(\mathrm{d}t), \end{aligned} \tag{7}$$

with N denoting the jump measure of the driving Lévy process L. Here, the $M^\diamond$- and the $N^\diamond$-integrals are in some sense the canonical extensions of the classical integrals with respect to a Lévy process and with respect to the random measure N to an integral with respect to the convoluted process M and with respect to N, respectively, in the Hitsuda-Skorokhod sense.

Let us now discuss the advantages over results that already exist in the literature and the generalising character of the generalised Itō formula (7):

- By choosing $f(t,s) = \mathbb{1}_{[0,t]}(s)$ as the kernel function, the Lévy-driven Volterra process M reduces to the underlying Lévy process L itself. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t f(t,s)^2\ \mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t 1\ \mathrm{d}s = 1$$

  and because of $\frac{\partial}{\partial t}f(t,s) = 0$, the integral

$$\int_{-\infty}^T\int_{\mathbb{R}_0}\int_{0\vee s}^T \left(G'\left(M(t)+xf(t,s)\right) - G'(M(t))\right) x\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ N^\diamond(\mathrm{d}x,\mathrm{d}s)$$

  vanishes. Consequently, the generalised Itō formula becomes the well known Itō formula for Lévy processes, that is

$$\begin{aligned} G(L(T)) = G(0) &+ \frac{\sigma^2}{2}\int_0^T G''(L(t-))\mathrm{d}t + \int_0^T G'(L(t-))\ L(\mathrm{d}t) \\ &+ \sum_{0<t\leq T}\left[G(L(t)) - G(L(t-)) - G'(L(t-))\Delta L(t)\right]. \end{aligned}$$

- In the case that L has no jump part, M also has no jumps and is a Gaussian process. Hence, the sum over the jumps of M and the triple integral vanish. Formula (7) reduces to the Itō formula for Gaussian processes, which is a well-established result, see for example [1], [14], and [7].

- If L is a pure jump Lévy process, we have $\sigma = 0$ and the first integral on the right-hand side of (7) vanishes. Such an Itō formula for pure jump Lévy-driven Volterra processes was obtained in [11] under rather strict assumptions on the kernel (compact support, among other restrictions) and the underlying Lévy process (existence of moments of all orders). We want to emphasise that we can weaken both the conditions on the kernel and the driving process. Especially, we are able to handle the case that M is a fractional Lévy process (via Mandelbrot-Van Ness representation).

Additionally, compared to [11], we can moderate the growth conditions on the function G if L contains a Gaussian part. This is due to the fact that the characteristic function of M (under change of measure) is rapidly decreasing in the presence of a Gaussian part.

The main results of this thesis are already published in two articles, jointly with C. Bender and R. Knobloch. Part I of this thesis is based on [10]:

> Christian Bender, Robert Knobloch, and Philip Oberacker. Maximal inequalities for fractional Lévy and related processes. *Stoch. Anal. Appl.*. 33(4):701-714, 2015.
> DOI: 10.1080/07362994.2015.1036167.

Part II of this thesis is based on [9]:

> Christian Bender, Robert Knobloch, and Philip Oberacker. A generalised Itō formula for Lévy-driven Volterra processes. *Stochastic Process. Appl.*, 125:2989–3022, 2015.
> DOI: 10.1016/j.spa.2015.02.009.

Part I

Path properties and maximal inequalities for convoluted processes

This part of the thesis is devoted to path properties and maximal inequalities of convoluted processes of the form

$$\int_{-\infty}^{t} f(t,s)\ X(\mathrm{d}s).$$

In Chapter 1 we will treat a quite general case where X is a two-sided martingale and f a Volterra type kernel fulfilling certain conditions which will be specified in Definition 1.2 and Theorem 1.3.

The key technique for us is to introduce an appropriate nondecreasing function $\varphi : [0,\infty) \to [1,\infty)$ which connects the behaviour of the function $s \mapsto f(t,s)$ and the underlying process X at $s = -\infty$ in the following way: On the one hand the function φ should increase "not too fast" to make sure that the conditions

$$\lim_{s\to\infty} f(t,-s)\varphi(s) = 0$$

and

$$\sup_{t\in[0,T]} \int_{-\infty}^{t} \left| \frac{\partial}{\partial s} f(t,s)\varphi(|s|) \right|^{1+\epsilon} \left(|s|^{2\epsilon} \vee 1\right)\ \mathrm{d}s < \infty$$

with some $\epsilon > 0$ are fulfilled. On the other hand the function φ should increase "fast enough" to guarantee that –assuming $X(t) \in \mathscr{L}^p(\mathbb{P})$ for every $t \in [0,T]$ and some $p > 1$– the expression

$$\sum_{n=0}^{\infty} \frac{\left\|X(2^{n+1}) - X(2^n)\right\|_p}{\varphi(2^n)}$$

is finite. Under these technical conditions (amongst others) we will show in Theorem 1.3 that the convolution integral $\int_{-\infty}^{t} f(t,s)\ X(\mathrm{d}s)$ exists (in an improper sense) and that the convoluted process inherits the càdlàg paths and the p-integrability of the underlying process X.

In Chapter 2 we particularise the results from Chapter 1 to the case that the driving process is a Lévy process. To be more precise, especially the situation in which the underlying process X is a centred Lévy process with at least a finite second moment is of particular interest for us, as we will develop a generalised Itō formula for the resulting convoluted processes in Part II of this thesis. This result will make use of the path properties and $\mathscr{L}^p(\mathbb{P})$-inequalities which are obtained in Part I.

In order to apply the results for the general case of a driving martingale X to a Lévy process L, the key element is to control the behaviour of the p-th moment of L as time increases. This will be done in Lemma 2.3, where we will show that given $p \geq 2$, $t \geq 1$, and $L(1) \in \mathscr{L}^p(\mathbb{P})$, there exists a constant C_p such that

$$\|L(t)\|_p \leq C_p\ t^{\frac{1}{2}}\ \|L(1)\|_p.$$

Chapter 1

The general case driven by a two-sided martingale

In this chapter we derive paths properties and a maximal inequality for certain modifications of processes of the form

$$\tilde{M}(t) = \int_{-\infty}^{t} f(t,s)\ X(\mathrm{d}s),\ 0 \le t \le T,$$

where X is a two-sided martingale, f is a deterministic kernel function, and $T > 0$ is fixed.

Definition 1.1 Let $\hat{X} := (\hat{X}(t))_{t\ge 0}$ be a càdlàg martingale starting at zero. We construct a two-sided process $X := (X(t))_{t\in\mathbb{R}}$ by taking two independent copies $(X_1(t))_{t\ge 0}$ and $(X_2(t))_{t\ge 0}$ of $\hat{X}$ and defining

$$X(t) := \begin{cases} X_1(t), & t \ge 0 \\ -X_2((-t)-), & t < 0. \end{cases} \tag{1.1}$$

Throughout this part of the thesis, let $\varphi : [0,\infty) \to [1,\infty)$ denote a nondecreasing function. We now introduce the following class of Volterra type kernels depending on φ:

Definition 1.2 Let $\tau \in [-\infty, 0]$. We denote by $\mathcal{K}(\varphi,\tau)$ the class of measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$ with $\mathrm{supp} f \subset [\tau,\infty)^2$ such that

(i) $\forall\, s > t \ge 0 : \quad f(t,s) = 0,$

(ii) the mapping $t \mapsto f(t,t)$ is continuous on $[0,T]$; moreover if $\tau > -\infty$ then also $t \mapsto f(t,\tau)$ is continuous on $[0,T]$,

(iii) for all $t \in [0,T]$ we have

$$\lim_{s\to\infty} f(t,-s)\varphi(s) = 0, \tag{1.2}$$

(iv) for every fixed $t \in [0,T]$ the function $s \mapsto f(t,s)$ is absolutely continuous on $[\tau, t]$ with density $\frac{\partial}{\partial s} f(t,\cdot)$, i.e.

$$f(t,s) = f(t,\tau) + \int_\tau^s \frac{\partial}{\partial u} f(t,u) \, \mathrm{d}u, \qquad \tau \le s \le t,$$

where $f(t,-\infty) := \lim_{x \to -\infty} f(t,x) = 0$, such that

(a) the function $t \mapsto \frac{\partial}{\partial s} f(t,s)$ is continuous on (s,∞) for λ-a.e. $s \in [\tau,\infty)$, where λ denotes the Lebesgue measure,

(b) there exists an $\epsilon > 0$ (independent of t) such that

$$\sup_{t \in [0,T]} \int_{-\infty}^t \left| \frac{\partial}{\partial s} f(t,s) \varphi(|s|) \right|^{1+\epsilon} \left(|s|^{2\epsilon} \vee 1 \right) \, \mathrm{d}s < \infty. \tag{1.3}$$

The function φ describes the behaviour of the kernel f and its density $\frac{\partial}{\partial s} f$ at $s = -\infty$. If it is connected to the $\mathscr{L}^p(\mathbb{P})$-norm (which we denote by $\|\cdot\|_p$) of the increments of the martingale X in an appropriate way (see (1.4)), we can show in the following theorem that the improper integral $\int_{-\infty}^t f(t,s) \, X(\mathrm{d}s)$ in the definition of $\tilde{M}$ exists and that there is a modification M of the convoluted process $\tilde{M}$ that inherits path properties and finite moments from the driving process X.

Theorem 1.3 *Let $f \in \mathcal{K}(\varphi,\tau)$ as well as $p > 1$ with $X(t) \in \mathscr{L}^p(\mathbb{P})$ for every $t \in \mathbb{R}$ and assume that*

$$\sum_{n=0}^\infty \frac{\left\| X(2^{n+1}) - X(2^n) \right\|_p}{\varphi(2^n)} < \infty. \tag{1.4}$$

Then the following assertions hold:

1. *The limit*

$$\tilde{M}(t) := \lim_{n \to \infty} \int_{-n}^t f(t,s) \, X(\mathrm{d}s), \quad 0 \le t \le T, \tag{1.5}$$

exists $\mathbb{P}$-a.s. and in $\mathscr{L}^p(\mathbb{P})$ and a modification of $\tilde{M}$ is given by

$$M(t) := f(t,t)X(t) - f(t,\tau)X(\tau) - \int_\tau^t X(s) \frac{\partial}{\partial s} f(t,s) \, \mathrm{d}s, \tag{1.6}$$

where $f(t,-\infty)X(-\infty) := \lim_{N \to \infty} f(t,-N)X(-N) = 0$ holds $\mathbb{P}$-a.s. and in $\mathscr{L}^p(\mathbb{P})$.

2. *The process M has càdlàg paths and its jumps are related to the jumps of the driver X by the formula*

$$\Delta M(t) = f(t,t) \Delta X(t).$$

3. *The maximal inequality*

$$
\begin{aligned}
&\left\| \sup_{t\in[0,T]} |M(t)| \right\|_p \\
&\leq \frac{p}{p-1} \sup_{t\in[0,T]} |f(t,t)| \|X(T)\|_p + \sup_{t\in[0,T]} \|X(\tau)f(t,\tau)\|_p \\
&\quad + \frac{2p}{p-1} \sup_{t\in[0,T]} \left(\int_\tau^t \varphi(|s|) \left| \frac{\partial}{\partial s} f(t,s) \right| \, \mathrm{d}s \right) \left(\|X(1)\|_p + \sum_{n=0}^\infty \frac{\left\| X(2^{n+1}) - X(2^n) \right\|_p}{\varphi(2^n)} \right) \\
&<\infty,
\end{aligned} \tag{1.7}
$$

holds, where $f(t,-\infty)X(-\infty) = 0$ *(cf. 1.).*

Remark 1.4 Note that by an application of Hölder's inequality we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} \int_\tau^t \varphi(|s|) \left| \frac{\partial}{\partial s} f(t,s) \right| \, \mathrm{d}s \\
&\leq \left(\sup_{t\in[0,T]} \int_{-\infty}^t \left| \frac{\partial}{\partial s} f(t,s) \varphi(|s|) \right|^{1+\epsilon} \left(|s|^{2\epsilon} \vee 1 \right) \, \mathrm{d}s \right)^{\frac{1}{1+\epsilon}} \left(\int_{-\infty}^T \left(|s|^{-2} \wedge 1 \right) \, \mathrm{d}s \right)^{\frac{\epsilon}{1+\epsilon}}
\end{aligned}
$$

and hence the finiteness of the right-hand side of (1.7) follows from the assumptions in Theorem 1.3, Definition 1.2(ii) and (1.3).

Remark 1.5

1. To treat kernels with compact support (i.e. the case $\tau > -\infty$), one can always choose $\varphi = 1$ and consider kernels f in the class $\mathcal{K}(1,\tau)$. In this situation we replace $X(t)$ by the process

$$
\tilde{X}(t) := \begin{cases} X(-(|\tau| \vee T)), & t < -(|\tau| \vee T) \\ X(t), & t \in [-(|\tau| \vee T), |\tau| \vee T] \\ X(|\tau| \vee T), & t > |\tau| \vee T, \end{cases}
$$

   which is constantly extended for $t \notin [-(|\tau| \vee T), |\tau| \vee T]$. This does not change the definition of M, but ensures that (1.4) is satisfied.

2. If f is sufficiently regular, the above relation between the jumps of M and X has already been proved in several papers, e.g. [11] (under additional compactness assumptions on the support of the kernel) and [35]. We emphasise that without some regularity assumptions M may fail to be continuous, even if f vanishes on the diagonal as shown by a counterexample in [26].

3. Motivated by the Mandelbrot-Van Ness representation of the fractional Brownian motion (cf. Proposition 7.2.6 in [38]) one can extend $\tilde{M}$ to the negative half line by setting

$$
\tilde{M}(t) := \int_{-\infty}^0 f(t,s) \; X(\mathrm{d}s), \quad t < 0.
$$

By slightly adapting the conditions on f in Definition 1.2 and the proof of Theorem 1.3 in an obvious way one observes that for $t < 0$ equation (1.6) becomes

$$M(t) = -f(t,\tau)X(\tau) - \int_\tau^0 X(s)\frac{\partial}{\partial s}f(t,s)\ \mathrm{d}s$$

and the maximal inequality (1.7) reads as follows

$$\begin{aligned}
&\left\| \sup_{t\in[-T,0]} |M(t)| \right\|_p \\
&\le \sup_{t\in[0,T]} \|X(\tau)f(t,\tau)\|_p \\
&\quad + \frac{2p}{p-1} \sup_{t\in[-T,0]} \left(\int_\tau^0 \varphi(|s|) \left| \frac{\partial}{\partial s} f(t,s) \right| \ \mathrm{d}s \right) \left(\|X(1)\|_p + \sum_{n=0}^\infty \frac{\left\| X(2^{n+1}) - X(2^n) \right\|_p}{\varphi(2^n)} \right).
\end{aligned}$$

We now prepare the proof of Theorem 1.3 by the following lemma.

Lemma 1.6 *Let $p > 1$ be as in Theorem 1.3. Then*

$$\left\| \sup_{s\in\mathbb{R}} \frac{|X(s)|}{\varphi(|s|)} \right\|_p \le \frac{2p}{p-1} \left(\|X(1)\|_p + \sum_{n=0}^\infty \frac{\left\| X(2^{n+1}) - X(2^n) \right\|_p}{\varphi(2^n)} \right) < \infty.$$

Proof Since Lemma 1.6 is only concerned with a distributional property of X, the construction of the two-sided process X entails that it suffices to consider X on the positive half line.

We first introduce the abbreviation

$$\mathfrak{L}_N := \sup_{s\in[0,2^N]} \frac{|X(s)|}{\varphi(s)} \quad (N \in \mathbb{N}).$$

By using Doob's inequality and the fact that the mapping $s \mapsto \varphi(s)^{-1}$ is bounded by 1, we infer $\mathfrak{L}_N \in \mathscr{L}^p(\mathbb{P})$. Drawing a distinction whether the supremum in the expression $\mathfrak{L}_N$ is attained on the set $[0, 2^{N-1}]$ or $[2^{N-1}, 2^N]$ and using the reverse triangle inequality as well as the fact that φ is nondecreasing we continue with the chain of estimates

$$\begin{aligned}
\mathfrak{L}_N &= (\mathfrak{L}_N - \mathfrak{L}_{N-1}) + \mathfrak{L}_{N-1} \\
&= \left(\left(\sup_{s\in[0,2^{N-1}]} \frac{|X(s)|}{\varphi(s)} \vee \sup_{s\in[2^{N-1},2^N]} \frac{|X(s)|}{\varphi(s)} \right) - \sup_{s\in[0,2^{N-1}]} \frac{|X(s)|}{\varphi(s)} \right) + \mathfrak{L}_{N-1} \\
&\le \left(\sup_{s\in[2^{N-1},2^N]} \frac{|X(s)|}{\varphi(s)} - \frac{|X(2^{N-1})|}{\varphi(2^{N-1})} \right) + \mathfrak{L}_{N-1} \\
&\le \sup_{s\in[2^{N-1},2^N]} \frac{\left| X(s) - X(2^{N-1}) \right|}{\varphi(2^{N-1})} + \mathfrak{L}_{N-1}.
\end{aligned}$$

Proceeding inductively we obtain

$$\mathfrak{L}_N \leq \sum_{n=1}^{N} \frac{\sup_{s\in[2^{n-1},2^n]} \left|X(s) - X(2^{n-1})\right|}{\varphi(2^{n-1})} + \sup_{s\in[0,1]} |X(s)|.$$

We now use Minkowski's inequality and Doob's inequality to deduce

$$\begin{aligned}
\|\mathfrak{L}_N\|_p &\leq \left\| \sup_{s\in[0,1]} |X(s)| \right\|_p + \sum_{n=1}^{N} \frac{\left\|\sup_{s\in[0,2^{n-1}]} |X(s+2^{n-1}) - X(2^{n-1})|\right\|_p}{\varphi(2^{n-1})} \\
&\leq \frac{p}{p-1} \left(\|X(1)\|_p + \sum_{n=1}^{N} \frac{\left\|X(2\cdot 2^{n-1}) - X(2^{n-1})\right\|_p}{\varphi(2^{n-1})} \right).
\end{aligned}$$

Since the series $\sum_{n=1}^{\infty} \frac{\left\|X(2^n)-X(2^{n-1})\right\|_p}{\varphi(2^{n-1})}$ is finite by assumption (1.4), it thus follows from the monotone convergence theorem that

$$\begin{aligned}
\left\| \sup_{s\in[0,\infty)} \frac{|X(s)|}{\varphi(s)} \right\|_p = \left\| \lim_{N\to\infty} \mathfrak{L}_N \right\|_p &= \lim_{N\to\infty} \|\mathfrak{L}_N\|_p \\
&\leq \frac{p}{p-1} \left(\|X(1)\|_p + \sum_{n=1}^{\infty} \frac{\left\|X(2^n) - X(2^{n-1})\right\|_p}{\varphi(2^{n-1})} \right) \\
&< \infty.
\end{aligned}$$

By symmetry in the construction of the two-sided process X the same inequality holds for the negative half line, which completes the proof. □

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3

1. For every $n \in \mathbb{N}$ we define

$$M_n(t) := \int_{\tau\vee -n}^{t} f(t,s)\ X(\mathrm{d}s),$$

which exists as a stochastic integral in the classical sense of integration with respect to a semimartingale by the continuity of $s \mapsto f(t,s)$, see Definition 1.2(iv). Since the above function is absolutely continuous, the standard integration by parts formula for Itō integrals yields for fixed t

$$\int_{\tau\vee -n}^{t} f(t,s)\ X(\mathrm{d}s) = f(t,t)X(t) - f(t,\tau\vee -n)X(\tau\vee -n) - \int_{\tau\vee -n}^{t} X(s)\frac{\partial}{\partial s} f(t,s)\ \mathrm{d}s. \tag{1.8}$$

In the case $\tau > -\infty$ we have $\tau \vee -n = \tau$ for n sufficiently large, which proves (1.6) in this case. If instead $\tau = -\infty$, we have

$$\begin{aligned}
|f(t,-n)X(-n)| &= |f(t,-n)\varphi(n)| \cdot \left| \frac{X(-n)}{\varphi(n)} \right| \\
&\leq \sup_{s\in(-\infty,t]} \frac{|X(s)|}{\varphi(|s|)} |f(t,-n)\varphi(n)|.
\end{aligned}$$

Since according to Lemma 1.6 the first factor on the above right-hand side is bounded in $\mathscr{L}^p(\mathbb{P})$ and therefore also bounded $\mathbb{P}$-a.s. and the second factor is deterministic and by (1.2) tends to 0 as $n \to \infty$, we deduce that

$$|f(t,-n)X(-n)| \to 0 \tag{1.9}$$

$\mathbb{P}$-a.s. and in $\mathscr{L}^p(\mathbb{P})$ as $n \to \infty$. Moreover, since $s \mapsto \frac{\partial}{\partial s} f(t,s)\varphi(|s|) \in \mathscr{L}^1(\lambda)$, cf. Remark 1.4, we obtain by Lemma 1.6 that

$$\int_{-\infty}^{t} \left| X(s) \frac{\partial}{\partial s} f(t,s) \right| \, \mathrm{d}s \leq \sup_{u \in (-\infty,t]} \left| \frac{X(u)}{\varphi(|u|)} \right| \int_{-\infty}^{t} \left| \frac{\partial}{\partial s} f(t,s)\varphi(|s|) \right| \, \mathrm{d}s < \infty$$

$\mathbb{P}$-a.s. and in $\mathscr{L}^p(\mathbb{P})$. Taking the limit as $n \to \infty$ in (1.8) and using the dominated convergence theorem thus proves the assertion.

2. In view of Definition 1.2(ii) and the assumption that X is càdlàg we only have to show that the third term on the right-hand side of (1.6) is continuous. For this purpose we define

$$\Upsilon(t,s) := X(s) \frac{\partial}{\partial s} f(t,s) \left(1 \vee |s|^2\right)$$

for $s, t \in \mathbb{R}$. By means of (1.3) and Lemma 1.6 we then obtain

$$\begin{aligned}
&\sup_{t \in [0,T]} \int_{\tau}^{t} |\Upsilon(t,s)|^{1+\epsilon} \, \frac{1}{1 \vee |s|^2} \mathrm{d}s \\
&\quad = \sup_{t \in [0,T]} \int_{\tau}^{t} \left| X(s) \frac{\partial}{\partial s} f(t,s) \right|^{1+\epsilon} \frac{(1 \vee |s|^2)^{1+\epsilon}}{(1 \vee |s|^2)} \, \mathrm{d}s \\
&\quad = \sup_{t \in [0,T]} \int_{\tau}^{t} \left| X(s) \frac{\partial}{\partial s} f(t,s) \right|^{1+\epsilon} (1 \vee |s|^{2\epsilon}) \, \mathrm{d}s \\
&\quad \leq \sup_{t \in [0,T]} \int_{\tau}^{t} \left| \varphi(|s|) \frac{\partial}{\partial s} f(t,s) \right|^{1+\epsilon} (1 \vee |s|^{2\epsilon}) \, \mathrm{d}s \cdot \sup_{s \in \mathbb{R}} \left(\frac{|X(s)|}{\varphi(|s|)} \right)^{1+\epsilon} \\
&\quad < \infty
\end{aligned}$$

$\mathbb{P}$-almost surely. Consequently, we can use the de la Vallée Poussin theorem to deduce that $(\mathbb{1}_{[\tau,t]}(\cdot)\Upsilon(t,\cdot))_{t \in [0,T]}$ is uniformly integrable with respect to the finite measure

$$\frac{1}{1 \vee |s|^2} \, \mathrm{d}s.$$

Now let $t \in [0,T]$ and choose an arbitrary sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n \to t$ as $n \to \infty$. The convergence $\Upsilon(t_n, s) \to \Upsilon(t,s)$ for λ-a.e. $s \in (-\infty, T]$, cf. Definition 1.2(iv)(a), together with the uniform integrability of $(\mathbb{1}_{[\tau,t]}(\cdot)\Upsilon(t,\cdot))_{t \in [0,T]}$ results in

$$\begin{aligned}
\lim_{n \to \infty} \int_{\tau}^{t_n} X(s) \frac{\partial}{\partial s} f(t_n, s) \, \mathrm{d}s &= \lim_{n \to \infty} \int_{\tau}^{T} \mathbb{1}_{[\tau,t_n]}(s)\Upsilon(t_n,s) \, \frac{1}{1 \vee |s|^2} \, \mathrm{d}s \\
&= \int_{\tau}^{t} X(s) \frac{\partial}{\partial s} f(t,s) \, \mathrm{d}s.
\end{aligned}$$

This implies that the mapping $t \mapsto \int_\tau^t X(s)\frac{\partial}{\partial s}f(t,s)\,\mathrm{d}s$ is continuous.

3. Due to the càdlàg paths of M (cf. 2.) the process M is separable and thus $\sup_{t\in[0,T]}|M(t)|^p$ is measurable. By means of the integration by parts formula (1.6), Minkowski's inequality, and Doob's inequality we obtain

$$\begin{aligned}\left\|\sup_{t\in[0,T]}|M(t)|\right\|_p &\leq \frac{p}{p-1}\sup_{t\in[0,T]}|f(t,t)|\|X(T)\|_p + \sup_{t\in[0,T]}\|X(\tau)f(t,\tau)\|_p \\ &\quad + \sup_{t\in[0,T]}\int_\tau^t \varphi(|s|)\left|\frac{\partial}{\partial s}f(t,s)\right|\,\mathrm{d}s\cdot\left\|\sup_{s\in\mathbb{R}}\frac{|X(s)|}{\varphi(|s|)}\right\|_p.\end{aligned}$$

The assertion is now a consequence of Definition 1.2(ii), (1.9), Remark 1.4, and Lemma 1.6. □

Chapter 2

The special case of Lévy-driven Volterra processes

In this chapter we particularise the main results of the previous chapter to the case when the driving martingale X is a Lévy process with some focus on fractional Lévy processes as introduced in [33].

We start with some introductory comments on Lévy processes in Section 2.1, where we also give some basic results. In Section 2.2 we specify the admitted underlying Lévy processes which are used to obtain the two-sided processes that we need to construct Lévy-driven Volterra processes. Finally, in Section 2.3 we show that the results from Chapter 1 are applicable to the Lévy-driven case.

2.1 Introductory comments on Lévy processes

In the remainder of Part I and in the whole of Part II of this thesis we will restrict ourselves to the case that the driving martingale is a Lévy process.

In our context a Lévy process[1] is a càdlàg real-valued stochastic process $(L_1(t))_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $L_1(0) = 0$ $\mathbb{P}$-a.s. that satisfies

1. L_1 has independent increments, i.e. for every $n \in \mathbb{N}$ and $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$ the random variables $L_1(t_0)$, $L_1(t_1) - L_1(t_0)$, $\ldots$, $L_1(t_n) - L_1(t_{n-1})$ are independent.
2. L_1 has stationary increments, i.e. the distribution of $L_1(t+h) - L_1(t)$ does not depend on t.
3. L_1 is stochastically continuous, i.e. for every $\epsilon > 0$ we have

$$\lim_{h\to 0} \mathbb{P}(|L_1(t+h) - L_1(t)| \geq \epsilon) = 0.$$

Note that some authors drop the assumption of càdlàg paths in the definition of a Lévy process as it can be shown that every Lévy process defined without the property of càdlàg paths has a unique modification which is càdlàg.

[1] In this introductory section we will use the (seemingly) inconvenient notation L_1 for a Lévy process instead of the more usual L because the letter L is reserved for the two-sided process which will be defined in the next section and which will be used far more often in the remainder of this thesis.

Because of their general importance and because we will make use of them we will now cite two fundamental results from the theory of Lévy processes. The first one is the Lévy-Khintchine representation:

Theorem 2.1 (Lévy-Khintchine representation) *Let $(L_1(t))_{t\geq 0}$ be a Lévy process. Then there exists a triplet (γ, σ, ν), called the* characteristic triplet *of the Lévy process L_1, consisting of constants $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and a sigma-finite measure ν on $\mathbb{R}_0$ which fulfils*

$$\int_{\mathbb{R}_0} \left(x^2 \wedge 1\right) \ \nu(\mathrm{d}x) < \infty$$

such that for every $t \geq 0$ the characteristic function $u \mapsto \mathbb{E}\left(e^{iuL_1(t)}\right)$ of $L_1(t)$ has the form

$$\mathbb{E}\left(e^{iuL_1(t)}\right) = e^{t\psi(u)}$$

with ψ given by

$$\psi(u) = iu\gamma - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}}\right) \ \nu(\mathrm{d}x).$$

In order to present the second fundamental result, given a Lévy process L_1 with characteristic triplet (γ, σ, ν), we now define a random measure N_1 by

$$N_1(\mathcal{A}, [a,b]) := \#\{s \in [a,b] : \ \Delta L_1(s) \in \mathcal{A}\}$$

for $0 \leq a \leq b$ and every Borel set $\mathcal{A} \subset \mathbb{R}_0$. N_1 is called the *jump measure* of L_1. Note that we have the relation

$$\nu(\mathcal{A}) = \mathbb{E}\left(\#\{s \in [0,1] : \ \Delta L_1(s) \in \mathcal{A}\}\right),$$

that is $\nu(\mathcal{A})$ is the expected number of jumps of L_1 over the unit interval $[0,1]$ whose sizes belong to $\mathcal{A}$. The measure $\nu(\mathrm{d}x)\ \mathrm{d}s$ is called the *compensator* and

$$\tilde{N}_1(\mathrm{d}x, \mathrm{d}s) := N_1(\mathrm{d}x, \mathrm{d}s) - \nu(\mathrm{d}x)\ \mathrm{d}s$$

is called the *compensated jump measure* of L_1. This enables us to state the Lévy-Itō decomposition:

Theorem 2.2 (Lévy-Itō decomposition) *Let $(L_1(t))_{t\geq 0}$ be a Lévy process with characteristic triplet (γ, σ, ν) and jump measure N_1. Then there exist a standard Brownian motion $W_1 = (W_1(t))_{t\geq 0}$ which is independent of N_1 such that*

$$L_1(t) = \gamma t + \sigma W_1(t) + \int_0^t \int_{\mathbb{R}\setminus[-1,1]} x \ N_1(\mathrm{d}x, \mathrm{d}s) + \int_0^t \int_{[-1,1]\setminus\{0\}} x \ \left[N_1(\mathrm{d}x, \mathrm{d}s) - \nu(\mathrm{d}x)\ \mathrm{d}s\right].$$

For more information on the theory of Lévy processes we refer e.g. to [13] and [27] as well as to [2], where stochastic analysis with respect to Lévy processes is treated.

2.2 Construction of Lévy-driven Volterra processes

Let $(L_1(t))_{t\geq 0}$ and $(L_2(t))_{t\geq 0}$ be two independent Lévy processes with càdlàg paths on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with characteristic triplet (γ, σ, ν), where $\sigma \geq 0$, ν is a Lévy measure on $\mathbb{R}_0$, that satisfies

$$\int_{\mathbb{R}_0} \left(x^2 \wedge x\right) \; \nu(\mathrm{d}x) < \infty, \tag{2.1}$$

and

$$\gamma = -\int_{\mathbb{R}\setminus[-1,1]} x \; \nu(\mathrm{d}x). \tag{2.2}$$

For $l = 1, 2$ note that (2.1) and (2.2) are equivalent to $L_l(t) \in \mathscr{L}^1(\mathbb{P})$ and $\mathbb{E}(L_l(1)) = 0$. Moreover, by using (2.2) we deduce

$$\gamma t = -t \int_{\mathbb{R}\setminus[-1,1]} x \; \nu(\mathrm{d}x) = -\int_0^t \int_{\mathbb{R}\setminus[-1,1]} x \; \nu(\mathrm{d}x) \; \mathrm{d}s.$$

Combining this with the Lévy-Itō decomposition we see that the processes $L_l(t)$ can be represented as

$$\begin{aligned} L_l(t) &= \gamma t + \sigma W_l(t) + \int_0^t \int_{\mathbb{R}\setminus[-1,1]} x \; N_l(\mathrm{d}x, \mathrm{d}s) + \int_0^t \int_{[-1,1]\setminus\{0\}} x \; [N_l(\mathrm{d}x, \mathrm{d}s) - \nu(\mathrm{d}x) \; \mathrm{d}s] \\ &= \sigma W_l(t) + \int_0^t \int_{\mathbb{R}\setminus[-1,1]} x \; [N_l(\mathrm{d}x, \mathrm{d}s) - \nu(\mathrm{d}x) \; \mathrm{d}s] + \int_0^t \int_{[-1,1]\setminus\{0\}} x \; [N_l(\mathrm{d}x, \mathrm{d}s) - \nu(\mathrm{d}x) \; \mathrm{d}s] \\ &= \sigma W_l(t) + \int_0^t \int_{\mathbb{R}_0} x \; \tilde{N}_l(\mathrm{d}x, \mathrm{d}s) \end{aligned}$$

where W_l is a standard Brownian motion and $\tilde{N}_l(\mathrm{d}x, \mathrm{d}s) = N_l(\mathrm{d}x, \mathrm{d}s) - \nu(\mathrm{d}x) \; \mathrm{d}s$ is the compensated jump measure of the Lévy process L_l.

We define a two-sided Lévy process $L := (L(t))_{t\in\mathbb{R}}$ by

$$L(t) := \begin{cases} L_1(t), & t \geq 0 \\ -L_2((-t)-), & t < 0 \end{cases}$$

and the two-sided Brownian motion $W := (W(t))_{t\in\mathbb{R}}$ is defined analogously.

Note that for $a, b \in \mathbb{R}$ with $a \leq b$ and any Borel set $\mathcal{A} \subset \mathbb{R}_0$ the jump measure $N(\mathrm{d}x, \mathrm{d}s)$ of the two-sided process L fulfils

$$\begin{aligned} N(\mathcal{A}, [a, b]) &:= \#\{s \in [a, b] : \; \Delta L(s) \in \mathcal{A}\} \\ &= N_1(\mathcal{A}, [a, b] \cap [0, \infty)) + N_2(\mathcal{A}, [-b, -a] \cap (0, \infty)) \end{aligned}$$

and hence

$$\mathbb{E}\left(N(\mathcal{A},[a,b])\right) = (b-a)\,\nu(\mathcal{A}).$$

In the cases $a \le b \le 0$ and $0 \le a \le b$ this assertion follows directly from the corresponding properties of the measures N_1 and N_2, respectively. In the case $a \le 0 \le b$ we get

$$\begin{aligned}
\mathbb{E}\left(N(\mathcal{A},[a,b])\right) &= \mathbb{E}\left(N_1(\mathcal{A},[a,b]\cap[0,\infty))\right) + \mathbb{E}\left(N_2(\mathcal{A},[-b,-a]\cap(0,\infty))\right) \\
&= \mathbb{E}\left(N_1(\mathcal{A},[0,b])\right) + \mathbb{E}\left(N_2(\mathcal{A},(0,-a])\right) \\
&= \nu(\mathcal{A})(b-0) + \nu(\mathcal{A})(-a-0) \\
&= (b-a)\,\nu(\mathcal{A}).
\end{aligned}$$

The compensated jump measure of the two-sided process L is defined as $\tilde{N}(\mathrm{d}x,\mathrm{d}s) := N(\mathrm{d}x,\mathrm{d}s) - \nu(\mathrm{d}x)\ \mathrm{d}s$. Furthermore, we assume that $\mathscr{F}$ is the completion of the σ-algebra generated by L.

Having the process L at hand we will consider the *Lévy-driven Volterra process*

$$M(t) = \int_{-\infty}^{t} f(t,s)\ L(\mathrm{d}s), \tag{2.3}$$

where f is an appropriate deterministic Volterra kernel, that is $f(t,s) = 0$ whenever $s > t \ge 0$.

In the literature, processes as in (2.3) are occasionally also referred to as *filtered Lévy processes* (see e.g. [17]) or *convoluted Lévy processes* (cf. [11]). However, as e.g. in [5] we think that *Lévy-driven Volterra processes* is the most apposite name for such processes. In the special case that $f(t,s) = g(t-s) - g(-s)$ for some function g such processes are also called *moving average processes*.

The prime example of Lévy-driven Volterra processes are *fractional Lévy processes*, where the integration kernel is given by

$$f_d(t,s) = \frac{1}{\Gamma(d+1)}\left((t-s)_+^d - (-s)_+^d\right).$$

Fractional Lévy processes exist e.g. for parameters $d \in (0,1/2)$, when the driving centred Lévy process L is square-integrable. In this case they have (up to some constant) the same second order structure as a fractional Brownian motion with Hurst parameter $H = 1/2 + d$, that is defining

$$M_d(t) = \int_{-\infty}^{t} f_d(t,s)\ L(\mathrm{d}s)$$

with a centred square-integrable pure jump Lévy process L, in [33] the formula

$$\operatorname{cov}\left(M_d(t), M_d(s)\right) = \frac{\mathbb{E}\left(L(1)^2\right)}{2\,\Gamma\left(2d+2\right)\sin\left(\pi\left(d+\frac{1}{2}\right)\right)}\left(|t|^{2d+1} + |s|^{2d+1} - |t-s|^{2d+1}\right)$$

is obtained. Note that in general fractional Lévy processes fail to be self-similar except in the special case that they are fractional Brownian motions and in general they are not semimartingales.

At the same time, these *fractional Lévy processes* have similar properties as the fractional Brownian motion such as Hölder continuous paths and stationary increments. Furthermore these processes exhibit *long memory* (in the sense of [33], where the proofs of the above statements can be found).

The motivation for the name fractional Lévy process is that it generalises the Mandelbrot-Van Ness representation of a fractional Brownian motion as an integral of the same kernel with respect to Brownian motion. However, note that there are different definitions of fractional Brownian motion as for example besides the Mandelbrot-Van Ness representation there is also the definition via the *Molchan-Golosov* representation

$$B_H(t) = \int_0^t z_H(t,s)\ B(\mathrm{d}s),$$

where B is Brownian motion and z_H is the so-called *Molchan-Golosov kernel*. Note that in this definition only a one-sided Brownian motion is needed since integration only takes place on the interval $[0,T]$.

In the case of a Brownian motion as a driving process, these two representations essentially lead to the same process. If we replace the Brownian motion by a Lévy process we will end up with different processes. For more details on the construction and the comparison of the different approaches we refer the reader to [41], where e.g. it is shown that the fractional Lévy processes obtained via Mandelbrot-Van Ness representation have stationary increments (as mentioned above), whereas the fractional Lévy processes obtained via Molchan-Golosov representation do not have stationary increments in general.

2.3 Results on Lévy-driven Volterra processes

In order to make the result of Theorem 1.3 applicable to the Lévy-driven case, given a kernel function f we have to guarantee the existence of a nondecreasing function $\varphi : [0,\infty) \to [1,\infty)$ such that $f \in \mathcal{K}(\varphi,\tau)$ and (1.4) is fulfilled.

As a first step we investigate the behaviour of the p-th moment of L as time approaches infinity in the following Lemma.

Lemma 2.3 *Let $p \geq 2$, $t \geq 1$ and $L(1) \in \mathscr{L}^p(\mathbb{P})$. Then there exist a constant C_p only depending on p such that*

$$\mathbb{E}\left(|L(t)|^p\right) \leq C_p\ t^{\frac{p}{2}}\ \mathbb{E}\left(|L(1)|^p\right).$$

Proof Note that for $l \geq 1$ and every Lévy process $\tilde{L}$ with $\tilde{L}(1) \in \mathscr{L}^l(\mathbb{P})$ and $n \in \mathbb{N}$ as well as $s \in [0,1)$ such that $t = n+s$ we infer by using Minkowski's inequality and the stationary increments of $\tilde{L}$

$$\begin{aligned}\left\|\tilde{L}(t)\right\|_l &= \left\|\left(\tilde{L}(n+s) - \tilde{L}((n-1)+s)\right) + \ldots + \left(\tilde{L}(s+1) - \tilde{L}(s)\right) + \tilde{L}(s)\right\|_l \\ &\leq (t+1) \sup_{u\in[0,1]} \left\|\tilde{L}(u)\right\|_l .\end{aligned}$$

Since $t \geq 1$ implies $(t+1)^l \leq (2t)^l$ the above leads to

$$\mathbb{E}\left(\left|\tilde{L}(t)\right|^l\right) = \left\|\tilde{L}(t)\right\|_l^l \leq 2^l t^l \sup_{u\in[0,1]} \left\|\tilde{L}(u)\right\|_l^l . \tag{2.4}$$

We now apply the Burkholder-Davis-Gundy inequality (see Theorem 48 in [36]) to the Lévy process L and deduce that there exists a constant $c_{p,1} > 0$ such that

$$\mathbb{E}\left(|L(t)|^p\right) \leq \mathbb{E}\left(\sup_{u\in[0,t]} |L(u)|^p\right) \leq c_{p,1}\mathbb{E}\left([L,L]_t^{\frac{p}{2}}\right). \tag{2.5}$$

Thus, using that the quadratic variation of a Lévy process is again a Lévy process (see [16], Example 8.5), choosing $\tilde{L}(t) := [L,L]_t$ and $l = \frac{p}{2}$, the inequalities (2.4) and (2.5) result in

$$\mathbb{E}\left(|L(t)|^p\right) \leq c_{p,1}\mathbb{E}\left([L,L]_t^{\frac{p}{2}}\right) \leq c_{p,1}2^{\frac{p}{2}} \sup_{u\in[0,1]} \mathbb{E}\left([L,L]_u^{\frac{p}{2}}\right) t^{\frac{p}{2}} = c_{p,1}2^{\frac{p}{2}}\mathbb{E}\left([L,L]_1^{\frac{p}{2}}\right) t^{\frac{p}{2}}. \tag{2.6}$$

We proceed with another application of the Burkholder-Davis-Gundy inequality and Doob's maximal inequality which lead to

$$\mathbb{E}\left([L,L]_1^{\frac{p}{2}}\right) \leq c_{p,2}\mathbb{E}\left(\sup_{u\in[0,1]} |L(u)|^p\right) \leq c_{p,2}\left(\frac{p}{p-1}\right)^p \mathbb{E}\left(|L(1)|^p\right)$$

for some $c_{p,2} > 0$. Plugging this into (2.6) results in

$$\mathbb{E}\left(|L(t)|^p\right) \leq c_{p,1}c_{p,2}2^{\frac{p}{2}}\left(\frac{p}{p-1}\right)^p t^{\frac{p}{2}}\ \mathbb{E}\left(|L(1)|^p\right).$$

Defining

$$C_p := c_{p,1}c_{p,2}2^{\frac{p}{2}}\left(\frac{p}{p-1}\right)^p$$

completes the proof. □

Theorem 1.3 in the Lévy-driven case roughly states that nice path and integrability properties of the Lévy process L are carried over to the convoluted process M for suitable kernel functions. The precise formulation in the situation that the driver is a centred Lévy process and has at least a finite second moment reads as follows.

Theorem 2.4 *Define $\varphi_q(s) = |s|^q \vee 1$. Let $p \geq 2$ and suppose that L is a centred Lévy process such that $L(1) \in \mathscr{L}^p(\mathbb{P})$. If $f \in \mathcal{K}(\varphi_q, -\infty)$ for some $q > 1/2$, then*

$$\tilde{M}(t) = \int_{-\infty}^t f(t,s)\ L(\mathrm{d}s),\ 0 \leq t \leq T,$$

exists as a Wiener integral and a càdlàg modification of $\tilde{M}$ is given by

$$M(t) := f(t,t)L(t) - \int_{-\infty}^t L(s)\frac{\partial}{\partial s}f(t,s)\ \mathrm{d}s.$$

This modification satisfies

$$\Delta M(t) = f(t,t)\Delta L(t)$$

and the following maximal inequality: There is a constant $C_{p,q}$ depending only on p and q such that

$$\left\| \sup_{t\in[0,T]} |M(t)| \right\|_p \leq C_{p,q}\|L(1)\|_p \left((T\vee 1)^{1/2} \sup_{t\in[0,T]} |f(t,t)| + \sup_{t\in[0,T]} \left(\int_{-\infty}^t \left| \frac{\partial}{\partial s} f(t,s) \right| (|s|^q \vee 1)\mathrm{d}s \right) \right)$$
$$< \infty.$$

Proof From the behaviour of $s \mapsto f(t,s)$ at $-\infty$ given by (1.2) with $\varphi = \varphi_q$ we deduce that there is some $\tilde{s} \leq -1$ such that $|f(t,s)| \leq |s|^{-q}$ for all $s \leq \tilde{s}$. Since $q > \frac{1}{2}$ we infer by using Definition 1.2(i) and the continuity of f in the s-variable (cf. Definition 1.2(iv)) the estimate

$$\int_{\mathbb{R}} |f(t,s)|^2 \, \mathrm{d}s \leq \int_{-\infty}^{\tilde{s}} |s|^{-2q} \, \mathrm{d}s + \int_{\tilde{s}}^t \sup_{s\in[\tilde{s},t]} |f(t,s)|^2 \, \mathrm{d}s < \infty$$

for all $t \in [0,T]$ and therefore $f(t,\cdot) \in \mathscr{L}^2(\mathbb{R})$.

Hence, $\tilde{M}(t)$ exists as a Wiener integral and

$$\tilde{M}(t) = \lim_{n\to\infty} \int_{-n}^t f(t,s) \; L(\mathrm{d}s)$$

in $\mathscr{L}^2(\mathbb{P})$. Using the stationary increments of L, Lemma 2.3, and the fact that $q > 1/2$ we deduce

$$\sum_{n=0}^{\infty} \frac{\left\|L(2^{n+1}) - L(2^n)\right\|_p}{\varphi_q(2^n)} = \sum_{n=0}^{\infty} \frac{\|L(2^n)\|_p}{\varphi_q(2^n)} \leq C_p^{\frac{1}{p}} \mathbb{E}(|L(1)|^p)^{1/p} \sum_{n=0}^{\infty} (2^n)^{1/2-q} < \infty. \tag{2.7}$$

Therefore, condition (1.4) is satisfied and so Theorem 1.3 applies with L in place of X. Plugging (2.7) into the right-hand side of (1.7) and changing from $\|L(T)\|_p$ to $\|L(1)\|_p$ in (1.7) by using Lemma 2.3 (in the case $T \geq 1$) yields the maximal inequality. □

Example 2.5 We now turn to the case of a fractional Lévy process

$$M_d(t) = \int_{-\infty}^t f_d(t,s) \; L(\mathrm{d}s)$$

with the kernel function

$$f_d(t,s) = \frac{1}{\Gamma(d+1)} \left((t-s)_+^d - (-s)_+^d \right)$$

for $d > 0$, where Γ denotes the Gamma function. The parameter d is related to the well-known Hurst parameter via $d = H - 1/2$. To apply Theorem 2.4 we show that $f_d \in \mathcal{K}(\varphi_q, -\infty)$, if $d+q < 1$. Is is easy to see, that Definition 1.2(i) and (ii) are fulfilled. We continue by calculating

$$\frac{\partial}{\partial s} f_d(t,s) = -\frac{1}{\Gamma(d+1)}\Big((t-s)_+^{d-1} - (-s)_+^{d-1}\Big)$$

and deduce that Definition 1.2(iv)(a) holds.

Moreover, by the mean value theorem we have for $t \in [0,T]$ and $s < 0$

$$\begin{aligned} \Gamma(d+1)|f_d(t,s)| &\leq dt|s|^{d-1}, \\ \Gamma(d+1)\left|\frac{\partial}{\partial s} f_d(t,s)\right| &\leq d(1-d)t|s|^{d-2}. \end{aligned} \tag{2.8}$$

The first inequality in (2.8) shows that (1.2) is satisfied with $\varphi = \varphi_q$, since $d+q-1<0$.

The critical points for checking for the integrability condition (1.3) are $s=t$, $s=0$, and $s \to -\infty$. Choosing $\epsilon < d/(1-d)$ we see that the integrability around $s=t$ and $s=0$ is guaranteed, since under this assumption the exponent of the terms $(t-s)_+$ and $(-s)_+$ appearing in the derivative $\frac{\partial}{\partial s} f_d(t,s)$ can be estimated as follows:

$$(d-1)(1+\epsilon) > (d-1)\left(1+\frac{d}{1-d}\right) = -1.$$

Using the second line of (2.8) we deduce that the term whose integrability for $s \to -\infty$ we have to check for is

$$\Big(|s|^{d-2}|s|^q\Big)^{1+\epsilon}|s|^{2\epsilon} = |s|^{d(1+\epsilon)+q(1+\epsilon)-2}$$

and we have

$$d(1+\epsilon)+q(1+\epsilon)-2 < -1$$

for $\epsilon < 1/(d+q) - 1$. Therefore, choosing $\epsilon < d/1-d \wedge (1/d+q - 1)$ we make sure that equation (1.3) is fulfilled.

Hence, we observe that Theorem 2.4 is applicable to fractional Lévy processes M_d for $0<d<1/2$, if the driving centred Lévy process has a finite second moment. The continuity of the fractional Lévy process follows from $f_d(t,t)=0$, but is well-known (see e.g. [33]). By the substitution $s=vt$ we obtain for $q>1/2$ such that $d+q<1$

$$\begin{aligned} &\Gamma(d+1)\int_{-\infty}^{t}\left|\frac{\partial}{\partial s} f_d(t,s)\right|(|s|^q \vee 1)\,\mathrm{d}s \\ \leq& t^{d+q}\int_{-\infty}^{1}\Big((1-v)_+^{d-1} - (-v)_+^{d-1}\Big)|v|^q\,\mathrm{d}v + t^d\int_{-\infty}^{1}\Big((1-v)_+^{d-1} - (-v)_+^{d-1}\Big)\,\mathrm{d}v. \end{aligned}$$

Therefore, in the situation $T \geq 1$ the maximal inequality in Theorem 2.4 can be simplified as follows:

If $q>1/2$ and $d+q<1$, there is a constant $C_{p,q,d}$ independent of the driving Lévy process L such that

$$\left\| \sup_{t\in[0,T]} |M_d(t)| \right\|_p \leq C_{p,q,d} \|L(1)\|_p \, T^{d+q}.$$

We conclude this section by proving a variant of Theorem 1.3 in which the driver L is a symmetric stable process with index of stability $\alpha \in (1,2)$. In this case the convoluted processes generalise fractional α-stable motions, for which we refer to [38] and [19].

Before stating the variant of Theorem 1.3, we briefly introduce the concept of so-called *L-integrals* as in [37]:

Using the Lévy process L we define a random measure on $\mathcal{B}(\mathbb{R})$ (which we will again denote by the letter L by a slight abuse of notation) via

$$L((a,b]) := L(b) - L(a)$$

for all $a, b \in \mathbb{R}$ with $a < b$. Now let

$$f(s) = \sum_{j=1}^{n} a_j \mathbb{1}_{A_j}(s), \quad s \in \mathbb{R},$$

for some $n \in \mathbb{N}$, constants $a_j \in \mathbb{R}$ and pairwise disjoint sets $A_j \in \mathcal{B}(\mathbb{R})$. One defines the L-integral of f over a set $A \in \mathcal{B}(\mathbb{R})$ as

$$\int_A f(s) \; L(\mathrm{d}s) := \sum_{j=1}^{n} a_j \; L(A \cap A_j).$$

A general measurable function $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called L-integrable, if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions as above such that $f_n(s) \to f(s)$ as n tends to infinity for a.e. $s \in \mathbb{R}$ and such that the sequence of random variables $\int_A f_n(s) \; L(\mathrm{d}s)$ converges in probability as n tends to infinity. In this case one defines

$$\int_A f(s) \; L(\mathrm{d}s) := \lim_{n\to\infty} \int_A f_n(s) \; L(\mathrm{d}s).$$

Adapting Proposition 1 in [19] (see also Theorem 3.3 in [37]) to our situation, we arrive at the following Lemma concerned with the L-integrability of a function f.

Lemma 2.6 *Let $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable function and let L be a Lévy process such that $\mathbb{E}\left(|L(1)|^p\right) < \infty$ for some $p > 0$. Furthermore, assume that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}_0} \left(|f(s)x|^2 \mathbb{1}_{\{|f(s)x| \leq 1\}} + |f(s)x|^p \mathbb{1}_{\{|f(s)x| > 1\}} \right) \; \nu(\mathrm{d}x) \; \mathrm{d}s < \infty$$

and

$$\int_{\mathbb{R}} |f(s)| \left| \int_{\mathbb{R}_0} x \left(\mathbb{1}_{\{|f(s)x| \leq 1\}} - \mathbb{1}_{\{|x| \leq 1\}} \right) \; \nu(\mathrm{d}x) \right| \; \mathrm{d}s < \infty.$$

Then f is L-integrable, we have

$$\mathbb{E}\left(\left|\int_{\mathbb{R}} f(s)\ L(\mathrm{d}s)\right|^p\right) < \infty,$$

and the mapping $f : L_{\phi_p}(\mathbb{R}) \to \mathscr{L}^p(\mathbb{P})$,

$$f \mapsto \int_{\mathbb{R}} f(s)\ L(\mathrm{d}s)$$

is continuous, where L_{ϕ_p} is the Musielak-Orlicz space *defined by*

$$L_{\phi_p}(\mathbb{R}) := \left\{f : f \text{ is } L\text{-integrable and } \int_{\mathbb{R}}\int_{\mathbb{R}_0} |f(s)x|^p \mathbb{1}_{\{|f(s)x|>1\}}\ \nu(\mathrm{d}x)\ \mathrm{d}s < \infty\right\}.$$

Proof Noting that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}_0} \left(|f(s)x|^2 \wedge 1\right)\ \mathrm{d}s &= \int_{\mathbb{R}}\int_{\mathbb{R}_0} \left(|f(s)x|^2\mathbb{1}_{\{|f(s)x|\le 1\}} + \mathbb{1}_{\{|f(s)x|>1\}}\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}_0} \left(|f(s)x|^2\mathbb{1}_{\{|f(s)x|\le 1\}} + |f(s)x|^p\mathbb{1}_{\{|f(s)x|>1\}}\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s,\end{aligned}$$

we see that the integrability assumptions in Lemma 2.6 imply the integrability assumptions in Proposition 1 in [19]. Consequently, the assertion follows directly from the latter result. □

We are now ready to state the following theorem, which -in contrast to Theorem 2.4- does not need a finite second moment of the driving Lévy process L.

Theorem 2.7 *Suppose that L is a symmetric α-stable Lévy process with $\alpha \in (1,2)$. If $f \in \mathcal{K}(\varphi_q, -\infty)$ for some $q > 1/\alpha$, then*

$$\tilde{M}(t) = \int_{-\infty}^{t} f(t,s)\ L(\mathrm{d}s),\ 0 \le t \le T,$$

exists as an L-integral in the sense of [37] as introduced above and a càdlàg modification of $\tilde{M}$ is given by

$$M(t) := f(t,t)L(t) - \int_{-\infty}^{t} L(s)\frac{\partial}{\partial s}f(t,s)\ \mathrm{d}s.$$

This modification satisfies

$$\Delta M(t) = f(t,t)\Delta L(t)$$

and the following maximal inequality: For every $p \in (1,\alpha)$ there is a constant $C_{p,q}$ depending only on p and q such that

$$\left\|\sup_{t\in[0,T]}|M(t)|\right\|_p \le C_{p,q}\|L(1)\|_p\left(T^{1/\alpha}\sup_{t\in[0,T]}|f(t,t)| + \sup_{t\in[0,T]}\left(\int_{-\infty}^{t}\left|\frac{\partial}{\partial s}f(t,s)\right|(|s|^q \vee 1)ds\right)\right).$$

Proof We first show the existence of the L-integral. As $f \in K(\varphi_q, -\infty)$, if follows from Definition 1.2(i) and (iv) as well as (1.2) that there is a constant $C(t) > 0$ depending on t such that

$$|f(t,s)| \leq C(t)(|s|^{-q} \wedge 1)$$

for every $s \in \mathbb{R}$. Noting that the Lévy measure of a symmetric α-stable process is given by $\nu(\mathrm{d}x) = A|x|^{-1-\alpha}\,\mathrm{d}x$ for some constant $A > 0$, we get for every $t \in [0,T]$ and $1 < p < \alpha$

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}_0} \left(|f(t,s)x|^2 \mathbb{1}_{\{|f(t,s)x|\leq 1\}} + |f(t,s)x|^p \mathbb{1}_{\{|f(t,s)x|>1\}}\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s \\
&\leq \quad A\int_{\mathbb{R}}\int_{\{|f(t,s)x|\leq 1\}\cap\{|x|\geq 1\}} |f(t,s)|^2\ |x|^{1-\alpha}\mathrm{d}x\ \mathrm{d}s \\
&\qquad +A\int_{\mathbb{R}}\int_{\{|f(t,s)x|\leq 1\}\cap\{0<|x|<1\}} |f(t,s)|^2\ |x|^{1-\alpha}\mathrm{d}x\ \mathrm{d}s \\
&\qquad +A\int_{\mathbb{R}}\int_{\{|f(t,s)x|>1\}} |f(t,s)|^p\ |x|^{p-1-\alpha}\mathrm{d}x\ \mathrm{d}s \\
&=: \quad \mathrm{I}(t) + \mathrm{II}(t) + \mathrm{III}(t).
\end{aligned} \tag{2.9}$$

To handle $\mathrm{I}(t)$ we assume $2-\alpha < \gamma < 2-1/q$ and use that $|f(t,s)x| \leq 1$ implies that $|x|^\gamma \leq |f(t,s)|^{-\gamma}$ as well as (1.2) and therefore deduce

$$\begin{aligned}
\mathrm{I}(t) &\leq A\int_{\mathbb{R}} |f(t,s)|^{2-\gamma}\ \mathrm{d}s \int_{\{|x|\geq 1\}} |x|^{1-\gamma-\alpha}\ \mathrm{d}x \\
&\leq A\ C(t)^{2-\gamma}\int_{\mathbb{R}} \left(|s|^{-q(2-\gamma)} \wedge 1\right)\ \mathrm{d}s \int_{\{|x|\geq 1\}} |x|^{1-\gamma-\alpha}\ \mathrm{d}x \\
&< \infty,
\end{aligned}$$

where the finiteness follows from the facts that $q(2-\gamma) > 1$ and $1-\gamma-\alpha < -1$. Similarly we get

$$\begin{aligned}
\mathrm{II}(t) &\leq A\ C(t)^2 \int_{\mathbb{R}} \left(|s|^{-2q} \wedge 1\right)\ \mathrm{d}s \int_{\{0<|x|<1\}} |x|^{1-\alpha}\ \mathrm{d}x \\
&< \infty,
\end{aligned}$$

since $2q > 2\frac{1}{\alpha} > 1$ and $1-\alpha > -1$. Finally, we see that $|f(t,s)x| > 1$ implies $|x| \geq C(t)^{-1}\,(|s|^q \vee 1)$ and therefore

$$\begin{aligned}
\mathrm{III}(t) &\leq A\ C(t)^p \int_{\mathbb{R}} \left(|s|^{-qp} \wedge 1\right) \int_{\{|x|\geq C(t)^{-1}(|s|^q\vee 1)\}} |x|^{p-1-\alpha}\ \mathrm{d}x\ \mathrm{d}s \\
&\leq 2A\ C(t)^\alpha \int_{\mathbb{R}} \left(|s|^{-qp} \wedge 1\right) \left(|s|^{-q(\alpha-p)} \wedge 1\right)\ \mathrm{d}s \\
&\leq 2A\ C(t)^\alpha \int_{\mathbb{R}} \left(|s|^{-q\alpha} \wedge 1\right)\ \mathrm{d}s \\
&< \infty,
\end{aligned}$$

since $q\alpha > 1$. We conclude that the right-hand side of (2.9) is finite.

We now want to show

$$\int_{\mathbb{R}} |f(t,s)| \left| \int_{\mathbb{R}_0} x \left(\mathbb{1}_{\{|f(t,s)x|\le 1\}} - \mathbb{1}_{\{|x|\le 1\}} \right) \ \nu(\mathrm{d}x) \right| \ \mathrm{d}s < \infty. \tag{2.10}$$

For that purpose we estimate

$$\int_{\mathbb{R}_0} |x| \mathbb{1}_{\{|f(t,s)x|\le 1 \wedge |x|>1\}} \ \nu(\mathrm{d}x) \le \int_{\mathbb{R}_0 \setminus [-1,1]} |x| \ \nu(\mathrm{d}x) < \infty$$

and therefore we can deduce

$$\int_{\mathbb{R}_0} x \mathbb{1}_{\{|f(t,s)x|\le 1 \wedge |x|>1\}} \ \nu(\mathrm{d}x) = 0 \tag{2.11}$$

because of the symmetry of the Lévy measure ν. Using the fact that $|f(t,s)x| > 1$ implies $|x|^{-1} < |f(t,s)|$ we get

$$\begin{aligned} \int_{\mathbb{R}_0} |x| \mathbb{1}_{\{|f(t,s)x|>1 \wedge |x|\le 1\}} \ \nu(\mathrm{d}x) &\le A \int_{\mathbb{R}_0} |x| \mathbb{1}_{\{|f(t,s)x|>1 \wedge |x|\le 1\}} \ |x|^{-1-\alpha} \mathrm{d}x \\ &\le |f(t,s)| A \int_{[-1,1]} |x|^{1-\alpha} \ \mathrm{d}x \\ &< \infty, \end{aligned}$$

with the finiteness following since $1-\alpha > -1$. Using the symmetry of the Lévy measure again this yields

$$\int_{\mathbb{R}_0} x \mathbb{1}_{\{|f(t,s)x|>1 \wedge |x|\le 1\}} \ \nu(\mathrm{d}x) = 0. \tag{2.12}$$

Since the expression

$$\mathbb{1}_{\{|f(t,s)x|\le 1\}} - \mathbb{1}_{\{|x|\le 1\}}$$

is unequal to zero if and only if one of the expressions

$$\mathbb{1}_{\{|f(t,s)x|\le 1 \wedge |x|>1\}} \text{ or } \mathbb{1}_{\{|f(t,s)x|>1 \wedge |x|\le 1\}}$$

is unequal to zero, we deduce from (2.11) and (2.12)

$$\int_{\mathbb{R}_0} x \left(\mathbb{1}_{\{|f(t,s)x|\le 1\}} - \mathbb{1}_{\{|x|\le 1\}} \right) \ \nu(\mathrm{d}x) = 0$$

and therefore (2.10) holds.

Now, Lemma 2.6 yields that $\tilde{M}(t)$ exists as an L-integral and since $\mathbb{1}_{[-n,t]}(s) f(t,\cdot) \to f(t,\cdot)$ in the Musielak-Orlicz space $L_{\phi_p}(\mathbb{R})$ as n tends to infinity we have

$$\tilde{M}(t) = \lim_{n\to\infty} \int_{-n}^{t} f(t,s) \ L(\mathrm{d}s)$$

in $\mathscr{L}^p(\mathbb{P})$.

The remainder of the proof is analogous to that of Theorem 2.4. Instead of Lemma 2.3 one can apply that

$$\|L(t)\|_p = t^{1/\alpha}\|L(1)\|_p$$

for all $t \geq 0$ by the self-similarity of the symmetric α-stable process. $\square$

Part II

A generalised Itō formula for Lévy-driven Volterra processes

Chapter 3

Set-up

In this part of the thesis we will use the results of Part I to prove a generalised Itō formula for Lévy-driven Volterra processes. For that purpose we will work with two-sided Lévy processes as introduced in Section 2.2. However, note that we replace assumption (2.1) by the stronger assumption

$$\int_{\mathbb{R}_0} x^2 \ \nu(\mathrm{d}x) < \infty. \tag{3.1}$$

Additionally, we have to restrict ourselves to a smaller class of kernel functions (compared to Definition 1.2 in Part I) as introduced in the following definition.

Definition 3.1 We denote by $\mathcal{K}$ the class of measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$ with $\mathrm{supp} f \subset [\tau, \infty)^2$ for some $\tau \in [-\infty, 0]$ such that

(i) $\forall\, s > t \geq 0 : \quad f(t,s) = 0,$

(ii) $f(0,\cdot) = 0 \quad$ Lebesgue-a.s.,

(iii) the function f is continuous on the set $\{(t,s) \in \mathbb{R}^2 : \ \tau \leq s \leq t\}$,

(iv) for all $0 < t \leq T$, $\{s \in \mathbb{R} : f(t,s) \neq 0\}$ is not a Lebesgue null set,

(v) for all $s \in \mathbb{R}$ the mapping $t \mapsto f(t,s)$ is continuously differentiable on the set (s,∞) and there exist some $C_0 > 0$ and $\beta, \gamma \in [0,1)$ with $\beta + \gamma < 1$ such that

$$\left| \frac{\partial}{\partial t} f(t,s) \right| \leq C_0 |s|^{-\beta} |t-s|^{-\gamma} \tag{3.2}$$

for all $t > s > -\infty$. Furthermore, for any $t \in [0,T]$ there exists some $\epsilon > 0$ such that

$$\sup_{s \in (-\infty,-1]} \left(\sup_{r \in [0 \vee (t-\epsilon),\, (t+\epsilon) \wedge T]} |f(r,s)| |s|^{\theta} \right) < \infty \tag{3.3}$$

for some fixed $\theta > (1-\gamma-\beta) \vee \frac{1}{2}$ which is independent of t.

(vi) for every fixed $t \in [0,T]$ the function $s \mapsto f(t,s)$ is absolutely continuous on $[\tau, t]$ with density $\frac{\partial}{\partial s} f(t,\cdot)$, i.e.

$$f(t,s) = f(t,\tau) + \int_\tau^s \frac{\partial}{\partial u} f(t,u) \, \mathrm{d}u, \qquad \tau \le s \le t,$$

where $f(t,-\infty) := \lim_{x\to-\infty} f(t,x) = 0$, such that

(a) the function $t \mapsto \frac{\partial}{\partial s} f(t,s)$ is continuous on (s,∞) for Lebesgue-a.e. $s \in [\tau,\infty)$,

(b) there exist $\eta > 0$ and $q > 1/2 + 5\eta/2$ (independent of t) such that

$$\sup_{t\in[0,T]} \int_{-\infty}^t \left| \frac{\partial}{\partial s} f(t,s) \right|^{1+\eta} (|s|^q \vee 1) \ \mathrm{d}s < \infty.$$

The next lemma provides some useful path and moment properties of the Lévy-driven Volterra process M. It is basically a version of Theorem 2.4 with the class $\mathcal{K}(\varphi_q, -\infty)$ of kernel functions replaced by the smaller class $\mathcal{K}$. Nevertheless, we present the result in its entirety to provide a better readability for the rest of this thesis.

Lemma 3.2 *Let $f \in \mathcal{K}$. We define a stochastic process $M := (M(t))_{t\in[0,T]}$ by*

$$M(t) = \int_{-\infty}^t f(t,s) \ L(\mathrm{d}s)$$

for every $t \in [0,T]$. Then there exists a modification of M (which we still denote by M and which is fixed from now on) such that:

1. *M has càdlàg paths.*
2. *The jumps of M fulfil*
$$\Delta M(t) = f(t,t)\Delta L(t), \quad t \in (0,T].$$
3. *Whenever $L(1) \in \mathscr{L}^p(\mathbb{P})$ for some $p \ge 2$ we have*
$$\sup_{t\in[0,T]} |M(t)| \in \mathscr{L}^p(\mathbb{P}).$$

Proof In the case $\tau > -\infty$ the assertion follows from Remark 1.5. In the case $\tau = -\infty$ it follows from Theorem 2.4 with the choice $\varphi_{q'}(s) = |s|^{q'} \vee 1$ for

$$\frac{1}{2} < q' < \theta$$

and

$$q' \le \frac{q-2\eta}{1+\eta}.$$

Here, the assumption $q' < \theta$ guarantees that condition (3.3) in Definition 3.1 implies condition (1.2) in Definition 1.2. On the other hand, the assumption $q' \le \frac{q-2\eta}{1+\eta}$ is equivalent to

$$q'(1+\eta) + 2\eta \le q$$

and therefore for $|s| \geq 1$

$$|s|^{q'(1+\eta)+2\eta} \leq |s|^q.$$

This ensures that the condition in Definition 3.1(vi)(b) implies the condition in Definition 1.2(iv)(b). For the above choice of q' to be possible we need to make sure that $\frac{1}{2} < \frac{q-2\eta}{1+\eta}$, which is guaranteed by the condition $q > \frac{1}{2} + \frac{5\eta}{2}$ in Definition 3.1(vi)(b). $\square$

Let us emphasise that in particular fractional Lévy processes (via the Mandelbrot-Van Ness representation) are included here. Indeed, the following lemma shows that the class $\mathcal{K}$ contains the kernels

$$f_d(t,s) = \frac{1}{\Gamma(d+1)} \left((t-s)_+^d - (-s)_+^d \right) \tag{3.4}$$

for $s, t \in \mathbb{R}$ and a fractional integration parameter $d \in (0, 1/2)$.

Lemma 3.3 *The function $f_d : \mathbb{R}^2 \to \mathbb{R}$, defined in (3.4), satisfies the assumptions in Definition 3.1 with $\tau = -\infty$.*

Proof It is easy to see that f_d satisfies conditions (i)-(iv) in Definition 3.1. Condition (v) follows from the following expression containing the derivative with respect to the first argument of f_d, i.e.

$$\frac{\partial}{\partial t} f_d(t,s) = \frac{1}{\Gamma(d+1)} d(t-s)_+^{d-1}.$$

If we choose $\beta = 0$ and $\gamma = 1-d$ we deduce that equation (3.2) in condition (v) is satisfied. Moreover, we infer by means of the mean value theorem that

$$\sup_{r \in [0 \vee (t-\epsilon),\, (t+\epsilon) \wedge T]} |f(r,s)| \leq \sup_{r \in [0 \vee (t-\epsilon),\, (t+\epsilon) \wedge T]} \sup_{u \in (-s, r-s)} \frac{1}{\Gamma(d+1)} dr|u|^{d-1} \leq \frac{1}{\Gamma(d+1)} d(t+\epsilon)|s|^{d-1}.$$

Hence, equation (3.3) in condition (v) holds with $\theta = 1-d$.

To show that condition (vi) holds we choose $\varphi_{q'}(s) = |s|^{q'} \vee 1$ for some

$$q' \in \left(\frac{1}{2}, \theta\right) = \left(\frac{1}{2}, 1-d\right)$$

as in the proof of Lemma 3.2. From Example 2.5 together with the choices

$$\eta = \epsilon \quad \text{and} \quad q = q'(1+\eta) + 2\eta > \frac{1}{2} + \frac{5\eta}{2}$$

we now deduce that condition (vi) if fulfilled. $\square$

Throughout the remainder of this thesis we use the following definition:

$$\mathcal{A}(\mathbb{R}) := \left\{ \xi : \mathbb{R} \to \mathbb{R} : \xi \text{ and } \mathcal{F}\xi \text{ are in } \mathscr{L}^1(\mathrm{d}u) \right\},$$

where $\mathcal{F}\xi$ denotes the Fourier transform of ξ. Note that the functions in $\mathcal{A}(\mathbb{R})$ are continuous and bounded.

Chapter 4

The Segal-Bargmann transform and Hitsuda-Skorokhod integrals

In this chapter we introduce two important concepts from stochastic analysis. The first one (treated in Section 4.1) is is the so-called *Segal-Bargmann transform* (subsequently also referred to as *S-transform*) which possesses an injectivity property that enables us to use the S-transform to identify random variables.

This injectivity property will be used in Section 4.2 to define *Hitsuda-Skorokhod integrals*. These integrals will appear later on in Chapter 5 in our Itō formula.

4.1 The Segal-Bargmann transform

In order to define the S-transform we start with some preparations. We first introduce a set Ξ by

$$\begin{aligned}\Xi := \operatorname{span}\{g : \mathbb{R}\times\mathbb{R}\to\mathbb{R} :\ & g(x,t) = \tilde{g}_1(x)\tilde{g}_2(t) \text{ for two measurable functions such that there} \\ & \text{exists an } n\in\mathbb{N} \text{ with } \operatorname{supp}(\tilde{g}_1)\subset[-n,-1/n]\cup\{0\}\cup[1/n,n],\ |\tilde{g}_1|\le n \text{ and } \tilde{g}_2\in\mathcal{S}\}.\end{aligned}$$

Here $\mathcal{S} := \mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing smooth functions, that is

$$\mathcal{S}(\mathbb{R}) := \left\{\phi\in C^\infty(\mathbb{R}) : \text{for all } m,n\in\mathbb{N}\cup\{0\} \text{ we have } \sup_{x\in\mathbb{R}}\left|x^m\frac{\mathrm{d}^n}{\mathrm{d}x^n}\phi(x)\right|<\infty\right\}$$

and for any function $h:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ we define $h^*:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$h^*(x,t) := xh(x,t)$$

for all $x,t\in\mathbb{R}$.

Remark 4.1 Let $g\in\Xi$ be given by $g(x,t)=\sum_{j=1}^N c_j\tilde{g}_{1,j}(x)\tilde{g}_{2,j}(t)$. Using the abbreviations

$$g_1(x) := \sum_{j=1}^N |c_j\tilde{g}_{1,j}(x)| \quad\text{and}\quad g_2(t) := \sum_{j=1}^N |\tilde{g}_{2,j}(t)|$$

we see easily that there exists an $n' \in \mathbb{N}$ such that $\operatorname{supp}(g_1) \subset [-n', -1/n'] \cup \{0\} \cup [1/n', n']$, $|g_1| \leq n'$, and that $\sup_{t \in \mathbb{R}} |g_2(t)p(t)|$ is finite for every polynomial p as well as

$$|g(x,t)| \leq g_1(x)g_2(t)$$

for every $x, t \in \mathbb{R}$. We will make use of this simple estimate in our subsequent calculations. $\diamondsuit$

We now introduce a measure μ on $\mathcal{B}(\mathbb{R}^2)$ by setting

$$\mu(A \times B) := \left(\int_{A \cap \mathbb{R}_0} x^2 \ \nu(\mathrm{d}x) + \sigma^2 \mathbb{1}_{\{0 \in A\}} \right) \cdot \lambda(B)$$

for all A, B in $\mathcal{B}(\mathbb{R})$. For every $n \in \mathbb{N}$ let I_n be the n-th order multiple Lévy-Itō integral with respect to

$$\Lambda(E) = \sigma \int_{\mathbb{R}} \mathbb{1}_E(0,s) \ W(\mathrm{d}s) + \int_{\mathbb{R}} \int_{\mathbb{R}_0} x \mathbb{1}_E(x,s) \ \tilde{N}(\mathrm{d}x, \mathrm{d}s), \tag{4.1}$$

where $E \in \mathcal{B}(\mathbb{R}^2)$ such that $\mu(E) < \infty$, see e.g. page 665 in [39].

For any $g \in \mathscr{L}^2(\mu(\mathrm{d}x, \mathrm{d}t))$ let $g^{\otimes n}$, $n \in \mathbb{N} \cup \{0\}$, be the n-fold tensor product of g and define a measure $\mathcal{Q}_g$ on $(\Omega, \mathscr{F})$ by the change of measure

$$\mathrm{d}\mathcal{Q}_g = \exp^{\diamond}(I_1(g))\mathrm{d}\mathbb{P}, \tag{4.2}$$

where the Radon-Nikodým derivative is the Wick exponential of the random variable $I_1(g)$:

$$\exp^{\diamond}(I_1(g)) := \sum_{n=0}^{\infty} \frac{I_n(g^{\otimes n})}{n!}. \tag{4.3}$$

Let us point out that it follows from

$$\mathbb{E}\left(\sum_{n=0}^{\infty} \frac{I_n(g^{\otimes n})}{n!} \right) = 1$$

that $\mathcal{Q}_g$ is a signed probability measure. In the following, $\mathbb{E}^{\mathcal{Q}_g}$ denotes the expectation under $\mathcal{Q}_g$.

We also mention that according to [39], Theorem 4.8, we have for $g \in \mathscr{L}^2(\mu(\mathrm{d}x, \mathrm{d}t))$ with $g^* \in \mathscr{L}^1(\mathbb{R} \times \mathbb{R}_0, \nu(\mathrm{d}x) \otimes \mathrm{d}t)$ that

$$\begin{aligned}\exp^{\diamond}(I_1(g)) = \exp &\left\{ \sigma \int_{\mathbb{R}} g(0,t) \ W(\mathrm{d}t) - \frac{\sigma^2}{2} \int_{\mathbb{R}_0} g(0,t)^2 \ \mathrm{d}t - \int_{\mathbb{R}} \int_{\mathbb{R}_0} g^*(x,t) \ \nu(\mathrm{d}x) \ \mathrm{d}t \right\} \\ &\cdot \prod_{t: \Delta L(t) \neq 0} (1 + g^*(\Delta L(t), t)),\end{aligned}$$

which equals the Doléans-Dade exponential of $I_1(g)$ at infinity.

Using Proposition 1.4 and formula (10.3) in [18] and the fact that the Brownian part and the jump part are independent, we infer by applying the Cauchy-Schwarz inequality that there exists a constant $e_g > 0$ (only depending on g) such that

$$\mathbb{E}^{\mathcal{Q}_g}\left(|X|\right) \leq \mathbb{E}\left(|X|^2\right)^{1/2} \cdot \mathbb{E}\left(\left|\sum_{n=0}^{\infty} \frac{I_n(g^{\otimes n})}{n!}\right|^2\right)^{1/2} \leq e_g \cdot \mathbb{E}\left(|X|^2\right)^{1/2} \tag{4.4}$$

holds for every $X \in \mathscr{L}^2(\mathbb{P})$.

We are now in the position to define the Segal-Bargmann transform on $\mathscr{L}^2(\mathbb{P})$.

Definition 4.2 For every $\varphi \in \mathscr{L}^2(\mathbb{P})$ its *Segal-Bargmann transform* $S\varphi$ is given as an integral transform on the set $\mathscr{L}^2(\mu(\mathrm{d}x, \mathrm{d}t))$ by

$$S\varphi(g) := \mathbb{E}^{\mathcal{Q}_g}(\varphi).$$

The following injectivity result for the S-transform provides us with a key property for both the definition of Hitsuda-Skorokhod integrals and the proof of the Itō formula.

Proposition 4.3 *Let φ, ψ be in $\mathscr{L}^2(\mathbb{P})$. If*

$$S\varphi(g) = S\psi(g)$$

for all $g \in \Xi$, then we have $\varphi = \psi$ $\mathbb{P}$-almost surely.

The proof of this result is based on the proof of Proposition 5.10 in [25].

Proof By the linearity of the S-transform, it is enough to show that

$$S\varphi(g) = 0$$

for all $g \in \Xi$ implies $\varphi = 0$ $\mathbb{P}$-almost surely. Therefore, we may start by assuming that $S\varphi(g) = 0$ holds for all $g \in \Xi$. By Theorem 2.2 in [31] the random variable φ can be written as an orthogonal direct sum

$$\varphi = \sum_{n=0}^{\infty} I_n(\phi_n)$$

with a unique series of kernel functions $\phi_n \in \hat{\mathscr{L}}^2\left(\left(\mathbb{R}^2\right)^n, \mu(\mathrm{d}x, \mathrm{d}t)^{\otimes n}\right)$. Here the hat-symbol indicates that each function ϕ_n is symmetric in its n tuples of variables

$$((x_1, t_1), \ldots, (x_n, t_n)) \in \left(\mathbb{R}^2\right)^n.$$

We endow $\hat{\mathscr{L}}^2\left(\left(\mathbb{R}^2\right)^n, (\mu(\mathrm{d}x, \mathrm{d}t))^{\otimes n}\right)$ with the norm

$$\|h\|_n = \left(\int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} h\left((x_1, t_1), \ldots, (x_n, t_n)\right)^2 \; \mu(\mathrm{d}x_1, \mathrm{d}t_1) \ldots \mu(\mathrm{d}x_n, \mathrm{d}t_n)\right)^{\frac{1}{2}}$$

for $h \in \hat{\mathscr{L}}^2\left(\left(\mathbb{R}^2\right)^n, (\mu(\mathrm{d}x, \mathrm{d}t))^{\otimes n}\right)$. Using formula (4.1) in [39] thus gives for every $\lambda \in \mathbb{R}$

$$0 = S\varphi(\lambda g) = \mathbb{E}\left(\sum_{n=0}^{\infty} I_n(\phi_n) \sum_{n=0}^{\infty} \frac{I_n(g^{\otimes n})}{n!}\lambda^n\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\phi_n, g^{\otimes n}\right)_n \lambda^n,$$

where $(\cdot,\cdot)_n$ denotes the scalar product in $\hat{\mathscr{L}}^2\left(\left(\mathbb{R}^2\right)^n, \mu(\mathrm{d}x,\mathrm{d}t)^{\otimes n}\right)$. As the above expression constitutes a power series in λ, we see $\phi_0 = 0$ and

$$\left(\phi_n, g^{\otimes n}\right)_n = 0 \tag{4.5}$$

for every $n \geq 1$ and $g \in \Xi$. We now want to show that (4.5) holds, if we take replace the condition $g \in \Xi$ by $g \in \mathscr{L}^2\left(\mathbb{R}^2, \mu(\mathrm{d}x,\mathrm{d}t)\right)$. For that purpose note that Ξ dense in $\mathscr{L}^2\left(\mathbb{R}^2, \mu(\mathrm{d}x,\mathrm{d}t)\right)$. Consequently, given a function $g \in \mathscr{L}^2\left(\mathbb{R}^2, \mu(\mathrm{d}x,\mathrm{d}t)\right)$ we can choose a sequence $(g_l)_{l\in\mathbb{N}}$ in Ξ such that $g_l \to g$ in $\mathscr{L}^2\left(\mathbb{R}^2, \mu(\mathrm{d}x,\mathrm{d}t)\right)$ as l tends to infinity. Using the estimate

$$\begin{aligned}\|g^{\otimes n} - g_l^{\otimes n}\|_n &\leq \|g^{\otimes n} - g\hat{\otimes} g_l^{\otimes n-1}\|_n + \|g\hat{\otimes} g_l^{\otimes n-1} - g_l^{\otimes n}\|_n \\ &\leq \|g\|_1 \, \|g^{\otimes n-1} - g_l^{\otimes n-1}\|_{n-1} + \|g - g_l\|_1 \, \|g_l^{\otimes n-1}\|_{n-1},\end{aligned}$$

with $\hat{\otimes}$ being the symmetric tensor product, and a proof by induction we see

$$g_l^{\otimes n} \to g^{\otimes n}$$

in $\hat{\mathscr{L}}^2\left(\left(\mathbb{R}^2\right)^n, \mu(\mathrm{d}x,\mathrm{d}t)^{\otimes n}\right)$ as l tends to infinity. Therefore, we can use an approximation argument to deduce that (4.5) implies

$$\left(\phi_n, g^{\otimes n}\right)_n = 0 \tag{4.6}$$

for every $n \geq 1$ and $g \in \mathscr{L}^2\left(\mathbb{R}^2, \mu(\mathrm{d}x,\mathrm{d}t)\right)$.

Using the polarisation identity for symmetric multilinear functionals (cf. [25], Appendix B.7), which yields that every element $g_1\hat{\otimes}\cdots\hat{\otimes}g_n \in \mathscr{L}^2\left(\mathbb{R}^2, \mu(\mathrm{d}x,\mathrm{d}t)\right)^{\hat{\otimes}n}$ can be expressed as

$$g_1\hat{\otimes}\cdots\hat{\otimes}g_n = \frac{1}{n!}\sum_{k=1}^{n}(-1)^{n-k}\sum_{j_1<\cdots<j_k}\left(g_{j_1} + \cdots + g_{j_k}\right)^{\otimes n},$$

we conclude that (4.6) implies that for all $n \geq 1$ the function ϕ_n is orthogonal to a total subset of $\hat{\mathscr{L}}^2\left(\left(\mathbb{R}^2\right)^n, \mu(\mathrm{d}x,\mathrm{d}t)^{\otimes n}\right)$ and therefore $\phi_n = 0$ for all $n \geq 1$. Combining this with the fact $\phi_0 = 0$ we deduce $\varphi = 0$ $\mathbb{P}$-almost surely. □

4.2 Hitsuda-Skorokhod integrals

Due to the injectivity property of the S-transform introduced in Proposition 4.3 we can use it to define integration with respect to Lévy-driven Volterra processes, thereby generalising the approach in [8] and [11]. We mention that, because Lévy-driven Volterra processes are in general not semimartingales (see the special cases of fractional Brownian motions and fractional Lévy processes), the classical integration theory for semimartingales in the Itō-sense is not available.

An advantage of the S-transform approach to stochastic integration is that it avoids the technicalities Malliavin calculus. The motivation for our approach of defining Hitsuda-Skorokhod integrals lies in the fact that under suitable integrability and predictability assumptions on the integrand they reduce to the well known stochastic integrals with respect to semimartingales and random measures, respectively.

Definition 4.4 Let $\mathcal{B} \subset [0, \infty)$ be a Borel set. Suppose the mapping $t \mapsto S(M(t))(g)$ is differentiable for every $g \in \Xi$, $t \in \mathcal{B}$, and let $X : \mathcal{B} \times \Omega \to \mathbb{R}$ be a stochastic process such that $X(t)$ is square-integrable for a.e. $t \in \mathcal{B}$. The process X is said to have a *Hitsuda-Skorokhod integral* with respect to M if there is a $\Phi \in \mathscr{L}^2(\mathbb{P})$ such that

$$S\Phi(g) = \int_{\mathcal{B}} S(X(t))(g) \frac{\mathrm{d}}{\mathrm{d}t} S(M(t))(g) \ \mathrm{d}t$$

holds for all $g \in \Xi$. As the S-transform is injective, Φ is unique and we write

$$\Phi = \int_{\mathcal{B}} X(t) \ M^{\diamond}(\mathrm{d}t).$$

Remark 4.5 A different approach of defining Skorokhod integrals with respect to fractional Lévy processes via white noise analysis can be found in [32]. If the fractional Lévy process is of finite p-variation, stochastic integrals with respect to this process can be defined pathwise as an improper Riemann-Stieltjes integral and have been considered in [21]. In the special case that M is a Lévy process, it can be shown with the techniques in [18] that our definition of Hitsuda-Skorokhod integrals coincides with the definition of Skorokhod integrals via the chaos decomposition. Related approaches to Skorokhod integrals via Malliavin calculus have been provided in recent articles, see [17] for the case of integration with respect to Poisson-driven Volterra processes as well as [3], [4], and [12] for integration with respect to *volatility modulated Lévy-driven Volterra processes.* $\diamond$

The following technical lemma will prove useful throughout various results that involve the continuity of integral expressions and the interchanging of differentiation and integration.

Lemma 4.6 *Let $F : \mathbb{R}^2 \to \mathbb{C}$ with supp $F \subset [\tau, \infty)^2$ for some $\tau \in [-\infty, 0]$, $F(t, \cdot) \in \mathscr{L}^1(\mathrm{d}s)$ for every $t \in [0, T]$, and let $\beta, \gamma \in [0, 1)$ with $\beta + \gamma < 1$. We define*

$$I_F(t) := \int_{-\infty}^{t} F(t, s) \ \mathrm{d}s.$$

(i) *Let the following set of conditions be satisfied:*

   (a) *For Lebesgue-a.e. $s \in \mathbb{R}$ the map $t \mapsto F(t, s)$ is continuous on the set $[0, T] \setminus \{s\}$.*

   (b) *For every $t \in [0, T]$ there exist an $\varepsilon > 0$ and a constant $\tilde{C} > 0$ such that*

$$\int_{-\infty}^{t-2\varepsilon} \sup_{r \in [0 \vee (t-\varepsilon),\, (t+\varepsilon) \wedge T]} |F(r, s)| \ \mathrm{d}s < \infty \tag{4.7}$$

   and

$$|F(r, s)| \leq \tilde{C} |s|^{-\beta} |r - s|^{-\gamma} \tag{4.8}$$

   for all $r \in [(t - \varepsilon) \vee 0,\, (t + \varepsilon) \wedge T]$ and $s \in [t - 2\varepsilon, r)$.

Then the function I_F is continuous on $[0,T]$.

(ii) *Let the following set of conditions be satisfied:*

(a) *The mapping $(s,t) \mapsto F(t,s)$ is continuous on the set $\{(t,s) \in \mathbb{R}^2 : \ \tau \le s \le t\}$.*

(b) *For Lebesgue-a.e. $s \in \mathbb{R}$ the map $t \mapsto F(t,s)$ is continuously differentiable on $[0,T] \setminus \{s\}$.*

(c) *For every $t \in [0,T]$ there exist an $\varepsilon > 0$ and a constant $\tilde{C} > 0$ such that*

$$\int_{-\infty}^{t-2\varepsilon} \sup_{r \in [(t-\varepsilon)\vee 0,\,(t+\varepsilon)\wedge T]} \left| \frac{\partial}{\partial r} F(r,s) \right| \, \mathrm{d}s < \infty \tag{4.9}$$

and

$$\left| \frac{\partial}{\partial r} F(r,s) \right| \le \tilde{C} |s|^{-\beta} |r-s|^{-\gamma} \tag{4.10}$$

for all $r \in [(t-\varepsilon) \vee 0, \, (t+\varepsilon) \wedge T]$ and $s \in [t-2\varepsilon, r)$.

Then the function I_F is continuously differentiable on $[0,T]$ with derivative

$$I_F'(t) = F(t,t) + \int_{-\infty}^{t} \frac{\partial}{\partial t} F(t,s) \, \mathrm{d}s, \quad t \in [0,T].$$

Proof (i) We start with the right continuity and therefore can assume $t \in [0,T)$. With $\varepsilon > 0$ as in assumption (i)(b) we write

$$\begin{aligned}
&I_F(t+h) - I_F(t) \\
&= \int_t^{t+h} F(t+h,s) \, \mathrm{d}s + \int_{-\infty}^{t-2\varepsilon} \left(F(t+h,s) - F(t,s)\right) \, \mathrm{d}s + \int_{t-2\varepsilon}^{t} \left(F(t+h,s) - F(t,s)\right) \, \mathrm{d}s \\
&=: \mathrm{I_i}(h) + \mathrm{II_i}(h) + \mathrm{III_i}(h).
\end{aligned}$$

For the first term we use the substitution $v := \frac{s}{t+h}$, which in view of (4.8) yields

$$\begin{aligned}
&\sup_{h \in (0, \varepsilon \wedge (T-t))} \int_t^{t+h} |F(t+h,s)|^q \, \mathrm{d}s \\
&\qquad \le \tilde{C}^q \sup_{h \in (0, \varepsilon \wedge (T-t))} \int_t^{t+h} s^{-\beta q} \, (t+h-s)^{-\gamma q} \, \mathrm{d}s \\
&\qquad = \tilde{C}^q \sup_{h \in (0, \varepsilon \wedge (T-t))} \int_{\frac{t}{t+h}}^{1} (t+h)(v(t+h))^{-\beta q} \, (t+h-v(t+h))^{-\gamma q} \, \mathrm{d}v \\
&\qquad = \tilde{C}^q \sup_{h \in (0, \varepsilon \wedge (T-t))} \left((t+h)^{1-(\beta+\gamma)q} \int_{\frac{t}{t+h}}^{1} v^{-\beta q} \, (1-v)^{-\gamma q} \, \mathrm{d}v \right) \\
&\qquad < \infty
\end{aligned} \tag{4.11}$$

for $1 < q < (\beta+\gamma)^{-1}$. Since $\mathbb{1}_{[t,t+h]}(s) F(t+h,s)$ converges to 0 pointwise for Lebesgue-a.e. s as $h \downarrow 0$, an application of the de la Vallée-Poussin theorem results in

$$\lim_{h\downarrow 0} \mathrm{I_i}(h) = 0.$$

For the term $\mathrm{II_i}(h)$ we consider

$$\int_{-\infty}^{t-2\varepsilon} \sup_{h\in(0,\varepsilon\wedge(T-t))} |F(t+h,s)-F(t,s)| \ \mathrm{d}s \leq 2\int_{-\infty}^{t-2\varepsilon} \sup_{r\in[t,(t+\varepsilon)\wedge T]} |F(r,s)| \ \mathrm{d}s < \infty,$$

where the finiteness follows from (4.7). Therefore, we can apply the dominated convergence theorem to obtain by means of (i)(a) that

$$\lim_{h\downarrow 0}\left(\mathrm{II_i}(h) - \int_{-\infty}^{t-2\varepsilon} F(t,s) \ \mathrm{d}s\right) \to 0.$$

For the term $\mathrm{III_i}(h)$ we estimate in the case $t>0$:

$$\begin{aligned}
\sup_{h\in(0,\varepsilon\wedge(T-t))} \int_{t-2\varepsilon}^{t} |F(t+h,s)-F(t,s)|^q \ \mathrm{d}s &\leq 2^q \sup_{r\in[t,(t+\varepsilon)\wedge T]} \int_{t-2\varepsilon}^{t} |F(r,s)|^q \ \mathrm{d}s \\
&\leq 2^q\tilde{C}^q \sup_{r\in[t,(t+\varepsilon)\wedge T]} \int_{t-2\varepsilon}^{t} |s|^{-\beta q} \, |r-s|^{-\gamma q} \ \mathrm{d}s \\
&= 2^q\tilde{C}^q \sup_{r\in[t,(t+\varepsilon)\wedge T]} \int_{\frac{t-2\varepsilon}{r}}^{\frac{t}{r}} r \, |rv|^{-\beta q} \, |r-rv|^{-\gamma q} \ \mathrm{d}v \\
&= 2^q\tilde{C}^q \sup_{r\in[t,(t+\varepsilon)\wedge T]} r^{1-(\beta+\gamma)q} \int_{\frac{t-2\varepsilon}{r}}^{\frac{t}{r}} |v|^{-\beta q} \, |1-v|^{-\gamma q} \ \mathrm{d}v \\
&< \infty,
\end{aligned}$$

where we used the substitution $v=\frac{s}{r}$ and (4.8). If $t=0$, we estimate by using again (4.8):

$$\begin{aligned}
&\sup_{h\in(0,\varepsilon\wedge T)} \int_{-2\varepsilon}^{0} |F(h,s)-F(0,s)|^q \ \mathrm{d}s \\
&\quad\leq \tilde{C}2^q \sup_{h\in(0,\varepsilon\wedge T)} \left(\int_{-2\epsilon}^{0} |-s|^{-\beta q}|h-s|^{-\gamma q} \ \mathrm{d}s + \int_{-2\epsilon}^{0} |-s|^{-\beta q}|-s|^{-\gamma q} \ \mathrm{d}s\right) \\
&\quad\leq \tilde{C}2^{q+1} \int_{-2\epsilon}^{0} |-s|^{-(\beta+\gamma)q} \ \mathrm{d}s,
\end{aligned}$$

which is finite for $1<q<(\beta+\gamma)^{-1}$. Another application of the de la Vallée-Poussin theorem therefore yields the convergence

$$\lim_{h\downarrow 0}\left(\mathrm{III_i}(h) - \int_{t-2\varepsilon}^{t} F(t,s) \ \mathrm{d}s\right) \to 0,$$

which shows the right continuity of I_F at t.

To prove the left continuity we assume $t\in(0,T]$ and write for $0<h<2\varepsilon$

$$
\begin{aligned}
&I_F(t) - I_F(t-h)\\
&= \int_{t-h}^{t} F(t,s)\ \mathrm{d}s + \int_{-\infty}^{t-2\varepsilon} \left(F(t,s) - F(t-h,s)\right)\ \mathrm{d}s + \int_{t-2\varepsilon}^{t-h} \left(F(t,s) - F(t-h,s)\right)\ \mathrm{d}s\\
&=: \mathrm{I}_{\mathrm{i}}'(h) + \mathrm{II}_{\mathrm{i}}'(h) + \mathrm{III}_{\mathrm{i}}'(h).
\end{aligned}
$$

Since $F(t,\cdot) \in \mathscr{L}^1(\mathrm{d}s)$ for every $t \in [0,T]$ we clearly have $\mathrm{I}_{\mathrm{i}}'(h) \to 0$ as $h \downarrow 0$ thanks to the dominated convergence theorem.

To deal with $\mathrm{II}_{\mathrm{i}}'(h)$ we estimate

$$
\int_{-\infty}^{t-2\varepsilon} \sup_{h\in(0,\varepsilon\wedge t)} |F(t,s) - F(t-h,s)|\ \mathrm{d}s \le 2\int_{-\infty}^{t-2\varepsilon} \sup_{r\in[(t-\varepsilon)\vee 0,t]} |F(r,s)|\ \mathrm{d}s < \infty,
$$

with the finiteness following from (4.7). Therefore, an application of the dominated convergence theorem yields $\mathrm{II}_{\mathrm{i}}'(h) \to 0$ as $h \downarrow 0$.

For the term $\mathrm{III}_{\mathrm{i}}'(h)$ we estimate for $1 < q < (\beta+\gamma)^{-1}$

$$
\begin{aligned}
&\sup_{h\in\left(0,\varepsilon\wedge\frac{t}{2}\right)} \int_{t-2\varepsilon}^{t-h} |F(t,s) - F(t-h,s)|^q\ \mathrm{d}s\\
&\qquad \le 2^{q-1}\left(\int_{t-2\varepsilon}^{t} |F(t,s)|^q\ \mathrm{d}s + \sup_{h\in\left(0,\varepsilon\wedge\frac{t}{2}\right)} \int_{t-2\varepsilon}^{t-h} |F(t-h,s)|^q\ \mathrm{d}s\right)\\
&\qquad \le 2^{q-1}\left(\int_{t-2\varepsilon}^{t} |F(t,s)|^q\ \mathrm{d}s + \sup_{r\in\left[\frac{t}{2}\vee(t-\varepsilon),t\right]} \int_{t-2\varepsilon}^{r} |F(r,s)|^q\ \mathrm{d}s\right)\\
&\qquad \le 2^q \sup_{r\in\left[\frac{t}{2}\vee(t-\varepsilon),t\right]} \int_{t-2\varepsilon}^{r} |F(r,s)|^q\ \mathrm{d}s\\
&\qquad \le 2^q\tilde{C}^q \sup_{r\in\left[\frac{t}{2}\vee(t-\varepsilon),t\right]} r^{1-(\beta+\gamma)q} \int_{\frac{t-2\varepsilon}{t}}^{1} |v|^{-\beta q}|1-v|^{-\gamma q}\ \mathrm{d}v\\
&\qquad < \infty,
\end{aligned}
$$

where we used (4.8) and the substitution $v = \frac{s}{r}$ again. Now we deduce by an application of the de la Vallée-Poussin theorem

$$
\lim_{h\downarrow 0} \mathrm{III}_{\mathrm{i}}'(h) = 0,
$$

which concludes the proof of Part (i).

(ii) We start with proving the differentiability from above. To this end we fix $t \in [0,T)$ and write

$$
\begin{aligned}
&\frac{I_F(t+h) - I_F(t)}{h}\\
&= \frac{1}{h}\int_{t}^{t+h} F(t+h,s)\ \mathrm{d}s + \int_{-\infty}^{t-2\varepsilon} \frac{F(t+h,s) - F(t,s)}{h}\ \mathrm{d}s + \int_{t-2\varepsilon}^{t} \frac{F(t+h,s) - F(t,s)}{h}\ \mathrm{d}s\\
&=: \mathrm{I}_{\mathrm{ii}}(h) + \mathrm{II}_{\mathrm{ii}}(h) + \mathrm{III}_{\mathrm{ii}}(h),
\end{aligned}
$$

where $\varepsilon > 0$ is chosen according to assumption (ii)(c). For the first term we derive by means of (ii)(a) that

$$|F(t,t) - \mathrm{I}_{\mathrm{ii}}(h)| \le \frac{1}{h}\int_t^{t+h} |F(t,t) - F(t+h,s)| \, \mathrm{d}s \le \sup_{s\in[t,t+h]} |F(t,t) - F(t+h,s)| \to 0$$

as $h \downarrow 0$. For the second term we use the mean value theorem for complex valued functions to see that

$$\int_{-\infty}^{t-2\varepsilon} \sup_{h\in(0,\varepsilon\wedge(T-t))} \left| \frac{F(t+h,s) - F(t,s)}{h} \right| \, \mathrm{d}s \le \int_{-\infty}^{t-2\varepsilon} \sup_{r\in[t,(t+\varepsilon)\wedge T]} \left| \frac{\partial}{\partial r} F(r,s) \right| \, \mathrm{d}s < \infty,$$

where the finiteness follows from (4.9). Therefore, we can apply the dominated convergence theorem to obtain

$$\lim_{h\downarrow 0} \mathrm{II}_{\mathrm{ii}}(h) = \int_{-\infty}^{t-2\varepsilon} \frac{\partial}{\partial t} F(t,s) \, \mathrm{d}s.$$

In order to tackle the third term we aim at applying the de la Vallée-Poussin theorem again. For this purpose we let $1 < q < (\beta+\gamma)^{-1}$ and deduce by using the mean value theorem for complex valued functions and (4.10) that

$$\begin{aligned}
&\sup_{h\in(0,\varepsilon\wedge(T-t))} \int_{t-2\varepsilon}^{t} \left| \frac{F(t+h,s) - F(t,s)}{h} \right|^q \, \mathrm{d}s \\
&\le \sup_{h\in(0,\varepsilon\wedge(T-t))} \int_{t-2\varepsilon}^{t} \sup_{r\in[t,t+h]} \left| \frac{\partial}{\partial r} F(r,s) \right|^q \, \mathrm{d}s \\
&\le \tilde{C}^q \sup_{h\in(0,\varepsilon\wedge(T-t))} \int_{t-2\varepsilon}^{t} \sup_{r\in[t,t+h]} |s|^{-\beta q} \, |r-s|^{-\gamma q} \, \mathrm{d}s \\
&\le \tilde{C}^q \sup_{h\in(0,\varepsilon\wedge(T-t))} \int_{t-2\varepsilon}^{t} |s|^{-\beta q} \, |t-s|^{-\gamma q} \, \mathrm{d}s \\
&< \infty,
\end{aligned}$$

where the finiteness follows from the facts $\beta q < 1$ and $\gamma q < 1$. Now the de la Vallée-Poussin theorem yields the uniform integrability of

$$\left(\frac{F(t+h,\cdot) - F(t,\cdot)}{h} \right)_{h\in(0,\varepsilon\wedge(T-t))}$$

on $[t-2\varepsilon, t]$. Consequently, we infer that

$$\lim_{h\downarrow 0} \mathrm{III}_{\mathrm{ii}}(h) = \int_{t-2\varepsilon}^{t} \frac{\partial}{\partial t} F(t,s) \, \mathrm{d}s,$$

which shows the differentiability from above with the desired right derivative.

For the differentiability from below we may assume $t \in (0,T]$ and write for $0 < h < 2\varepsilon$

$$
\begin{aligned}
&\frac{I_F(t) - I_F(t-h)}{h} \\
&= \frac{1}{h}\int_{t-h}^{t} F(t,s)\ \mathrm{d}s + \int_{-\infty}^{t-2\varepsilon} \frac{F(t,s) - F(t-h,s)}{h}\ \mathrm{d}s + \int_{t-2\varepsilon}^{t-h} \frac{F(t,s) - F(t-h,s)}{h}\ \mathrm{d}s \\
&=: \mathrm{I}'_{\mathrm{ii}}(h) + \mathrm{II}'_{\mathrm{ii}}(h) + \mathrm{III}'_{\mathrm{ii}}(h).
\end{aligned}
$$

Writing

$$
\begin{aligned}
\lim_{h\downarrow 0}\left|\frac{1}{h}\int_{t-h}^{t} F(t,s)\ \mathrm{d}s - F(t,t)\right| &= \lim_{h\downarrow 0}\left|\frac{1}{h}\int_{t-h}^{t} F(t,s) - F(t,t)\ \mathrm{d}s\right| \\
&\le \lim_{h\downarrow 0}\frac{1}{h}\int_{t-h}^{t} \sup_{r\in[t-h,t]} |F(t,r) - F(t,t)|\ \mathrm{d}s \\
&\le \lim_{h\downarrow 0} \sup_{r\in[t-h,t]} |F(t,r) - F(t,t)| = 0,
\end{aligned}
$$

where we used the continuity of the function $(s,t) \mapsto F(t,s)$ on the set $\{(t,s) \in \mathbb{R}^2 :\ \tau \le s \le t\}$, we see that

$$
\lim_{h\downarrow 0}\frac{1}{h}\int_{t-h}^{t} F(t,s)\ \mathrm{d}s = F(t,t).
$$

To deal with $\mathrm{II}'_{\mathrm{ii}}(h)$ we use the mean value theorem for complex valued functions and get the chain of estimates

$$
\int_{-\infty}^{t-2\varepsilon} \sup_{h\in(0,\varepsilon\wedge t)} \left|\frac{F(t,s) - F(t-h,s)}{h}\right|\ \mathrm{d}s \le \int_{-\infty}^{t-2\varepsilon} \sup_{r\in[(t-\varepsilon)\vee 0,t]} \left|\frac{\partial}{\partial r}F(r,s)\right|\ \mathrm{d}s < \infty,
$$

with the finiteness following from (4.9). Consequently, we can use the dominated convergence theorem to deduce

$$
\lim_{h\downarrow 0}\int_{-\infty}^{t-2\varepsilon} \frac{F(t,s) - F(t-h,s)}{h}\ \mathrm{d}s = \int_{-\infty}^{t-2\varepsilon} \frac{\partial}{\partial t}F(t,s)\ \mathrm{d}s.
$$

Finally, for the term $\mathrm{III}'_{\mathrm{ii}}(h)$ we want to apply the de la Vallée-Poussin theorem one last time. Just as above we choose $1 < q < (\beta+\gamma)^{-1}$ and infer by using the mean value theorem for complex valued functions and the estimate (4.10) that

$$
\begin{aligned}
&\sup_{h\in\left(0,\varepsilon\wedge\frac{t}{2}\right)}\int_{t-2\varepsilon}^{t-h}\left|\frac{F(t,s)-F(t-h,s)}{h}\right|^q \ \mathrm{d}s\\
&\qquad\le \sup_{h\in\left(0,\varepsilon\wedge\frac{t}{2}\right)}\int_{t-2\varepsilon}^{t-h}\sup_{r\in[t-h,t]}\left|\frac{\partial}{\partial r}F(r,s)\right|^q \ \mathrm{d}s\\
&\qquad\le \tilde{C}^q \sup_{h\in\left(0,\varepsilon\wedge\frac{t}{2}\right)}\int_{t-2\varepsilon}^{t-h}|s|^{-\beta q}\,|t-h-s|^{-\gamma q}\ \mathrm{d}s\\
&\qquad= \tilde{C}^q \sup_{h\in\left(0,\varepsilon\wedge\frac{t}{2}\right)}\int_{\frac{t-2\varepsilon}{t-h}}^{1}(t-h)|v(t-h)|^{-\beta q}\,|t-h-v(t-h)|^{-\gamma q}\ \mathrm{d}v\\
&\qquad= \tilde{C}^q \sup_{h\in\left(0,\varepsilon\wedge\frac{t}{2}\right)}\left((t-h)^{1-(\beta+\gamma)q}\int_{\frac{t-2\varepsilon}{t-h}}^{1}|v|^{-\beta q}\,|1-v|^{-\gamma q}\ \mathrm{d}v\right)\\
&\qquad< \infty,
\end{aligned}
$$

where the finiteness follows again from the facts $\beta q < 1$ and $\gamma q < 1$. Now the de la Vallée-Poussin theorem yields the uniform integrability of

$$
\left(\mathbb{1}_{[t-2\varepsilon,t-h]}(s)\frac{F(t,\cdot)-F(t-h,\cdot)}{h}\right)_{h\in\left(0,\varepsilon\wedge\frac{t}{2}\right)}
$$

on $[t-2\varepsilon, t]$. Consequently, we infer that

$$
\lim_{h\downarrow 0}\mathrm{III}'_{\mathrm{ii}}(h) = \int_{t-2\varepsilon}^{t}\frac{\partial}{\partial t}F(t,s)\ \mathrm{d}s,
$$

which shows the differentiability from below of the function I_F with the claimed derivative.

By combining the differentiability shown above and (i) we conclude that every function F fulfilling conditions (ii)(a)-(ii)(c) is continuously differentiable on $[0,T]$, since in this case (i) is applicable to the derivative $\frac{\partial}{\partial t}F$. This concludes the proof of Lemma 4.6. □

We next derive the explicit form of the derivative of the S-transform of $M(t)$. This result particularly yields that the differentiability condition on the mapping $t\mapsto S(M(t))(g)$ in Definition 4.4 is fulfilled for kernel functions $f\in\mathcal{K}$.

Lemma 4.7 *For all $f\in\mathcal{K}$ and $g\in\Xi$ the mapping $t\mapsto S(M(t))(g)$ is continuously differentiable on the set $[0,T]$ with derivative*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}S(M(t))(g) =& \sigma\left(f(t,t)g(0,t)+\int_{-\infty}^{t}\frac{\partial}{\partial t}f(t,s)g(0,s)\ \mathrm{d}s\right)\\
&+ f(t,t)\int_{\mathbb{R}_0}xg^*(x,t)\ \nu(\mathrm{d}x)+\int_{-\infty}^{t}\int_{\mathbb{R}_0}\frac{\partial}{\partial t}f(t,s)xg^*(x,s)\ \nu(\mathrm{d}x)\ \mathrm{d}s.
\end{aligned}
$$

Remark 4.8 Note that the finiteness of the integrals appearing in the above lemma is guaranteed by Definition 3.1(v). ◊

Proof By the isometry of Lévy-Itō integrals we obtain

$$\begin{aligned} S(M(t))(g) &= \sigma \int_{-\infty}^{t} f(t,s)g(0,s)\ \mathrm{d}s + \int_{-\infty}^{t}\int_{\mathbb{R}_0} f(t,s)xg^*(x,s)\ \nu(\mathrm{d}x)\ \mathrm{d}s \\ &=: \mathrm{I}(t) + \mathrm{II}(t), \end{aligned} \tag{4.12}$$

cf. e.g. Section 3.1 of [8] and Example 3.6 in [11].

We now want to apply Lemma 4.6 to $\mathrm{I}(t)$ with $F_{\mathrm{I}}(t,s) := f(t,s)g(0,s)$. It is easy to check that conditions (a) and (b) of Lemma 4.6(ii) are satisfied. Since $\sup_{s\in\mathbb{R}} |g(0,s)| < \infty$ (cf. Remark 4.1), we deduce from (3.2) that (4.10) is fulfilled for every $r > s > -\infty$. Moreover, by using (3.2) and the rapid decrease of $s \mapsto g(0,s)$ we get for $t \in [0,T]$ and arbitrary $\epsilon > 0$

$$\begin{aligned} \int_{-\infty}^{t-2\epsilon} \sup_{r\in[(t-\epsilon)\vee 0,\,(t+\epsilon)\wedge T]} \left|\frac{\partial}{\partial r} F_{\mathrm{I}}(r,s)\right|\ \mathrm{d}s &= \int_{-\infty}^{t-2\epsilon} \sup_{r\in[(t-\epsilon)\vee 0,\,(t+\epsilon)\wedge T]} \left|\frac{\partial}{\partial r} f(r,s)g(0,s)\right|\ \mathrm{d}s \\ &\le C_0 \int_{-\infty}^{t-2\epsilon} |s|^{-\beta} \sup_{r\in[(t-\epsilon)\vee 0,\,(t+\epsilon)\wedge T]} |r-s|^{-\gamma}|g(0,s)|\ \mathrm{d}s \\ &\le C_0 \int_{-\infty}^{t-2\epsilon} |s|^{-\beta}|t-\epsilon-s|^{-\gamma}|g(0,s)|\ \mathrm{d}s \\ &< \infty, \end{aligned}$$

where C_0, β, and γ are given by (3.2). This shows that (4.9) holds and thus Lemma 4.6 is applicable, which results in $t \mapsto \mathrm{I}(t)$ being continuously differentiable on $[0,T]$.

To deal with $\mathrm{II}(t)$ we recall from Remark 4.1 that every $g \in \Xi$ can be written in the form $g(x,t) = \sum_{j=1}^{N} c_j \tilde{g}_{1,j}(x)\tilde{g}_{2,j}(t)$. We set

$$F_{\mathrm{II}}(t,s) := \int_{\mathbb{R}_0} f(t,s)xg^*(x,s)\ \nu(\mathrm{d}x) = \sum_{j=1}^{N} c_j \left(\int_{\mathbb{R}_0} x^2 \tilde{g}_{1,j}(x)\ \nu(\mathrm{d}x)\right) f(t,s)\tilde{g}_{2,j}(s),$$

where $\tilde{g}_{2,j} \in \mathcal{S}$ for all $1 \le j \le N$. Hence, each of the finitely many summands in F_{II} is of the same form as F_{I}. Consequently, $t \mapsto \mathrm{II}(t)$ is continuously differentiable. Therefore, in view of (4.12) the mapping $t \mapsto S(M(t))(g)$ is continuously differentiable on $[0,T]$. □

We proceed by introducing a Hitsuda-Skorokhod integral with respect to an appropriate random measure that will enable us to establish a connection between the Hitsuda-Skorokhod integral with respect to the Lévy-driven Volterra process M and the classical integral with respect to the underlying Lévy process L (see Theorem 4.11 below).

Definition 4.9 Let $\mathcal{B} \subset \mathbb{R}$ be a Borel set and $X : \mathbb{R}\times\mathcal{B}\times\Omega \to \mathbb{R}$ be a random field such that $X(x,t) \in \mathscr{L}^2(\mathbb{P})$ for $\nu(\mathrm{d}x)\otimes\mathrm{d}t$-a.e. (x,t). The *Hitsuda-Skorokhod integral* of X with respect to the random measure

$$\Lambda(\mathrm{d}x,\mathrm{d}t) = \sigma\delta_0(\mathrm{d}x)\otimes W(\mathrm{d}t) + x\tilde{N}(\mathrm{d}x,\mathrm{d}t),$$

(cp. (4.1)), where δ_0 denotes the Dirac measure in 0, is said to exist in $\mathscr{L}^2(\mathbb{P})$, if there is a random variable $\Phi \in \mathscr{L}^2(\mathbb{P})$ that satisfies

$$S\Phi(g) = \sigma \int_{\mathcal{B}} S(X(0,t))(g)\ g(0,t)\ \mathrm{d}t + \int_{\mathcal{B}} \int_{\mathbb{R}_0} S(X(x,t))(g) g^*(x,t)\ x\ \nu(\mathrm{d}x)\ \mathrm{d}t$$

for all $g \in \Xi$. In this case, by Proposition 4.3 the random variable Φ is unique and we write

$$\Phi = \int_{\mathcal{B}} \int_{\mathbb{R}} X(x,t)\ \Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}t).$$

Remark 4.10 Let $a \in [0,T]$ and $X : \mathbb{R} \times [a,T] \times \Omega \to \mathbb{R}$ be a predictable random field satisfying

$$\mathbb{E}\left(\int_a^T X(0,t)^2\ \mathrm{d}t + \int_a^T \int_{\mathbb{R}_0} X(x,t)^2\ \nu(\mathrm{d}x)\ \mathrm{d}t \right) < \infty.$$

1. Assume that $\sigma > 0$ and let X be given by

$$X(x,\cdot) = \begin{cases} \frac{1}{\sigma} Y(\cdot), & x = 0 \\ 0, & x \neq 0 \end{cases}$$

for some stochastic process $Y : [a,T] \times \Omega \to \mathbb{R}$. Since X is predictable, we infer from Theorem 3.1 in [8] that the Hitsuda-Skorokhod integral $\int_a^T \int_{\mathbb{R}} X(x,t)\ \Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}t)$ exists and satisfies

$$\int_a^T \int_{\mathbb{R}} X(x,t)\ \Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}t) = \int_a^T Y(t)\ W(\mathrm{d}t),$$

where the last integral is the classical stochastic integral with respect to the Brownian motion W. Note that it follows from the calculations in Section 3.1 of [8] that

$$g(0,t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g(0,s)\ \mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t} S(W(t))(g)$$

and hence we have

$$S\left(\int_a^T \int_{\mathbb{R}} X(x,t)\ \Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}t) \right)(g) = \int_a^T S(Y(t))(g)\ \frac{\mathrm{d}}{\mathrm{d}t} S(W(t))(g)\ \mathrm{d}t.$$

2. If X fulfils $X(0,\cdot) \equiv 0$, then it follows from Theorem 3.5 in [11] that

$$\int_a^T \int_{\mathbb{R}} X(x,t)\ \Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}t) = \int_a^T \int_{\mathbb{R}_0} x X(x,t)\ \tilde{N}(\mathrm{d}x, \mathrm{d}t),$$

where the last integral is the classical stochastic integral with respect to the compensated Poisson jump measure $\tilde{N}$.

3. According to 1. and 2. we have

$$\int_a^T \int_{\mathbb{R}} X(x,t)\ \Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}t) = \sigma \int_a^T X(0,t)\ W(\mathrm{d}t) + \int_a^T \int_{\mathbb{R}_0} x X(x,t)\ \tilde{N}(\mathrm{d}x, \mathrm{d}t).$$

$\Diamond$

Now we are in the position to state the connection between the Hitsuda-Skorokhod integral with respect to M and the Itō integral with respect to the driving Lévy process.

Theorem 4.11 *Suppose that X is a predictable process such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X(t)|^2\right)<\infty \tag{4.13}$$

and let $f\in\mathcal{K}$. Then

$$\int_0^T X(t)\ M^\diamond(\mathrm{d}t) \tag{4.14}$$

exists, if and only if

$$\int_{-\infty}^T\int_{\mathbb{R}}\int_{0\vee s}^T\frac{\partial}{\partial t}f(t,s)X(t)\ \mathrm{d}t\ \Lambda^\diamond(\mathrm{d}x,\mathrm{d}s) \tag{4.15}$$

exists. In this case

$$\int_0^T X(t)\ M^\diamond(\mathrm{d}t)=\int_0^T f(t,t)X(t)\ L(\mathrm{d}t)+\int_{-\infty}^T\int_{\mathbb{R}}\int_{0\vee s}^T\frac{\partial}{\partial t}f(t,s)X(t)\ \mathrm{d}t\ \Lambda^\diamond(\mathrm{d}x,\mathrm{d}s). \tag{4.16}$$

Proof

Note that $\int_0^T f(t,t)X(t)\ L(\mathrm{d}t)$ exists in $\mathscr{L}^2(\mathbb{P})$ because of the continuity of $t\mapsto f(t,t)$ and (4.13). By the previous remark its S-transform is given by

$$\begin{aligned}&S\left(\int_0^T f(t,t)X(t)\ L(\mathrm{d}t)\right)(g)\\&\quad=\sigma\int_0^T S(X(t))(g)f(t,t)g(0,t)\ \mathrm{d}t+\int_0^T\int_{\mathbb{R}_0}S(X(t))(g)f(t,t)xg^*(x,t)\ \nu(\mathrm{d}x)\ \mathrm{d}t.\end{aligned}$$

Using assumption (4.13) and (3.2) we obtain the estimate

$$\mathbb{E}\left(\left(\int_{0\vee s}^T\frac{\partial}{\partial t}f(t,s)X(t)\ \mathrm{d}t\right)^2\right)\le\mathbb{E}\left(\sup_{t\in[0,T]}|X(t)|^2\right)\left(\int_{0\vee s}^T\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\right)^2<\infty.$$

Thus, we can apply Definition 4.9 to $\int_{0\vee s}^T\frac{\partial}{\partial t}f(t,s)X(t)\ \mathrm{d}t$. Assuming the existence of (4.15), we therefore infer from Definition 4.4, Lemma 4.7, and Fubini's theorem that (4.14) exists and satisfies (4.16). Analogous arguments yield the converse implication. $\square$

Considering the jump measure N instead of the compensated jump measure $\tilde{N}$ naturally leads to the following definition by adding the S-transform of the integral with respect to the compensator.

Definition 4.12 Let $\mathcal{B}\subset\mathbb{R}$ be a Borel set and $X:\mathbb{R}_0\times\mathcal{B}\times\Omega\to\mathbb{R}$ be a random field such that $X(x,t)\in\mathscr{L}^2(\mathbb{P})$ for $\nu(\mathrm{d}x)\otimes\mathrm{d}t$-a.e. (x,t). The *Hitsuda-Skorokhod integral* of X with respect to the jump measure $N(\mathrm{d}x, \mathrm{d}t)$ is said to exist in $\mathscr{L}^2(\mathbb{P})$, if there is a (unique) random variable $\Phi \in \mathscr{L}^2(\mathbb{P})$ that satisfies

$$S\Phi(g) = \int_{\mathcal{B}} \int_{\mathbb{R}_0} S(X(x,t))(g)(1 + g^*(x,t)) \; \nu(\mathrm{d}x) \; \mathrm{d}t$$

for all $g \in \Xi$. We write

$$\Phi = \int_{\mathcal{B}} \int_{\mathbb{R}_0} X(x,t) \; N^{\diamond}(dx, dt).$$

Chapter 5

Generalised Itō formulas

This chapter is devoted to formulating precisely and to proving our generalised Itō formula. The following theorem is the main result of this part of the thesis:

Theorem 5.1 *Let $f \in \mathcal{K}$ and $G \in C^2(\mathbb{R})$. Additionally, assume that one of the following assumptions is fulfilled:*

(i) $\sigma > 0$, G, G', and G'' are of polynomial growth with degree $q \geq 0$, that is

$$|G^{(l)}(x)| \leq C_{pol}(1 + |x|^q) \quad \textit{for every } x \in \mathbb{R} \textit{ and } l = 0, 1, 2$$

with a constant $C_{pol} > 0$, and

$$L(1) \in \mathscr{L}^{2q+2}(\mathbb{P}).$$

(ii) $G, G', G'' \in \mathcal{A}(\mathbb{R})$ and $L(1) \in \mathscr{L}^2(\mathbb{P})$.

Then the following generalised Itō formula

$$\begin{aligned}
G(M(T)) = G(0) &+ \frac{\sigma^2}{2} \int_0^T G''(M(t)) \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^t f(t,s)^2 \mathrm{d}s \right) \mathrm{d}t \\
&+ \sum_{0<t\leq T} \big[G(M(t)) - G(M(t-)) - G'(M(t-))\Delta M(t) \big] \\
&+ \int_{-\infty}^T \int_{\mathbb{R}} \int_{0\vee s}^T G'\left(M(t) + xf(t,s)\right) \frac{\partial}{\partial t} f(t,s) \ \mathrm{d}t \ \Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}s) \\
&+ \int_{-\infty}^T \int_{\mathbb{R}_0} \int_{0\vee s}^T \big(G'\left(M(t) + xf(t,s)\right) - G'(M(t)) \big) x \frac{\partial}{\partial t} f(t,s) \ \mathrm{d}t \ \nu(\mathrm{d}x) \ \mathrm{d}s \\
&+ \int_0^T G'(M(t-)) f(t,t) \ L(\mathrm{d}t)
\end{aligned} \tag{5.1}$$

holds $\mathbb{P}$-almost surely. In particular, all the terms in (5.1) exist in $\mathscr{L}^2(\mathbb{P})$.

In order to prove Theorem 5.1 we start with a heuristic argumentation that gives a rough outline of the steps of our proof and motivates the auxiliary results that we shall prove below. Suppose the fundamental theorem of calculus enables us to write

$$S(G(M(T)))(g) = G(0) + \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} S(G(M(t)))(g)\ \mathrm{d}t. \tag{5.2}$$

Subsequently, by making use of the Fourier inversion theorem in the spirit of [30] we would obtain

$$S(G(M(t)))(g) = \mathbb{E}^{\mathcal{Q}_g}\big(G(M(t))\big) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}G)(u) \mathbb{E}^{\mathcal{Q}_g}\big(e^{iuM(t)}\big)\ \mathrm{d}u. \tag{5.3}$$

Differentiating the right-hand side of (5.3) with respect to t, using some standard manipulations of the Fourier transform, plugging the resulting formula for $\frac{\mathrm{d}}{\mathrm{d}t}S(G(M(t)))(g)$ into (5.2) and using the injectivity of the S-transform would then give an explicit expression for $G(M(T))$ leading to a generalised Itō formula.

Our approach to prove Theorem 5.1 is based on several auxiliary results. More precisely, following the above motivation we derive explicit expressions for the characteristic function $\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)$ under $\mathcal{Q}_g$ as well as its derivative $\frac{\partial}{\partial t}\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)$ in Proposition 5.3 and Lemma 5.4.

In (the proof of) Proposition 5.7 we will show that the integral appearing in (5.3) is well-defined and that the mapping $t \mapsto S(G(M(t)))(g)$ is indeed differentiable. We then complete the proof of Theorem 5.1 by using the explicit expression for $\frac{\partial}{\partial t}\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)$ and the injectivity of the S-transform.

Let us now follow our approach by providing the characteristic function of $M(t)$. The following result was obtained in Proposition 2.7 of [37]. Nonetheless, in our situation we prefer to give a more elementary proof.

Lemma 5.2 *For every $f \in \mathcal{K}$ and $t \in [0,T]$ we have*

$$\begin{aligned}
&\mathbb{E}\left(e^{iuM(t)}\right) \\
&\quad = \exp\left(-\frac{\sigma^2 u^2}{2}\int_{-\infty}^t f(t,s)^2\ \mathrm{d}s + \int_{-\infty}^t \int_{\mathbb{R}_0} \left(e^{iuxf(t,s)} - 1 - iuxf(t,s)\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s\right).
\end{aligned}$$

Proof Note that $(L^r(t))_{t\in\mathbb{R}}$, given by

$$L^r(t) := L(t) - L(r) = \sigma\left(W(t) - W(r)\right) + \int_r^t \int_{\mathbb{R}_0} x\left[N(\mathrm{d}x,\mathrm{d}s) - \nu(\mathrm{d}x)\ \mathrm{d}s\right]$$

for any $r \le t$, is a semimartingale with characteristics

$$\left(0,\ \sigma^2(t-r),\ \mathbb{1}_{[r,t]}(s)\ \nu(\mathrm{d}x)\right).$$

Consequently, the characteristic function of $L^r(t)$ is

$$\mathbb{E}\left(e^{iuL^r(t)}\right) = \exp\left(-\frac{\sigma^2u^2}{2}(t-r) + \int_r^t \int_{\mathbb{R}_0} \left(e^{iux} - 1 - iux\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s\right). \tag{5.4}$$

Let $(f_{N,+}(t,\cdot))_{N\in\mathbb{N}}$ and $(f_{N,-}(t,\cdot))_{N\in\mathbb{N}}$ be sequences of step functions

$$f_{N,\pm}(t,s) := \sum_{j=1}^{N} \mathbb{1}_{(t_j^N, t_{j+1}^N]}(s)\tau_{j,\pm}^N(t)$$

such that $f_{N,\pm}(t,\cdot) \uparrow f_\pm(t,\cdot)$ λ-a.e. as $N \to \infty$, where f_+ and f_- are the positive and the negative part of f and define

$$\tau_j^N(t) := \tau_{j,+}^N(t) - \tau_{j,-}^N(t) \quad \text{and} \quad f_N(t,\cdot) := f_{N,+}(t,\cdot) - f_{N,-}(t,\cdot)$$

for all $N \in \mathbb{N}$, $j = 1, \ldots, N$, and $t \geq 0$. This implies that $f_N(t,s)^2 \leq f(t,s)^2$ for λ-a.e. $s < t$ and $N \in \mathbb{N}$ as well as $f_N(t,\cdot) \to f(t,\cdot)$ in $\mathscr{L}^2(\lambda)$ as $N \to \infty$. Resorting to the Itō isometry and the independence of the increments of L this results in

$$\begin{aligned}
&\mathbb{E}\left(e^{iuM(t)}\right) \\
&\quad = \mathbb{E}\left(\exp\left(iu\ \mathscr{L}^2\text{-}\lim_{N\to\infty} \int_{-\infty}^{t} f_N(t,s)\ L(\mathrm{d}s)\right)\right) \\
&\quad = \mathbb{E}\left(\lim_{k\to\infty} \prod_{j=1}^{N_k} e^{iu\tau_j^{N_k}(t)(L(t_{j+1})-L(t_j))}\right) \\
&\quad = \lim_{k\to\infty} \prod_{j=1}^{N_k} \mathbb{E}\left(e^{iu\tau_j^{N_k}(t)L^{t_j}(t_{j+1})}\right),
\end{aligned}$$

for some sequence $(N_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$, where we used the dominated convergence theorem in the last step. Using (5.4) we see that the last expression equals

$$\begin{aligned}
&\lim_{k\to\infty} \exp\left(\sum_{j=1}^{N_k}\left(-\frac{\sigma^2\left(u\tau_j^{N_k}(t)\right)^2}{2}(t_{j+1}-t_j) + \int_{t_j}^{t_{j+1}}\int_{\mathbb{R}_0}\left(e^{iux\tau_j^{N_k}(t)} - 1 - iux\tau_j^{N_k}(t)\right)\nu(\mathrm{d}x)\mathrm{d}s\right)\right) \\
&= \lim_{k\to\infty} \exp\left(-\frac{\sigma^2u^2}{2}\int_{-\infty}^{t} f_{N_k}(t,s)^2\mathrm{d}s + \int_{-\infty}^{t}\int_{\mathbb{R}_0}\left(e^{iuxf_{N_k}(t,s)} - 1 - iuxf_{N_k}(t,s)\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s\right) \\
&= \exp\left(-\frac{\sigma^2u^2}{2}\int_{-\infty}^{t} f(t,s)^2\mathrm{d}s + \int_{-\infty}^{t}\int_{\mathbb{R}_0}\left(e^{iuxf(t,s)} - 1 - iuxf(t,s)\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s\right).
\end{aligned}$$

Note that because of the fact that $\sup_{N\in\mathbb{N}} f_N(t,s)^2 \leq f(t,s)^2$, the interchanging of the limit and the integrals in the last step is justified by the dominated convergence theorem, since we have the estimate

$$\begin{aligned}
&\int_{-\infty}^{t} \sup_{k\in\mathbb{N}} f_{N_k}(t,s)^2\mathrm{d}s + \int_{-\infty}^{t}\int_{\mathbb{R}_0} \sup_{k\in\mathbb{N}}\left|e^{iuxf_{N_k}(t,s)} - 1 - iuxf_{N_k}(t,s)\right|\ \nu(\mathrm{d}x)\ \mathrm{d}s \\
&\quad \leq \int_{-\infty}^{t} f(t,s)^2\mathrm{d}s + \sup_{z\in\mathbb{R}_0}\left|\frac{e^{iuz} - 1 - iuz}{z^2}\right| u^2 \int_{-\infty}^{t}\int_{\mathbb{R}_0} x^2 f(t,s)^2\ \nu(\mathrm{d}x)\ \mathrm{d}s \\
&\quad < \infty.
\end{aligned}$$

This completes the proof. $\square$

The following proposition is concerned with the characteristic function of $M(t)$ under the signed measure $\mathcal{Q}_g$.

Proposition 5.3 *Let $f \in \mathcal{K}$, $g \in \Xi$, and $t \in [0,T]$. Then*

$$\begin{aligned}
&\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) \\
&= \exp\Bigg(iu \int_{-\infty}^{t} \sigma^2 f(t,s) g(0,s)\ \mathrm{d}s + iu \int_{-\infty}^{t} \int_{\mathbb{R}_0} x f(t,s) g^*(x,s)\ \nu(\mathrm{d}x)\ \mathrm{d}s \\
&\qquad - \frac{\sigma^2 u^2}{2} \int_{-\infty}^{t} f(t,s)^2\ \mathrm{d}s + \int_{-\infty}^{t} \int_{\mathbb{R}_0} \left(e^{iuxf(t,s)} - 1 - iuxf(t,s)\right)(1 + g^*(x,s))\ \nu(\mathrm{d}x)\ \mathrm{d}s \Bigg).
\end{aligned} \tag{5.5}$$

Proof Note that

$$M(t) = I_1(f(t,\cdot)). \tag{5.6}$$

Approximating $f(t,\cdot)$ by the sequence of functions $(f_n(t,x,s))_{n\in\mathbb{N}}$ defined via

$$f_n(t,x,s) := \mathbb{1}_{[-n,n]}(x)\mathbb{1}_{[-n,t]}(s) f(t,s)$$

and using Theorem 5.3 in [28] we see that $\mathbb{P}$-a.s. the equation

$$e^{I_1(iuf_n(t,\cdot))} = \mathbb{E}\left(e^{I_1(iuf_n(t,\cdot))}\right)\exp^{\diamond}\left(I_1\left(k_{t,n}\right)\right)$$

holds with

$$\begin{aligned}
k_{t,n}(x,s) &:= \mathbb{1}_{\{0\}}(x) iu f_n(t,x,s) + \mathbb{1}_{\mathbb{R}_0}(x)\frac{e^{iuxf_n(t,x,s)} - 1}{x} \\
&= \mathbb{1}_{[-n,t]}(s)\mathbb{1}_{[-n,n]}(x) k_t(x,s),
\end{aligned}$$

where k_t is given by

$$k_t(x,s) := \mathbb{1}_{\{0\}}(x) iu f(t,s) + \mathbb{1}_{\mathbb{R}_0}(x)\frac{e^{iuxf(t,s)} - 1}{x}.$$

Note that the results in [28] are applicable, because by construction of f_n we can switch from the original Lévy process with Lévy measure ν to an auxiliary one with Lévy measure $\nu_n(A) = \nu(A \cap [-n,n])$, for which the moment conditions assumed in [28] are satisfied.

In view of (4.2) this yields

$$\mathbb{E}^{\mathcal{Q}_g}\left(e^{I_1(iuf_n(t,\cdot))}\right) = \mathbb{E}\left(e^{I_1(iuf_n(t,\cdot))}\right)\mathbb{E}\Big(\exp^{\diamond}(I_1(g))\exp^{\diamond}(I_1(k_{t,n}))\Big).$$

By the isometry for multiple Lévy-Itō integrals and (4.3) we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{Q}_g}\left(e^{I_1(iuf_n(t,\cdot))}\right)\\
&= \mathbb{E}\left(e^{I_1(iuf_n(t,\cdot))}\right)\exp\left(\int_{-n}^{t}\int_{-n}^{n} g(x,s)k_t(x,s)\ \mu(\mathrm{d}x,\mathrm{d}s)\right).
\end{aligned} \tag{5.7}
$$

Resorting to (5.6) and (5.7), taking the limit as $n \to \infty$ and using the dominated convergence theorem results in

$$
\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) = \mathbb{E}\left(e^{iuM(t)}\right)\exp\left(\int_{-\infty}^{t}\int_{\mathbb{R}} g(x,s)k_t(x,s)\ \mu(\mathrm{d}x,\mathrm{d}s)\right). \tag{5.8}
$$

We continue by calculating

$$
\begin{aligned}
&\int_{-\infty}^{t}\int_{\mathbb{R}} g(x,s)k_t(x,s)\ \mu(\mathrm{d}x,\mathrm{d}s)\\
&\quad= iu\int_{\mathbb{R}_0}\sigma^2 f(t,s)g(0,s)\ \mathrm{d}s \quad + \int_{-\infty}^{t}\int_{\mathbb{R}_0}\left(e^{iuxf(t,s)}-1\right)g^*(x,s)\ \nu(\mathrm{d}x)\ \mathrm{d}s\\
&\quad= iu\int_{\mathbb{R}_0}\sigma^2 f(t,s)g(0,s)\ \mathrm{d}s \quad + iu\int_{-\infty}^{t}\int_{\mathbb{R}_0} xf(t,s)g^*(x,s)\ \nu(\mathrm{d}x)\ \mathrm{d}s\\
&\qquad\qquad + \int_{-\infty}^{t}\int_{\mathbb{R}_0}\left(e^{iuxf(t,s)}-1-iuxf(t,s)\right)g^*(x,s)\ \nu(\mathrm{d}x)\ \mathrm{d}s.
\end{aligned} \tag{5.9}
$$

Plugging the formula for $\mathbb{E}(e^{iuM(t)})$ from Lemma 5.2 and (5.9) into (5.8) shows that (5.5) holds. □

In the spirit of Lemma 4.7 we now derive a formula for the derivative of the S-transform of $e^{iuM(t)}$.

Lemma 5.4 *Let $f \in \mathcal{K}$ and $g \in \Xi$. Then the map $t \mapsto \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)$ is continuously differentiable on $[0,T]$ with derivative*

$$
\begin{aligned}
&\frac{\partial}{\partial t}\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)\\
&= \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)\cdot\Bigg[\ iu\frac{\mathrm{d}}{\mathrm{d}t}S(M(t))(g)\\
&\qquad - \frac{\sigma^2u^2}{2}\left(f(t,t)^2 + 2\int_{-\infty}^{t}\frac{\partial}{\partial t}f(t,s)\cdot f(t,s)\ \mathrm{d}s\right)\\
&\qquad + \int_{\mathbb{R}_0}\left(e^{iuxf(t,t)}-1-iuxf(t,t)\right)(1+g^*(x,t))\ \nu(\mathrm{d}x)\\
&\qquad + \int_{-\infty}^{t}\int_{\mathbb{R}_0}\left(iux\frac{\partial}{\partial t}f(t,s)\left(e^{iuxf(t,s)}-1\right)(1+g^*(x,s))\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s\Bigg].
\end{aligned}
$$

Proof By the differentiability of the exponential function we only have to prove the differentiability of the terms in the exponential of (5.5). The first two of these summands are easily identified as the terms that occur in (4.12) and are already treated in Lemma 4.7.

To show the continuous differentiability of the third summand in the exponential of (5.5) we want to apply Lemma 4.6(ii) to the function

$$F_{\mathrm{III}}(t,s) := f(t,s)^2.$$

The continuity condition in Lemma 4.6(ii)a) follows directly from Definition 3.1(iii). Using Definition 3.1(v) yields the continuous differentiability of the mapping $t \mapsto F_{\mathrm{III}}(t,s)$ (condition (ii)b)). By combining equation (3.2) and (3.3) in Definition 3.1(v) and assuming without loss of generality $t-2\epsilon \geq -1$ we derive

$$\begin{aligned}
&\int_{-\infty}^{t-2\varepsilon} \sup_{r\in[(t-\varepsilon)\vee 0,\,(t+\varepsilon)\wedge T]} \left|\frac{\partial}{\partial r} F_{\mathrm{III}}(r,s)\right| \,\mathrm{d}s = 2\int_{-\infty}^{t-2\varepsilon} \sup_{r\in[(t-\varepsilon)\vee 0,\,(t+\varepsilon)\wedge T]} \left|\frac{\partial}{\partial r} f(r,s)\ f(r,s)\right| \,\mathrm{d}s \\
&\quad\leq \tilde{C}_0 \Bigg(\int_{-\infty}^{-1} \sup_{r\in[(t-\varepsilon)\vee 0,\,(t+\varepsilon)\wedge T]} |s|^{-\beta}|r-s|^{-\gamma}|s|^{-\theta} \,\mathrm{d}s \\
&\qquad\qquad + \int_{-1}^{t-2\varepsilon} \sup_{r\in[(t-\varepsilon)\vee 0,\,(t+\varepsilon)\wedge T]} \left||s|^{-\beta}|r-s|^{-\gamma}\ f(r,s)\right| \,\mathrm{d}s \Bigg) \\
&\quad\leq \tilde{C}_0 \Bigg(\sup_{r\in[(t-\varepsilon)\vee 0,\,(t+\varepsilon)\wedge T]} \sup_{u\in(-\infty,-1]} \left|\frac{r}{u}-1\right|^{-\gamma} \int_{-\infty}^{-1} |s|^{-(\beta+\gamma+\theta)} \,\mathrm{d}s \\
&\qquad\qquad + \sup_{r\in[(t-\varepsilon)\vee 0,\,(t+\varepsilon)\wedge T]} \sup_{u\in[-1,t-2\epsilon]} \left||r-u|^{-\gamma}\ f(r,u)\right| \int_{-1}^{t-2\varepsilon} |s|^{-\beta} \,\mathrm{d}s \Bigg) \\
&\quad< \infty
\end{aligned}$$

for some constant $\tilde{C}_0 > 0$. Note that the finiteness follows from the continuity of f (Definition 3.1(iii)) and the facts $\beta+\gamma+\theta > 1$ and $\beta < 1$. Therefore, condition (4.9) in Lemma 4.6(ii)c) is fulfilled. Finally, using (3.2) and again the continuity of f we get

$$\left|\frac{\partial}{\partial r} F_{\mathrm{III}}(r,s)\right| \leq \left(\sup_{v\in[(t-\varepsilon)\vee 0,\,(t+\varepsilon)\wedge T]} \sup_{u\in[t-2\epsilon,v]} |f(v,u)|\, C_0 \right) |s|^{-\beta}|r-s|^{-\gamma}$$

for all $r \in [(t-\varepsilon)\vee 0,\, (t+\varepsilon)\wedge T]$ and $s \in [t-2\varepsilon, r)$. We conclude that (4.10) holds.

Consequently, the assumptions of Lemma 4.6(ii) are satisfied which yields that the mapping $t \mapsto I_{F_{\mathrm{III}}}(t)$ is continuously differentiable on $[0,T]$ with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{F_{\mathrm{III}}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} f(t,s)^2 \,\mathrm{d}s = f(t,t)^2 + 2\int_{-\infty}^{t} \frac{\partial}{\partial t} f(t,s)\cdot f(t,s) \,\mathrm{d}s.$$

To deal with the fourth summand in the exponential of (5.5) we define

$$F_{\mathrm{IV}}(t,s) := \int_{\mathbb{R}_0} \left(e^{iuxf(t,s)} - 1 - iuxf(t,s)\right)(1+g^*(x,s))\ \nu(\mathrm{d}x).$$

To check for the continuity condition in Lemma 4.6(ii)a) we choose a sequence $(t_n,s_n)_{n\in\mathbb{N}}$ with $(t_n,s_n) \to (t,s)$ as $n \to \infty$. Without loss of generality we may assume $|t-t_n| \leq 1$ and $|s-s_n| \leq 1$ for all $n \in \mathbb{N}$. In order to apply the dominated convergence theorem to the expression

$$F_{\mathrm{IV}}(t_n,s_n) = \int_{\mathbb{R}_0} \left(e^{iuxf(t_n,s_n)} - 1 - iuxf(t_n,s_n)\right)(1+g^*(x,s_n))\ \nu(\mathrm{d}x) \tag{5.10}$$

we define

$$D := \sup_{z\in\mathbb{R}_0}\left|\frac{e^{iz}-1-iz}{z^2}\right| < \infty$$

and write by using Remark 4.1

$$\begin{aligned}
&\int_{\mathbb{R}_0}\sup_{n\in\mathbb{N}}\left|\left(e^{iuxf(t_n,s_n)} - 1 - iuxf(t_n,s_n)\right)(1+g^*(x,s_n))\right|\ \nu(\mathrm{d}x)\\
&\quad\le D\int_{\mathbb{R}_0}\sup_{n\in\mathbb{N}}\left(\left(u^2x^2f(t_n,s_n)^2\right)(1+|g^*(x,s_n)|)\right)\ \nu(\mathrm{d}x)\\
&\quad\le Du^2\int_{\mathbb{R}_0}\sup_{h_1,h_2\in[-1,1]}\left(\left(x^2f(t+h_1,s+h_2)^2\right)(1+|g^*(x,s+h_2)|)\right)\ \nu(\mathrm{d}x)\\
&\quad\le Du^2\sup_{h_1,h_2\in[-1,1]}\left(f(t+h_1,s+h_2)^2\right)\int_{\mathbb{R}_0}x^2\Big(1+g_1(x)\sup_{h_2\in[-1,1]}g_2(s+h_2)\Big)\ \nu(\mathrm{d}x)\\
&\quad<\infty,
\end{aligned} \tag{5.11}$$

where the finiteness follows from the continuity of f and the fact that the expression in the brackets in the last integral is bounded by Remark 4.1. In view of (5.11) the pointwise convergence in n of the integrand of (5.10) shows the continuity of the function F_{IV}.

Using again Remark 4.1 and (3.2) we get for $t\in[0,T]$, $s<t$ with $s\neq 0$, and an arbitrary $\epsilon\in(0,t-s)$ the following chain of estimates that will be useful below:

$$\begin{aligned}
&\int_{\mathbb{R}_0}\sup_{r\in[0\vee(t-\epsilon),\,(t+\epsilon)\wedge T]}\left|ux\frac{\partial}{\partial r}f(r,s)\left(e^{iuxf(r,s)}-1\right)\right||1+g^*(x,s)|\ \nu(\mathrm{d}x)\\
&\le 2u^2\sup_{y\in\mathbb{R}_0}\left|\frac{e^{iy}-1}{y}\right|\int_{\mathbb{R}_0}x^2(1\vee g_1^*(x))\ \nu(\mathrm{d}x)\\
&\qquad\qquad\sup_{r\in[0\vee(t-\epsilon),\,(t+\epsilon)\wedge T]}|f(r,s)|\cdot C_0|s|^{-\beta}|t-\epsilon-s|^{-\gamma}(1\vee g_2(s))\\
&<\infty,
\end{aligned} \tag{5.12}$$

where C_0,β, and γ are given by (3.2).

In order to show the differentiability of the function F_{IV}, we use the mean value theorem for complex valued functions to obtain the estimate

$$\begin{aligned}
&\int_{\mathbb{R}_0}\sup_{h\in(-\epsilon,\epsilon)\cap(-t,T-t)\setminus\{0\}}\left|\frac{e^{iuxf(t+h,s)}-e^{iuxf(t,s)}-iux(f(t+h,s)-f(t,s))}{h}(1+g^*(x,s))\right|\ \nu(\mathrm{d}x)\\
&\le\int_{\mathbb{R}_0}\sup_{r\in[0\vee(t-\epsilon),\,(t+\epsilon)\wedge T]}\left|ux\frac{\partial}{\partial r}f(r,s)\left(e^{iuxf(r,s)}-1\right)\right||1+g^*(x,s)|\ \nu(\mathrm{d}x),
\end{aligned}$$

which is finite by (5.12) for $t \in [0,T]$, $s < t$ with $s \neq 0$, and $\epsilon \in (0, t-s)$. Therefore an application of the dominated convergence theorem yields the differentiability of F_{IV} with respect to its first variable and

$$\frac{\partial}{\partial t} F_{\mathrm{IV}}(t,s) = iu \int_{\mathbb{R}_0} x \frac{\partial}{\partial t} f(t,s) \left(e^{iuxf(t,s)} - 1 \right) (1 + g^*(x,s))\, \nu(\mathrm{d}x). \tag{5.13}$$

By another application of the dominated convergence theorem in view of (5.12) we also get the continuity of the derivative $\frac{\partial}{\partial t} F_{\mathrm{IV}}(t,s)$ for $s \neq 0$ and $t \in [0,T] \setminus \{s\}$.

To check (4.9) in Lemma 4.6(ii) for F_{IV} we fix $t \in [0,T]$ and choose $C_0, \beta, \gamma, \theta$ and $\epsilon \in (0, {(t+1)}/{2})$ such that (3.2) and (3.3) hold. Integrating (5.12) with respect to s from $-\infty$ to $t - 2\epsilon$ thus results in

$$\begin{aligned}
&\int_{-\infty}^{t-2\epsilon} \int_{\mathbb{R}_0} \sup_{r \in [0 \vee (t-\epsilon),\, (t+\epsilon) \wedge T]} \left| ux \frac{\partial}{\partial r} f(r,s) \left(e^{iuxf(r,s)} - 1 \right) \right| |1 + g^*(x,s)|\, \nu(\mathrm{d}x)\, \mathrm{d}s \\
&\leq 2u^2 \sup_{y \in \mathbb{R}_0} \left| \frac{e^{iy} - 1}{y} \right| \int_{\mathbb{R}_0} x^2 (1 \vee |g_1^*(x)|)\, \nu(\mathrm{d}x) \\
&\qquad \int_{-\infty}^{t-2\epsilon} \sup_{r \in [0 \vee (t-\epsilon),\, (t+\epsilon) \wedge T]} \left(|f(r,s)| \cdot C_0 |s|^{-\beta} |r-s|^{-\gamma} \right) (1 \vee g_2(s))\, \mathrm{d}s \\
&\leq 2u^2 C_0 \sup_{y \in \mathbb{R}_0} \left| \frac{e^{iy} - 1}{y} \right| \int_{\mathbb{R}_0} x^2 (1 \vee |g_1^*(x)|)\, \nu(\mathrm{d}x) \\
&\qquad \cdot \Bigg(\sup_{v \in (-\infty,-1]} \sup_{r \in [0 \vee (t-\epsilon),\, (t+\epsilon) \wedge T]} \left(|f(r,v)| |v|^{\theta} \right) \int_{-\infty}^{-1} |s|^{-(\theta+\beta)} |t - \epsilon - s|^{-\gamma} (1 \vee g_2(s))\, \mathrm{d}s \\
&\qquad\quad + \sup_{v \in [-1, t-2\epsilon]} \sup_{r \in [0 \vee (t-\epsilon),\, (t+\epsilon) \wedge T]} |f(r,v)| \int_{-1}^{t-2\epsilon} |s|^{-\beta} |t - \epsilon - s|^{-\gamma} (1 \vee g_2(s))\, \mathrm{d}s \Bigg) \\
&< \infty.
\end{aligned}$$

Note that the finiteness follows from Remark 4.1, (3.2), (3.3), and Definition 3.1(iii). Hence, (4.9) is fulfilled.

To check (4.10) we use (5.13) and write for $t \in [0,T]$, $\epsilon \in (0, {(t+1)}/{2})$, $r \in [(t-\varepsilon) \vee 0,\, (t+\varepsilon) \wedge T]$, and $s \in [t - 2\epsilon, r)$:

$$\left| \frac{\partial}{\partial r} F_{\mathrm{IV}}(r,s) \right| \leq |u| \int_{\mathbb{R}_0} x \left(e^{iuxf(r,s)} - 1 \right) (1 + g^*(x,s))\, \nu(\mathrm{d}x) \left| \frac{\partial}{\partial r} f(r,s) \right| \leq \tilde{C} |s|^{-\beta} |r-s|^{-\gamma}$$

with β and γ given by (3.2) and $\tilde{C}$ defined as follows:

$$\tilde{C} := C_0 u^2 \sup_{y \in \mathbb{R}_0} \left| \frac{e^{iy} - 1}{y} \right| \sup_{u \in [(t-\varepsilon) \vee 0,\, t+\varepsilon]} \sup_{v \in [t-2\epsilon, t+\epsilon]} |f(u,v)| \int_{\mathbb{R}_0} x^2 (1 + g^*(x,s))\, \nu(\mathrm{d}x) < \infty,$$

where the finiteness results from Definition 3.1(iii).

Therefore, the assumptions of Lemma 4.6(ii) are satisfied and thus we conclude by using the expression for the derivative obtained in (5.13) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}\int_{\mathbb{R}_0}\left(e^{iuxf(t,s)}-1-iuxf(t,s)\right)(1+g^*(x,s))\ \nu(\mathrm{d}x)\ \mathrm{d}s\\
&\quad=\int_{\mathbb{R}_0}\left(e^{iuxf(t,t)}-1-iuxf(t,t)\right)(1+g^*(x,t))\ \nu(\mathrm{d}x)\\
&\qquad+\int_{-\infty}^{t}\int_{\mathbb{R}_0}\left(iux\frac{\partial}{\partial t}f(t,s)\left(e^{iuxf(t,s)}-1\right)(1+g^*(x,s))\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s
\end{aligned}
$$

and this derivative is continuous on $[0,T]$. □

The following result generalises the first part of Proposition 4.2 in [29].

Proposition 5.5 *Suppose that L has a nontrivial Gaussian part (i.e. $\sigma > 0$) and that the moment condition*

$$
\int_{\mathbb{R}_0} x^n\ \nu(\mathrm{d}x) < \infty \quad \textit{for all } n \geq 2
$$

is fulfilled. Then for every $t \in (0,T]$ the mapping $u \mapsto \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)$ is a Schwartz function on $\mathbb{R}$.

Proof The proof is divided into two parts. The first part deals with the derivative of the map $u \mapsto \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)$, which is then used in the second part to prove the assertion.

Part I First we show by induction that the above mapping is smooth with j-th derivative, $j \in \mathbb{N} \cup \{0\}$, given by

$$
\frac{\mathrm{d}^j}{\mathrm{d}u^j}\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) = \mathbb{E}^{\mathcal{Q}_g}\left(i^j M(t)^j e^{iuM(t)}\right). \tag{5.14}
$$

For $j = 0$ the assertion is trivial. Now let the statement hold for some $k \in \mathbb{N}$. For the purpose of interchanging differentiation and integration we consider

$$
\begin{aligned}
\sup_{h\in\mathbb{R}_0}\left|i^k M(t)^k \frac{e^{i(u+h)M(t)}-e^{iuM(t)}}{h}\right| &= \left|M(t)^k\right| \sup_{h\in\mathbb{R}_0}\left|M(t)\frac{e^{ihM(t)}-1}{hM(t)}\right|\cdot\left|e^{iuM(t)}\right|\\
&\leq \left|M(t)^{k+1}\right| \sup_{x\in\mathbb{R}_0}\left|\frac{e^{ix}-1}{x}\right|.
\end{aligned}
$$

Since the last supremum is finite, the term on the right-hand side is bounded by $D|M(t)^{k+1}|$ for some $D > 0$. In the light of (4.4) this yields

$$
\mathbb{E}^{\mathcal{Q}_g}\left(\sup_{h\in\mathbb{R}_0}\left|i^k M(t)^k \frac{e^{i(u+h)M(t)}-e^{iuM(t)}}{h}\right|\right) \leq \mathbb{E}^{\mathcal{Q}_g}\left(D\left|M(t)^{k+1}\right|\right) \leq e_g\, D\, \mathbb{E}\left(\left|M(t)^{2(k+1)}\right|\right)^{1/2},
$$

which is finite thanks to Lemma 3.2, where e_g is a constant depending only on g. Therefore, we can apply the dominated convergence theorem in order to interchange differentiation and integration and obtain

$$
\begin{aligned}
\frac{\mathrm{d}^{k+1}}{\mathrm{d}u^{k+1}}\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) &= \frac{\mathrm{d}}{\mathrm{d}u}\mathbb{E}^{\mathcal{Q}_g}\left(i^k M(t)^k e^{iuM(t)}\right)\\
&= \lim_{h\to 0}\mathbb{E}^{\mathcal{Q}_g}\left(i^k M(t)^k \frac{e^{i(u+h)M(t)}-e^{iuM(t)}}{h}\right)\\
&= \mathbb{E}^{\mathcal{Q}_g}\left(i^{k+1}M(t)^{k+1}e^{iuM(t)}\right),
\end{aligned}
$$

where the first equality follows from the induction hypothesis. This proves (5.14) and hence finishes the first part of the proof.

Part II It remains to show that for all $m,n \in \mathbb{N}\cup\{0\}$ the expression

$$
\left|u^n \frac{\mathrm{d}^m}{\mathrm{d}u^m}\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)\right| \tag{5.15}
$$

is bounded in u. In view of Proposition 5.3 we start by writing

$$
\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) = \exp\left(-\frac{\sigma^2}{2}\int_{-\infty}^t f(t,s)^2 \ \mathrm{d}s \cdot u^2\right)\cdot R_{g,t}(u) \tag{5.16}
$$

with $R_{g,t}(u)$ given by

$$
R_{g,t}(u) = \exp\left(iu\int_{-\infty}^t \sigma^2 f(t,s)g(0,s)\ \mathrm{d}s\right)\cdot \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM_j(t)}\right),
$$

where the process M_j is constructed analogously to M by using the characteristic triple $(\gamma,0,\nu)$ instead of (γ,σ,ν). Applying the arguments of Part I to the process $(M_j(t))_{t\in[0,T]}$ we infer that $R_{g,t}$ has bounded derivatives of every order. Since $\frac{\sigma^2}{2}\int_{-\infty}^t f(t,s)^2\ \mathrm{d}s > 0$ according to Definition 3.1(iv), the mapping

$$
u \mapsto \exp\left(-\frac{\sigma^2}{2}\int_{-\infty}^t f(t,s)^2\ \mathrm{d}s\cdot u^2\right)
$$

is a Schwartz function and thus (5.15) is bounded in u, which completes the proof. □

Remark 5.6 Let $\sigma > 0$ and $a_0 \in (0,T]$. Note that

$$
\sup_{s\in[a_0,T]}\left|\exp\left(-\frac{\sigma^2}{2}\int_{-\infty}^s f(s,r)^2\ \mathrm{d}r\cdot u^2\right)\right| = \exp\left(-\frac{\sigma^2}{2}\int_{-\infty}^{s_0} f(s_0,r)^2\ \mathrm{d}r\cdot u^2\right) \le \exp\left(-cu^2\right)
$$

for some $s_0 \in [a_0,T]$ and $c>0$, cf. Definition 3.1(iv). Furthermore, it follows from (4.4) that

$$
\sup_{u\in\mathbb{R}}\sup_{r\in[a_0,T]}\left|\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM_j(r)}\right)\right| \le \sup_{u\in\mathbb{R}}\sup_{r\in[a_0,T]}\mathbb{E}\left(\left|e^{iuM_j(r)}\exp^{\diamond}(I_1(g))\right|\right) \le e_g. \tag{5.17}
$$

Thus, we see in view of (5.16) that the function

$$u \mapsto \sup_{s\in[a_0,T]} \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(s)}\right)$$

is an element of $\mathscr{L}^2(\mathrm{d}u)$ in the situation that L has a nontrivial Gaussian part. $\Diamond$

Our starting point in the proof of Theorem 5.1 is the equation

$$S(G(M(T)))(g) = S(G(M(a)))(g) + \int_a^T \frac{\mathrm{d}}{\mathrm{d}t} S(G(M(t)))(g)\ \mathrm{d}t,$$

which follows from the fundamental theorem of calculus if the function $t \mapsto S(G(M(t)))(g)$ is continuously differentiable. Therefore, we first prove the existence of $\frac{\mathrm{d}}{\mathrm{d}t}S(G(M(t)))(g)$.

Proposition 5.7 *Let $f \in \mathcal{K}$, $g \in \Xi$, as well as $G \in C^2(\mathbb{R})$ such that one of the following assumptions holds:*

a) $\sigma > 0$, G has compact support, and $a \in (0,T]$,

b) $G, G', G'' \in \mathcal{A}(\mathbb{R})$ and $a = 0$.

Then $S(G(M(\cdot)))(g)$ is continuously differentiable on $[a,T]$ with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} S(G(M(t)))(g) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}G)(u) \frac{\partial}{\partial t} \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)\ \mathrm{d}u.$$

Proof Let $t \in [a,T]$. Recall that the Fourier transform of a function $f \in \mathscr{L}^1(\mathbb{R})$ is denoted by $\mathcal{F}f$. Note that it follows from Theorem 8.22 in [22] that also under condition $a)$ we have $\mathcal{F}G \in \mathscr{L}^1(\mathrm{d}u)$. By means of the Fourier inversion theorem we deduce both in case $a)$ and $b)$ that

$$S(G(M(t)))(g) = \mathbb{E}^{\mathcal{Q}_g}(G(M(t))) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}G)(u) \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)\ \mathrm{d}u. \tag{5.18}$$

Hence, we have

$$\begin{aligned} &\sqrt{2\pi}\frac{S(G(M(t+h)))(g) - S(G(M(t)))(g)}{h} \\ &= \int_{\mathbb{R}} (\mathcal{F}G)(u) \frac{\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t+h)}\right) - \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)}{h}\ \mathrm{d}u. \end{aligned} \tag{5.19}$$

Moreover, resorting to Lemma 5.4 we infer from the mean value theorem for complex valued functions that

$$\begin{aligned} &\sup_{h\in[-t/2,T-t]\setminus\{0\}} \left| (\mathcal{F}G)(u) \frac{\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t+h)}\right) - \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)}{h} \right| \\ &\le |(\mathcal{F}G)(u)| \cdot \sup_{r\in[t/2,T]} \left| \frac{\partial}{\partial r} \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(r)}\right) \right| \\ &= |(\mathcal{F}G)(u)| \cdot \sup_{r\in[t/2,T]} \left| \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(r)}\right) (\mathrm{I}(r) + \mathrm{II}(r) + \mathrm{III}(r) + \mathrm{IV}(r)) \right|, \end{aligned}$$

where the terms I(r), II(r), III(r) and IV(r) are given by

$$
\begin{aligned}
\mathrm{I}(r) &= iu\frac{\mathrm{d}}{\mathrm{d}r}S(M(r))(g),\\
\mathrm{II}(r) &= -\frac{\sigma^2 u^2}{2}\left(f(r,r)^2 + 2\int_{-\infty}^{r}\frac{\partial}{\partial r}f(r,s)\cdot f(r,s)\ \mathrm{d}s\right),\\
\mathrm{III}(r) &= \int_{\mathbb{R}_0}\left(e^{iuxf(r,r)} - 1 - iuxf(r,r)\right)(1+g^*(x,r))\ \nu(\mathrm{d}x),
\end{aligned}
$$

and

$$
\mathrm{IV}(r) = \int_{-\infty}^{r}\int_{\mathbb{R}_0}\left(iux\frac{\partial}{\partial r}f(r,s)\left(e^{iuxf(r,s)}-1\right)(1+g^*(x,s))\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s.
$$

Hence,

$$
\begin{aligned}
&\sup_{h\in[-t/2,T-t]\setminus\{0\}}\left|(\mathcal{F}G)(u)\frac{\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t+h)}\right)-\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)}{h}\right|\\
&\le \sup_{r\in[t/2,T]}\left|\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(r)}\right)\right|\\
&\quad\times\sup_{r\in[t/2,T]}\Bigg(\left|(\mathcal{F}G')(u)\right|\cdot\left|\frac{\mathrm{d}}{\mathrm{d}r}S(M(r))(g)\right|\\
&\qquad+\left|(\mathcal{F}G'')(u)\right|\cdot\Bigg[-\frac{\sigma^2}{2}\left(f(r,r)^2+2\int_{-\infty}^{r}\frac{\partial}{\partial r}f(r,s)\cdot f(r,s)\ \mathrm{d}s\right)\\
&\qquad+\sup_{y\in\mathbb{R}_0}\left|\frac{e^{iy}-1-iy}{y^2}\right|\int_{\mathbb{R}_0}x^2 f(r,r)^2\left|1+g^*(x,r)\right|\ \nu(\mathrm{d}x)\\
&\qquad+\sup_{y\in\mathbb{R}_0}\left|\frac{e^{iy}-1}{y}\right|\int_{-\infty}^{r}\int_{\mathbb{R}_0}\left(x^2\left|\frac{\partial}{\partial r}f(r,s)\right|\cdot|f(r,s)|\left|1+g^*(x,s)\right|\right)\ \nu(\mathrm{d}x)\ \mathrm{d}s\Bigg]\Bigg)\\
&= \sup_{r\in[t/2,T]}\left|\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(r)}\right)\right|\cdot\Bigg(\left|(\mathcal{F}G')(u)\right|\cdot\sup_{r\in[t/2,T]}\left|\frac{\mathrm{d}}{\mathrm{d}r}S(M(r))(g)\right|\\
&\qquad+\left|(\mathcal{F}G'')(u)\right|\cdot\sup_{r\in[t/2,T]}\mathfrak{E}(r)\Bigg),
\end{aligned}
\tag{5.20}
$$

where $\mathfrak{E}(r)$ denotes the expression in the square brackets. We now prove that under assumption a) or b) the left-hand side of (5.20) is in $\mathscr{L}^1(\mathrm{d}u)$:

a) If $\sigma > 0$ and G has compact support, then in particular $G', G'' \in \mathscr{L}^1 \cap \mathscr{L}^2(\mathrm{d}u)$ and $\mathcal{F}G', \mathcal{F}G'' \in \mathscr{L}^2(\mathrm{d}u)$. Since $r \mapsto \mathfrak{E}(r)$ is continuous (cf. Lemma 5.4), we deduce that

$$
\sup_{r\in[t/2,T]}\mathfrak{E}(r) < \infty. \tag{5.21}
$$

According to Lemma 4.7 also $r \mapsto \frac{\mathrm{d}}{\mathrm{d}r}S(M(r))(g)$ is continuous and thus

$$\sup_{r\in[t/2,T]}\left|\frac{\mathrm{d}}{\mathrm{d}r}S(M(r))(g)\right|<\infty. \tag{5.22}$$

As $t/2 \geq a/2 > 0$ we infer in view of Remark 5.6 (with the choice $a_0 = t/2$) that both factors on the right-hand side of (5.20) are elements of $\mathscr{L}^2(\mathrm{d}u)$. By means of the Cauchy-Schwarz inequality it follows that the mapping

$$u \mapsto \sup_{h\in[-t/2,T-t]\setminus\{0\}}\left|(\mathcal{F}G)(u)\frac{\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t+h)}\right)-\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)}{h}\right| \tag{5.23}$$

is an element of $\mathscr{L}^1(\mathrm{d}u)$.

b) Note that if $G', G'' \in \mathcal{A}(\mathbb{R})$, then $\mathcal{F}G', \mathcal{F}G'' \in \mathscr{L}^1(\mathrm{d}u)$. Therefore, in the light of (5.17), (5.21), and (5.22) we conclude that (5.23) holds.

In both cases we are now able to apply the dominated convergence theorem to (5.19) and thus we infer that $t \mapsto S(G(M(t)))(g)$ is differentiable and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}S(G(M(t)))(g) &= \lim_{h\to 0}\frac{S(G(M(t+h)))(g)-S(G(M(t)))(g)}{h}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\mathcal{F}G)(u)\lim_{h\to 0}\frac{\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t+h)}\right)-\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)}{h}\,\mathrm{d}u\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\mathcal{F}G)(u)\frac{\partial}{\partial t}\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)\,\mathrm{d}u.
\end{aligned}$$

The continuity of the derivative can be proven by using Lemma 5.4 and the estimates in (5.20). □

We are now in a position to give the proof of our main result.

Proof of Theorem 5.1 The proof is divided into two parts. In the first part we show that the auxiliary generalised Itō formula

$$\begin{aligned}
G(M(T)) = G(M(a)) &+ \frac{\sigma^2}{2}\int_a^T G''(M(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t f(t,s)^2\mathrm{d}s\right)\mathrm{d}t\\
&+ \sum_{a<t\leq T}\left[G(M(t))-G(M(t-))-G'(M(t-))\Delta M(t)\right]\\
&+ \int_{-\infty}^T\int_{\mathbb{R}}\int_{a\vee s}^T G'\left(M(t)+xf(t,s)\right)\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \Lambda^{\diamond}(\mathrm{d}x,\mathrm{d}s)\\
&+ \int_{-\infty}^T\int_{\mathbb{R}_0}\int_{a\vee s}^T \left(G'\left(M(t)+xf(t,s)\right)-G'(M(t))\right)x\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \nu(\mathrm{d}x)\ \mathrm{d}s\\
&+ \int_a^T G'(M(t-))f(t,t)\ L(\mathrm{d}t)
\end{aligned} \tag{5.24}$$

holds if

a) $\sigma > 0$, G has compact support, and $a \in (0, T]$, or

b) $G, G', G'' \in \mathcal{A}(\mathbb{R})$ and $a = 0$.

In the second part we are concerned with proving Theorem 5.1 (i), i.e. we deal with the situation that $\sigma > 0$ and G, G', and G'' are of polynomial growth. To this end, we will approximate such functions G by functions that satisfy a).

Part I Let a) or b) above be satisfied. By means of Proposition 5.7 and Lemma 5.4 we obtain

$$\begin{aligned} S(G(M(T)))(g) - S(G(M(a)))(g) &= \int_a^T \frac{\mathrm{d}}{\mathrm{d}t} S(G(M(t)))(g)\ \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_a^T \left(\mathrm{I}(t) + \mathrm{II}(t) + \mathrm{III}(t) + \mathrm{IV}(t)\right)\ \mathrm{d}t \end{aligned} \tag{5.25}$$

with the terms $\mathrm{I}(t)$, $\mathrm{II}(t)$, $\mathrm{III}(t)$ and $\mathrm{IV}(t)$ given by

$$\begin{aligned} \mathrm{I}(t) &= \int_{\mathbb{R}} (\mathcal{F}G)(u) \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) \cdot iu \frac{\mathrm{d}}{\mathrm{d}t} S(M(t))(g)\ \mathrm{d}u, \\ \mathrm{II}(t) &= \int_{\mathbb{R}} (\mathcal{F}G)(u) \frac{-\sigma^2 u^2}{2} \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) \left(f(t,t)^2 + 2\int_{-\infty}^t \frac{\partial}{\partial t} f(t,s) f(t,s)\ \mathrm{d}s\right) \mathrm{d}u, \\ \mathrm{III}(t) &= \int_{\mathbb{R}} (\mathcal{F}G)(u) \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) \int_{\mathbb{R}_0} \left(e^{iuxf(t,t)} - 1 - iuxf(t,t)\right)(1 + g^*(x,t))\ \nu(\mathrm{d}x)\ \mathrm{d}u, \end{aligned}$$

and

$$\begin{aligned} \mathrm{IV}(t) = \int_{\mathbb{R}} (\mathcal{F}G)(u) \mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right) \cdot \int_{-\infty}^t \int_{\mathbb{R}_0} \left(iux \frac{\partial}{\partial t} f(t,s) \left(e^{iuxf(t,s)} - 1\right)\right) \\ \times (1 + g^*(x,s))\ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}u. \end{aligned}$$

We exemplarily give the argument for $\mathrm{III}(t)$. Using the estimate

$$\left| e^{iuxf(t,t)} - 1 - iuxf(t,t) \right| \leq Du^2 x^2 f(t,t)^2$$

for some constant $D > 0$, and every $u \in \mathbb{R}$ and $x \in \mathbb{R}_0$ as well as the formula

$$(\mathcal{F}G)(u) u^2 = -(\mathcal{F}G'')(u)$$

we deduce

$$\begin{aligned} &\int_{\mathbb{R}} |(\mathcal{F}G)(u)| \left|\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)\right| \int_{\mathbb{R}_0} \left|e^{iuxf(t,t)} - 1 - iuxf(t,t)\right| |(1 + g^*(x,t))|\ \nu(\mathrm{d}x)\ \mathrm{d}u \\ &\leq D \int_{\mathbb{R}} \left|(\mathcal{F}G)(u)u^2\right| \left|\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)\right| \int_{\mathbb{R}_0} x^2 f(t,t)^2 |(1 + g^*(x,t))|\ \nu(\mathrm{d}x)\ \mathrm{d}u \\ &= D \int_{\mathbb{R}} \left|(\mathcal{F}G'')(u)\right| \left|\mathbb{E}^{\mathcal{Q}_g}\left(e^{iuM(t)}\right)\right| \mathrm{d}u \int_{\mathbb{R}_0} x^2 f(t,t)^2 |(1 + g^*(x,t))|\ \nu(\mathrm{d}x) \\ &< \infty. \end{aligned}$$

This implies that we can apply Fubini's theorem to the term III(t) which yields

$$\mathrm{III}(t) = \int_{\mathbb{R}_0} \int_{\mathbb{R}} (\mathcal{F}G)(u) \left(e^{iuxf(t,t)} - 1 - iuxf(t,t) \right) (1 + g^*(x,t)) \cdot \mathbb{E}^{\mathcal{Q}_g} \left(e^{iuM(t)} \right) \, \mathrm{d}u \; \nu(\mathrm{d}x).$$

By using standard manipulations of the Fourier transform we derive

$$\begin{aligned} \mathrm{III}(t) = \int_{\mathbb{R}_0} \int_{\mathbb{R}} \mathcal{F} \left(G(\cdot + xf(t,t)) - G(\cdot) - xf(t,t)G'(\cdot) \right)(u) \\ \times \mathbb{E}^{\mathcal{Q}_g} \left(e^{iuM(t)} \right) (1 + g^*(x,t)) \, \mathrm{d}u \; \nu(\mathrm{d}x). \end{aligned}$$

Consequently, by applying (5.18) we obtain

$$\mathrm{III}(t) = \int_{\mathbb{R}_0} \sqrt{2\pi} S \left(G(M(t) + xf(t,t)) - G(M(t)) - xf(t,t)G'(M(t)) \right)(g) \; (1 + g^*(x,t)) \; \nu(\mathrm{d}x).$$

For the terms corresponding to I(t), II(t) and IV(t) similar techniques apply and result in

$$\mathrm{I}(t) = \sqrt{2\pi} S(G'(M(t)))(g) \frac{\mathrm{d}}{\mathrm{d}t} S(M(t))(g)$$

and

$$\begin{aligned} \mathrm{II}(t) &= \frac{\sigma^2}{2} \left(f(t,t)^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial t} f(t,s) \cdot f(t,s) \, \mathrm{d}s \right) \int_{\mathbb{R}} (\mathcal{F}G'')(u) \mathbb{E}^{\mathcal{Q}_g} \left(e^{iuM(t)} \right) \, \mathrm{d}u \\ &= \frac{\sigma^2}{2} \left(f(t,t)^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial t} f(t,s) \cdot f(t,s) \, \mathrm{d}s \right) \sqrt{2\pi} S(G''(M(t)))(g) \end{aligned}$$

as well as

$$\mathrm{IV}(t) = \int_{-\infty}^t \int_{\mathbb{R}_0} x \frac{\partial}{\partial t} f(t,s) \sqrt{2\pi} S(G'(M(t) + xf(t,s)) - G'(M(t)))(g) \cdot (1 + g^*(x,s)) \; \nu(\mathrm{d}x) \; \mathrm{d}s.$$

Therefore, plugging the above expressions into (5.25) we obtain

$$\begin{aligned} & S(G(M(T)))(g) - S(G(M(a)))(g) \\ & = \int_a^T S(G'(M(t)))(g) \frac{\mathrm{d}}{\mathrm{d}t} S(M(t))(g) \, \mathrm{d}t \\ & \qquad + \int_a^T \frac{\sigma^2}{2} \left(f(t,t)^2 + 2 \int_{-\infty}^t \frac{\partial}{\partial t} f(t,s) \cdot f(t,s) \, \mathrm{d}s \right) S(G''(M(t)))(g) \, \mathrm{d}t \\ & \qquad + \int_a^T \int_{\mathbb{R}_0} S \Big(G(M(t) + xf(t,t)) - G(M(t)) - xf(t,t)G'(M(t)) \Big)(g) \cdot (1 + g^*(x,t)) \; \nu(\mathrm{d}x) \, \mathrm{d}t \end{aligned}$$

$$+\int_a^T \int_{-\infty}^t \int_{\mathbb{R}_0} x\frac{\partial}{\partial t}f(t,s)S(G'(M(t)+xf(t,s))-G'(M(t)))(g)\cdot(1+g^*(x,s))\ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t$$
$$=: \mathrm{I}^* + \mathrm{II}^* + \mathrm{III}^* + \mathrm{IV}^*.$$

Using the boundedness of G' and the explicit form of $\frac{\mathrm{d}}{\mathrm{d}t}S(M(t))(g)$ derived in Lemma 4.7 as well as the fact that the mapping $t \mapsto \frac{\mathrm{d}}{\mathrm{d}t}S(M(t))(g)$ is continuous on $[0,T]$ we can apply Fubini's theorem which yields the following equality for I^*

$$\begin{aligned}
&\int_a^T S(G'(M(t)))(g)\frac{\mathrm{d}}{\mathrm{d}t}S(M(t))(g)\ \mathrm{d}t\\
&= \int_{-\infty}^T \int_{a\vee s}^T S(G'(M(t)))(g)\sigma\frac{\partial}{\partial t}f(t,s)g(0,s)\ \mathrm{d}t\ \mathrm{d}s\\
&\quad + \int_a^T S(G'(M(t)))(g)\sigma f(t,t)g(0,t)\ \mathrm{d}t\\
&\quad + \int_a^T S(G'(M(t)))(g)f(t,t)\int_{\mathbb{R}_0} xg^*(x,t)\ \nu(\mathrm{d}x)\ \mathrm{d}t\\
&\quad + \int_a^T S(G'(M(t)))(g)\int_{-\infty}^t \int_{\mathbb{R}_0} \frac{\partial}{\partial t}f(t,s)xg^*(x,s)\ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t\\
&=: \mathrm{I}_1^* + \mathrm{I}_2^* + \mathrm{I}_3^* + \mathrm{I}_4^*.
\end{aligned}$$

Resorting to Remark 4.10 we obtain

$$\mathrm{I}_2^* = S\left(\sigma\int_a^T G'(M(t-))f(t,t)\ W(\mathrm{d}t)\right)(g)$$

as well as

$$\mathrm{I}_3^* = S\left(\int_a^T \int_{\mathbb{R}_0} G'(M(t-))f(t,t)x\ \tilde{N}(\mathrm{d}x,\mathrm{d}t)\right)(g),$$

where we used that both integrals exist separately and reduce to classical stochastic integrals, because t is $\mathcal{Q}_g$-a.s. not a jump time and therefore $S(G'(M(t-)))(g) = S(G'(M(t)))(g)$ and because of the predictability of the integrands. In particular, we infer

$$\mathrm{I}_2^* + \mathrm{I}_3^* = S\left(\int_a^T G'(M(t-))f(t,t)\ L(\mathrm{d}t)\right)(g).$$

In view of the boundedness of G'' and the continuity of the mapping

$$t \mapsto f(t,t)^2 + 2\int_{-\infty}^t \frac{\partial}{\partial t}f(t,s)\cdot f(t,s)\ \mathrm{d}s$$

(cf. Lemma 4.7) we can use Fubini's theorem to write the term II^* as

$$\mathrm{II}^* = S\left(\frac{\sigma^2}{2}\int_a^T \left(f(t,t)^2 + 2\int_{-\infty}^t \frac{\partial}{\partial t}f(t,s)\cdot f(t,s)\ \mathrm{d}s\right)G''(M(t))\ \mathrm{d}t\right)(g).$$

For the third term we obtain

$$
\begin{aligned}
\mathrm{III}^* &= S\left(\int_a^T \int_{\mathbb{R}_0} \big(G(M(t-)+xf(t,t)) - G(M(t-)) - xf(t,t)G'(M(t-))\big)\ N^{\diamond}(\mathrm{d}x,\mathrm{d}t)\right)(g)\\
&= S\left(\sum_{a<t\leq T} G(M(t)) - G(M(t-)) - G'(M(t-))\Delta M(t)\right)(g).
\end{aligned}
$$

Note that the predictability of the integrand implies that the Hitsuda-Skorokhod integral is an ordinary integral with respect to the jump measure and hence the second equality above holds by means of Lemma 3.2.

Using the boundedness of G' and G'', the mean value theorem, and Lemma 4.6(i), with $F(t,s) = \left|f(t,s)\frac{\partial}{\partial s}f(t,s)\right|$, in order to justify the application of Fubini's theorem we deduce that

$$
\begin{aligned}
\mathrm{IV}^* &= \int_a^T \int_{-\infty}^t \int_{\mathbb{R}_0} x\frac{\partial}{\partial t}f(t,s)S(G'(M(t)+xf(t,s)) - G'(M(t)))(g)\cdot(1+g^*(x,s))\ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t\\
&= \int_{-\infty}^T \int_{\mathbb{R}_0} \int_{a\vee s}^T x\frac{\partial}{\partial t}f(t,s)S\left(G'\left(M(t)+xf(t,s)\right)\right)(g)\cdot g^*(x,s)\ \mathrm{d}t\ \nu(\mathrm{d}x)\ \mathrm{d}s\\
&\quad - \int_a^T \int_{-\infty}^t \int_{\mathbb{R}_0} x\frac{\partial}{\partial t}f(t,s)S\left(G'\left(M(t)\right)\right)(g)\cdot g^*(x,s)\ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t\\
&\quad + S\left(\int_{-\infty}^T \int_{\mathbb{R}_0} \int_{a\vee s}^T x\frac{\partial}{\partial t}f(t,s)\left(G'(M(t)+xf(t,s)) - G'(M(t))\right)\ \mathrm{d}t\ \nu(\mathrm{d}x)\ \mathrm{d}s\right)(g)\\
&=: \mathrm{IV}_1^* - \mathrm{I}_4^* + \mathrm{IV}_3^*.
\end{aligned}
$$

Now note that the mappings $t\mapsto f(t,t)$ and

$$
t\mapsto f(t,t)^2 + 2\int_{-\infty}^t \frac{\partial}{\partial t}f(t,s)\cdot f(t,s)\ \mathrm{d}s
$$

are continuous on $[0,T]$ (cf. Definition 3.1(iii) and Lemma 4.6(ii) applied to the function $F(t,s) = f(t,s)^2$, respectively). Combining this with the fact that G' and G'' are bounded, the expressions

$$
\frac{\sigma^2}{2}\int_a^T \left(f(t,t)^2 + 2\int_{-\infty}^t \frac{\partial}{\partial t}f(t,s)\cdot f(t,s)\ \mathrm{d}s\right)G''(M(t))\ \mathrm{d}t
$$

and

$$
\int_a^T G'(M(t-))f(t,t)\ L(\mathrm{d}t)
$$

are elements of $\mathscr{L}^2(\mathbb{P})$. From the calculations on pp.77 ff and pp.75 ff in the special case that we do not consider differences, that is, we consider the expressions $\delta_0 G$, $\delta_0 G'$, $D_{1,0}(x,t,s)$ and $D_{2,0}(x,t,s)$ with $G_0 = G_0' := 0$ and $q=0$, we deduce that also

$$\sum_{a<t\leq T} G(M(t)) - G(M(t-)) - G'(M(t-))\Delta M(t)$$

and

$$\int_{-\infty}^{T} \int_{\mathbb{R}_0} \int_{a\vee s}^{T} x \frac{\partial}{\partial t} f(t,s) \left(G'(M(t) + xf(t,s)) - G'(M(t)) \right) \, \mathrm{d}t \; \nu(\mathrm{d}x) \; \mathrm{d}s$$

are elements of $\mathscr{L}^2(\mathbb{P})$ thanks to the boundedness of G, G' and G'' and the $\mathscr{L}^2(\mathbb{P})$-assumptions on L. Therefore we obtain in view of (5.25) that

$$\begin{aligned}
&\mathrm{I}_1^* + \mathrm{IV}_1^* \\
&\quad = S(G(M(T)) - G(M(a)))(g) \\
&\qquad - S\left(\frac{\sigma^2}{2} \int_a^T \left(f(t,t)^2 + 2 \int_{-\infty}^{t} \frac{\partial}{\partial t} f(t,s) \cdot f(t,s) \; \mathrm{d}s \right) G''(M(t)) \; \mathrm{d}t \right)(g) \\
&\qquad - S\left(\sum_{a<t\leq T} G(M(t)) - G(M(t-)) - G'(M(t-))\Delta M(t) \right)(g) \\
&\qquad - S\left(\int_{-\infty}^{T} \int_{\mathbb{R}_0} \int_{a\vee s}^{T} x \frac{\partial}{\partial t} f(t,s) \left(G'(M(t) + xf(t,s)) - G'(M(t)) \right) \; \mathrm{d}t \; \nu(\mathrm{d}x) \; \mathrm{d}s \right)(g) \\
&\qquad - S\left(\int_a^T G'(M(t-)) f(t,t) \; L(\mathrm{d}t) \right)(g),
\end{aligned} \tag{5.26}$$

with all the arguments of the S-transforms on the right-hand side being elements of $\mathscr{L}^2(\mathbb{P})$.

By linearity of the S-transform $\mathrm{I}_1^* + \mathrm{IV}_1^*$ equals the S-transform of some $\Phi \in \mathscr{L}^2(\mathbb{P})$ and can thus, by Definition 4.9, be written as

$$S\left(\int_{-\infty}^{T} \int_{\mathbb{R}} \int_{a\vee s}^{T} \frac{\partial}{\partial t} f(t,s) \left(G'(M(t) + xf(t,s)) \right) \; \mathrm{d}t \; \Lambda^{\diamond}(\mathrm{d}x, \mathrm{d}s) \right)(g).$$

Hence, reordering the terms in (5.26) and resorting to the injectivity property of the S-transform (see Proposition 4.3) this results in the change of variable formula (5.24).

<u>Part II</u> We now consider the case that $\sigma > 0$ and G, G', and G'' are of polynomial growth of degree q, that is we have

$$\left| G^{(l)}(x) \right| \leq C_{pol} \left(1 + |x|^q \right) \qquad \text{for every } x \in \mathbb{R} \text{ and } l = 0, 1, 2.$$

Our approach is to approximate such G by functions fulfilling condition a) above as we know from Part I of this proof that the auxiliary generalised Itō formula (5.24) holds true for these functions in the case $a \in (0, T]$. To this end, let $(G_n)_{n\in\mathbb{N}}$ be a sequence of functions defined by

$$G_n(x) = \begin{cases} G(x), & |x| \leq n, \\ G(x)\varphi(|x| - n), & n < |x| < n+1, \\ 0, & |x| \geq n+1, \end{cases} \tag{5.27}$$

where the function $\varphi : (0,1) \to \mathbb{R}$ is given by

$$\varphi(y) = \exp\left(-\left(\frac{y}{1-y}\right)^3\right).$$

Lemma 5.8 *The functions G_n, $n \in \mathbb{N}$, as defined in* (5.27) *fulfil the following conditions:*

1. $G_n \in C^2(\mathbb{R})$,
2. $G_n(x) = G(x),\ x \in [-n,n]$,
3. G_n *has compact support, and*
4. $|G_n(x)| + |G_n'(x)| + |G_n''(x)| \le C'_{pol}(1+|x|^q)$ *for all* $x \in \mathbb{R}$,

where the constant C'_{pol} is given by

$$C'_{pol} := 7\, C_{pol} \max\left\{ \sup_{y\in(0,1)} \varphi(y),\ \sup_{y\in(0,1)} \varphi'(y),\ \sup_{y\in(0,1)} \varphi''(y) \right\}.$$

Proof The conditions *2.* and *3.* are obviously fulfilled. Using the symmetry of the function

$$x \mapsto \exp\left(-\left(\frac{|x|-n}{n+1-|x|}\right)^3\right)$$

and the substitution $|x| - n \to y$ we see that it is sufficient to check the conditions *1.* and *4.* for the mapping

$$y \mapsto \begin{cases} G(n), & y = 0, \\ G(y+n)\varphi(y), & y \in (0,1), \\ 0, & y = 1, \end{cases}$$

with one-sided derivatives in 0 and 1. As the function φ is an element of $C^\infty((0,1))$, the product $G\varphi$ is in $C^2\left((0,1)\right)$. It remains to check the one-sided derivatives and the boundedness.
Calculating the derivatives φ' and φ'' yields

$$\varphi'(y) = -\frac{3y^2}{(1-y)^4} \exp\left(-\left(\frac{y}{1-y}\right)^3\right)$$

and

$$\varphi''(y) = -\frac{3y(2y^4 - y^3 + 4y - 2)}{(1-y)^8} \exp\left(-\left(\frac{y}{1-y}\right)^3\right).$$

Using this expressions we obtain

$$
\begin{aligned}
&\lim_{y\to 0}\varphi(y) &&= 1, && \\
&\lim_{y\to 0}\varphi'(y) &&= \lim_{y\to 0}\varphi''(y) &&= 0, \\
&\lim_{y\to 1}\varphi(y) &&= \lim_{y\to 1}\varphi'(y) &&= \lim_{y\to 1}\varphi''(y) = 0
\end{aligned}
\tag{5.28}
$$

and deduce by making use of the continuity of φ and its derivatives that the expressions

$$
\sup_{y\in(0,1)}\varphi(y),\quad \sup_{y\in(0,1)}\varphi'(y)\qquad\text{and}\qquad \sup_{y\in(0,1)}\varphi''(y)
$$

are finite.

In the last step we calculate the derivatives of the product $G\varphi$ as follows:

$$
\begin{aligned}
(G\varphi)' &= G'\varphi + G\varphi', \\
(G\varphi)'' &= G''\varphi + 2G'\varphi' + G\varphi''
\end{aligned}
\tag{5.29}
$$

By means of the behaviour of the derivatives (5.28) we conclude

$$
\begin{aligned}
&\lim_{y\to 0}(G\varphi)(y) &&= G(n), && \\
&\lim_{y\to 0}(G\varphi)'(y) &&= G'(n), && \\
&\lim_{y\to 0}(G\varphi)''(y) &&= G''(n), && \\
&\lim_{y\to 1}(G\varphi)(y) &&= \lim_{y\to 1}(G\varphi)'(y) &&= \lim_{y\to 1}(G\varphi)''(y) = 0,
\end{aligned}
$$

which gives $G\varphi \in C^2([0,1])$. Summing up and collecting terms we finally deduce

$$
|(G\varphi)(y)| + \left|(G\varphi)'(y)\right| + \left|(G\varphi)''(y)\right| \leq C'_{pol}(1+|y|^q), \quad y\in[0,1].
$$

□

In Part I we showed that the auxiliary generalised Itō formula

$$
\begin{aligned}
G_n(M(T)) = {} & G_n(M(1/n)) + \frac{\sigma^2}{2}\int_{1/n}^{T} G_n''(M(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} f(t,s)^2\mathrm{d}s\right)\mathrm{d}t \\
& + \sum_{1/n<t\leq T}\left[G_n(M(t)) - G_n(M(t-)) - G_n'(M(t-))\Delta M(t)\right] \\
& + \int_{-\infty}^{T}\int_{\mathbb{R}}\int_{1/n\vee s}^{T} G_n'\left(M(t)+xf(t,s)\right)\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \Lambda^{\diamond}(\mathrm{d}x,\mathrm{d}s) \\
& + \int_{-\infty}^{T}\int_{\mathbb{R}_0}\int_{1/n\vee s}^{T}\left(G_n'\left(M(t)+xf(t,s)\right) - G_n'(M(t))\right)x\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \nu(\mathrm{d}x)\ \mathrm{d}s \\
& + \int_{1/n}^{T} G_n'(M(t-))f(t,t)\ L(\mathrm{d}t)
\end{aligned}
\tag{5.30}
$$

holds for any $n \in \mathbb{N}$ such that $1/n \le T$. It remains to show that this formula also holds true in the limit as $n \to \infty$. For this purpose we need to interchange the limits and the integrals on the right-hand side of (5.30).

Let us first deal with the penultimate term on the right-hand side of (5.30). For this purpose we use the mean value theorem to find some

$$z_0 \in \big(M(t) \wedge (M(t) + xf(t,s)),\ M(t) \vee (M(t) + xf(t,s))\big)$$

such that

$$\begin{aligned}|G_n'(M(t) + xf(t,s)) - G_n'(M(t))| &= |G_n''(z_0)| \cdot |xf(t,s)| \\ &\le C_{pol}'(1 + |z_0|^q) \cdot |xf(t,s)| \\ &\le C_{pol}'(1 + |M(t)|^q + |M(t) + xf(t,s)|^q) \cdot |xf(t,s)|.\end{aligned} \tag{5.31}$$

Using the same technique and the fact $C_{pol}' \ge C_{pol}$ we also get the estimate

$$\begin{aligned}|G'(M(t) + xf(t,s)) - G'(M(t))| &\le C_{pol}(1 + |z_0|^q) \cdot |xf(t,s)| \\ &\le C_{pol}'(1 + |M(t)|^q + |M(t) + xf(t,s)|^q) \cdot |xf(t,s)|.\end{aligned} \tag{5.32}$$

In order to apply the dominated convergence theorem we introduce the abbreviation

$$D_{1,n}(x,t,s) := G'(M(t) + xf(t,s)) - G'(M(t)) - \big(G_n'(M(t) + xf(t,s)) - G_n'(M(t))\big)\,\mathbb{1}_{[1/n,T]}(t)$$

for $x \in \mathbb{R}_0$, $s \in \mathbb{R}$, and $t \in [0,T]$. By using (5.31), (5.32), and Jensen's inequality we get the following chain of estimates

$$\begin{aligned}
&\mathbb{E}\left(\left(\int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} \sup_{n\in\mathbb{N}} \left| D_{1,n}(x,t,s) x \frac{\partial}{\partial t} f(t,s)\right| \ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t\right)^2\right) \\
&\le \mathbb{E}\left(\left(\int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} 2C_{pol}'\,(1 + |M(t)|^q + |M(t) + xf(t,s)|^q) \cdot \left|x^2 f(t,s) \frac{\partial}{\partial t} f(t,s)\right| \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t\right)^2\right) \\
&\le \mathbb{E}\left(\left(\int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} K_1\,(1 + |M(t)|^q + |xf(t,s)|^q) \cdot \left|x^2 f(t,s) \frac{\partial}{\partial t} f(t,s)\right| \ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t\right)^2\right) \\
&\le K_1^2 \mathbb{E}\Bigg(\Bigg(\sup_{t\in[0,T]} (1 + |M(t)|^q) \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} \left|x^2 f(t,s) \frac{\partial}{\partial t} f(t,s)\right| \ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t \\
&\qquad\qquad + \int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} |xf(t,s)|^q \cdot \left|x^2 f(t,s) \frac{\partial}{\partial t} f(t,s)\right| \ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t\Bigg)^2\Bigg) \\
&\le 2K_1^2 \mathbb{E}\left(\left(1 + \sup_{t\in[0,T]} |M(t)|^q\right)^2\right) \left(\int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} x^2 \left|f(t,s) \frac{\partial}{\partial t} f(t,s)\right| \ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t\right)^2 \\
&\qquad\qquad + 2K_1^2 \left(\int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} |x|^{q+2} |f(t,s)|^{q+1} \cdot \left|\frac{\partial}{\partial t} f(t,s)\right| \ \nu(\mathrm{d}x)\ \mathrm{d}s\ \mathrm{d}t\right)^2
\end{aligned} \tag{5.33}$$

for a suitable constant $K_1 > 0$.

We now want to apply Lemma 4.6(i) to prove that the right-hand side of (5.33) is finite. For that purpose we define for any $t \in [0,T]$, $s \le t$, and $q \ge 0$

$$F_q(t,s) := \int_{\mathbb{R}_0} |x|^{q+2} |f(t,s)|^{q+1} \cdot \left| \frac{\partial}{\partial t} f(t,s) \right| \, \nu(\mathrm{d}x).$$

Estimate (3.2) then yields that for every $t \in [0,T]$ there is an $\epsilon \in (0, {}^{t+1}/_{2})$ such that

$$|F_q(r,s)| \le C_0 \int_{\mathbb{R}_0} |x|^{q+2} \, \nu(\mathrm{d}x) \cdot |f(r,s)|^{q+1} \cdot |s|^{-\beta} \, |r-s|^{-\gamma}$$

for $r \in [(t-\epsilon) \vee 0, \, (t+\epsilon) \wedge T]$ and $s < r$. In particular, (4.8) is fulfilled with

$$\tilde{C} := C_0 \sup_{(r,v) \in [0,T] \times [0,T]} |f(r,v)|^{q+1} \int_{\mathbb{R}_0} |x|^{q+2} \, \nu(\mathrm{d}x).$$

By means of Definition 3.1(iii) and (v), for all $s \in \mathbb{R}$ the mapping $t \mapsto F_q(t,s)$ is continuous on (s, ∞). Moreover, for any fixed $t \in [0,T]$ as well as $C_0, \beta, \gamma, \theta$, and $\epsilon \in (0, {}^{(t+1)}/_{2})$ such that (3.2) and (3.3) hold, we infer in view of Definition 3.1(iii) that

$$\begin{aligned}
&\int_{-\infty}^{t-2\epsilon} \sup_{r \in [0 \vee (t-\epsilon), \, (t+\epsilon) \wedge T]} |F_q(r,s)| \, \mathrm{d}s \\
&\le C_0 \int_{\mathbb{R}_0} |x|^{q+2} \, \nu(\mathrm{d}x) \left(\sup_{u \in (-\infty,-1]} \sup_{r \in [0 \vee (t-\epsilon), \, (t+\epsilon) \wedge T]} \Big(|f(r,u)| |u|^{\theta} \Big)^{q+1} \right) \\
&\qquad\qquad\qquad\qquad \cdot \int_{-\infty}^{-1} |s|^{-((q+1)\theta+\beta)} |t-s|^{-\gamma} \, \mathrm{d}s \\
&\qquad + C_0 \int_{\mathbb{R}_0} |x|^{q+2} \, \nu(\mathrm{d}x) \cdot \sup_{u \in [-1,t]} \sup_{r \in [0 \vee (t-\epsilon), \, (t+\epsilon) \wedge T]} |f(r,u)|^{q+1} \int_{-1}^{t-2\epsilon} |s|^{-\beta} |t-s|^{-\gamma} \, \mathrm{d}s \\
&< \infty,
\end{aligned}$$

and hence (4.7) holds. Note that here we have used the assumptions on the moments of the Lévy process L.

In view of the continuity of f and $\frac{\partial}{\partial t} f(\cdot, s)$ for Lebesgue-a.e. $s \in \mathbb{R}$, condition (i)a) in Lemma 4.6 is satisfied and consequently Lemma 4.6 implies that for every $q \ge 0$ the mapping

$$t \mapsto \int_{-\infty}^{t} F_q(t,s) \, \mathrm{d}s$$

is continuous and thus

$$\int_0^T \int_{-\infty}^{t} F_q(t,s) \, \mathrm{d}s \, \mathrm{d}t \le T \sup_{t \in [0,T]} \int_{-\infty}^{t} F_q(t,s) \, \mathrm{d}s < \infty.$$

Combining this with Lemma 3.2 and (5.33) we see that

$$\mathbb{E}\left(\left(\int_0^T \int_{-\infty}^t \int_{\mathbb{R}_0} \sup_{n\in\mathbb{N}} \left| D_{1,n}(x,t,s) x \frac{\partial}{\partial t} f(t,s) \right| \ \nu(\mathrm{d}x) \ \mathrm{d}s \ \mathrm{d}t \right)^2\right) < \infty. \tag{5.34}$$

Furthermore, observe that

$$|D_{1,n}(x,t,s)| \to 0$$

$\mathbb{P}$-a.s. for every $x \in \mathbb{R}_0$, $t \in (0,T]$, and $s \in \mathbb{R}$ as $n \to \infty$. In the light of (5.34) we conclude by means of Fubini's theorem and the dominated convergence theorem

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-\infty}^T \int_{\mathbb{R}_0} \int_{1/n \vee s}^T \left(G_n'\left(M(t) + xf(t,s)\right) - G_n'(M(t))\right) x \frac{\partial}{\partial t} f(t,s) \ \mathrm{d}t \ \nu(\mathrm{d}x) \ \mathrm{d}s \\
&\quad = \lim_{n\to\infty} \int_{1/n}^T \int_{-\infty}^t \int_{\mathbb{R}_0} \left(G_n'\left(M(t) + xf(t,s)\right) - G_n'(M(t))\right) x \frac{\partial}{\partial t} f(t,s) \ \nu(\mathrm{d}x) \ \mathrm{d}s \ \mathrm{d}t \\
&\quad = \int_{-\infty}^T \int_{\mathbb{R}_0} \int_{0 \vee s}^T \left(G'\left(M(t) + xf(t,s)\right) - G'(M(t))\right) x \frac{\partial}{\partial t} f(t,s) \ \mathrm{d}t \ \nu(\mathrm{d}x) \ \mathrm{d}s
\end{aligned} \tag{5.35}$$

in $\mathscr{L}^2(\mathbb{P})$.

To handle the third term of the right-hand side of (5.30) we use Lemma 3.2 and the shorthands

$$\delta G_n(x,t) := G(x) - G_n(x)\mathbb{1}_{(1/n,T]}(t) \text{ and } \delta G_n'(x,t) := G'(x) - G_n'(x)\mathbb{1}_{(1/n,T]}(t)$$

to write

$$\begin{aligned}
&\mathbb{E}\left(\left(\sum_{0<t\leq T} \left[\delta G_n(M(t),t) - \delta G_n(M(t-),t) - \delta G_n'(M(t-),t)\Delta M(t)\right]\right)^2\right) \\
&= \mathbb{E}\left(\left(\sum_{0<t\leq T} \left[\delta G_n(M(t-) + \Delta L(t) f(t,t), t) - \delta G_n(M(t-),t) - \delta G_n'(M(t-),t)\Delta L(t) f(t,t)\right]\right)^2\right) \\
&= \mathbb{E}\left(\left(\int_0^T \int_{\mathbb{R}_0} \left[\delta G_n(M(t-) + xf(t,t),t) - \delta G_n(M(t-),t)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - \delta G_n'(M(t-),t) x f(t,t)\right] \ N(\mathrm{d}x,\mathrm{d}t)\right)^2\right)
\end{aligned} \tag{5.36}$$

After introducing the abbreviation

$$D_{2,n}(x,t) := \delta G_n(M(t-) + xf(t,t),t) - \delta G_n(M(t-),t) - \delta G_n'(M(t-),t) x f(t,t) \tag{5.37}$$

for $x \in \mathbb{R}_0$ and $t \in [0,T]$ we continue in view of the Itō isometry for Lévy processes and Jensen's inequality

$$
\begin{aligned}
&\mathbb{E}\left(\left(\int_0^T\int_{\mathbb{R}_0} D_{2,n}(x,t)\ N(\mathrm{d}x,\mathrm{d}t)\right)^2\right)\\
&\quad\leq 2\mathbb{E}\left(\left(\int_0^T\int_{\mathbb{R}_0} D_{2,n}(x,t)\ \tilde{N}(\mathrm{d}x,\mathrm{d}t)\right)^2\right)+2\mathbb{E}\left(\left(\int_0^T\int_{\mathbb{R}_0} D_{2,n}(x,t)\ \nu(\mathrm{d}x)\ \mathrm{d}t\right)^2\right) \qquad (5.38)\\
&\quad\leq 2\int_0^T\int_{\mathbb{R}_0}\mathbb{E}\left(D_{2,n}(x,t)^2\right)\ \nu(\mathrm{d}x)\ \mathrm{d}t+2\mathbb{E}\left(\left(\int_0^T\int_{\mathbb{R}_0} |D_{2,n}(x,t)|\ \nu(\mathrm{d}x)\ \mathrm{d}t\right)^2\right).
\end{aligned}
$$

In order to apply the dominated convergence theorem we thus consider the two expressions

$$
\mathrm{I} := \int_0^T\int_{\mathbb{R}_0}\mathbb{E}\left(\sup_{n\in\mathbb{N}} D_{2,n}(x,t)^2\right)\ \nu(\mathrm{d}x)\ \mathrm{d}t
$$

and

$$
\mathrm{II} := \mathbb{E}\left(\left(\int_0^T\int_{\mathbb{R}_0}\sup_{n\in\mathbb{N}}|D_{2,n}(x,t)|\ \nu(\mathrm{d}x)\ \mathrm{d}t\right)^2\right).
$$

Note that by making use of the mean value theorem we can find

$$
z_1, z_{1,n} \in \big(M(t-)\wedge(M(t-)+xf(t,t)), M(t-)\vee(M(t-)+xf(t,t))\big)
$$

such that

$$
G(M(t-)+xf(t,t)) - G(M(t-)) = G'(z_1)xf(t,t)
$$

and

$$
G_n(M(t-)+xf(t,t)) - G_n(M(t-)) = G_n'(z_{1,n})xf(t,t).
$$

Plugging these expressions into (5.37), using the polynomial bound of G and G_n as well as Jensen's inequality yields

$$
\begin{aligned}
\sup_{n\in\mathbb{N}}|D_{2,n}(x,t)| \leq{}& \left|G'(z_1)xf(t,t)-G'(M(t-))xf(t,t)\right|\\
&+\sup_{n\in\mathbb{N}}\left|G_n'(z_{1,n})xf(t,t)-G_n'(M(t-))xf(t,t)\right| \qquad (5.39)\\
\leq{}& K_2|xf(t,t)|\left(1+|M(t-)|^q+|xf(t,t)|^q\right)
\end{aligned}
$$

for some constant $K_2>0$. In view of this estimate we continue by applying Jensen's inequality

$$
\begin{aligned}
\mathrm{I} &= \int_0^T \int_{\mathbb{R}_0} \mathbb{E}\left(\sup_{n\in\mathbb{N}} D_{2,n}(x,t)^2\right) \ \nu(\mathrm{d}x)\ \mathrm{d}t \\
&\leq K_2^2 \int_0^T \int_{\mathbb{R}_0} x^2 f(t,t)^2 \mathbb{E}\left((1+|M(t)|^q + |xf(t,t)|^q)^2\right) \ \nu(\mathrm{d}x)\ \mathrm{d}t \\
&\leq K_3 \int_0^T \int_{\mathbb{R}_0} x^2 f(t,t)^2 \mathbb{E}\left(1+\sup_{r\in(0,T]} |M(r)|^{2q} + |xf(t,t)|^{2q}\right) \ \nu(\mathrm{d}x)\ \mathrm{d}t \\
&< \infty
\end{aligned}
$$

for a constant $K_3 > 0$ by using the estimate (5.39), where the finiteness follows again from the moment assumptions on L.

To handle II we make a Taylor expansion of first order of $G(M(t-)+xf(t,t))$ at $M(t-)$ to find some

$$
z_2, z_{2,n} \in \big(M(t-) \wedge (M(t-)+xf(t,t)), M(t-) \vee (M(t-)+xf(t,t))\big)
$$

such that

$$
G(M(t-)+xf(t,t)) - G(M(t-)) - G'(M(t-))xf(t,t) = \frac{1}{2}G''(z_2)x^2 f(t,t)^2
$$

and

$$
G_n(M(t-)+xf(t,t)) - G_n(M(t-)) - G_n'(M(t-))xf(t,t) = \frac{1}{2}G_n''(z_{2,n})x^2 f(t,t)^2.
$$

Plugging these into (5.37), using the polynomial bound of G'' and G_n'' as well as again Jensen's inequality results in

$$
\sup_{n\in\mathbb{N}} |D_{2,n}(x,t)| \leq K_4 x^2 f(t,t)^2 \left(1+|M(t-)|^q + |xf(t,t)|^q\right)
$$

for some constant $K_4 > 0$. Using this estimate leads to

$$
\begin{aligned}
\mathrm{II} &\leq \mathbb{E}\left(\left(K_4 \int_0^T \int_{\mathbb{R}_0} x^2 f(t,t)^2 \left(1+|M(t-)|^q + |xf(t,t)|^q\right) \ \nu(\mathrm{d}x)\ \mathrm{d}t\right)^2\right) \\
&\leq 4K_4 \mathbb{E}\left(1+\sup_{t\in[0,T]} M(t)^{2q}\right)\left(\int_0^T f(t,t)^2\ \mathrm{d}t \int_{\mathbb{R}_0} x^2\ \nu(\mathrm{d}x)\right)^2 \\
&\quad + 2K_4 \left(\int_0^T |f(t,t)|^{q+2}\ \mathrm{d}t \int_{\mathbb{R}_0} |x|^{q+2}\ \nu(\mathrm{d}x)\right)^2 \\
&< \infty.
\end{aligned}
$$

Since

$$
|D_{2,n}(x,t)| \to 0
$$

$\mathbb{P}$-a.s. for every $x \in \mathbb{R}_0$ and $t \in (0,T]$ as $n \to \infty$ we see by means of the dominated convergence theorem that the right-hand side of (5.38) converges to 0 and according to (5.36) we conclude

$$\begin{aligned}\lim_{n\to\infty} \sum_{1/n<t\leq T} &\big[G_n(M(t)) - G_n(M(t-)) - G_n'(M(t-))\Delta M(t)\big] \\ &= \sum_{0<t\leq T} \big[G(M(t)) - G(M(t-)) - G'(M(t-))\Delta M(t)\big]\end{aligned}$$

in $\mathscr{L}^2(\mathbb{P})$. Similar but simpler arguments lead to

$$\lim_{n\to\infty} \frac{\sigma^2}{2} \int_{1/n}^T G_n''(M(t)) \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^t f(t,s)^2 \mathrm{d}s\right) \mathrm{d}t = \frac{\sigma^2}{2} \int_0^T G''(M(t)) \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^t f(t,s)^2 \mathrm{d}s\right) \mathrm{d}t,$$

$$\lim_{n\to\infty} \int_{1/n}^T G_n'(M(t-)) f(t,t)\ L(\mathrm{d}t) = \int_0^T G'(M(t-)) f(t,t)\ L(\mathrm{d}t),$$

$$\lim_{n\to\infty} G_n(M(T)) = G(M(T)),$$

as well as

$$\lim_{n\to\infty} G_n(M(1/n)) = G(0)$$

in $\mathscr{L}^2(\mathbb{P})$.

It remains to consider the $\Lambda^\diamond$-integral. Note in view of (4.4) and the convergence in $\mathscr{L}^2(\mathbb{P})$ shown above that the right-hand side of (5.26) (with $a = 1/n$ and G replaced by G_n) converges to the right-hand side of (5.26) (with $a = 0$) as $n \to \infty$. By applying arguments as in (5.33) and (5.35) we can deduce that

$$\begin{aligned}\lim_{n\to\infty} S &\left(\int_{-\infty}^T \int_{\mathbb{R}} \int_{1/n\vee s}^T \frac{\partial}{\partial t} f(t,s) G_n'\left(M(t) + x f(t,s)\right)\ \mathrm{d}t\ \Lambda^\diamond(\mathrm{d}x, \mathrm{d}s)\right)(g) \\ &= S\left(\int_{-\infty}^T \int_{\mathbb{R}} \int_{0\vee s}^T \frac{\partial}{\partial t} f(t,s) G'(M(t) + x f(t,s))\ \mathrm{d}t\ \Lambda^\diamond(\mathrm{d}x, \mathrm{d}s)\right)(g),\end{aligned}$$

which in view of the linearity of the S-transform completes the proof of (5.1). $\square$

Under the additional assumption that all the occurring terms exist as elements of $\mathscr{L}^2(\mathbb{P})$ we can prove the following variant of Theorem 5.1:

Theorem 5.9 *Let $f \in \mathcal{K}$ and $G \in C^2(\mathbb{R})$. Additionally, assume that one of the following assumptions is fulfilled:*

(i) $\sigma > 0$, G, G', and G'' are of polynomial growth with degree $q \geq 0$, that is

$$|G^{(l)}(x)| \leq C_{pol}(1 + |x|^q) \quad \textit{for every } x \in \mathbb{R} \textit{ and } l = 0,1,2$$

with a constant $C_{pol} > 0$, and

$$L(1) \in \mathscr{L}^{2q+2}(\mathbb{P}).$$

(ii) $G, G', G'' \in \mathcal{A}(\mathbb{R})$ *and* $L(1) \in \mathscr{L}^2(\mathbb{P})$*.*

Then the following generalised Itō formula

$$\begin{aligned} G(M(T)) = G(0) &+ \frac{\sigma^2}{2}\int_0^T G''(M(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t f(t,s)^2 \mathrm{d}s\right)\mathrm{d}t \\ &+ \sum_{0<t\leq T}\left[G(M(t)) - G(M(t-)) - G'(M(t-))\Delta M(t)\right] \\ &+ \int_{-\infty}^T\int_{\mathbb{R}_0}\int_{0\vee s}^T \left(G'\left(M(t)+xf(t,s)\right) - G'(M(t))\right) x\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ N^{\diamond}(\mathrm{d}x,\mathrm{d}s) \\ &+ \int_0^T G'(M(t))\ M^{\diamond}(\mathrm{d}t), \end{aligned} \tag{5.40}$$

holds $\mathbb{P}$*-almost surely, provided all terms there exist in* $\mathscr{L}^2(\mathbb{P})$*.*

Proof In order to show the variant (5.40) of our generalised Itō formula, we assume that all terms occurring in (5.40) exist in $\mathscr{L}^2(\mathbb{P})$. Since the first three summands of the right-hand sides of our generalised Itō formulas coincide, we only have to show that the sum of the last two summands in (5.40) equals the sum of the last three summands in (5.1).

For that purpose, we apply the S-transform to the last two summands in (5.40) and use Definition 4.12 and Theorem 4.11 to deduce for $g \in \Xi$

$$\begin{aligned} &S\left(\int_{-\infty}^T\int_{\mathbb{R}_0}\int_{0\vee s}^T \left(G'\left(M(t)+xf(t,s)\right) - G'(M(t))\right) x\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ N^{\diamond}(\mathrm{d}x,\mathrm{d}s)\right)(g) \\ &\quad + S\left(\int_0^T G'(M(t))\ M^{\diamond}(\mathrm{d}t)\right)(g) \\ &= \int_{-\infty}^T\int_{\mathbb{R}_0}\int_{0\vee s}^T S\left(G'\left(M(t)+xf(t,s)\right) - G'(M(t))\right)(g)\ x\frac{\partial}{\partial t}f(t,s)\ (1+g^*(x,s))\ \mathrm{d}t\ \nu\mathrm{d}x\ \mathrm{d}s \\ &\quad + S\left(\int_0^T G'(M(t-))f(t,t)\ L(\mathrm{d}t)\right)(g) + S\left(\int_{-\infty}^T\int_{\mathbb{R}}\int_{0\vee s}^T G'(M(t))\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \Lambda^{\diamond}(\mathrm{d}x,\mathrm{d}s)\right)(g) \\ &= \int_{-\infty}^T\int_{\mathbb{R}_0}\int_{0\vee s}^T S\left(G'\left(M(t)+xf(t,s)\right) - G'(M(t))\right)(g)\ x\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \nu\mathrm{d}x\ \mathrm{d}s \\ &\quad + \int_{-\infty}^T\int_{\mathbb{R}_0}\int_{0\vee s}^T S\left(G'\left(M(t)+xf(t,s)\right) - G'(M(t))\right)(g)\ x\frac{\partial}{\partial t}f(t,s)\ g^*(x,s)\ \mathrm{d}t\ \nu\mathrm{d}x\ \mathrm{d}s \\ &\quad + S\left(\int_0^T G'(M(t-))f(t,t)\ L(\mathrm{d}t)\right)(g) + S\left(\int_{-\infty}^T\int_{\mathbb{R}}\int_{0\vee s}^T G'(M(t))\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \Lambda^{\diamond}(\mathrm{d}x,\mathrm{d}s)\right)(g). \end{aligned} \tag{5.41}$$

Using the definition of the $\Lambda^{\diamond}$-integral (Definition 4.9) we see that the second and the fourth expression on the right-hand side of (5.41) can be combined as follows:

$$
\begin{aligned}
&\int_{-\infty}^{T}\int_{\mathbb{R}_0}\int_{0\vee s}^{T} S\left(G'\left(M(t)+xf(t,s)\right)-G'(M(t))\right)(g)\ x\frac{\partial}{\partial t}f(t,s)\ g^*(x,s)\ \mathrm{d}t\ \nu\mathrm{d}x\ \mathrm{d}s\\
&\quad+S\left(\int_{-\infty}^{T}\int_{\mathbb{R}}\int_{0\vee s}^{T} G'(M(t))\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \Lambda^{\diamond}(\mathrm{d}x,\mathrm{d}s)\right)(g)\\
&=\int_{-\infty}^{T}\int_{\mathbb{R}_0}\int_{0\vee s}^{T} S\left(G'\left(M(t)+xf(t,s)\right)-G'(M(t))\right)(g)\ x\frac{\partial}{\partial t}f(t,s)\ g^*(x,s)\ \mathrm{d}t\ \nu\mathrm{d}x\ \mathrm{d}s\\
&\quad+\int_{-\infty}^{T}\int_{\mathbb{R}_0}\int_{0\vee s}^{T} S\left(G'(M(t))\right)(g)\ x\frac{\partial}{\partial t}f(t,s)\ g^*(x,s)\ \mathrm{d}t\ \nu(\mathrm{d}x)\ \mathrm{d}s\\
&\quad+\sigma\int_{-\infty}^{T}\int_{0\vee s}^{T} S\left(G'(M(t))\right)(g)\frac{\partial}{\partial t}f(t,s)\ g(0,s)\ \mathrm{d}t\ \mathrm{d}s\\
&=\int_{-\infty}^{T}\int_{\mathbb{R}_0}\int_{0\vee s}^{T} S\left(G'\left(M(t)+xf(t,s)\right)\right)(g)\ x\frac{\partial}{\partial t}f(t,s)\ g^*(x,s)\ \mathrm{d}t\ \nu\mathrm{d}x\ \mathrm{d}s\\
&\quad+\sigma\int_{-\infty}^{T}\int_{0\vee s}^{T} S\left(G'(M(t))\right)(g)\frac{\partial}{\partial t}f(t,s)\ g(0,s)\ \mathrm{d}t\ \mathrm{d}s\\
&=S\left(\int_{-\infty}^{T}\int_{\mathbb{R}}\int_{0\vee s}^{T} G'\left(M(t)+xf(t,s)\right)\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \Lambda^{\diamond}(\mathrm{d}x,\mathrm{d}s)\right)(g).
\end{aligned}
$$

Plugging this into (5.41) we arrive at

$$
\begin{aligned}
&S\left(\int_{-\infty}^{T}\int_{\mathbb{R}_0}\int_{0\vee s}^{T}\left(G'\left(M(t)+xf(t,s)\right)-G'(M(t))\right)x\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ N^{\diamond}(\mathrm{d}x,\mathrm{d}s)\right)(g)\\
&\quad+S\left(\int_{0}^{T} G'(M(t))\ M^{\diamond}(\mathrm{d}t)\right)(g)\\
&=S\left(\int_{-\infty}^{T}\int_{\mathbb{R}}\int_{0\vee s}^{T} G'\left(M(t)+xf(t,s)\right)\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \Lambda^{\diamond}(\mathrm{d}x,\mathrm{d}s)\right)(g)\\
&\quad+S\left(\int_{-\infty}^{T}\int_{\mathbb{R}_0}\int_{0\vee s}^{T} G'\left(M(t)+xf(t,s)\right)-G'(M(t))\ x\frac{\partial}{\partial t}f(t,s)\ \mathrm{d}t\ \nu\mathrm{d}x\ \mathrm{d}s\right)(g)\\
&\quad+S\left(\int_{0}^{T} G'(M(t-))f(t,t)\ L(\mathrm{d}t)\right)(g).
\end{aligned}
$$

Finally, using the injectivity of the S-transform (cf. Proposition 4.3) we see that both variants of our generalised Itō formula coincide $\mathbb{P}$-almost surely. □

Bibliography

[1] Elisa Alòs, Olivier Mazet, and David Nualart. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, 29(2):766–801, 2001.

[2] David Applebaum. *Lévy processes and stochastic calculus. 2nd ed.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2009.

[3] Ole E. Barndorff-Nielsen, Fred Espen Benth, Jan Pedersen, and Almut E. D. Veraart. On stochastic integration for volatility modulated Lévy-driven Volterra processes. *Stochastic Process. Appl.*, 124(1):812–847, 2014.

[4] Ole E. Barndorff-Nielsen, Fred Espen Benth, and Benedykt Szozda. On stochastic integration for volatility modulated Brownian-driven Volterra processes via white noise analysis. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 17(2):1450011, 28pp, 2014.

[5] Ole E. Barndorff-Nielsen, Fred Espen Benth, and Almut E. D. Veraart. Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. *Bernoulli*, 19(3):803–845, 2013.

[6] Albert Benassi, Serge Cohen, and Jacques Istas. On roughness indices for fractional fields. *Bernoulli*, 10(2):357–373, 2004.

[7] Christian Bender. An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. *Stochastic Process. Appl.*, 104(1):81–106, 2003.

[8] Christian Bender. An S-transform approach to integration with respect to a fractional Brownian motion. *Bernoulli*, 9(6):955–983, 2003.

[9] Christian Bender, Robert Knobloch, and Philip Oberacker. A generalised Itō formula for Lévy-driven Volterra processes. *Stochastic Process. Appl.*, 125:2989–3022, 2015.

[10] Christian Bender, Robert Knobloch, and Philip Oberacker. Maximal inequalities for fractional Lévy and related processes . *Stoch. Anal. Appl.*, 33(4):701–714, 2015.

[11] Christian Bender and Tina Marquardt. Stochastic calculus for convoluted Lévy processes. *Bernoulli*, 14(2):499–518, 2008.

[12] Fred Espen Benth and André Süß. Integration Theory for infinite dimensional volatility modulated Volterra processes. *To appear in Bernoulli*, 2015.

[13] Jean Bertoin. *Lévy processes.* Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1996.

[14] Francesca Biagini, Bernt Øksendal, Agnès Sulem, and Naomi Wallner. An introduction to white-noise theory and Malliavin calculus for fractional Brownian motion. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460(2041):347–372, 2004. Stochastic analysis with applications to mathematical finance.

[15] Jaya P. N. Bishwal. Maximum quasi-likelihood estimation in fractional Levy stochastic volatility model. *J. Math. Finance*, 1(3):58–62, 2011.

[16] Rama Cont and Peter Tankov. *Financial modelling with jump processes.* Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[17] Laurent Decreusefond and Nicolas Savy. Anticipative calculus with respect to filtered Poisson processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(3):343–372, 2006.

[18] Giulia Di Nunno, Bernt Øksendal, and Frank Proske. *Malliavin calculus for Lévy processes with applications to finance.* Universitext. Springer-Verlag, Berlin, 2009.

[19] Sebastian Engelke and Jeannette H. C. Woerner. A unifying approach to fractional Lévy processes. *Stoch. Dyn.*, 13(2):1250017, 19 pp, 2013.

[20] Holger Fink. Conditional characteristic functions of Molchan-Golosov fractional Lévy processes with application to credit risk. *J. Appl. Probab.*, 50(4):983–1005, 2013.

[21] Holger Fink and Claudia Klüppelberg. Fractional Lévy-driven Ornstein-Uhlenbeck processes and stochastic differential equations. *Bernoulli*, 17(1):484–506, 2011.

[22] Gerald B. Folland. *Real analysis.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999.

[23] Yaozhong Hu and Bernt Øksendal. Fractional white noise calculus and applications to finance. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 6(1):1–32, 2003.

[24] Claudia Klüppelberg and Muneya Matsui. Generalized fractional Lévy processes with fractional Brownian motion limit. *to appear in Adv. Appl. Prob.*, 47, 2015.

[25] Hui-Hsiung Kuo. *White noise distribution theory.* Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996.

[26] Stanisław Kwapień, Michael B. Marcus, and Jan Rosiński. Two results on continuity and boundedness of stochastic convolutions. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(5):553–566, 2006.

[27] Andreas E. Kyprianou. *Fluctuations of Lévy processes with applications. Introductory lectures. 2nd ed.* Universitext. Springer-Verlag, Berlin, 2014.

[28] Yuh-Jia Lee and Hsin-Hung Shih. The Segal-Bargmann transform for Lévy functionals. *J. Funct. Anal.*, 168(1):46–83, 1999.

[29] Yuh-Jia Lee and Hsin-Hung Shih. Donsker's delta function of Lévy process. *Acta Appl. Math.*, 63(1-3):219–231, 2000.

[30] Yuh-Jia Lee and Hsin-Hung Shih. Itô formula for generalized Lévy functionals. In *Quantum information II*, pages 87–105. World Sci. Publ., River Edge, NJ, 2000.

[31] Yuh-Jia Lee and Hsin-Hung Shih. Analysis of generalized Lévy white noise functionals. *J. Funct. Anal.*, 211(1):1–70, 2004.

[32] Xuebin Lü and Wanyang Dai. Stochastic integration for fractional Levy process and stochastic differential equation driven by fractional Levy noise. *Acta Math. Sci. Ser. A Chin. Ed.*, , A33(6):1022–1034, 2013, (Englisch version: arXiv:1307.4173).

[33] Tina Marquardt. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli*, 12(6):1099–1126, 2006.

[34] Yuliya S. Mishura. *Stochastic calculus for fractional Brownian motion and related processes*, volume 1929 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.

[35] Leonid Mytnik and Eyal Neuman. Sample path properties of Volterra processes. *Commun. Stoch. Anal.*, 6(3):359–377, 2012.

[36] Philip E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2005. Second edition. Version 2.1.

[37] Balram S. Rajput and Jan Rosiński. Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields*, 82(3):451–487, 1989.

[38] Gennady Samorodnitsky and Murad S. Taqqu. *Stable non-Gaussian random processes. Stochastic models with infinite variance*. Stochastic Modeling. Chapman & Hall, New York, 1994.

[39] Hsin-Hung Shih. The Segal-Bargmann transform for Lévy white noise functionals associated with non-integrable Lévy processes. *J. Funct. Anal.*, 255(3):657–680, 2008.

[40] Albert N. Shiryaev. *Essentials of stochastic finance*, volume 3 of *Advanced Series on Statistical Science & Applied Probability*. World Scientific Publishing Co., Inc., River Edge, NJ, 1999. Facts, models, theory, Translated from the Russian manuscript by N. Kruzhilin.

[41] Heikki Tikanmäki and Yuliya Mishura. Fractional Lévy processes as a result of compact interval integral transformation. *Stoch. Anal. Appl.*, 29(6):1081–1101, 2011.

[42] Michael Unser, Pouya D. Tafti, and Qiyu Sun. A unified formulation of Gaussian versus sparse stochastic processes - part I: continuous-domain theory. *IEEE Trans. Inform. Theory*, 60(3):1945–1962, 2014.

[43] Robert L. Wolpert and Murad S. Taqqu. Fractional Ornstein-Uhlenbeck Lévy processes and the Telecom process: Upstairs and downstairs. *Signal Processing*, 85(8):1523–1545, 2005.