The Unirationality of Hurwitz Spaces of Hexagonal Curves of Small Genus

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Saarbrücken, Januar 2013

(Florian Geiß)

Meinen Eltern.

Abstract

The main subject of this thesis is the geometry of the Hurwitz space $\mathscr{H}_{g,k}$. We give a computer-aided proof of the unirationality of $\mathscr{H}_{g,k}$ in the cases $k = 6$ and $5 \leq g \leq 31$ or $g = 34, 35, 36, 39, 40, 45$ and $k = 7$ and $6 \leq g \leq 12$. We show along examples for small values of g that our approach also covers the classically known cases of the unirationality of $\mathscr{H}_{g,k}$ for $k \leq 5$. As an immediate conclusion from our result we obtain the unirationality of the Severi variety $\mathscr{V}_{g,d}$ for $g \leq 13$ and $d = \lceil 2/3g + 2 \rceil$.
We consider two applications of this result for $k = 6$. First we study hexagonal curves with a view towards the theory of Gorenstein ideals of codimension 4. In the cases covered by the unirationality construction we consider the general hexagonal canonical curve as subvariety of the rational normal scroll of dimension 5 which is spanned by the special linear series and show that it has the expected syzygies.
In the second application we utilize the unirationality construction for the case $g = 10$ to prove the existence of stable Ulrich bundles of rank 3 on a general cubic hypersurface in $\mathbf{P}^4$. In this context we give a computer-aided proof of the vanishing of cohomology groups of certain extensions.

Zusammenfassung

Die vorliegende Arbeit befasst sich mit der Geometrie des Hurwitzraumes $\mathscr{H}_{g,k}$. Wir geben einen computergestützten Beweis der Unirationalität von $\mathscr{H}_{g,k}$ in den Fällen $k = 6$ und $5 \leq g \leq 31$ oder $g = 34, 35, 36, 39, 40, 45$ sowie $k = 7$ und $6 \leq g \leq 12$ und zeigen exemplarisch an kleinen Werten für g, dass unser Ansatz auch die klassischen Fälle der Unirationalität von $\mathscr{H}_{g,k}$ für $k \leq 5$ abdeckt. Als unmittelbare Folgerung unseres Ergebnisses erhalten wir die Unirationalität der Severi-Varietät $\mathscr{V}_{g,d}$ für $g \leq 13$ und $d = \lceil 2/3g + 2 \rceil$.
Wir betrachten zwei Anwendungen des Resultates für $k = 6$. Zunächst untersuchen wir die hexagonalen Kurven im Hinblick auf die Theorie von Gorenstein Idealen von Kodimension 4. Wir zeigen in den durch die Konstruktion abgedeckten Fällen, dass die allgemeine hexagonale kanonische Kurve, aufgefasst als Untervarietät des 5-dimensionalen rationalen Scrolls, welcher von der speziellen Linearschar aufgespannt wird, die erwarteten Syzygien besitzt.
In der zweiten Anwendung nutzen wir die Unirationalitätskonstruktion für $g = 10$ zum zum Nachweis der Existenz stabiler Ulrichbündel von Rang 3 auf einer allgemeinen kubischen Hyperfläche in $\mathbf{P}^4$. Weiterhin geben wir in diesem Zusammenhang einen computergestützten Beweis zur Verschwindung von Kohomologiegruppen von bestimmten Extensionen.

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Florian Geiß — Saarbrücken, Januar 2013

Contents

1 Introduction and Outline of the Results

The Hurwitz Space

The main subject of this thesis is the study of the geometry of the Hurwitz space $\mathscr{H}_{g,k}$. This space parametrizes simply branched k-sheeted covers of the projective line $\mathbf{P}^1$ by smooth curves of genus g.

It was Riemann in his famous work [Rie57] who developed the idea to express curves in this way in order to count their number of moduli. Since then the Hurwitz spaces played a crucial role in the understanding of the moduli of curves.

A simply branched cover $f : C \to \mathbf{P}^1(\mathbf{C})$ has precisely $\omega = 2g + 2k + 2$ distinct branch points $Q_1, \ldots, Q_\omega$ which together with the monodromy action of the fundamental group $\pi_1(\mathbf{P}^1(C) \setminus \{Q_1, \ldots, Q_\omega\})$ on the sheets determine f uniquely. Hurwitz [Hur91] showed that the resulting finite map $\mathscr{H}_{g,k} \to \mathrm{Sym}_\omega(\mathbf{P}^1(\mathbf{C})) \setminus \Delta$ is an unramified covering. Together with a combinatorial study of the fibers by Clebsch [Cle72] this shows that $\mathscr{H}_{g,k}$ is a smooth and connected hence irreducible quasi-projective variety. This result was later generalized to any characteristic $p > g + 1$ by Fulton [Ful69].

The natural forgetful map $\pi : \mathscr{H}_{g,k} \to \mathscr{M}_g$ relates the geometry of the Hurwitz space to the one of the moduli space $\mathscr{M}_g$ of curves of genus g. A particularly interesting case is when this map is dominant which happens as soon as $k \geq (g + 2)/2$. An immediate consequence is that $\mathscr{M}_g$ is also irreducible and Riemanns count can be regarded as determining the dimension of $\mathscr{H}_{g,k}$ and the fiber dimension of π. On the other hand, for $k < \lceil (g + 2)/2 \rceil$, a general element in $\mathscr{H}_{g,k}$ admits exactly one special pencil, hence $\mathscr{H}_{g,k}$ maps birationally onto the locus $\mathscr{M}^1_{g,k}$ of k-gonal curves in $\mathscr{M}_g$ These subvarieties form a stratification of the moduli space, see [Far01]:

$$\mathscr{M}^1_{g,2} \subset \mathscr{M}^1_{g,3} \subset \ldots \subset \mathscr{M}_g.$$

Recall that a variety V unirational if there is a dominant rational map from some projective space $\mathbf{P}^N \dashrightarrow V$. It is desirable to have a unirational moduli space since this allows a parametrization of the objects in free parameters in terms of the rational map. A classical result due to B. Segre [Seg28] asserts that for $k \geq 3$ the general k-gonal curve of genus g can be realized as a plane curve of certain degree n with an $(n - k)$-fold point and δ ordinary double points and no other singularities. In [AC81] Arbarello and Cornalba proved that in the cases where the linear system $L(n; (n - k)p, 2q_1, \ldots, 2q_\delta)$ is not empty for a general choice of points p, $q_1, \ldots, q_\delta$ the general curve in this system is irreducible and has precisely the singularities as desired. With the well-known result on hyperelliptic curves understood they obtain in this way that

$$\mathscr{H}_{g,k} \text{ is unirational for } \begin{cases} k \leq 5 \text{ and } g \geq k-1, \\ k = 6 \text{ and } 5 \leq g \leq 10 \text{ or } g = 12, \\ k = 7 \text{ and } g = 7. \end{cases} \tag{1.1}$$

As a consequence, $\mathscr{M}_g$ is also unirational for $g \leq 10$. We want to mention that the unirationality of $\mathscr{M}_g$ was in stages extended with different methods by Sernesi, Chan and Ran and finally Verra to all cases $g \leq 14$.

For a projective algebraic variety V the Kodaira dimension $\kappa(V)$ is a birational invariant that measures "how rational" V is. It is defined to be the largest dimension of the image of a desingularization $\tilde{V}$ of V under the pluricanonical map $|nK_{\tilde{V}}|$ for $n \gg 0$. If V is a unirational variety then this implies $\kappa(V) = -\infty$. On the other extreme when $\kappa(V)$ equals the dimension of V which is the maximal possible value then V is said to be of general type and there is no rational curve through a general point of V.

In [HM82] Mumford and Harris constructed the space of admissible covers $\overline{\mathscr{H}}_{g,k}$ which is a modular compactification of the Hurwitz space and enjoys the property that there is a map $\overline{\pi} : \overline{\mathscr{H}}_{g,k} \to \overline{\mathscr{M}}_g$ extending the map π to the moduli space $\overline{\mathscr{M}}_g$ of stable curves of genus g. In a sequence of papers Mumford and Harris [HM82], Harris [Har84], Eisenbud and Harris [EH87] and Farkas [Far09] showed that $\overline{\mathscr{M}}_g$ is of general type for $g = 22$ or $g \geq 24$. This implies that

$$\overline{\mathscr{H}}_{g,k} \text{ is not unirational for } g = 22 \text{ or } g \geq 24 \text{ and } k \geq [(g+2)/2]\,. \tag{1.2}$$

In the light of these classical results it is an interesting question to determine in the range between (1.1) and (1.2) the birational type of $\mathscr{H}_{g,k}$, resp. $\mathscr{M}^1_{g,k}$, and moreover, to ask when this space is unirational.

The Main Result

The main result of this work is the following theorem.

Theorem 1.3. *$\mathscr{H}_{g,k}$ is unirational for*

(i) $k = 6$ *and* $5 \leq g \leq 31$ *or* $g = 33, 34, 35, 36, 39, 40, 45$*,*

(ii) $k = 7$ *and* $6 \leq g \leq 12$*.*

Our basic strategy is to consider a curve C in $\mathscr{H}_{g,k}$ under the embedding $C \to \mathbf{P}^1 \times \mathbf{P}^2$ which is given by the special pencil of degree k and a linear series $\mathfrak{g}^2_d$ of certain degree d and to study this algebro-geometric situation.

For most of the cases in (i) our parametrization is based on the observation that a general 6-gonal curve in $\mathbf{P}^1 \times \mathbf{P}^2$ can be linked in two steps to the union of a rational curve and a collection of lines. It turns out that for small genera this process can be reversed by starting with a general rational curve and general lines. The remaining cases are constructed in a different way. Under some reasonable assumptions the truncated vanishing ideal has a resolution of length 3 and the curve can be recovered from a module associated to this resolution. In a similar way we also regain, in principle, the classical results for curves of gonality $k = 3, 4, 5$.

In many unirationality proofs the laborious part is usually to show that the parametrization in focus is dominant. We simplify this step by the use of computer algebra. To show that the described constructions yield a dominant map to the Hurwitz space, we only need to carry out the construction for a single curve over a finite field. This computation is passed to *Macaulay2* [GS]. Semicontinuity then ensures that all assumptions we made actually hold for an open dense subset of $\mathscr{H}_{g,k}$ in characteristic zero.

The unirationality of the 7-gonal curves of genera 11 and 12 yields another proof of the unirationality of $\mathscr{M}_g$ for theses cases. Taking also into account the result by Chang and Ran [CR84] on the existence of a unirational component in the Hilbert scheme of spatial models of genus 13 curves which dominates $\mathscr{M}_{13}$ and using that the embedding line bundle is Brill-Noether dual to a plane model, we obtain the following statement.

Corollary 1.4. *The Severi-Variety $\mathscr{V}_{d,g}$ of plane irreducible nodal curves of genus g and degree $d = \lceil \frac{2}{3}g + 2 \rceil$ is unirational for $g \leq 13$.*

The main result is presented in the first two chapters of this thesis.

Unirational Subvarieties of $\mathscr{H}_{g,6}$.

There is an intrinsic way to express elements in $\mathscr{H}_{g,k}$ as Gorenstein ideals of codimension $k+1$ which arises as follows. For $k \geq 3$ and an element $f : C \to \mathbf{P}^1$ in $\mathscr{H}_{g,k}$, C not hyperelliptic, we consider the canonical embedding $C \subset \mathbf{P}^{g-1}$. The divisors in the pencil $H^0(f^*\mathscr{O}_{\mathbf{P}^1}(1))$ sweep out a rational normal scroll $X \subset \mathbf{P}^{g-1}$ containing the curve. The well understood structure theory for Gorenstein ideals of codimension ≤ 3 allows an alternate proof of the unirationality of $\mathscr{H}_{g,k}$ for $k \leq 5$, see [Sch86]. For example, in the case $k = 5$ the resolution of $\mathscr{I}_{C/X}$ is described by the Buchsbaum-Eisenbud complex [BE77]. The unirational parametrization is given by the free choice of the entries of a certain skew-symmetric matrix whose pfaffians form a set of generators of $\mathscr{I}_{C/X}$.

Although there is no general structure theory for Gorenstein Ideals of codimension ≥ 4, one is led to ask whether an analogous construction can be found at least for curves of higher gonality.

We consider for $k = 6$ the Gulliksen-Negard complex [GN72] which resolves certain Gorenstein ideals of codimension 4. The explicit description of hexagonal curves in computer algebra allows the study of the curves on the scroll in the canonical embedding. We obtain the following computational result

Theorem 1.5. *For g as in Theorem 1.3 (i), the general 6-gonal curve, considered as curve on the associated rational normal scroll, does not admit a resolution by a Gulliksen-Negard complex.*

Restricting to direct sums of line bundles, we obtain the following statement.

Theorem 1.6. *For $g = 2m + 1$, $m \geq 2$ there is a unirational subvariety of $V \subset \mathscr{H}_{g,6}$ of dimension $3m + 9$ with the property that the generic element, considered as curve on the associated rational normal scroll, is resolved by a Gulliksen-Negard complex.*

Stable Ulrich Bundles on Cubic Threefolds

The last chapter of this work focusses on ACM bundles. These are vector bundles with no intermediate cohomology groups. More precisely, we focus on such ACM bundles which have the maximal possible number of global sections, called Ulrich bundles. Horrocks theorem characterizes the ACM bundles on projective space as precisely those bundles that split as a direct sum of line bundles. Thus, it is interesting to ask for Ulrich bundles as they are – informally speaking – the ”simplest” vector bundles a projective variety can have.

There are numerous results on the existence of Ulrich sheaves (possibly of high rank) on certain classes of varieties, e.g. arbitrary curves, Veronese embeddings, del Pezzo surfaces and Segre products of varieties that admit Ulrich sheaves [ESW03], complete intersections [HUB91] and Fano varieties [PL10]. However, in general the following problem remains unsolved.

Problem 1.7. *Does every variety $X \subset \mathbf{P}^n$ have an Ulrich sheaf? If so, what is the smallest possible rank for such a sheaf?*

We want to mention that Eisenbud and Schreyer conjecture that the answer to the first part of this problem is affirmative. This is motivated by Boij-Soederberg theory as the existence of an Ulrich sheaf implies that the cone of cohomology tables of coherent sheaves of the variety equals the one of the projective space. The connection between Ulrich bundles and curves is established by the Serre correspondence. It is classically known that a general cubic threefold $X \subset \mathbf{P}^4$ contains an elliptic normal curve which yields the existence of rank 2 stable Ulrich bundles on X. We utilize our construction for $k = 6$ and $g = 10$ to prove the following theorem.

Theorem 1.8. *On the general cubic threefold $Y \subset \mathbf{P}^4$ there exists a stable Ulrich bundle of rank 3.*

Moreover, the unirationality enters in the proofs of a number of vanishing theorems for certain cohomology groups for curves on the general cubic threefold which are rather technical so we do not formulate them here. These results are a crucial ingredient to the proof of the following theorem.

Theorem 1.9 (Casanellas, G., Hartshorne, Schreyer)**.** *For any $r \geq 2$, the moduli space of stable rank r Ulrich bundles on a general cubic threefold Y in $\mathbf{P}^4$ is non-empty and smooth of dimension $r^2 + 1$. Furthermore, it has an open subset for which the restriction to a hyperplane section gives an unramified dominant map to the moduli of stable bundles on the cubic surface.*

Publications and Software Packages

The first two chapters are in parts based on the paper

- F. Geiß: *The Unirationality of Hurwitz Spaces of 6-gonal curves of small genus,* Doc. Math. 17 (2012), 627 – 661.

The last chapter is based on

- F. Geiß and F.-O. Schreyer: *ACM curves of small degree on cubic three folds,* Appendix to the paper M. Casanellas, R. Hartshorne: *Stable Ulrich bundles.* Int. J. Math. 23.

The software for *Macaulay2* developed in the course of this work will be part of the *randomCurves*-packages which implements various unirationality constructions for moduli spaces of curves. This package will be available through upcoming releases of *Macaulay2.*

- H.C. von Bothmer, F. Geiß and F.-O. Schreyer: *Random Curves. A collection of Macaulay2 packages for the construction of curves,* manuscript in preparation.

A current version of these packages is also available at [vBGS13].

Notation

If not otherwise mentioned, we work over an arbitrary field $\mathbf{k}$ of characteristic 0 with emphasis on the case of complex numbers. In our computational approaches we will also work over some finite fields which, by semicontinuity arguments, will lead to analogous results in characteristic 0.

2 Preliminaries

This chapter is devoted to the development and presentation of the theory required to formulate and prove our main results. Section 1 sketches the construction of the Hurwitz space, the main object in focus. In Section 2 we summarize important results from Brill-Noether theory on linear series on algebraic curves and develop a criterium for the irreducibility of certain plane curves. In Section 3 we turn to our setup in multiprojective space and consider multigraded free resolutions. In Section 4 we turn to the construction of modules over standard multigraded rings. Section 5 reviews the required theory of linkage in the setting of curves in $\mathbf{P}^1 \times \mathbf{P}^2$.

2.1 The Hurwitz Space: Construction and Properties.

Let us recall the construction of the Hurwitz space and its elementary properties. We follow [ACG11, § 21.11]. We will work in this section over the complex numbers.

Simply Branched Covers. Let C be a smooth curves of genus g. A k-sheeted covering

$$f : C \to \mathbf{P}^1$$

is called *simply branched* if for every ramification point $P \in C$ the ramification index $e_P = \mathrm{length}(\Omega_{C/\mathbf{P}^1})_P + 1 = 2$ and no two ramification points lie over the same point of $\mathbf{P}^1$. The divisor

$$R_f = \sum_{P \in C} (e_P - 1).P \in \mathrm{Div}(C)$$

is called the *ramification divisor* of f and the divisor

$$B_f = f_*(R_f) \in \mathrm{Div}(\mathbf{P}^1)$$

is called the *branch divisor* of f. By the Riemann-Hurwitz formula

$$w := \deg(B_f) = 2k + 2g - 2.$$

As f is simply branched B_f is the sum of w distinct points. Thus we can think of B_f as an element in the open subscheme $\mathbf{P}_w = \mathrm{Sym}_w(\mathbf{P}^1) \smallsetminus \Delta$ where

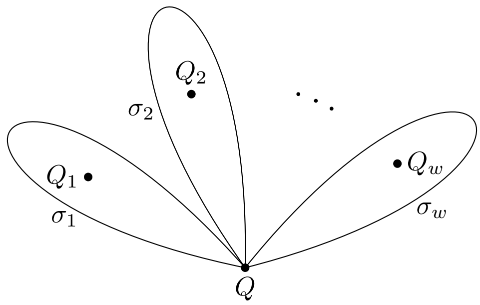

Figure 2.1: Generators of the fundamental group $\pi_1(Q, \mathbf{P}^1 \setminus \{Q_1 \ldots, Q_w\})$.

Δ denotes the closed subscheme of points in $\mathrm{Sym}_w(\mathbf{P}^1)$ with at least two identical summands.

The Hurwitz Space. The key idea in the construction of the Hurwitz space is to identify a simply branched covering with its branch divisor and some additional combinatorial data. For an element $B = Q_1 + \ldots + Q_w \in \mathbf{P}_w$ let

$$\mathscr{H}_{g,k}(B) = \{f : C \to \mathbf{P}^1 k\text{-sheeted simply branched with } B_f = B\}/\sim$$

where two simply branched coverings $f \sim f'$ if there is an isomorphism of curves $\phi : C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\ & \mathbf{P}^1 & \end{tikzcd}$$

For a fixed point $Q \in \mathbf{P}^1$ outside of the support of B we consider the system of generators $\sigma_1, \ldots, \sigma_w$ of $\pi = \pi_1(Q, \mathbf{P}^1 \setminus \{Q_1 \ldots, Q_w\})$ as sketched in Figure 2.1. For a simply branched k-sheeted covering $f : C \to \mathbf{P}^1$ with $B_f = B$ the fiber $f^{-1}(Q)$ is a reduced scheme of length k. For a point $P \in f^{-1}(Q)$ a loop $\sigma \in \pi$ has a unique lifting to $C \setminus R_f$ with starting point P and endpoint $P_\sigma \in f^{-1}(Q)$. In this way we define the monodromy action

$$\mu_f : \pi \to \left\langle \text{Permutations of } f^{-1}(Q) \right\rangle, \; \sigma \mapsto \{P \mapsto P_\sigma\}.$$

Thus, fixing a numeration of the points in the fiber $f^{-1}(Q)$ we can associate to an element $f \in \mathscr{H}_{g,k}(B)$ a group homomorphism from π into the symmetric group $\mathfrak{S}_n$. This map is canonical up to inner automorphism (i.e. up to enumerating the points in a different way). Hence, we obtain a map

$$\mathscr{H}_{g,k}(B) \to \mathrm{Hom}^{ext}(\pi, \mathfrak{S}_n). \tag{2.1}$$

where the right hand side is the group of homomorphisms $\pi \to \mathfrak{S}_n$ up to conjugation with elements in $\mathfrak{S}_n$.

Theorem 2.2 (Riemanns Existence Theorem). *The map (2.1) is injective. Its image consists of those classes which are induced by irreducible representations ξ such that $\tau_i = \xi(\sigma_i)$ is a transposition for $i = 1, \ldots, w$ and $\prod \tau_i = 1$.*

Hence, $\mathscr{H}_{g,k}(B)$ can be identified with the subset $\mathscr{G}_{g,k}$ of $\mathfrak{S}_n$ consisting of tuples of conjugacy classes $(\tau_1, \ldots, \tau_w)$ of transpositions τ_i which generate a transitive subgroup of $\mathfrak{S}_n$ and satisfy $\prod \tau_i = 1$. We define the Hurwitz space on the level of sets as

$$\mathscr{H}_{g,k} := \coprod_{B \in \mathbf{P}_w} \mathscr{H}_{g,k}(B).$$

In other words, the fibers of the natural map $\mathscr{H}_{g,k} \to \mathbf{P}_w$ can be identified with the finite set $\mathscr{G}_{g,k}$. Hence, $\mathscr{H}_{g,k}$ can be equipped with structure of a smooth complex manifold.

Theorem 2.3 (Lüroth, Clebsch, Hurwitz). *$\mathscr{H}_{g,k}$ is connected.*

Sketch of Proof. The idea of the proof is to show that for a given branch divisor $B = b_1 + \ldots + b_w \in \mathbf{P}_w$ the fundamental group $\pi_1(\mathbf{P}_w, B)$ acts transitively on the fiber $\mathscr{H}_{g,k}(B)$ under the map $\mathscr{H}_{g,k} \to \mathbf{P}_w$. More precisely, we consider the subgroup generated by the paths

$$\Gamma_i = b_1 + \ldots + b_{i-1} + \gamma_i(t) + \gamma_i'(t) + b_{i+2} + \ldots + b_w$$

in $\mathbf{P}_w$ where $\gamma_i, \gamma_i' : [0,1] \to \mathbf{P}^1 \setminus \{b_1, \ldots, b_{i-1}, b_{i+2}, \ldots, b_w\}$ are paths with $\gamma_i(0) = \gamma_i'(1) = b_i$ and $\gamma_i(1) = \gamma_i'(0) = b_{i+1}$. It is easy to see that Γ_i acts on elements in $\mathscr{G}_{g,k}$ in the following way:

$$\Gamma_i \cdot (\tau_1, \ldots, \tau_w) = (\tau_1, \ldots, \tau_{i-1}, \tau_i\tau_{i+1}\tau_i, \tau_i, \tau_{i+2}, \ldots \tau_w).$$

Finally, a combinatorial calculation shows that the orbit of any element under the actions of the Γ_i contains the element

$$(\underbrace{(12), (12), \ldots, (12)}_{2g+2 \text{ times}}, (23), (23), (34), (34), \ldots, (k-1\ k), (k-1\ k)).$$

$\square$

Corollary 2.4. *The Hurwitz space $\mathscr{H}_{g,k}$ is a smooth quasi-projective variety of dimension $2g+2k-5$.*

2.2 Brill-Noether Theory

In this section we resume the central facts of the theory of linear series on algebraic curves. In our exposition we follow [ACGH85].

The Brill-Noether Loci. Let C be a smooth curve of genus g. A linear series $\mathfrak{g}^r_d$ on C is a datum $(\mathscr{L}, V)$ consisting of a line bundle $\mathscr{L} \in \mathrm{Pic}^d(C)$ on C of degree d together with a vector space $V \subset H^0(C, \mathscr{L})$ of dimension $r+1$. A $\mathfrak{g}^r_d$ is base point free if the sections in V do not have a common zero. In this case the $\mathfrak{g}^r_d$ gives rise to a map $C \to \mathbf{P}^r = \mathbf{P}V$ by mapping a point $x \in C$ to the hyperplane of all sections vanishing in x. The *Brill-Noether locus* parametrizes those line bundles on C that admit a $\mathfrak{g}^r_d$. It can be defined on the level of sets as

$$W^r_d(C) = \{\mathscr{L} \in \mathrm{Pic}^d(C) \mid h^0(\mathscr{L}) \geq r+1\}.$$

We can equip this set with a scheme structure as follows. We fix a Poincaré line bundle $\mathcal{L}$ of degree d for C, i.e. a universal line bundle on $C \times \mathrm{Pic}^d(C)$, and an effective divisor E on C of sufficiently large degree m. We denote with $\nu : C \times \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$ the projection onto the second factor. By [ACGH85, Ch. IV §3], the Brill-Noether locus can be realized as the degeneracy locus of the natural map of vector bundles

$$\gamma : \nu_*\mathcal{L}(E \times \mathrm{Pic}^d(C)) \to \nu_*(\mathcal{L}(E \times \mathrm{Pic}^d(C))/\mathcal{L})$$

where $\nu_*\mathcal{L}(E \times \mathrm{Pic}^d(C))$ and $\nu_*(\mathcal{L}(E \times \mathrm{Pic}^d(C))/\mathcal{L})$ are of rank $d+m-g+1$ and m, respectively. Thus, we have a natural scheme structure

$$W^r_d(C) = X^{m+d-g-r}(\gamma).$$

From this definition, we see that the "expected dimension" of $W^r_d(C)$ is the *Brill-Noether number*

$$\rho(g,r,d) = g - (r+1)(g-d+r). \tag{2.5}$$

Moreover, a study of the local geometry of the Brill-Noether locus yields the following accessible criterion for smoothness at a given point $\mathscr{L}$.

Theorem 2.6. *Let C be a smooth curve of genus g.*

(i) Let $\mathscr{L} \in W^r_d(C) \setminus W^{r+1}_d(C)$. The tangent space to $W^r_d(C)$ at $\mathscr{L}$ is

$$T_{\mathscr{L}} W^r_d(C) = (\mathrm{im}\, \mu_{\mathscr{L}})^{\perp}$$

where

$$\mu_{\mathscr{L}} : H^0(C, \mathscr{L}) \otimes H^0(C, \omega_C \otimes \mathscr{L}^{-1}) \to H^0(C, \omega_C)$$

is the Petri-map. Hence, $W^r_d(C)$ is smooth of dimension ρ at $\mathscr{L}$ if and only if $\mu_{\mathscr{L}}$ is injective.

(ii) Let $\mathscr{L}$ be a point of $W^{r+1}_d(C)$. Then

$$T_{\mathscr{L}} W^r_d(C) = T_{\mathscr{L}} \mathrm{Pic}^d(C).$$

In particular, if $W^r_d(C)$ has the expected dimension ρ and $r > d - g$ then $\mathscr{L}$ is a singular point of $W^r_d(C)$.

Proof. [ACGH85, Proposition 4.2]. □

The definition of $W^r_d(C)$ allows also to determine its fundamental class in $\mathrm{Pic}^d(C)$ using Porteous formula. Let θ denote the class of the theta divisor in the cohomology ring of $\mathrm{Pic}^d(C) \cong J(C)$.

Theorem 2.7. *Let C be a smooth curve of genus g such that $W^r_d(C)$ is either empty or of expected dimension ρ. Then its fundamental class is given by*

$$[W^r_d(C)] = \prod_{\alpha=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!} \theta^{(r+1)(g-d+r)}.$$

Proof. [ACGH85, Ch. VII, Proposition 4.4]. □

Linear Series on the k-gonal Curve. Let us note that in most cases the presence of a special pencil on the curve does not affect the geometry of the Brill-Noether loci in a pathological way.

Theorem 2.8 (Coppens, Martens)**.** *Let C be a general k-gonal curve of genus g and r, d positive integers such that $d - g < r \leq k - 2$ and $\rho(g, r, d) \geq 0$. Then the Brill-Noether locus $W^r_d(C)$ has an irreducible component of expected dimension $\rho(g, r, d)$. A general element of this component is base point free.*

Proof. [CM99]. □

In particular, we see that for $k \geq 4$ the general k-gonal curve of genus g has an expected plane model of degree $d = \lceil \frac{2}{3}g + 2 \rceil$.

Plane Models of Curves. We recall the following result on minimal resolutions of points in the plane.

Proposition 2.9. *Let Δ be a collection of δ general points in $\mathbf{P}^2$ and let k be maximal under the condition $\varepsilon = \delta - \binom{k+1}{2} \geq 0$. Then the minimal free resolution of $\mathscr{O}_\Delta$ is of the form*

$$0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{O}_{\mathbf{P}^2} \to \mathscr{O}_\Delta \to 0$$

with locally free sheaves

(i) $\mathscr{F} = \mathscr{O}(-k)^{k+1-\varepsilon}$ *and* $\mathscr{G} = \mathscr{O}(-k-1)^{k-2\varepsilon} \oplus \mathscr{O}(-k-2)^{\varepsilon}$ *if* $2\varepsilon \leq k$,

(ii) $\mathscr{F} = \mathscr{O}(-k)^{k+1-\varepsilon} \oplus \mathscr{O}(-k-1)^{2\varepsilon-k}$ *and* $\mathscr{G} = \mathscr{O}(-k-2)^{\varepsilon}$ *else.*

Proof. [Gae51]. □

We also note the following simple but useful criterion for the irreducibility of plane curves. For this, recall that a variety over a field $\mathbf{k}$ is called *absolutely irreducible* if it is irreducible as a variety over the algebraic closure $\overline{\mathbf{k}}$.

Proposition 2.10. *Let C be a reduced plane curve of degree d with $\delta \leq \frac{d(d-3)}{2}$ ordinary double points and no other singularities. If the singular locus Δ of C has a resolution as in Proposition 2.9 then C is absolutely irreducible.*

Proof. Assume that C decomposes into two curves C_1 and C_2 of degree d_1 and d_2 defined by homogeneous polynomials f_1 and f_2. By assumption, C_1 and C_2 intersect transversely in $d_1 \cdot d_2$ distinct points and we have $\delta - d_1 d_2 \geq 0$. First, we reduce to the case $d_1, d_2 \leq k+1$ where $k = \lceil (\sqrt{9+8\delta} - 3)/2 \rceil$ is the minimal degree of generators of I_Δ.

Assume f_1, say, has degree strictly larger than $k+1$. As $C_1 \cap C_2 \subset \Delta$ we have $I_\Delta \subset (f_1, f_2)$. Since I_Δ is minimally generated in degree k and $k+1$ we already have $I_\Delta \subset (f_2)$ which would imply $C_2 \subset \Delta$, absurd.

Now, to exclude the case $d_1, d_2 \leq k+1$ we compute a lower bound for dimension of the space of homogeneous polynomials of degree k passing through Δ. A polynomial of the form $sf_1 + tf_2$ of degree k lies in I_Δ if it vanishes at the remaining $\delta - d_1 d_2$ points. Hence,

$$\begin{aligned} h^0(\mathscr{I}_\Delta(k)) &\geq \binom{k-d_1+2}{2} + \binom{k-d_2+2}{2} - \delta + d_1 d_2 \\ &= \binom{k+2}{2} - \delta + \left[\binom{k+2}{2} + \binom{d-1}{2} - (dk+1) \right] \end{aligned}$$

On the other hand, $h^0(\mathscr{I}_\Delta(k)) = k+1-\varepsilon = \binom{k+2}{2} - \delta$. The assumption on δ and d implies $d \geq k+3$. Setting $d = k+3+r$ we compute

$$\binom{k+2}{2} + \binom{d-1}{2} - (dk+1) = \frac{1}{2}r^2 + \frac{3}{2}r + 1 \geq 1,$$

contradiction. □

Proposition 2.11. *Let C be a smooth curve of genus $g \geq 3$ with $|D|$ a base point free $\mathfrak{g}^2_d$, $d = \lceil \frac{2g}{3} + 2 \rceil$, such that the image of C under the associated map is a plane curve with $\delta = \binom{d-1}{2} - g$ ordinary double points and no other singularities. If the singular locus Δ has a resolution as in Proposition 2.9 then $|D|$ is a smooth point in $W^2_d(C)$.*

Proof. By adjunction, the Petri map for $\mathscr{O}(D)$ can be identified with

$$H^0(\mathbf{P}^2, \mathscr{O}(1)) \otimes H^0(\mathbf{P}^2, \mathscr{I}_\Delta(d-4)) \to H^0(\mathbf{P}^2, \mathscr{I}_\Delta(d-3)).$$

Under the given assumptions the minimal degree of generators of I_Δ is precisely $k = d-4$. As $2\varepsilon = 4\left\lceil \frac{2g}{3} \right\rceil - 2g - 2 \geq \left\lceil \frac{2g}{3} \right\rceil - 2 = k$ we are in case (ii) of (2.9). It follows that the Petri map is injective since there are no linear relations among the generators I_Δ of degree k. $\square$

2.3 Resolutions in Multiprojective Space

We turn to resolutions of sheaves and modules in multiprojective space. In analogy to graded free resolutions in projective space we can study multigraded resolutions of modules over the Cox ring of cartesian products of projective spaces. We answer the question which line bundles should occur in a multigraded resolution of a coherent sheaf by writing down a generating set of the bounded derived category using Beilinson's monads. We will make use of this when studying linkage and resolutions of ideal sheaves of curves in $\mathbf{P}^1 \times \mathbf{P}^2$ in forthcoming sections.

Line Bundles and Cohomology. Let $n_1, \ldots, n_k$ be a list of positive integers. We abbreviate

$$\mathbf{P} := \mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_k} = \mathbf{P}(V_1) \times \ldots \times \mathbf{P}(V_k).$$

Let $\pi_i : \mathbf{P} \to \mathbf{P}^{n_i}$ denote the projection onto the i-th factor. For an integer vector $\mathbf{a} := (a_1, \ldots a_k)$ we set

$$\mathscr{O}_{\mathbf{P}}(\mathbf{a}) := \mathscr{O}_{\mathbf{P}^{n_1}}(a_1) \boxtimes \ldots \boxtimes \mathscr{O}_{\mathbf{P}^{n_k}}(a_k) = \pi_1^* \mathscr{O}_{\mathbf{P}^{n_1}}(a_1) \otimes \ldots \otimes \pi_n^* \mathscr{O}_{\mathbf{P}^{n_k}}(a_k).$$

Then $\mathrm{Pic}(\mathbf{P}) = \{\mathscr{O}_{\mathbf{P}}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{Z}^k\} = \mathbf{Z}^k$. The cohomology of the line bundles on $\mathbf{P}$ can be computed with the Künneth formula.

Proposition 2.12 (Künneth Formula)**.** *Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on projective schemes X and Y, respectively. Then*

$$H^k(X \times Y, \mathscr{F} \boxtimes \mathscr{G}) = \bigoplus_{i+j=k} H^i(X, \mathscr{F}) \otimes H^j(Y, \mathscr{G}).$$

Proof. [Kem80, Section 4]. $\square$

Remark 2.13. We note for later reference that the intermediate cohomology groups of a line bundle on $\mathbf{P} = \mathbf{P}^1 \times \mathbf{P}^2$ decompose as

$$H^1(\mathbf{P}, \mathscr{O}_{\mathbf{P}}(a,b)) = H^1(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(a)) \otimes H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(b))$$

and

$$H^2(\mathbf{P}, \mathscr{O}_{\mathbf{P}}(a,b)) = H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(a)) \otimes H^2(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(b)).$$

We also need the following formulation for locally free sheaves.

Corollary 2.14. *Let $\mathscr{F}_1, \mathscr{G}_1$ be locally free sheaves on X and let $\mathscr{F}_2, \mathscr{G}_2$ be locally free sheaves on Y. Then*

$$\operatorname{Ext}^k(\mathscr{F}_1 \boxtimes \mathscr{G}_1, \mathscr{F}_2 \boxtimes \mathscr{G}_2) \cong \bigoplus_{i+j=k} \operatorname{Ext}^i(\mathscr{F}_1, \mathscr{G}_1) \otimes \operatorname{Ext}^j(\mathscr{F}_2, \mathscr{G}_2)$$

Proof. As the projections onto the factors are flat we have

$$\operatorname{Ext}^k(\mathscr{F}_1 \boxtimes \mathscr{F}_2, \mathscr{G}_1 \boxtimes \mathscr{G}_2) = \operatorname{Ext}^k(\mathscr{O}_{X \times Y}, (\mathscr{F}_1^\vee \otimes \mathscr{G}_1) \boxtimes (\mathscr{F}_2^\vee \otimes \mathscr{G}_2)).$$

□

Multigraded Beilinson Monads. Recall that the objects of the bounded derived category $\mathcal{D}^b(X)$ of a scheme X are bounded complexes of coherent sheaves on X and the morphisms are maps of complexes which are identified if homotopic. Two complexes are isomorphic if there exists a quasi-isomorphism between them. Thus a coherent sheaf $\mathscr{F}$ on X (considered as complex concentrated in a single cohomological degree) and any finite resolution or monad of $\mathscr{F}$ are isomorphic objects in $\mathcal{D}^b(X)$. For a rigorous definition and a proof of the existence of $\mathcal{D}^b(X)$ we refer to [GM03].

Definition 2.15. *A collection $(\mathscr{E}_0, \ldots, \mathscr{E}_n)$ of coherent sheaves on X is called a* full strongly exceptional collection *if the following conditions are satisfied*

1. $\operatorname{Hom}(\mathscr{E}_k, \mathscr{E}_j) = 0$ *for* $j < k$ *and* $\operatorname{Ext}^i(\mathscr{E}_k, \mathscr{E}_j) = 0$ *for all* k, j *and* $i \geq 1$,
2. $\mathscr{E}_0, \ldots, \mathscr{E}_n$ *generate* $\mathcal{D}^b(X)$.

In general, it is a challenging question to determine whether there is a fully strong exceptional sequence for a given scheme X, see for example [Kap88]. Beilinson's classical result [Beĭ78] asserts that $\{\mathscr{O}_{\mathbf{P}^n}(-i)\}_{i=1}^n$ and $\{\Omega^i_{\mathbf{P}^n}(i)\}_{i=1}^n$ are full strongly exceptional sequences for $\mathbf{P}^n$. We can generalize this result to multiprojective space. To this end, note that $\Omega_{\mathbf{P}} = \pi_1^* \Omega_{\mathbf{P}^{n_1}} \otimes \ldots \otimes \pi_n^* \Omega_{\mathbf{P}^{n_k}}$, so it makes sense to define similarly

$$\Omega^{\mathbf{a}}_{\mathbf{P}}(\mathbf{a}) := \Omega^{a_1}_{\mathbf{P}^{n_1}}(a_1) \boxtimes \ldots \boxtimes \Omega^{a_n}_{\mathbf{P}^{n_k}}(a_n)$$

for any vector $\mathbf{a} = (a_1, \ldots, a_k)$ of non-negative integers.

Lemma 2.16. *Let $\mathbf{a}, \mathbf{b} \in \mathbf{Z}^k$ with $0 \leq a_i, b_i \leq n_i$ for all $i = 1, \ldots, k$ and let $j > 0$. Then*

$$\begin{aligned} \operatorname{Hom}(\mathscr{O}_{\mathbf{P}}(-\mathbf{a}), \mathscr{O}_{\mathbf{P}}(-\mathbf{b})) &= \textstyle\bigotimes_{i=1}^k \operatorname{Sym}^{a_i - b_i}(V_i) \\ \operatorname{Ext}^j(\mathscr{O}_{\mathbf{P}}(-\mathbf{a}), \mathscr{O}_{\mathbf{P}}(-\mathbf{b})) &= 0 \end{aligned}$$

and

$$\begin{aligned}\operatorname{Hom}(\Omega^{\mathbf{a}}_{\mathbf{P}}(\mathbf{a}), \Omega^{\mathbf{b}}_{\mathbf{P}}(\mathbf{b})) &= \textstyle\bigotimes_{i=1}^{k} \bigwedge^{a_i-b_i}(V_i^*)\\ \operatorname{Ext}^j(\Omega^{\mathbf{a}}_{\mathbf{P}}(\mathbf{a}), \Omega^{\mathbf{b}}_{\mathbf{P}}(\mathbf{b})) &= 0.\end{aligned}$$

Proof. We use Corollary 2.14 to reduce the statements to a single factor. Let $\mathbf{P}^n = \mathbf{P}(V)$. The statements for the sheaves $\mathscr{O}_{\mathbf{P}^n}(-a)$ are trivial to check. For the remaining statements consider the short exact sequences

$$0 \to \Omega^a(a) \to \bigwedge^a V \otimes \mathscr{O}_{\mathbf{P}^n} \to \Omega^{a-1}(a) \to 0$$

arising from the Koszul complex (see [Eis95, Chapter 17.5]). □

Proposition 2.17 (Resolution of the Diagonal)**.** *The diagonal* Δ *in* $\mathbf{P} \times \mathbf{P}$ *has the following locally free resolution*

$$0 \to \mathscr{O}_{\mathbf{P}}(\mathbf{n}) \boxtimes \Omega^{\mathbf{n}}_{\mathbf{P}^n}(\mathbf{n}) \to \ldots \to \bigoplus_{\substack{|\mathbf{a}|=j\\ \mathbf{a}\geq \mathbf{0}}} \mathscr{O}_{\mathbf{P}}(\mathbf{a}) \boxtimes \Omega^{\mathbf{a}}_{\mathbf{P}^n}(\mathbf{a}) \to \ldots \to \mathscr{O}_{\mathbf{P}\times\mathbf{P}} \to \mathscr{O}_{\Delta} \to 0$$

where $\mathbf{n} = (n_1, \ldots, n_k)$.

Proof. Recall that the diagonal $\mathscr{O}_{\Delta_i}$ on $\mathbf{P}^{n_i} \times \mathbf{P}^{n_i}$ has a locally free resolution

$$\begin{aligned}&0 \to \mathscr{O}_{\mathbf{P}^{n_i}}(-n_i) \boxtimes \Omega^{n_i}_{\mathbf{P}^{n_i}}(n_i) \to \ldots\\ &\ldots \to \mathscr{O}_{\mathbf{P}^{n_i}}(-1) \boxtimes \Omega^{1}_{\mathbf{P}^{n_i}}(1) \to \mathscr{O}_{\mathbf{P}^{n_i}\times\mathbf{P}^{n_i}} \to \mathscr{O}_{\Delta_i} \to 0\end{aligned}$$

If $\rho_i : \mathbf{P} \times \mathbf{P} \to \mathbf{P}^{n_i} \times \mathbf{P}^{n_i}$ denotes the projection then

$$\mathscr{O}_{\Delta} = \rho_1^* \mathscr{O}_{\Delta_1} \otimes \ldots \otimes \rho_n^* \mathscr{O}_{\Delta_r}.$$

Hence, the product of the pullbacks of the resolutions of the $\mathscr{O}_{\Delta_i}$ resolves $\mathscr{O}_{\Delta}$. Reordering of the factors completes the proof. □

Theorem 2.18 (Beilinson)**.** *The sequences*

$$\begin{aligned}\mathcal{B}_I = \{ \quad & \mathscr{O}_{\mathbf{P}}, \mathscr{O}_{\mathbf{P}}(-1,0,\ldots,0), \ldots, \mathscr{O}_{\mathbf{P}}(-n_1, 0\ldots,0),\\ & \ldots\\ & \mathscr{O}_{\mathbf{P}}(0,-n_2,\ldots,-n_k), \ldots, \mathscr{O}_{\mathbf{P}}(-n_1,-n_2,\ldots,-n_k) \quad \}\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_{II} = \{ \quad & \Omega_{\mathbf{P}}, \Omega_{\mathbf{P}}^{(1,0,\ldots,0)}(1,0,\ldots,0), \ldots, \Omega_{\mathbf{P}}^{(n_1,0,\ldots,0)}(n_1, 0\ldots,0),\\ & \ldots\\ & \Omega_{\mathbf{P}}^{(0,n_2,\ldots,n_k)}(0,n_2,\ldots,n_k), \ldots, \Omega_{\mathbf{P}}^{(n_1,n_2,\ldots,n_k)}(n_1,n_2,\ldots,n_k) \quad \}\end{aligned}$$

form full strongly exceptional collections for $\mathbf{P}$*, respectively.*

Proof. By Lemma 2.16, it remains only to show that $\mathcal{B}_I$ and $\mathcal{B}_{II}$ generate $\mathcal{D}^b(\mathbf{P})$. From the resolution of the diagonal we see that $\mathscr{O}_\Delta$ and thus any object of the form $\mathscr{O}_\Delta \otimes L\pi_2^* F$ for $F \in \mathrm{Ob}(\mathcal{D}^b(\mathbf{P}))$ lies in the fully triangulated subcategory of $\mathcal{D}^b(\mathbf{P} \times \mathbf{P})$ which is generated by sheaves of the form $\Omega^{\mathbf{a}}(\mathbf{a}) \boxtimes G$ for $G \in \mathrm{Ob}(\mathcal{D}^b(\mathbf{P}))$. Applying the projection formula we see that

$$F = R\pi_{1*}(\mathscr{O}_\Delta \otimes L\pi_2^* F).$$

Thus F belongs to the subcategory generated by the $\Omega^{\mathbf{a}}(\mathbf{a})$. An analogous argument for objects $\mathscr{O}_\Delta \otimes L\pi_1^* F$ yields the result for $\mathcal{B}_I$. □

Let $\mathscr{F}$ be a coherent sheaf on $\mathbf{P}$. Recall that a *monad* for $\mathscr{F}$ is a bounded complex of coherent sheaves $\mathcal{K}^\bullet : \ldots \to \mathcal{K}^{-1} \to \mathcal{K}^0 \to \mathcal{K}^1 \to \ldots$ with cohomology

$$H^i(\mathcal{K}^\bullet) = \begin{cases} \mathscr{F} & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$$

Theorem 2.18 states that there are monads for $\mathscr{F}$ whose terms are direct sums of elements in $\mathcal{B}_I$ and $\mathcal{B}_{II}$ respectively. Let us turn this in a more explicit statement by identifying the Betti numbers of these monads with certain cohomology groups associated to $\mathscr{F}$.

Theorem 2.19 (Beilinson Monads)**.** *For any coherent sheaf $\mathscr{F}$ on $\mathbf{P}$ there exist monads $\mathcal{K}_I^\bullet$ and $\mathcal{K}_{II}^\bullet$ which are unique up to homotopy with terms*

$$\mathcal{K}_I^i \cong \bigoplus_{j \in \mathbf{Z}} \bigoplus_{|\mathbf{a}|=j} H^{i+j}(\mathbf{P}, \mathscr{F} \otimes \Omega_{\mathbf{P}}^{\mathbf{a}}(\mathbf{a})) \otimes \mathscr{O}_{\mathbf{P}}(-\mathbf{a})$$

and

$$\mathcal{K}_{II}^i \cong \bigoplus_{j \in \mathbf{Z}} \bigoplus_{|\mathbf{a}|=j} H^{i+j}(\mathbf{P}, \mathscr{F} \otimes \mathscr{O}_{\mathbf{P}}(-\mathbf{a})) \otimes \Omega^{\mathbf{a}}(\mathbf{a}).$$

Proof. We will prove the statement for $\mathcal{K}_I^\bullet$ and leave the analogous argument for the second monad to the interested reader. Clearly, by what has been shown before, there is a monad $\mathcal{K}_I^\bullet$ for $\mathscr{F}$ whose terms are direct sums of copies of $\mathscr{O}_{\mathbf{P}}, \ldots, \mathscr{O}_{\mathbf{P}}(-n_1, \ldots, -n_k)$, i.e. $\mathcal{K}_I^i = \bigoplus_{\mathbf{a} \in \mathbf{Z}^k} \mathscr{O}(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$. To identify the Betti numbers $\beta_{i,\mathbf{a}}$ of $\mathcal{K}_I^\bullet$ we consider the spectral sequences $'E$ and $''E$ which approximate the hypercohomology of the complex $\mathcal{K}_I^\bullet \otimes \Omega_{\mathbf{P}}^{\mathbf{a}}(\mathbf{a})$, see [GM03, Ch. III.7]. These spectral sequences have E_2 pages

$$\begin{aligned} {}'E_2^{pq} &= H^p(\mathbf{P}, H^q(\mathcal{K}_I^\bullet \otimes \Omega^{\mathbf{a}}(\mathbf{a}))) \\ {}''E_2^{pq} &= H^q(H^p(\mathbf{P}, \mathcal{K}_I^\bullet \otimes \Omega^{\mathbf{a}}(\mathbf{a}))) \end{aligned}$$

As $\Omega_{\mathbf{P}}^{\mathbf{a}}(\mathbf{a})$ is locally free the only non vanishing cohomology of the tensored complex is $H^0(\mathcal{K}_I^\bullet \otimes \Omega^{\mathbf{a}}(\mathbf{a})) = \mathscr{F} \otimes \Omega_{\mathbf{P}}^{\mathbf{a}}(\mathbf{a})$. Thus $'E$ collapses on the second page with

$$'E_2^{pq} = {}'E_\infty^{pq} = \begin{cases} H^p(\mathbf{P}, \mathscr{F} \otimes \Omega^{\mathbf{a}}_{\mathbf{P}}(\mathbf{a})) & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}$$

Next, note that by the Künneth formula we have

$$\dim_{\mathbf{k}} H^p(\mathbf{P}, \mathcal{K}_I^i \otimes \Omega^{\mathbf{a}}(\mathbf{a})) = \begin{cases} \beta_{i,\mathbf{a}} & \text{if } p = |\mathbf{a}| \\ 0 & \text{else.} \end{cases}$$

We can assume that all maps of the form $\mathscr{O}_{\mathbf{P}}(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \to \mathscr{O}_{\mathbf{P}}(-\mathbf{a})^{\beta_{i+1,\mathbf{a}}}$ in $\mathcal{K}_I^\bullet$ are zero. Indeed, if this is not the case we obtain by Gaussian elimination a direct sum of trivial complexes $0 \to \mathscr{O}_{\mathbf{P}}(-\mathbf{a}) \xrightarrow{1} \mathscr{O}_{\mathbf{P}}(-\mathbf{a}) \to 0$. The quotient $\tilde{\mathcal{K}}_I^\bullet$ of $\mathcal{K}_I^\bullet$ by these complexes is again a monad for $\mathscr{F}$. Thus, $''E$ also collapses on the second page with

$$''E_\infty^{pq} = {}''E_2^{pq} = \begin{cases} H^p(\mathcal{K}_I^q \otimes \Omega^{\mathbf{a}}(\mathbf{a})) & \text{if } p = |\mathbf{a}| \\ 0 & \text{else} \end{cases}$$

Finally, we get $h^p(\mathscr{F} \otimes \Omega^{\mathbf{a}}(\mathbf{a})) = h^j(\mathcal{K}^{p-j} \otimes \Omega^{\mathbf{a}}(\mathbf{a})) = \beta_{p-j,\mathbf{a}}$ which concludes the proof. □

Remark 2.20. The Beilinson monads for a coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n = \mathbf{P}(V)$ can be obtained by applying certain functors on a doubly infinite complex $\mathbf{T}(\mathscr{F})$ defined over the exterior algebra $\bigwedge(V^*)$. $\mathbf{T}(\mathscr{F})$ is called the *Tate resolution* of $\mathscr{F}$ and it turns out to be suitable object for the construction of $\mathscr{F}$ in many cases, see [EFS03] and [DE02]. However, there seems to be no straightforward generalization to multigraded cases.

Problem 2.21. *Find an appropriate counterpart of the Tate resolution for the cases of coherent sheaves on weighted projective space and multiprojective space.*

Multigraded Hilbert Series. We denote by

$$R = \bigoplus_{\mathbf{a} \in \mathbf{Z}^k} H^0(\mathbf{P}, \mathscr{O}_{\mathbf{P}}(\mathbf{a})) \cong \mathbf{k}[x_{10}, \ldots, x_{1n_1}, \ldots, x_{k0}, \ldots, x_{kn_k}]$$

the Cox ring of $\mathbf{P}$. R is naturally $\mathbf{Z}^k$-graded with $\deg(x_{ij}) = \mathbf{e}_i \in \mathbf{Z}^k$ for all $1 \leq i \leq k$ and $0 \leq j \leq n_i$. Let $M = \bigoplus_{\mathbf{a} \in \mathbf{Z}^n} M_{\mathbf{a}}$ be a finitely generated $\mathbf{Z}^k$-graded module over R. Then M admits a multigraded minimal free resolution

$$0 \to F_r \to \ldots \to F_1 \to M \to 0$$

with finitely generated free modules $F_i = \bigoplus_{\mathbf{a} \in \mathbf{Z}^k} R(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$. The minimal resolution $F_\bullet$ is uniquely determined up to isomorphisms of graded complexes. Thus, the dimensions of the graded pieces $\beta_{i,\mathbf{a}}$ depend only on M. They are called the *(multigraded) Betti numbers* of M. The formal Laurent series

$$H_M := \sum_{\mathbf{a}\in\mathbf{Z}^k} \dim_{\mathbf{k}}(M_{\mathbf{a}}) \cdot s^{\mathbf{a}} \in \mathbf{Q}[[s_1,\dots,s_k]][s_1^{-1},\dots,s_k^{-1}].$$

is called *multigraded Hilbertseries of* M. The following well-known result shows in which way H_M encodes the Betti numbers of M.

Lemma 2.22. *The Hilbert series factors as*

$$H_M = \frac{\mathrm{HN}_M}{\prod_{i=1}^k (1-s_i)^{n_i+1}}$$

with numerator

$$\mathrm{HN}_M = \sum_{\mathbf{a}\in\mathbf{Z}^n} \left(\sum_{i=0}^r (-1)^i \beta_{i,\mathbf{a}} \right) s^{\mathbf{a}}.$$

The Laurent polynomial HN_M *is called the* Hilbert numerator *of* M.

Proof. The Hilbert series of R is

$$\sum_{\mathbf{a}\in\mathbf{Z}^k_{\geq 0}} \left(\prod_{i=1}^k \binom{n_i+a_i}{n_i} \right) s^{\mathbf{a}} = \frac{1}{\prod_{i=1}^k (1-s_i)^{n_i+1}}.$$

and for a free module $F = \bigoplus_{i=j}^m R(-\mathbf{a}_j)^{\beta_{\mathbf{a}_j}}$ we have

$$H_F = \sum_{i=j}^m \beta_{\mathbf{a}_j} H_{R(-\mathbf{a}_j)} = \frac{\sum_{j=1}^m \beta_{\mathbf{a}_j} s^{\mathbf{a}_j}}{\prod_{i=1}^k (1-s_i)^{n_i+1}}.$$

From the definition of the Hilbert series it is easy to see that

$$H_M = \sum_{i=1}^r (-1)^i H_{F_i}$$

where H_{F_i} denotes the Hilbert series of the i-th term in the minimal free resolution of M. From this the result follows immediately. □

As M is finitely generated, we may assume by shifting degrees that M is generated in non-negative degrees in which case HN_M is a polynomial. Under the assumption that M has a *natural resolution*, i.e. for all $\mathbf{a} \in \mathbf{Z}^n$ there is at most one i with $\beta_{i,\mathbf{a}} \neq 0$, we can read off the Betti numbers from the Hilbert numerator. In particular, it is then possible to explicitly compute the Betti numbers if the Hilbert function h_M of M is known. To this end, we consider for a sufficiently large $r \gg 0$ the polynomial

$$\tilde{H}_M = \sum_{\mathbf{a}\in\mathbf{Z}^k, |\mathbf{a}|\leq r} h_M(\mathbf{a})s^{\mathbf{a}}.$$

The product of $\tilde{H}_M$ with the denominator of H_M can be written as

$$\tilde{H}_M \cdot \prod_{i=1}^{k}(1-s_i)^{n_i+1} = \mathrm{HN}_M + P$$

where P involves only terms of degree strictly larger than r.

2.4 Divided Powers and Macaulays Inverse System

Macaulay [Mac94] introduced the concept of describing modules over the polynomial ring as annihilators under the partial derivative or contraction action. In this section we develop the theory of inverse systems for modules over standard multigraded rings. In our treatment we follow the descriptions of Eisenbud [Eis95], Iarrobino and Kanev [IK99] and Kunte [Kun08].

Divided Power Algebra. As before, let $R = \mathbf{k}[x_{10},\ldots,x_{1n_1},\ldots,x_{k0},\ldots,x_{kn_k}]$ denote the multigraded Cox ring of $\mathbf{P} = \mathbf{P}^{n_1}\times\ldots\times\mathbf{P}^{n_k}$ with degrees $\deg(x_{ij}) = \mathbf{e}_i \in \mathbf{Z}^k$ for $i = 1,\ldots,k$ and $j = 1,\ldots,n_k$. Let D denote the graded dual of R, that is

$$D = \bigoplus_{\mathbf{a}\in\mathbf{Z}^k} \mathrm{Hom}_{\mathbf{k}}(R_{\mathbf{a}},\mathbf{k}) = \bigoplus_{\mathbf{a}\in\mathbf{Z}^k} D_{-\mathbf{a}}.$$

Let us fix some notation. For a multi-tuple of non-negative integers $A = (A_1,\ldots,A_k) = ((a_{10},\ldots,a_{1n_1}),\ldots,(a_{k0},\ldots,a_{kn_k}))$ we write

$$|A| = (|A_1|,\ldots,|A_k|) = (\sum_{j=0}^{n_1} a_{1,j},\ldots,\sum_{j=0}^{n_k} a_{k,j})$$

and abbreviate $x^A = x_{10}^{a_{10}}\cdots x_{1n_1}^{a_{1n_1}}\cdots x_{kn_k}^{a_{kn_k}}$. For the $\mathbf{k}$-basis $\{x^A \mid |A| = \mathbf{a}\}$ of $R_{\mathbf{a}}$ we denote by $\{X^{[A]} \mid |A| = \mathbf{a}\}$ the dual basis of $D_{-\mathbf{a}}$. Moreover, we extend the notation to arbitrary integer values by setting $X^{[A]} = 0$ if one of the $a_{ij} < 0$. If $\mathrm{char}(\mathbf{k}) = 0$ then a ring structure on D is given by the multiplication

$$X^{[A]} \cdot X^{[B]} = \frac{(A+B)!}{A!B!} X^{[A+B]}.$$

Equipped with this multiplication D is called the *divided power algebra.*

Remark 2.23. Eisenbud [Eis95, Appendix A2] gives a more conceptual approach to the algebra structure which allows to extend the algebra structure to any field.

Definition 2.24. *For all* $\mathbf{a}, \mathbf{b} \in \mathbf{N}_0^k$ *we define the contraction map*

$$\circ : R_{\mathbf{a}} \times D_{-\mathbf{b}} \to D_{-\mathbf{b}+\mathbf{a}}, (\varphi, f) \mapsto \varphi \circ f = \begin{cases} 0 & \textit{for } \mathbf{b} < \mathbf{a}, \\ \psi \mapsto f(\varphi\psi) & \textit{else.} \end{cases}$$

The contraction map equips the divided power algebra D with the structure of a graded R-module, hence the negativ grading. Note that D is not finitely generated as an R-module.

Example 2.25. Let $R = \mathbf{k}[x_0, x_1]$ be the coordinate ring of $\mathbf{P}^1$ and consider $\varphi = x_0 + 2x_1$ and $f = X_0^2 + X_0X_1 + X_1^2$. Then $\varphi \circ f = 3X_0 + 3X_1$.

Remark 2.26. If $\operatorname{char} \mathbf{k} = 0$ then the contraction map is up to scalars equivalent to the action of R on D as higher partial differential operators, i.e. for $\varphi \in R$ and $f \in D$ we consider the action given by $\varphi(\frac{\partial}{\partial X_{10}}, \dots, \frac{\partial}{\partial X_{kn_k}})f$. In more detail, for any multivectors A, B with $a_{ij} \geq b_{ij}$ we have

$$x^B \circ X^{[A]} = X^{[A-B]} = \frac{1}{\prod_{ij}(a_{ij} - b_{ij})!} \cdot \frac{\partial^{b_{10}}}{\partial^{b_{10}} X_{10}} \cdots \frac{\partial^{b_{nn_k}}}{\partial^{b_{nn_k}} X_{nn_k}} X^{[A]}.$$

Macaulays Inverse System. We now turn to the description of modules. To this end we extend the contraction map to a pairing of modules in the following way.

Definition 2.27. *Let* $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbf{Z}^k$ *be a collection of degrees and consider the modules* $F = \bigoplus_{i=1}^r D(\mathbf{a}_i)$ *and* $G = \bigoplus_{i=1}^r R(-\mathbf{a}_i)$.

(i) We define the pairing of R*-modules*

$$F \times G \to D, f, \varphi \mapsto \langle f, \varphi \rangle := \sum_{i=1}^n \varphi_i \circ f_i.$$

(ii) For a submodule $N \subset F$ *we define the* annihilator of N in R *as*

$$\operatorname{Ann}_R(N) := \{\varphi \in G \mid \langle f, \varphi \rangle = 0 \textit{ for all } f \in N\} \subset G.$$

(iii) For a submodule $M \subset G$ *we define the* inverse system of M *as*

$$M^\perp := \{f \in F \mid \langle f, \varphi \rangle = 0 \textit{ for all } \varphi \in M\} \subset F.$$

As the pairing defined in (i) is R-linear, the annihilator $\operatorname{Ann}_R(N)$ is a R-module. Assume $N \subset F$ is finitely generated as a R-module. Then a minimal system of generators $P^1, \dots, P^s \in F$ of N can be regarded as the columns of a matrix $P \in \operatorname{Hom}_R\left(\bigoplus_{j=1}^r R(\mathbf{b}_j), G\right)$ with $P(e_j) = P^j$. In this case we also write $\operatorname{Ann}_R(P) := \operatorname{Ann}_R(N)$.

Modules of Finite Length. If M is a module over R of finite length, then the graded dual $\operatorname{Hom}_{\mathbf{k}}(M, \mathbf{k})$ is finitely generated as an R-module. Following Kunte [Kun08], we consider minimal systems of generators

$$\bigoplus_{i=1}^r R(\mathbf{a}_i) \xrightarrow{\alpha} M \to 0$$

and

$$\bigoplus_{j=1}^s R(-\mathbf{b}_j) \xrightarrow{\beta} \operatorname{Hom}_{\mathbf{k}}(M, \mathbf{k}) \to 0$$

for M and $\operatorname{Hom}_{\mathbf{k}}(M, k)$ respectively. We apply $\operatorname{grHom}_{\mathbf{k}}(—, \mathbf{k})$ to the latter to obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\beta^*"] & \bigoplus_{i=1}^r D(\mathbf{a}_i) \\ 0 & M \arrow[l] \arrow[u, "\mathrm{id}"] & \bigoplus_{j=1}^s R(\mathbf{b}_j) \arrow[l, "\alpha"'] \arrow[u, "P"'] \end{tikzcd}$$

where P is defined as the composition of β^* and α. In summation, we get the following proposition.

Proposition 2.28. *Let M be a graded R-module of finite length. Then there are degrees $\mathbf{a}_1, \ldots, \mathbf{a}_s, \mathbf{b}_1, \ldots, \mathbf{b}_r \in \mathbf{Z}^k$ and a matrix $P \in \operatorname{Hom}_R(\bigoplus R(\mathbf{b}_j), \bigoplus D(\mathbf{a}_i))$ such that B*

$$M \cong M(P) := \bigoplus R(\mathbf{b}_j) \Big/ \operatorname{Ann}_R(P).$$

Other Modules. In general it is not possible to set up an equivalence as above if M is not of finite length. However, we are in good shape if we can "split up" M into a part of finite length and one of infinite length which is easy to describe.

Proposition 2.29. *Let M be a finitely generated graded R-module with minimal free presentation*

$$\bigoplus_{j=1}^s R(-\mathbf{b}_j) \xrightarrow{A} \bigoplus_{j=1}^r R(-\mathbf{c}_j) \to M \to 0.$$

Assume that there is a degree $\mathbf{a} \in \mathbf{Z}^k$ such that $\mathbf{b}_i - \mathbf{c}_j \geq \mathbf{a}$ for all i, j and such that the quotient $M/M_{\geq \mathbf{a}}$ is of finite length. Then there is a collection

$\mathbf{d}_1, \ldots, \mathbf{d}_t \in \mathbf{Z}^k$ *of degrees and a matrix* $P \in \operatorname{Hom}(\oplus R(\mathbf{c}_j), \oplus R(\mathbf{d}_j))$ *such that* $\operatorname{Ann}_R(P)_{\mathbf{b}_j}$ *is generated by the columns of* A *in these degrees.*

Proof. By (2.28) there is a $P \in \operatorname{Hom}(\oplus R(\mathbf{c}_j), \oplus R(\mathbf{d}_j))$ with $M(P) = M/M_{\geq \mathbf{a}}$. Now by the assumption on the degrees of the minimal presentation, we see that Hilbert functions of $\operatorname{Ann}_R(P)$ and $\operatorname{im} A$ coincide in the degrees $\mathbf{b}_j$. □

Remark 2.30. (i) The existence of a degree $a \in \mathbf{Z}^k$ such that the quotient $M/M_{\geq \mathbf{a}}$ is of finite length is trivially fulfilled when $k = 1$ but not otherwise. A remedy is to form the quotient by a sufficiently large sum of modules of the form $M_{\mathbf{a}_1} + \ldots + M_{\mathbf{a}_j}$, but we will not need this generalisation.

(ii) The critical point of reversing (2.28) and (2.29) in order to construct a certain module M is of course to find a suitable matrix P in the first place. We will be concerned with this construction in section 3.2.

Example 2.31. Let $R = \mathbf{k}[x_0, x_1, y_0, y_1, y_2]$ be the bigraded Cox Ring of $\mathbf{P}^1 \times \mathbf{P}^2$ with degrees $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$ and consider the module given by the presentation

$$R(-1,0)^{\oplus 3} \oplus R(0,-1)^{\oplus 5} \xrightarrow{(A|B)} R^3 \to M \to 0$$

with

$$(A|B) = \left(\begin{array}{ccc|ccccc} x_0 & 0 & 0 & y_0 & y_1 & y_2 & 0 & 0 \\ 0 & x_1 & 0 & 0 & y_0 & y_1 & y_2 & 0 \\ 0 & 0 & x_0 + x_1 & 0 & 0 & y_0 & y_1 & y_2 \end{array} \right).$$

The Hilbert function $h_M(i, j)$ of M has nonzero values

$$\begin{array}{c|cccl} j & & & & \\ \uparrow & & & & \\ & 3 & & & \\ & 4 & 1 & & \\ & 3 & 3 & 3 & \cdots \\ \hline & & & & \to i \end{array}$$

where the dots indicate that $h_M(i, 0) = 3$ for all $i \geq 3$. In particular, M is not a finite module but $M_{\geq (0,1)}$ is. $N = M_{\geq (2,0)}$. Informally speaking, the module N encodes essentially the information of A as can be seen from its presentation

$$R(-3,0)^{\oplus 3} \oplus R(-2,-1)^{\oplus 9} \xrightarrow{(A|\mathrm{Id}\cdot y_0|\mathrm{Id}\cdot y_1|\mathrm{Id}\cdot y_2)} R(-2,0)^{\oplus 3} \to N \to 0.$$

The Hilbert function of $M_0 = M/N$ is

$$\begin{array}{c|ll} j & & \\ & 3 & \\ & 4 & 1 \\ & 3 & 3 \\ \hline & & & i \end{array}$$

On the other hand, we can express $M_0 = R^{\oplus 3}/\mathrm{Ann}_R(P)$ where

$$P = \begin{pmatrix} X_1Y_2 & Y_1^{[2]} & Y_1Y_2 & Y_2^{[2]} \\ -X_0Y_1 & -Y_0Y_1 & -2Y_1^{[2]} - Y_0Y_2 & -Y_1Y_2 \\ X_0Y_0 - X_1Y_0 & Y_0^{[2]} & Y_0Y_1 & Y_1^{[2]} \end{pmatrix}.$$

Note that the colums of P correspond to the socle elements of M in degrees $(1,1)$ and $(0,2)$. We see that the columns both A and of B can be recovered (up to $\mathbf{k}$-linear combinations) as the $\mathbf{k}$-bases of $\mathrm{Ann}_R(P)_{(1,0)}$ and of $\mathrm{Ann}_R(P)_{(0,1)}$.

The computation of the annihilator and the inverse system can be implemented in *Macaulay2*.

Code 2.32. We illustrate the calculations along the previous example. We start by loading the package and the define the Cox ring.

```
i1 : loadPackage"RandomGonalCurves";
     R=QQ[x_0,x_1,y_0..y_2,Degrees=>{2:{1,0},3:{0,1}}];
```

Note that we will use this ring also to express elements in divided power algebra. We define module M via the presentation by $(A|B)$ and compute the Hilbert function:

```
i2 : A=matrix{{x_0,0,0},{0,x_1,0},{0,0,x_0+x_1}};
     B=matrix{{y_0,y_1,y_2,0,0},{0,y_0,y_1,y_2,0},{0,0,y_0,y_1,y_2}};
     M=coker(A|B);
     matrix apply(4,i->apply(4,j->hilbertFunction({i,j},M)))

o2 = | 3 4 3 0 |
     | 3 1 0 0 |
     | 3 0 0 0 |
     | 3 0 0 0 |
```

We compute bases of $M^\perp$ in bidegrees $(1,1)$ and $(0,2)$:

```
i3 : P1=getComplementBasis({1,1},A|B);
     P2=getComplementBasis({0,2},A|B);
     P=P1|P2
```

```
o3 = {-1, -1} | -x_1y_2         y_1^2   y_1y_2         y_2^2   |
     {-1, -1} | x_0y_1          -y_0y_1 -2y_1^2-y_0y_2 -y_1y_2 |
     {-1, -1} | -x_0y_0+x_1y_0 y_0^2   y_0y_1         y_1^2   |
```

We can recover A and B (up to $\mathbf{k}$-linear combinations of the columns) by computing $\mathrm{Ann}_R(P)$ in degrees $(1,0)$ and $(0,1)$:

```
i4 : P'=transpose sum(rank source P,i->homomorphism(P_{i}));
     getAnnihilatorBasis({1,0},P')|getAnnihilatorBasis({0,1},P')

o4 = {-1, 0} | x_0 0   0       y_0 y_1 0   0   y_2 0   0   |
     {-1, 0} | 0   x_1 0       0   0   y_0 y_2 y_1 0   0   |
     {-1, 0} | 0   0   x_0+x_1 0   0   0   0   y_0 y_1 y_2 |
```

2.5 Liaison Theory

In this section we collect the tools for our liaison construction.

Notation. We start by precising the objects in consideration. By a *curve* in $\mathbf{P}^1 \times \mathbf{P}^2$ we mean an equidimensional subschema of codimension 2 which is locally a complete intersection.

Definition 2.33. *A curve C is* geometrically linked *to a curve C' in $\mathbf{P}^1 \times \mathbf{P}^2$ by a complete intersection X if C and C' have no common component and $C \cup C' = X$ on the level of schemes.*

Geometric linkage is a specialization of algebraic linkage:

Definition 2.34. *Two curves C and C' are* algebraically linked *by a complete intersection X if the following holds*

1. $\mathscr{I}_C/\mathscr{I}_X \cong \mathcal{H}om_{\mathscr{O}_\mathbf{P}}(\mathscr{O}_{C'}, \mathscr{O}_X)$.
2. $\mathscr{I}_{C'}/\mathscr{I}_X \cong \mathcal{H}om_{\mathscr{O}_\mathbf{P}}(\mathscr{O}_{C}, \mathscr{O}_X)$.

Indeed it is not hard to see that if C and C' are linked geometrically then they are also linked algebraically. Conversely, if the schemes C and C' are algebraically linked and have no common components then they are also linked geometrically.

Numerics. We can compute the degree and genus of linked curves as follows.

Proposition 2.35 (Exact sequence of Liaison)**.** *Let C be a curve of bidegree (d_1, d_2) linked to C' by a complete intersection X. Let (a_1, b_1) and (a_2, b_2) denote the bidegrees of the forms defining X and let $a = a_1 + a_2$ and $b = b_1 + b_2$.*

(i) There is a short exact sequence

$$0 \to \omega_C \to \omega_X \to \mathscr{O}_C(a-2, b-3) \to 0.$$

(ii) C' has bidegree

$$(d_1', d_2') = (b_1 b_2 - d_1, a_1 b_2 + a_2 b_1 - d_2)$$

and for arithmetic genus we have the equation

$$p_a(C') - p_a(C) = (d_1 - d_1')(a-2) + (d_2' - d_2)(b-3).$$

Proof. Consider the long exact sequence arising from the standard exact sequence

$$0 \to \mathscr{I}_{C/X} \to \mathscr{O}_X \to \mathscr{O}_C \to 0$$

by applying $\mathcal{H}om(-, \omega_{\mathbf{P}})$. Clearly, $\mathcal{E}xt^1(\mathscr{I}_C, \omega_{\mathbf{P}}) = 0$ and we obtain

$$0 \to \omega_C \to \omega_X \to \mathcal{E}xt^2(\mathscr{I}_C, \omega_{\mathbf{P}}) \to 0$$

As C is linked to C' via X we have $\mathcal{E}xt^2(\mathscr{I}_C, \omega_{\mathbf{P}}) = \mathscr{O}_{C'}(a-2, b-3)$. The formula for the genus follows immediately. For α the class of the pullback of a point in $\mathbf{P}^1$ and β the pullback of the class of a line in $\mathbf{P}^2$ we have $[C]+[C'] = [X] = (b_1 b_2)\beta^2 + (a_1 b_2 + a_2 b_1)\alpha\beta$ in the Chow ring of $\mathbf{P}$. $\square$

The Mapping Cone Construction. The classical mapping cone construction [PS74] for a resolution of the linked curve works in our setting only under certain additional assumptions. We recall the following local result.

Proposition 2.36. *Let R be regular local ring, $I \subset R$ an ideal such that R/I is a Cohen-Macaulay ring of codimension d. Let $f = (f_1, \ldots, f_d)$ be a regular sequence contained in I. Let $F_\bullet$ be a projective resolution of R/I and $G_\bullet$ a projective resolution of R/f and let $\alpha : F_\bullet \to G_\bullet$ be the morphism of complexes induced by the inclusion $f \subset I$. Let I' be the ideal obtained under linkage of I via f. Then a projective resolution of R/I' is given by the mapping cone of $\alpha^\vee : G_\bullet^\vee \to F_\bullet^\vee$.*

Proof. This is [PS74, Proposition 2.6]. For the convenience of the reader we give the proof here. Consider the commutative diagram

$$\begin{tikzcd}
H_0(F^\vee) \arrow[r, "\varphi"] \arrow[d, "\sim"] & H_0(G^\vee) \arrow[d] \\
\operatorname{Ext}^d_R(R/I, R) \arrow[r, "\operatorname{Ext}^d_R(p, R)"] & \operatorname{Ext}^d_R(R/f, R)
\end{tikzcd}$$

with the map between the Ext groups is induced by the projection $p : R/f \to R/I$. There is an isomorphism of functors $\mathrm{Ext}^d_R(\bullet, R) \cong \mathrm{Hom}_{R/f}(\bullet, R/f)$ on the category of finitely generated R/f modules. Hence we obtain a commutative diagram

$$\begin{tikzcd} H_0(F^\vee) \arrow[r, "\varphi"] \arrow[d, "\sim"] & H_0(G^\vee) \arrow[d, "\sim"] \\ \mathrm{Hom}(R/I, R/f) \arrow[r, "\mathrm{Hom}(p, R/f)"] & \mathrm{Hom}(R/f, R/f) \end{tikzcd}$$

which proves the result. □

Let us introduce one more bit of notation. For a sheaf $\mathscr{F}$ on $\mathbf{P}$ and some bidegree (a, b) we set $H^i_{\geq(a,b)}(\mathscr{F}) := \bigoplus_{i\geq a, j\geq b} H^i(\mathscr{F}(i, j))$.

Proposition 2.37 (Mapping Cone)**.** *Let C be a curve in $\mathbf{P}$ linked to a curve C' via a complete intersection X defined by forms of bidegrees (a_i, b_i). We set $a_0 = \min(a_1, a_2)$ and $b_0 = \min(b_1, b_2)$. Suppose*

$$0 \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{O}_{\mathbf{P}} \to \mathscr{O}_C \to 0$$

is a resolution of $\mathscr{O}_C$ by locally free sheaves such that the map $\mathscr{F}_1 \to \mathscr{I}_C$ is onto on global sections in bidegree (a_0, b_0). Let $\mathscr{G}_\bullet$ denote the minimal free resolution of $\mathscr{O}_X$. Then the mapping cone $[\mathscr{F}^\vee(-a, -b)_\bullet \to \mathscr{G}^\vee(-a, -b)_\bullet]$ is a (in general not minimal) locally free resolution of $\mathscr{O}_{C'}$.

Proof. By assumption, the truncated vanishing ideal $I_{C,\geq(a_0,b_0)}$ has the locally free resolution

$$0 \to \Gamma_{\geq(a_0,b_0)}(\mathscr{F}_2) \to \Gamma_{\geq(a_0,b_0)}(\mathscr{F}_1) \to I_{C,\geq(a_0,b_0)} \to 0.$$

By the projectivity of $\Gamma_*(\mathscr{G}_1)$ and $\Gamma_*(\mathscr{G}_2)$ the inclusion $\iota : I_X \to I_{C,\geq(a_0,b_0)}$ extends to a map of sheafified complexes $\iota : \mathscr{G}_\bullet \to \mathscr{F}_\bullet$. The twisted dual mapping cone $\mathscr{M}_\bullet = [\mathscr{F}^\vee_\bullet \otimes \mathscr{O}_{\mathbf{P}}(-a, -b) \to \mathscr{G}^\vee_\bullet \otimes \mathscr{O}_{\mathbf{P}}(-a, -b)]$ yields a long exact homology sequence

$$\ldots \to H_1(\mathscr{M}_\bullet) \to H_0(\mathscr{F}^\vee_\bullet(-a, -b)) \to H_0(\mathscr{G}^\vee_\bullet(-a - b)) \to H_0(\mathscr{M}_\bullet) \to 0$$

As $\mathscr{F}$ resolves $\mathscr{O}_C$ we have

$$H_0(\mathscr{F}^\vee(-a, -b)) = \mathscr{E}xt^2(\mathscr{O}_C, \mathscr{O}_{\mathbf{P}}(a, b)) = \omega_C(2 - a, 3 - b) = \mathscr{I}_{C'}/\mathscr{I}_X$$

and $H_0(\mathscr{G}^\vee(-a, -b)) = \omega_X(2 - a, 3 - b) = \mathscr{O}_X$ as X is a complete intersection (and hence Gorenstein). Finally, Proposition 2.36 shows that the map $\mathscr{I}_{C'/X} \to \mathscr{O}_X$ coming from the homology sequence is the canonical inclusion. □

3 Unirationality Results

In this chapter we present the main result of this thesis. Our method of proof is to show for each case (k,g) the existence of a unirational component H of a Hilbert scheme of curves C of genus g and bidegree $\left(k, \lceil \frac{2g}{3}+2 \rceil\right)$ in the multiprojective space $\mathbf{P} = \mathbf{P}^1 \times \mathbf{P}^2$ and satisfying certain Zariski-open "good" properties. We then show that H dominates the Hurwitz space $\mathscr{H}_{g,k}$.

The unirationality is established by giving a construction in free parameters of the general curve C with "good" properties. This is done in two different ways depending on the particular case (k,g). Our first construction relies on linkage of curves in $\mathbf{P}$ and covers the cases $k=6$ and $5 \le g \le 28$ and $g = 30, 31, 33, 35, 36, 40, 45$. The second construction expresses the curve C as the dependency locus of a vector bundle on $\mathbf{P}$ which in turn can be constructed from a certain deficiency module. This construction yields for $k=6$ the cases $5 \le g \le 25$ and $g = 27, 28, 29, 30, 33, 34, 39$. Moreover, we obtain for $k=7$ the cases $6 \le g \le 12$. Basically, we also recover the classical cases of the unirationality for $3 \le k \le 5$ and $g \ge k-1$. Despite some redundancy in the outcome we want to outline both constructions as they are rather different in nature and may serve as a basis for future work extending the presented results.

We establish a posteriori the existence of curves with "good" properties by implementing the construction in *Macaulay2* and computing a single curve over a finite field. We remark that in this point our approach differs substantially from many classical unirationality proofs for moduli spaces. The advantage of this implementation to us is that we can easily produce a general curve in order to study applications.

3.1 Construction via Liaison

For $g \ge 5$, let $f : C \to \mathbf{P}^1$ be an element of $\mathscr{H}_{g,6}$ and let $\mathscr{O}(D_1) = f^*\mathscr{O}_{\mathbf{P}^1}(1)$ be the 6-gonal bundle. We assume that C has a line bundle $\mathscr{O}(D_2)$ such that $|D_2|$ is a complete base point free $\mathfrak{g}^2_d$ with $d = d(g) = \left\lceil \frac{2g}{3}+2 \right\rceil$ minimal under the condition that the Brill-Noether number $\rho(g,2,d) \ge 0$. Suppose further that the map

$$\varphi : C \xrightarrow{|D_1|,|D_2|} \mathbf{P}H^0(\mathscr{O}(D_1)) \times \mathbf{P}H^0(\mathscr{O}(D_2)) = \mathbf{P}$$

is an embedding. In particular, this is the case when we assume that the plane model has only ordinary double points and no other singularities and for any node p the points in the preimage of p under $C \to \mathbf{P}^2$ are not identified under the map to $\mathbf{P}^1$. Hence, we will identify C with its image under φ. Furthermore, we assume that the map $H^0(\mathscr{O}_{\mathbf{P}}(a,3)) \to H^0(\mathscr{O}_C(a,3))$ is of maximal rank for all $a \geq 1$. To simplify matters, assume $g \equiv 0\ (12)$ for the moment. By the maximal rank assumption, we have

$$a_{\mathrm{Cubic}} := \min\{a \mid H^0(\mathscr{I}_C(a,3)) \neq 0\} = \frac{g}{4}$$

and $h^0(\mathscr{I}_C(a_{\mathrm{Cubic}},3)) = 3$. Let $X = V(f_1, f_2)$ be the complete intersection defined by two general sections $f_i \in H^0(\mathscr{I}_C(a_i, b_i))$ of bidegrees $(a_1, b_1) = (a_2, b_2) = (a_{\mathrm{Cubic}}, 3)$. The curve C', obtained by liaison of C by X, is smooth of bidegree $(3, \frac{5}{6}g - 2)$ and genus $g' = \frac{g}{2} - 3$ with $h^0(\mathscr{I}_{C'}(a_{\mathrm{Cubic}}, 3)) \geq 2$.

The geometric situation is understood best when thinking of C as a family of collections of plane points over $\mathbf{P}^1$. We expect the general fiber of C to be a collection of 6 points in $\mathbf{P}^2$ which are cut out by 4 cubics. We expect a finite number ℓ of distinguished fibers where the points lie on a conic as this is a codimension 1 condition on the points. Since the residual three points under liaison are collinear exactly in the distinguished fibers we can compute ℓ by examining the geometry of C'. The projection of C' to $\mathbf{P}^2$ yields a divisor D_2' of degree $d' > g' + 2$. Our claim is that $\ell = d' - (g' + 2)$. Indeed, the image of C' under the associated map

$$\psi : C' \to \mathbf{P}^1 \times \mathbf{P}H^0(C', \mathscr{O}(D_2')) = \mathbf{P}^1 \times \mathbf{P}^{d'-g'}$$

lies on the graph of the projection $S \to \mathbf{P}^1$ where S is a 3-dimensional scroll of degree $d' - g' - 2$ swept out by the 3-gonal series $|D_1'|$, i.e.

$$\psi(C') \subset \mathbf{P}^1 \times S = \bigcup_{D \in |D_1'|} \{D\} \times \overline{D}.$$

See [Sch86] for a proof of this fact. C' is obtained from $\psi(C')$ by projection from a linear subspace $\mathbf{P}^1 \times V \subset \mathbf{P}^1 \times \mathbf{P}^{d'-g'}$ of codimension 3. A general space V intersects S in precisely $d' - g' - 2$ points lying in distinct fibers over $\mathbf{P}^1$. Clearly, under the projection the points of $D \in |D_1'|$ are mapped to 3 collinear points if and only if V meets the corresponding fiber of S.

To keep things neat, we consider again the case $g \equiv 0\ (12)$ which implies $\ell = \frac{1}{3}g - 1$. Suppose further that $\ell \equiv 1\ (3)$. If we assume that $H^0(\mathscr{O}_{\mathbf{P}}(a,2)) \to H^0(\mathscr{O}_{C'}(a,2))$ is of maximal rank for all $a \geq 1$ then

$$a_{\mathrm{Conic}} = \min\{a \mid H^0(\mathscr{I}_{C'}(a,2)) \neq 0\} = \frac{g' + 2\ell + 1}{3}$$

and $h^0(\mathscr{I}_{C'}(a_{\mathrm{Conic}},2)) = 2$. Let $X' = V(f_1', f_2')$ be defined by two general forms $f_i' \in H^0(\mathscr{I}(a_i', b_i'))$ of bidegrees $(a_1', b_1') = (a_2', b_2') = (a_{\mathrm{Conic}}, 2)$ and let C'' denote the curve that is linked to C' via X'. The general fiber of C''

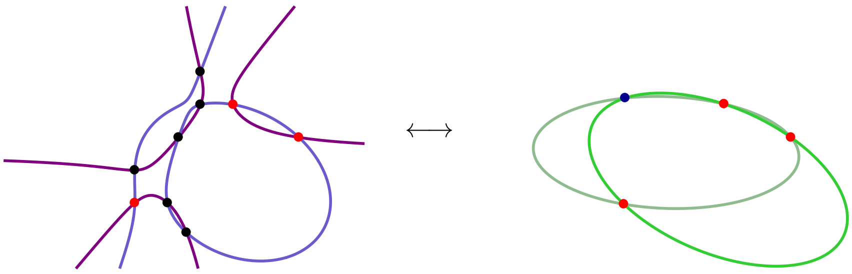

Figure 3.1: Liaison in the general fiber.

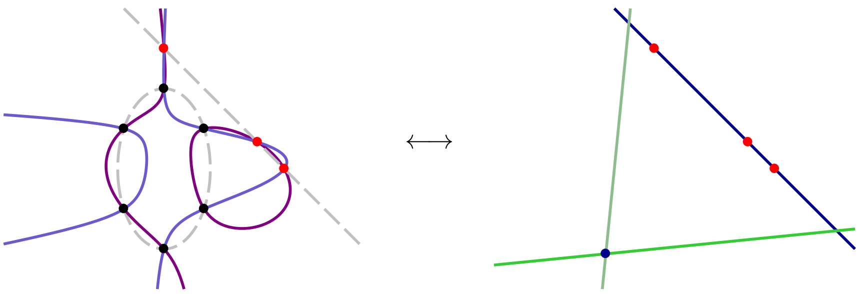

Figure 3.2: Liaison in a special fiber.

consists of a single point. In a distinguished fiber the conics of the complete intersection are reducible and have the line spanned by the points of the fiber of C' as a common factor. Hence, C'' is a rational curve together with ℓ lines. The rational curve has degree

$$d'' = \frac{g' + 2\ell - 2}{3} = \frac{7}{18}g - \frac{7}{3}.$$

Turning things around we see that the difficulty lies in reversing the first linkage step. Indeed, a simple counting argument shows that for any g, the union of ℓ general lines in $\mathbf{P}$ and the graph of a general rational normal curve of degree d'' we have

$$\min\{a \in \mathbf{Z} | \; H^0(\mathscr{I}_{C''}(a,2)) \neq 0\} = \left\lceil \frac{2d''+3\ell}{5} \right\rceil - 1 \leq a_{\text{Conic}}.$$

Hence, we always obtain a trigonal curve C' as desired. For general choices of C'' and X' we expect that the map $H^0(\mathscr{O}_{\mathbf{P}}(a,3)) \to H^0(\mathscr{O}_{C'}(a_{\text{Cubic}},3))$ is of maximal rank. In the case $g \equiv 0(12)$, this yields $h^0(\mathscr{I}_{C'}(a_{\text{Cubic}},3)) = -\frac{g}{4}+12$, hence $g < 48$. Checking all congruency classes of g, we expect C' can be linked to a general curve C exactly in the cases

$$5 \leq g \leq 28 \text{ or } g = 30, 31, 33, 35, 36, 40, 45. \tag{3.1}$$

Table 3.1 lists the appearing numbers for all values of g in (3.1).

Summarizing, we obtain for g among (3.1) the following unirational construction for curves in $\mathscr{H}_{g,6}$:

Construction Method 3.2.

1. We start with a general rational curve of degree d'' in $\mathbf{P}$ together with a collection of ℓ general lines. Call the union C''.
2. We choose two general forms $f_i' \in H^0(\mathscr{I}_{C''}(a_i', b_i'))$, $i = 1, 2$, that define a complete intersection X' and obtain a trigonal curve $C' = \overline{X' \smallsetminus C''}$ of degree d' and genus g'.
3. We choose two general forms $f_i \in H^0(\mathscr{I}_{C'}(a_i, b_i))$, $i = 1, 2$, that define a complete intersection X and obtain a 6-gonal curve $C = \overline{X \smallsetminus C'}$.

It remains to show that the construction actually yields a parametrization of the Hurwitz spaces.

3.2 Proof of The Dominance

Theorem 3.3. *For all (g,d) as in Table 3.1, there is a unirational component H_g of the Hilbert scheme $\mathrm{Hilb}_{(6,d),g}(\mathbf{P})$ of curves in $\mathbf{P}$ of bidegree $(6,d)$ and genus g. The generic point of H_g corresponds to a smooth absolutely irreducible curve C such that the map $H^0(\mathscr{O}_{\mathbf{P}}(a,3)) \to H^0(\mathscr{O}_C(a,3))$ is of maximal for all $a > 1$.*

Proof. The crucial part is to prove the existence of a curve with the desired properties. Code 3.5 implements the construction above for any given value of g in (3.1) and establishes the existence of a smooth and absolutely irreducible curve C_p of given genus and bidegree defined over a prime field $\mathbf{F}_p$. This computation can be regarded as the reduction of a computation over $\mathbf{Q}$ which yields some curve C_0. This curve is already defined over the rationals, since all construction steps invoke only Groebner basis computations.

g	d	$(a_1,b_1),(a_2,b_2)$	g'	d'	$(a'_1,b'_1),(a'_2,b'_2)$	ℓ	d''
5	6	$(2,3),(2,3)$	2	6	$(3,2),(2,2)$	2	2
6	6	$(2,3),(1,3)$	0	3	$(1,2),(1,2)$	1	0
7	7	$(2,3),(2,3)$	1	5	$(2,2),(2,2)$	2	1
8	8	$(3,3),(2,3)$	2	7	$(3,2),(3,2)$	3	2
9	8	$(2,3),(2,3)$	0	4	$(2,2),(2,2)$	2	2
10	9	$(3,3),(3,3)$	4	9	$(4,2),(4,2)$	3	4
11	10	$(3,3),(3,3)$	2	8	$(4,2),(4,2)$	4	4
12	10	$(3,3),(3,3)$	3	8	$(4,2),(3,2)$	3	3
13	11	$(4,3),(3,3)$	4	10	$(5,2),(4,2)$	4	4
14	12	$(4,3),(4,3)$	5	12	$(6,2),(5,2)$	5	5
15	12	$(4,3),(4,3)$	6	12	$(5,2),(5,2)$	4	4
16	13	$(4,3),(4,3)$	4	11	$(5,2),(5,2)$	5	4
17	14	$(5,3),(5,3)$	8	16	$(7,2),(7,2)$	6	6
18	14	$(5,3),(4,3)$	6	13	$(6,2),(6,2)$	5	6
19	15	$(5,3),(5,3)$	7	15	$(7,2),(7,2)$	6	7
20	16	$(6,3),(5,3)$	8	17	$(8,2),(8,2)$	7	8
21	16	$(5,3),(5,3)$	6	14	$(7,2),(6,2)$	6	6
22	17	$(6,3),(6,3)$	10	19	$(9,2),(8,2)$	7	8
23	18	$(6,3),(6,3)$	8	18	$(9,2),(8,2)$	8	8
24	18	$(6,3),(6,3)$	9	18	$(8,2),(8,2)$	7	7
25	19	$(7,3),(6,3)$	10	20	$(9,2),(9,2)$	8	8
26	20	$(7,3),(7,3)$	11	22	$(10,2),(10,2)$	9	9
27	20	$(7,3),(7,3)$	12	22	$(10,2),(10,2)$	8	10
28	21	$(7,3),(7,3)$	10	21	$(10,2),(10,2)$	9	10
30	22	$(8,3),(7,3)$	12	23	$(11,2),(10,2)$	9	10
31	23	$(8,3),(8,3)$	13	25	$(12,2),(11,2)$	10	11
33	24	$(8,3),(8,3)$	12	24	$(11,2),(11,2)$	10	10
35	26	$(9,3),(9,3)$	14	28	$(13,2),(13,2)$	12	12
36	26	$(9,3),(9,3)$	15	28	$(13,2),(13,2)$	11	13
40	29	$(10,3),(10,3)$	16	31	$(15,2),(14,2)$	13	14
45	32	$(11,3),(11,3)$	18	34	$(16,2),(16,2)$	14	16

Table 3.1: Numerical data for all cases of the linkage construction

By semicontinuity, C_0 is also smooth, absolutely irreducible and of maximal rank.

Again, by semicontinuity, there exists a nonempty Zariski open neighborhood $U \subset \mathrm{Hilb}_{(6,d),g}(\mathbf{P})$ of points corresponding to smooth absolutely irreducible curves that fulfill the maximal rank condition. Let $\mathbf{A}^N$ be the space of parameters for all the choices made in the construction, i.e. the space of coefficients of the polynomials defining C'' and the complete intersections X and X'. The construction then translates to a rational map $\mathbf{A}^N \dashrightarrow U$ defined over $\mathbf{Q}$ and we set H_g to be the closure of the image of this map. □

It remains to show that H_g parametrizes the Hurwitz space.

Theorem 3.4. *For g among (3.1) and H_g as in Theorem 3.3 there is a dominant rational map*

$$H_g \dashrightarrow \mathscr{H}_{g,6}.$$

This implies that $\mathscr{H}_{g,6}$ is unirational.

Proof. Using Code 3.5 again, we check for any given value of g in (3.1) there is a point in H_g corresponding to a smooth absolutely irreducible curve $C \subset \mathbf{P}$ such that the projection onto $\mathbf{P}^1$ is simply branched and the bundle $L_2 = \varphi^* \mathscr{O}_{\mathbf{P}}(0,1)$ is a smooth point in the corresponding $W^2_d(C)$. By semicontinuity, the locus of curves with this property is open and dense in H_g. Hence, we have a rational map $H_g \dashrightarrow \mathscr{H}_{g,6}$. The locus of curves in $\mathscr{H}_{g,6}$ having a smooth component of the Brill-Noether locus of expected dimension is also open and contains the image of $[C]$ under this map. Since $\mathscr{H}_{g,6}$ is irreducible this locus is dense. This proves the theorem. □

3.3 Computational Verification

The following Code for *Macaulay2* [GS] realizes the unirational construction of a 6-gonal curve of genus g as in (3.1) over a finite field with random choices for all parameters.

In order to explain the single steps in the computation, we also print the most relevant parts of the output for the example case $g = 24$.

Code 3.5. We start with the following initialization:

```
i1 : Fp=ZZ/32009; -- a finite field
     S=Fp[x_0,x1,y_0..y_2,Degrees=>{2:{1,0},3:{0,1}}];
        -- Cox-ring of P^1 x P^2
     m=ideal basis({1,1},S);
        -- irrelevant ideal
     setRandomSeed("HurwitzSpaces");
        -- initialization of the random number generator
```

The following functions handle the numerics of the construction:

```
i2 : expHilbFuncIdealSheaf=(g,d,a)->
        max(0,(a_0+1)*(a_1+2)*(a_1+1)/2-(a_0*d_0+a_1*d_1+1-g))
        -- expected number of sections of the ideal sheaf

     linkedGenus=(g,d,F,G)->(
        pX:=binomial(F_0+G_0-1,1)*binomial(F_1+G_1-1,2)-
            (F_0-1)*binomial(F_1-1,2)-(G_0-1)*binomial(G_1-1,2);
            -- genus of the complete intersection
        pX-d_0*(F_0+G_0-2)-d_1*(F_1+F_1-3)-1+g)
        -- genus of the linked curve

     linkedDegree=(g,d,F,G)->{F_1*G_1-d_0,F_0*G_1+G_0*F_1-d_1}
        -- bidegree of the linked curve
```

The first step is to determine the degree d'' of the rational curve and the number of lines ℓ. We start by computing the bidegrees of the forms that define the complete intersection for the linkage to the trigonal curve:

```
i3 : g=24;
     d={6,ceiling(-g/3+g+2)};
        -- choose the second degree Brill-Noether general
     a=for i from 0 do
         if expHilbFuncIdealSheaf(g,d,{i,3})!=0 then break i;
        -- find the minimal value a s.t. H^0(IC(a,3)) nonzero
     if expHilbFuncIdealSheaf(g,d,{a,3})==1 then
        fX={{a+1,3},{a,3}} else fX={{a,3},{a,3}};
        -- choose bidegrees of forms for the complete intersection
     (d,fX)

o3 = ({6, 18}, {{6, 3}, {6, 3}})
```

The genus and degree of the trigonal curve and the number of lines:

```
i4 : g'=linkedGenus(g,d,fX_0,fX_1);
     d'=linkedDegree(g,d,fX_0,fX_1);
     l=d'_1-g'-2;
     (g',d',l)

o4 = (9,{3,18},7)
```

We compute the bidegrees for the complete intersection for the linkage to the rational curve

```
i5 : b=for i from 0 do
         if expHilbFuncIdealSheaf(g',d',{i,2})!=0 then break i;
     if expHilbFuncIdealSheaf(g',d',{b,2})==1 then
        fX'={{b+1,2},{b,2}} else fX'={{b,2},{b,2}};
     d''=linkedDegree(g'+2*l,d'+{0,l},fX'_0,fX'_1);
     dRat={{ceiling(d''_1/2),1},{floor(d''_1/2),1}};
     (d'',fX')

o5 = ({1, 7},{{8, 2}, {8, 2}})
```

The second step is the actual construction: First, we choose a rational curve and random lines and compute the saturated vanishing ideal $I_{C''}$ of their union:

```
i6 : ICrat=saturate(ideal random(S^1,S^(-dRat)),m);
     ILines=apply(l,i->ideal random(S^1,S^{{-1,0},{0,-1}}));
     time IC''=saturate(intersect(ILines|{ICrat}),ideal(x_0*y_0));
        -- used 1.29537 seconds
```

Next, we choose random forms in $I_{C''}$ of degree b (resp. of $b+1$) that define the complete intersection X' and compute the saturated vanishing ideal $I_{C'}$ of the trigonal curve C'.

```
i7 : IX'=ideal(gens IC'' * random(source gens IC'',S^(-fX')));
     IC'=IX':ICrat;
     time scan(l,i->IC'=IC':ILines_i);
     time IC'sat=saturate(IC',ideal(x_0*y_0));
        -- used 2.06236 seconds
        -- used 23.7319 seconds
```

In the final step, we compute the vanishing ideal of the 6-gonal curve C by linking C' with a complete intersection X given by random forms in $I_{C'}$ of degree a (resp. $a+1$).

```
i8 : IX=ideal(gens IC'sat * random(source gens IC'sat,S^(-fX)));
     time IC=IX:IC';
     time ICsat=saturate(IC,ideal(x_0*y_0));
        -- used 15.7815 seconds
        -- used 3.84807 seconds
```

We check that C is of maximal rank in the degrees $(a, 3)$ by looking at the minimal generators of the saturated vanishing ideal: For $h^0(\mathscr{I}_C(a, 3))$ we expect under maximal rank the values $(0, 0, 0, 0, 0, 0, 3, 7, 15, \ldots)$. Since $h^0(\mathscr{I}_C(a, 2)) = 0$ for all a we expect 3 minimal generators in degree $(6, 3)$ and 1 generator in degree $(7, 3)$:

```
i9 : tally degrees ideal mingens gb ICsat

o9 = Tally{{0, 18} => 1}
           {1, 14} => 5
           {1, 15} => 4
           {2, 8} => 2
           {2, 9} => 8
           {3, 6} => 9
           {4, 4} => 2
           {4, 5} => 8
           {5, 4} => 7
           {6, 3} => 3
           {7, 3} => 1
```

In order to check irreducibility, we compute the plane model Γ of C:

```
i10 : Sel=Fp[x_0,x_1,y_0..y_2,MonomialOrder=>Eliminate 2];
         -- elimininination order
      R=Fp[y_0..y_2]; -- coordinate ring of P^2
      IGammaC=sub(ideal selectInSubring(1,gens gb sub(ICsat,Sel)),R);
         -- ideal of the plane model
```

We check that Γ is a curve of desired degree and genus and its singular locus Δ consists only of ordinary double points:

```
i11 : distinctPoints=(J)->(
         singJ:=minors(2,jacobian J)+J;
         codim singJ==3)

i12 : IDelta=ideal jacobian IGammaC + IGammaC; -- singular locus
      distinctPoints(IDelta)

o12 = true

i13 : delta=degree IDelta;
      dGamma=degree IGammaC;
      gGamma=binomial(dGamma-1,2)-delta;
      (dGamma,gGamma)==(d_1,g)

o13 = true
```

We compute the free resolution of I_Δ:

```
i14 : time IDelta=saturate IDelta;
      betti res IDelta
         -- used 55.063 seconds

             0 1 2
o14 = total: 1 8 7
          0: 1 . .
          1: . . .
          2: . . .
          3: . . .
          4: . . .
          5: . . .
          6: . . .
          7: . . .
          8: . . .
          9: . . .
         10: . . .
         11: . . .
```

```
12: . . .
13: . 8 .
14: . . 7
```

This is the resolution as expected. Hence, C is absolutely irreducible by (2.9) and $\mathscr{O}(D_2)$ is a smooth point of the Brill-Noether locus by (2.11).
It remains to verify that C is actually smooth and simply branched. We compute the vanishing ideal $I_B \subset K[x_0, x_1]$ of the locus B in $\mathbf{P}^1$ of points with non-reduced fiber.

```
i15 : gensICsat=flatten entries mingens ICsat;
      Icubics=ideal select(gensICsat,f->(degree f)_1==3);
         -- select the cubic forms
      Jacobian=diff(matrix{{y_0}..{y_2}},gens Icubics);
         -- compute the jacobian w.r.t. to vars of P^2
      IGraphB=minors(2,Jacobian)+Icubics;
      time IGraphBsat=saturate(IGraphB,ideal(x_0*y_0));
         -- used 60.2963 seconds
```

We check that the fibers over B are disjoint from the preimages of the double points of the plane model. This shows that C is smooth. To speed up the computation we compute only the saturation at (x_0, y_0) instead of the full irrelevant ideal of $\mathbf{P}$. All that remains to check for this is that we do not remove points that lie in $V(x_0, y_0)$.

```
i16 : dim(sub(IDelta,S)+IGraphBsat)==0
      ISing0=sub(IDelta,S)+IGraphBsat;
      time ISing=saturate(ISing0,ideal(S_0*S_2));
      degree ISing==0

o16 = true

i17 : dim(ISing0+ideal(S_0*S_2))<=1
        -- the ideal of a point in R has dimension 2.

o17 = true
```

Finally, we verify that B is reduced of expected degree $2g + 10$ and hence that C is simply branched.

```
i18 : time IGraphBsat=saturate(IGraphB,ideal(x_0*y_0));
      gensIGraphBsat=flatten entries mingens IGraphBsat;
      IB=ideal select(gensIGraphBsat,f->(degree f)_1==0);
      degree radical IB==2*g+10

o18 = true
```

It takes approximately 5 hours CPU-time on a 2.4 GHz processor to check all cases.

Remark 3.6. We want to point out two issues concerning the computational verification:

(i) The restriction to finite fields in the *Macaulay2* computation in the appendix is only due to limitations in computational power. For very small values of g, i.e. $g \leq 15$, it is still possible to compute examples over the rationals if all coefficients are chosen among integers of small absolute value.

(ii) The reduction of C_0 modulo p gives a curve C_p with desired properties for p in an open part of $\mathrm{Spec}(\mathbf{Z})$. Hence, the main theorem is also true in almost all characteristics p. One way to extend it to all prime numbers would be to keep track of all denominators in a computation over the rationals and check case by case the primes where a bad reduction happens. This is computationally also out of reach at the moment.

3.4 Constructions via Free Resolutions and Deficiency Modules

In [CR84] Chang and Ran prove the unirationality of the moduli space $\mathscr{M}_g$ in the cases $g = 11, 12, 13$ by exploiting the fact that the general curve C of genus g can be recovered from a module associated to a spatial model of C. More precisely, for a curve C in $\mathbf{P}^3$ they consider the Hartshorne-Rao module $M = \bigoplus_{i\in\mathbf{Z}} H^1(\mathbf{P}^3, \mathscr{I}_{C/\mathbf{P}^3}(i))$. The unirationality of $\mathscr{M}_g$ then follows from the observation that in the considered cases the space of Hartshorne-Rao modules is unirational. This seminal result motivates to establish a similar theory for curves in $\mathbf{P}^1 \times \mathbf{P}^2$ to prove unirationality for $\mathscr{H}_{g,k}$. It turns out that a certain submodule of the dual of the Hartshorne-Rao module is an appropriate substitute.

Resolutions of Curves in $\mathbf{P}^1 \times \mathbf{P}^2$. As in the preceeding section we impose a number of conditions on our curve. Let $k \geq 5$. Let $C \in \mathscr{H}_{g,k}$ and let $\mathscr{O}(D_1)$ denote the special pencil. Assume that C satisfies the condition

(G1) Let d be the smallest integer subject to the condition that the Brill-Noether number $\rho(g, 2, d) \geq 0$. There is a line bundle $\mathscr{O}(D_2)$ on C of degree d such that $|D_2|$ is a complete base point free $\mathfrak{g}^2_d$ and such that the map
$$C \xrightarrow{|D_1|,|D_2|} \mathbf{P}^1 \times \mathbf{P}^2$$
is an embedding.

We want to use Theorem 2.19 to compute a resolution of $\mathscr{I}_{C/\mathbf{P}}$. To this end we compute the Beilinson monad $\mathcal{K}_I^\bullet$ for the ideal sheaf twisted by some $\mathscr{O}_{\mathbf{P}}(s,t)$. To obtain a resolution we need to twist in such a way that all terms

$\mathcal{K}_I^i = 0$ for $i > 0$. The following assumptions allow us to derive a fairly simple resolution of $\mathscr{I}_C$.

(G2) For a general point $P = V(f) \in \mathbf{P}^1$ the fiber $C_p = Z \subset \mathbf{P}^2$ is a reduced scheme of length k with minimal free resolution as in (2.9).

(G3) C is of maximal rank.

The minimal degree of generators of I_Z is $t = \lceil (\sqrt{9+8k} - 3)/2 \rceil$. Let s be minimal under the condition that the Hilbert function

$$h^0(\mathscr{I}_{C/\mathbf{P}}(s,t)) = \binom{s+1}{1}\binom{t+2}{2} - (sk + td + 1 - g) > 0.$$

We also require the following vanishing of cohomology groups

(G4) $H^0(\mathscr{I}_Z(s-1,t) \otimes \pi_2^*\Omega^1_{\mathbf{P}^2}(1)) = H^0(\mathscr{I}_Z(s,t) \otimes \pi_2^*\Omega^1_{\mathbf{P}^2}(1)) = 0.$

Definition 3.7. *A curve $C \in \mathscr{H}_{g,k}$ that satisfies (G1)–(G4) is a* good *curve.*

As before the conditions above are Zariski-open and hence the existence of single good curve implies that the set of good curves is dense in $\mathscr{H}_{g,k}$.

Proposition 3.8. *Let $C \in \mathscr{H}_{g,k}$ be a good curve of genus g and $\mathscr{I}'_C = \mathscr{I}_C(s,t)$ with s and t as above. Then*

$$0 \to \mathscr{F}_3 \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{I}'_C \to 0 \tag{3.9}$$

with

$$\begin{aligned}
\mathscr{F}_1 &= H^0(\mathscr{I}'_C) \otimes \mathscr{O}_{\mathbf{P}} \oplus H^1(\mathscr{I}'_C(-1,0)) \otimes \mathscr{O}_{\mathbf{P}}(-1,0) \\
&\quad \oplus H^1(\mathscr{I}'_C \otimes \pi_2^*\Omega^1_{\mathbf{P}^2}(1)) \otimes \mathscr{O}_{\mathbf{P}}(0,-1) \\
\mathscr{F}_2 &= H^1(\mathscr{I}'_C(-1,0) \otimes \pi_2^*\Omega^1_{\mathbf{P}^2}(1)) \otimes \mathscr{O}_{\mathbf{P}}(-1,-1) \\
&\quad \oplus H^1(\mathscr{I}'_C(0,-1)) \otimes \mathscr{O}_{\mathbf{P}}(0,-2) \\
\mathscr{F}_3 &= H^1(\mathscr{I}'_C(-1,-1)) \otimes \mathscr{O}_{\mathbf{P}}(-1,-2)
\end{aligned}$$

is a minimal free resolution of $\mathscr{I}'_C$.

Proof. We consider the monad $\mathcal{K}_I^\bullet(\mathscr{I}'_C)$ from (2.19). As (3.9) is a subcomplex of this monad and we only need to show the vanishing of all cohomology groups not occurring in the resolution. An elementary but lengthly computation which we omit at this point shows that the divisor $D = sD_1 + tD_2$ has degree $> 2g-2$ for $k \geq 5$. Thus D is non special which implies the vanishing $H^i(\mathscr{I}'_C \otimes \mathscr{F}) = 0$ for $i = 2,3$ and any sheaf $\mathscr{F} \in \mathcal{B}_I$. Moreover, $H^1(\mathscr{I}'_C) = 0$ by the assumption of maximal rank. Thus, $\mathcal{K}_I^i(\mathscr{I}'_C) = 0$ for $i > 0$ and $\mathcal{K}_I^0 \cong \mathscr{F}_1$. Furthermore, from the choice of s and t we see that also $H^0(\mathscr{I}'_C(-1,0)) = H^0(\mathscr{I}'_C(0,-1)) = H^0(\mathscr{I}'_C(-1,-1)) = 0$. Condition (G4) states precisely the vanishing of the remaining groups. □

In practice it is more feasible to determine the Betti numbers $\beta_{i,(j,k)}$ of this resolution by computing the Hilbert numerator of the truncated ideal sheaf

$$I_{C,\geq(s,t)} = \bigoplus_{s\geq a, t\geq b} H^0(\mathscr{I}_C(a,b))$$

as illustrated in the following example.

Example 3.10. Let $C \subset \mathbf{P}^1 \times \mathbf{P}^2$ be a good curve of bidegree $d = (6, 14)$ and genus $g = 17$. Then $(s,t) = (5,3)$. The Hilbert numerator of the truncated vanishing ideal $I' := I_{\geq(5,3)}$ of C is given by

$$\mathrm{HN}_{I_{\geq(5,3)}} = ({s_1}^5 s_2)^3(6s_1s_2^2 - 11s_1s_2 - 6s_2^2 + 8s_2 + 4).$$

Hence, the minimal free resolution $F_\bullet$ of I' is of the form

$$0 \to R(-1,-2)^{\oplus 6} \to R(-1,-1)^{\oplus 11} \oplus R(0,-2)^{\oplus 6} \to R^4 \oplus R(0,-1)^{\oplus 8} \to I' \to 0.$$

The Truncated Deficiency Module. Let us now introduce the key object for the second unirationality construction.

Definition 3.11. *Let C be a good curve and let*

$$0 \to F_3 \to F_2 \to F_1 \to I_{\geq(s,t)} \to 0$$

be the minimal free resolution of the truncated vanishing ideal obtained by applying $\Gamma_{\geq(s,t)}$ on (3.9). We call the module

$$K := \mathrm{coker}(F_2^\vee \to F_3^\vee)$$

the truncated deficiency module *of C.*

Sheafifying the minimal free resolution and twisting with $\mathscr{O}(s,t)$ we obtain again (3.9). Let $\mathscr{E} = \mathrm{coker}(\mathscr{F}_3 \to \mathscr{F}_2)$. Then, by Serre's duality

$$K = \mathrm{Ext}^1_*(\mathscr{E}, \mathscr{O}_{\mathbf{P}}) \cong \mathrm{Hom}_{\mathbf{k}}(H^2_*(\mathscr{E}(-2,-3)), \mathbf{k}).$$

Now let $M = H^1_*(\mathscr{I}_{C/\mathbf{P}})$ be the Hartshorne-Rao module of C. As $\mathscr{F}_1$ is a direct sum of copies of the line bundles $\mathscr{O}_{\mathbf{P}}$, $\mathscr{O}_{\mathbf{P}}(-1,0)$ and $\mathscr{O}_{\mathbf{P}}(0,-1)$ we have $H^2(\mathscr{E}(a,b)) \cong H^1(\mathscr{I}'_C(a,b))$ for $(a,b) \leq (-1,-1)$ and thus

$$K \cong (\mathrm{Hom}_{\mathbf{k}}(M,\mathbf{k})(2-s, 3-t))_{\geq(-1,-2)}\,. \tag{3.12}$$

As we will see in a minute the module K will be a somewhat "simpler" object than the Hartshorne-Rao module M. Nevertheless, the curve C can be recovered from K.

Proposition 3.13. *$\mathscr{E}^\vee$ is the second sheafified syzygy module of K. For a general map $\varphi \in \mathrm{Hom}(\mathscr{F}_1^\vee, \mathscr{E}^\vee)$ the associated Eagon-Northcott complex defined by the minors of φ the resolves the twisted ideal sheaf $\mathscr{I}_{C'}(s,t)$ of a codimension 2-subscheme C' in $\mathbf{P}$ with Hilbert polynomial $p_{C'}(s,t) = sk + td + 1 - g$.*

Proof. The presentation of K extends to a free resolution of K which can easily be seen to be minimal. Thus the second syzygy module fits into the exact sequence

$$0 \to N \to R(-1,0)^{\beta_{1,(0,2)}} \oplus R(0,-1)^{\beta_{1,(1,1)}} \to R^{\beta_{2,(1,2)}} \to K \to 0$$

Sheafifying and dualizing shows that $\tilde{N}^\vee = \mathscr{E}$. The Eagon-Northcott complex [Eis05, Appendix B] associated to a random map is of the form

$$0 \to \mathscr{E} \otimes \mathscr{L} \to \mathscr{F}_1 \otimes \mathscr{L} \to \mathscr{I}_{C'} \to 0$$

where $\mathscr{L} = \bigwedge^{\mathrm{rk}\mathscr{F}_1} \mathscr{F}_1^\vee \otimes \bigwedge^{\mathrm{rk}\mathscr{E}} \mathscr{E} = \mathscr{O}(-s,-t)$. □

We want to relate this object to the preceding construction by studying how the deficiency modules behave under linkage: Assume for simplicity that C is linked to C' by a complete intersection X defined by two forms of the same bidegree (s,t). By (2.37), the twisted mapping cone of the dual of following map of complexes

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{E}(-s,-t) & \longrightarrow & \mathscr{F}_1(-s,-t) & \longrightarrow & \mathscr{O}_{\mathbf{P}} \\
& \uparrow & & \uparrow & & \| \\
0 \to & \mathscr{O}_{\mathbf{P}}(-2s,-2t) & \to & \mathscr{O}_{\mathbf{P}}(-s,-t)^{\oplus 2} & \to & \mathscr{O}_{\mathbf{P}}
\end{array}$$

yields a projective resolution of $\mathscr{I}_{C'}$. We can partially minimalize this resolution and end up with

$$0 \to \mathscr{F}_1^\vee(-s,-t) \to \mathscr{E}^\vee(-s,-t) \oplus \mathscr{O}(-s,-t)^{\oplus 2} \to \mathscr{I}_{C'} \to 0.$$

Twisting back with $\mathscr{O}_{\mathbf{P}}(s,t)$ and considering the corresponding long exact sequence in cohomology, we see that

$$H^2(\mathscr{E}(a-2,b-3)) = H^1(\mathscr{E}^\vee(a,b))^\vee = H^1(\mathscr{I}_{C'}(s+a,t+b))$$

for $(a,b) \geq (s-1,t-2)$ or $(a,b) \leq (s-2,t-2)$. If M' denotes the deficiency module of C' then

$$K \cong M'(s,t)_{\geq(-1,-2)}. \tag{3.14}$$

Structure of the Deficiency Module. From the identification (3.12) we see that

$$h_K(i,j) = h^1(\mathscr{I}_C(s-2-i,t-3-j)) \text{ for } i,j \geq 0.$$

Evaluating this expression under the assumption of maximal rank immediately yields the following numerical information about the Hilbert function of the deficiency module of a good curve C.

Proposition 3.15. *With the notation from above, the Hilbert function of the deficiency module K of a good curve C is of following the form.*

(i) Binomial Coefficient Case. *For $k = \binom{t+1}{2}$ the Hilbert function of K has nonzero values $h_{i,j}$ at the following positions*

$$
\begin{array}{l|lllllll}
j & & & & & & & \\
 & h_{0,r} \cdots h_{i_r,r} & & & & & & \\
 & \vdots & & & & & & \\
 & h_{0,2} & \cdots & h_{i_2,2} & & & & \\
 & h_{0,1} & & \cdots & h_{i_1,1} & & & \\
 & \ell \quad \ell \quad \cdots & & & \cdots & \ell & \cdots & \\
\hline
 & & & & & & & \rightarrow i
\end{array}
$$

with $r \leq r_0$ where r_0 only depends on k, $\ell = h_{0,0} = h_{i,0}$ for all i and $i_1 > \ldots > i_r \geq 0$ is a strictly decreasing sequence of integers. Furthermore, for a fixed $j = 1, \ldots, r$ the difference functions $\partial h_{i,j}$ are constant.

(ii) Non Binomial Coefficient Case. *If $k = \binom{t+1}{2} + \varepsilon$ for $t + 1 > \varepsilon > 0$ the Hilbert function of K has nonzero values $h_{i,j}$ at most at the following positions*

$$
\begin{array}{l|lllll}
j & & & & & \\
 & h_{0,r} \cdots h_{i_r,r} & & & & \\
 & \vdots & & & & \\
 & h_{0,2} & \cdots & h_{i_1,1} & & \\
 & h_{0,0} & & \cdots & h_{i_0,0} & \\
\hline
 & & & & & \rightarrow i
\end{array}
$$

where $h_{i+1,0} - h_{i,0} = \varepsilon$ for $i = 0, \ldots, i_0 - 1$. where $j_1 > \ldots > j_r \geq 0$ is a strictly decreasing sequence of integers. The difference functions $\partial h_{i,j}$ are constant for a fixed $j = 1, \ldots, r$.

Remark 3.16. (i) We see that the truncated deficiency module K is of finite length if k is not a binomial coefficient of the form $\binom{t+1}{2}$.

(ii) If k is a binomial coefficient then the nonfinite part of K is supported only on (x_0, x_1). Note that in this case ℓ is just number of special fibers as discussed in the liaison construction. This can be seen as follows.

Let $\mathscr{F}_\bullet \to \mathscr{I}'_C$ be the resolution (3.9). For a point in $P \in \mathbf{P}^1$ the complex $\mathscr{F}_\bullet \otimes k(P)$ resolves the twisted ideal sheaf $\mathscr{I}_{C_p/\mathbf{P}^2}(t)$ of the fiber. Now if $k = \binom{t+1}{2}$ then $\beta_{2,(1,2)} = \beta_{1,(0,2)} = \ell$ and the differential $\mathscr{O}_{\mathbf{P}}(-1,-2)^\ell \to \mathscr{O}_{\mathbf{P}}(0,-2)^\ell$ in $\mathscr{F}_\bullet$ is represented by a $\ell \times \ell$-square matrix A^{T} with linear entries in x_0, x_1. Thus, if P lies in the scheme $V(\det(A))$ which has length ℓ then A^{T} drops rank which in turn implies that $H^0(\mathscr{I}_{C_P/\mathbf{P}^2}(t))$ has a nontrivial section.

Example 3.17. We consider the curve $C \subset \mathbf{P}$ from Example 3.10. The Hilbert function of the module $M = H^1_*(\mathscr{I}_C)$ is

$$
\begin{array}{r|lllllll}
\ddots & & & & & & & \\
\cdots\ 16 & 7 & & & & & & \\
\cdots\ 12 & 8 & 4 & & & & & \\
\cdots\ \mathbf{6} & \mathbf{6} & \mathbf{6} & \mathbf{6} & 6 & 6 & 6 & \cdots \\
 & \mathbf{1} & \mathbf{4} & \mathbf{7} & 10 & 13 & 16 & \cdots \\
 & & & \mathbf{3} & 8 & 13 & 18 & \cdots \\
\hline
 & & & & & 6 & 12 & \cdots \\
 & & & & & & & \ddots
\end{array}
\quad (j \uparrow,\ i \to)
$$

and the Hilbert function of $K \otimes R(1,2)$ is

$$
\begin{array}{|lllll}
3 & & & & \\
7 & 4 & 1 & & \\
6 & 6 & 6 & 6 & 6\ \cdots \\
\hline
\end{array}
\quad (j \uparrow,\ i \to)
$$

Construction of the Deficiency Module via Divided Powers. The truncated deficiency module K is determined by the morphisms

$$R(-1,0)^{\beta_{1,(0,2)}} \oplus R(0,-1)^{\beta_{1,1,1}} \xrightarrow{(A,B)} R^{\beta_{2,(1,2)}} \to K \to 0$$

of the minimal presentation (3.11). Our construction aims for choosing A and B in free parameters such that K has the desired Hilbert function. With the exception of only a few cases, a generic choice of A and B will not do.

We use Macaulays inverse system to find A and B as desired. More precisely, we construct a matrix P in divided powers such that $\mathrm{im}A + \mathrm{im}B \subset \mathrm{Ann}_R(P)$. The following dimension count indicates that chances are better when we pick the matrix A first, then try to construct P subject to the condition $\operatorname{im} A \subset \operatorname{Ann}_R(P)$ and finally construct B. Indeed, for a general element $P \in \operatorname{Hom}(R(-a,-b), D^\ell)$ we have

$$h_{\operatorname{Ann}_R(P)}(i,j) = \max(0, \ell\binom{i+1}{1}\binom{j+2}{2} - \binom{a-i+1}{1}\binom{b-j+2}{2}).$$

Thus for $b = 1$ we have $h_{\operatorname{Ann}_R(P)}(1,0) = \max(0, 2\ell - 3a)$ but on the other hand $h_{\operatorname{Ann}_R(P)}(0,1) = \max(0, 3\ell - a - 1)$. In many cases we will have $2\ell - 3a < \ell$ thus we will not be able to write down a matrix A in free parameters such that condition $\operatorname{im}_R(A) \subset \operatorname{Ann}_R(P)$ is satisfied. In other words, for the incidence correspondence Z of triples (C, A, B) of good curves C and matrices A and B forming a minimal presentation of the deficiency module of C with projections

$$\begin{array}{ccc} Z & \xrightarrow{\quad \pi_A \quad} & \operatorname{Hom}(R(-1,0)^\ell, R^\ell)_{(0,0)} \\ \pi_B \big\downarrow & & \\ \operatorname{Hom}(R(0,-1)^{\beta_{1,(1,1)}}, R^\ell)_{(0,0)} & & \end{array}$$

we expect π_A to be surjective but not π_B. Thus, we apply (2.29) on K with the degree $\mathbf{a} = (i_1, 0)$. By what was just said, this choice of $\mathbf{a}$ is close at hand as the socle elements in degrees (i_j, j) for $j > 1$ generate the corresponding module in divided powers.

Lemma 3.18. *A and B are uniquely determined up to* $\mathbf{k}$*-linear transformations by a matrix* $P \in \operatorname{Hom}(\bigoplus_{j=1}^r R(i_j, j)^{h_{i_j,j}}, D^{h_{00}})$ *with*

$$R^{h_{00}}/\operatorname{Ann}_R(P) \cong K/K_{\geq (i_1,0)}.$$

Example 3.19. Let $C \subset \mathbf{P}$ be as in Example 3.10. For a general choice of morphisms A and B the cokernel of $(A,B) : F_2^\vee \to F_3^\vee$ has the Hilbert function

$$\begin{array}{l|lllll} j & & & & & \\ & 3 & & & & \\ & 7 & 0 & 0 & & \\ & 6 & 6 & 6 & 6 & 6 \ \cdots \\ \hline & & & & & \rightarrow i \end{array}$$

Thus, a general choice for A and B does not lead to a curve as desired. In the language of Macaulays inverse system the condition for A and B to satisfy is the existence of a morphism $P \in \operatorname{Hom}(R(-1,-1)^{\oplus 2} \oplus R(-1,-2), D^6)$ such that $\operatorname{Ann}_R(P) \subset \operatorname{im}_R(A) \cap \operatorname{im}_R(B)$.

Putting everything together, we obtain the following construction method.

Construction Method 3.20.

1. Construct a deficiency module K with Hilbert function h_K as described in Proposition 3.15:
   (a) Pick a general element $A \in \mathrm{Hom}(R(-1,0)^{\beta_{1,(0,2)}}, R^{\beta_{2,(1,2)}})$.
   (b) For bidegrees (a_i, b_i) and integers m_i suitable, construct a matrix $P \in \mathrm{Hom}\left(\bigoplus_{i=1}^{r} R(-a_i,-b_i)^{m_i}, D^{\beta_{2,(1,2)}}\right)$ in free parameters such that $\mathrm{Ann}_R(P) \subset \mathrm{im}_R(A)$.
   (c) We pick a general morphism $B \in \mathrm{Hom}(R(0,-1)^{\beta_{1,(1,1)}}, R^{\beta_{2,(1,2)}})$ under the conditions that $\mathrm{Ann}_R(P) \subset \mathrm{im}_R(B)$.

2. Compute the beginning of a minimal free resolution

$$\ldots \to F \xrightarrow{\psi} F_2^\vee(-1,-2) \to F_3^\vee(-1,-2) \to K \to 0$$

   and choose a random element $\varphi \in \mathrm{Hom}(F_1^\vee(-1,-2), F)$. By (3.13) we have $\ker(F_1^\vee \xrightarrow{\psi\circ\phi} F_2^\vee) \cong R^1$ and the entries of the syzygies of $\psi \circ \phi$ generate the ideal I of a codimension 2 subscheme of $\mathbf{P}$.

3. Verify that the resulting curve defined by the saturation of I is an element in $\mathscr{H}_{g,k}$.

Clearly, Step 2 is staightforward and Step 3 is equivalent to the verification in Code 3.5 so we will only focus on Step 1 in the following.

3.4.1 Hexagonal Curves

Let us start by considering the construction of K in the setting of curves of gonality $k = 6$ and small values of g. The expected Hilbert function of the module K is of the form

$$\begin{array}{c|cccccccc}
j & & & & & & & & \\
 & m+5b & \cdots & m+5 & m & & & & \\
 & n+3a & \cdots & & & \cdots & n+3 & n & \\
 & \ell & \cdots & \ell & \ell & \cdots & & & \\
\hline
 & & & & b & & & a & i
\end{array}$$

with integers $1 \le n \le 3$, $1 \le m \le 5$ and $b \le a$ depending on g. Table 3.2 lists the values of these numbers as well as h_K for $5 \le g \le 49$. The difficulty of the construction of the deficiency modules of 6-gonal curves increases with the genus g of the curve. We recover the cases $5 \le g \le 25$, $g = 27, 28, 30, 33$ obtained by the linkage approach of Section 3.1 and obtain the additional cases $g = 29, 34, 39$. Let us discuss the cases grouped by the difficulty of their construction of K:

The cases $5 \leq g \leq 12$ or $g = 18$. In these cases we have $a = b = m = 0$. Thus the Hilbert function of K has the form

$$\begin{array}{c|cccc} j & & & & \\ & n & & & \\ & \ell & \ell & \ell & \cdots \\ \hline & & & & \to i \end{array}$$

We see that P is trivial and the construction of K is unconstrained, i.e. a general choice for the matrices A and B yields a deficiency module as desired.

The cases $g = 15, 16, 17, 20, 21, 22, 25, 27, 30$. We have $b = 0$ and $m = 3(n + 3a - \ell)$. Accordingly h_K has nonzero values

$$\begin{array}{c|ccccc} j & & & & & \\ & m & & & & \\ & n+3a & \cdots & n+3 & n & \\ & \ell & \cdots & \ell & \ell & \cdots \\ \hline & & & & a & \to i \end{array}$$

with $n \in \{1, 2, 3\}$. We start with a general matrix A and construct P by picking general elements $P_1, \ldots, P_n \in \mathrm{Ann}(\mathrm{im}A)_{(-n,-1)}$ and $P'_1 \ldots, P'_{3-n} \in \mathrm{Ann}(\mathrm{im}A)_{(-n-1,-1)}$. The module in divided powers generated by these elements will have Hilbert function

$$\begin{array}{cccc|c} i \;\; -a & & & & \\ \hline \ell & \ell & \cdots & \ell & \\ n & n+3 & \cdots & n+3a & \\ & & & & j \end{array}$$

We see that B is uniquely determined (up to $\mathbf{k}$-linear transformations) by the $(n + 3a)$ elements in $\mathrm{Ann}_R(P)_{(0,-1)}$. Note that we expect that for a general B the expected Hilbert function of $\mathrm{coker}(B)$ as a module over $\mathbf{k}[y_0, y_1, y_2]$ is $(\ell, n + 3a, m, 0, 0 \ldots)$. Thus it remains only to verify the open condition that a general choice for B actually has this Hilbert function which can be done utilizing *Macaulay2*.

The cases $g = 14, 19, 23, 24, 28, 29, 33, 34, 39$. Let us sketch the constraints for a construction of K in the general case first. Assume K is a module with presentation matrix (A, B) as desired. Let $M := (\mathrm{im}\ A)^{\perp}$ and let

$$P_1, \ldots, P_n \in M_{(a,1)}, \;\; P'_1, \ldots, P'_{(3-n)} \in M_{(a-1,1)}$$

and

$$Q_1,\dots,Q_m \in M_{(b,2)},\ \ Q'_1,\dots,Q'_{(5-m)} \in M_{(b-1,2)}$$

be generators corresponding to the socle elements of K and let N be the R-module generated by these elements. Then, in particular, the Hilbert function of N takes values

$$h_N(-a+b,-1) = n+3(a-b) \text{ and } h_N(-a+b+1,-1) = n+3(a-b+1).$$

Hence, there are relations among the generators in these degrees, i.e. matrices $\Psi_1 \in \mathbf{k}^{(n+3(m+a-b))\times 3m}$ such that

$$\left[y_0Q_1,\dots,y_2Q_m, x_0^{a-b}P_1,\dots,x_1^{a-b}P_1,\dots,x_1^{a-b-1}P'_{n-3}\right]\cdot\Psi_1 = 0 \tag{3.21}$$

and $\Psi_2 \in \mathbf{k}^{(n+3(a-b+1))\times(15-3m)}$ with

$$\left[y_0Q'_1,\dots,y_2Q'_{5-m}, x_0^{a-b+1}P_1,\dots,x_1^{a-b+1}P_1,\dots,x_1^{a-b}P'_{n-3}\right]\cdot\Psi_2 = 0. \tag{3.22}$$

At the moment, the only way we know how to write down such modules in free parameters is by choosing Ψ_1 and Ψ_2 generic and picking the elements P_i, P'_i and Q_i, Q'_i subject to these relations. But already (3.21) constrains the cases which can be constructed in this way. To count parameters we think of the tuples $((P,P',Q),\Psi_1)$ satisfying (3.21) as elements of an incidence correspondence Z_1 with natural projections

$$\begin{array}{ccc} Z_1 & \xrightarrow{\quad\pi_2\quad} & \mathbf{G}(3m,3(m+a-b)) \\ \pi_1\downarrow & & \\ \mathbf{G}(n,3\ell)\times\mathbf{G}(3-n,3\ell-2n)\times\mathbf{G}(m,6\ell) & & \end{array}$$

The fiber over a point $\Psi_1 \in \mathbf{G}(m,6\ell)$ is a linear space of expected dimension

$$\begin{aligned}\dim \pi_2^{-1}(\Psi_1) &= \dim\mathbf{G}(n,3\ell)+\dim\mathbf{G}(3-n,3\ell-2n)+\dim\mathbf{G}(m,6\ell)-9m\ell \\ &= 9\ell-9-3m\ell-m^2\end{aligned}$$

which can only be positive when $m \le 2$. Now taking into account the additional restriction (3.22), an analogous computation shows that the we expect the fiber over a general pair (Ψ_1,Ψ_2) to be empty. This leaves us with the cases $m \le 2$ and either $b=0$ or $m \le 2$, $b=1$ and $5m+1 = 3(n+3a-\ell)$ which are precisely the ones listed above. We demonstrate the construction in *Macaulay2* in one example case.

Code 3.23. We construct the truncated deficiency module in the case $g=39$. We start by choosing a generic matrix $A \in \mathrm{Hom}(R(0,-1)^{12}, R^{12})$.

```
i1 : loadPackage"RandomGonalCurves";
     S=(ZZ/10007)[x_0,x_1,y_0..y_2,Degrees=>{2:{1,0},3:{0,1}}];
```

```
i2 : A:=random(S^12,S^{12:{-1,0}});
```

We compute $\mathbf{k}$-bases of the involved graded parts of $(\mathrm{im}(A)^{\perp})$.

```
i3 : V41=getComplementBasis({4,1},A);
     V31=getComplementBasis({3,1},A);
     V12=getComplementBasis({1,2},A);
```

For the sake of simplicity, we introduce a ring of coefficients parametrizing the $\mathbf{k}$-vector spaces $(\mathrm{im}(A)^{\perp})_{(4,1)}$, $(\mathrm{im}(A)^{\perp})_{(3,1)}$ and $(\mathrm{im}(A)^{\perp})_{(1,2)}$ which allows us to conveniently write down the generic elements in these spaces.

```
i4 : C=Fp[a_0..a_35,b_0..b_35,c_0..c_35,d_0..d_71];
     CS=C**S;
     P1gen=sum(36,i->a_i*sub(homomorphism(V41_{i}),CS));
     P2gen=sum(36,i->b_i*sub(homomorphism(V41_{i}),CS));
     P1'gen=sum(36,i->c_i*sub(homomorphism(V31_{i}),CS));
     Q1gen=sum(72,i->d_i*sub(homomorphism(V12_{i}),CS));
```

Next, we choose a random matrix Ψ and solve for elements satisfying equation (3.21):

```
i5 : M=contract(basis({0,3,0},CS),P1gen)|
       contract(basis({0,3,0},CS),P2gen)|
       contract(basis({0,2,0},CS),P1'gen)|
       contract(basis({0,0,1},CS),Q1gen);
     Psi=random(Fp^14,Fp^3);
     N=M*Psi;
     rel0=ideal apply(flatten entries N, f->contract(basis({0,1,1},CS),f));
     rel1=transpose mingens sub(rel0,C);
     sols=transpose syz sub(contract(vars C,rel1),Fp);
```

We pick a random solution and construct the corresponding module K:

```
i6 : sol=random(Fp^1,Fp^(rank target sols))*sols;
     P=map(S^{2:{4,1},1:{3,1},1:{1,2}},S^12,
        transpose sub(P1gen|P2gen|P1'gen|Q1gen,sol|vars S));
     B0=getAnnihilatorBasis({0,1},P);
     B:=getRandomSubspace(22,B0);
     K:=coker(A|B);
     matrix apply(toList(0..5),i->
                 apply(toList(0..2),j->hilbertFunction({i,j},K)))

o6 = | 12 14 6 |
     | 12 11 1 |
     | 12 8  0 |
     | 12 5  0 |
     | 12 2  0 |
     | 12 0  0 |
```

We construct the curve from K. This is Step 2 from (3.20):

```
i7 : resK:=res K;
     tally degrees resK_2
     F0=S^{19:{-1,-1},4:{-1,-2}};
     IC=ideal syz(resK.dd_2 *random(resK_2,F0));
     time ICsat=saturate(IC,ideal(S_0*S_2));
       -- used 26.2516 seconds
```

Clearly, in terms of computational resources the bottle neck is Step 3 from (3.20), the verification of the curve:

```
i8 : time certifyGonalCurve(ICsat,6,39,S)
       -- used 716.015 seconds

o8 = true
```

3.4.2 Heptagonal Curves.

The cases $6 \leq g \leq 10$ or $g = 12$. The construction of the deficiency module works analogous to the 6-gonal cases. We note that for $g = 12$ the construction is again unconstrained.

The case $g = 11$. We want to discuss this case in more detail as the construction slightly differs from the previous cases. We will give a construction in terms of the strands of a bigraded free resolution which arises from a different truncation. From a Hilbert numerator computation analogous to (3.10) we expect that the truncated vanishing ideal $I' := I_{C,\geq(3,3)}$ has the minimal free natural resolution

$$\begin{aligned} 0 \to S(-2,-3) \to S(-2,-2)^{\oplus 3} \oplus F_2 \to S(-2,-1)^{\oplus 3} \oplus F_1 \\ \to S(-2,0) \oplus F_0 \to I' \to 0 \end{aligned}$$

with free modules

$$\begin{aligned} F_0 &= S(-1,0)^{\oplus 2} \oplus S(0,-1)^{\oplus 9} \\ F_1 &= S(-1,-1)^{\oplus 7} \oplus S(0,-2)^{\oplus 4} \oplus S(0,-3)^{\oplus 1} \\ F_2 &= S(-1,-3)^{\oplus 2}. \end{aligned}$$

It suffices to construct

$$0 \to F_2 \xrightarrow{\psi} F_1 \xrightarrow{\varphi} F_0 \to I'' \to 0.$$

as the sheafifications $\tilde{I'}$ and $\tilde{I''}$ will coincide. This follows from the fact that the Koszul complex resolves the ideal (y_0, y_1, y_2) which is supported on an irrelevant part in $\mathbf{P}$. For the deficiency module $K = \operatorname{coker}(F_1^\vee \to F_2^\vee)$ we

g	c	h_K	g	c	h_K	c	g	h_K
5	u	2 2 2 $\cdots$	20	u	6 3 7 7 7 $\cdots$	35	o	5 13 10 7 4 1 12 12 12 12 12 $\cdots$
6	u	1 1 1 $\cdots$	21	u	5 2 6 6 6 $\cdots$	36	o	4 12 9 6 3 11 11 11 11 11 $\cdots$
7	u	1 2 2 $\cdots$	22	u	3 8 5 2 7 7 7 $\cdots$	37	o	3 12 9 6 3 12 12 12 12 12 $\cdots$
8	u	1 3 3 $\cdots$	23	u	2 8 5 2 8 8 8 $\cdots$	38	o	7 2 15 12 9 6 3 13 13 13 13 13 $\cdots$
9	u	2 2 $\cdots$	24	u	1 7 4 1 7 7 7 $\cdots$	39	u	6 1 14 11 8 5 2 12 12 12 12 12 $\cdots$
10	u	3 3 3 $\cdots$	25	u	7 4 1 8 8 8 $\cdots$	40	o	5 14 11 8 5 2 13 13 13 13 13 $\cdots$
11	u	3 4 4 $\cdots$	26	o	4 10 7 4 1 9 9 9 $\cdots$	41	o	9 4 17 14 11 8 5 2 14 14 14 14 14 14 $\cdots$
12	u	2 3 3 $\cdots$	27	u	3 9 6 3 8 8 8 $\cdots$	42	o	3 13 10 7 4 1 13 13 13 13 13 13 $\cdots$
13	u	2 4 4 $\cdots$	28	u	2 9 6 3 9 9 9 $\cdots$	43	o	7 2 16 13 10 7 4 1 14 14 14 14 14 14 $\cdots$
14	u	1 5 2 5 5 $\cdots$	29	u	6 1 12 9 6 3 10 10 10 10 10 $\cdots$	44	o	6 1 16 13 10 7 4 1 15 15 15 15 15 15 $\cdots$
15	u	4 1 4 4 $\cdots$	30	u	8 5 2 9 9 9 9 9 $\cdots$	45	o	5 15 12 9 6 3 14 14 14 14 14 $\cdots$
16	u	4 1 5 5 $\cdots$	31	o	4 11 8 5 2 10 10 10 10 10 $\cdots$	46	o	9 4 18 15 12 9 6 3 15 15 15 15 15 15 $\cdots$
17	u	3 7 4 1 6 6 6 $\cdots$	32	o	3 11 8 5 2 11 11 11 11 11 $\cdots$	47	o	8 3 18 15 12 9 6 3 16 16 16 16 16 16 $\cdots$
18	u	3 5 5 5 $\cdots$	33	u	2 10 7 4 1 10 10 10 10 10 $\cdots$	48	o	7 2 17 14 11 8 5 2 15 15 15 15 15 15 $\cdots$
19	u	1 6 3 6 6 6 $\cdots$	34	u	6 1 13 10 7 4 1 11 11 11 11 11 $\cdots$	49	o	6 1 17 14 11 8 5 2 16 16 16 16 16 16 $\cdots$

Table 3.2: Expected Hilbert functions of the deficiency modules of 6-gonal curves for $5 \leq g \leq 49$ with labels u for understood cases and o for open cases of the construction of K.

expect the Hilbert function

$$\begin{array}{l|llllll} j \uparrow & & & & & & \\ & 1 & & & & & \\ & 5 & 1 & & & & \\ & 6 & 5 & 4 & 3 & 2 & 1 \\ & 2 & 3 & 4 & 5 & 6 & 7\cdots \\ \hline & & & & & & \rightarrow i \end{array}$$

To start with, we consider the strand

$$S(-1,-3)^{\oplus 2} \xrightarrow{\psi_1} S(-1,-1)^{\oplus 7} \xrightarrow{\varphi_1} S(-1,0)^{\oplus 2}.$$

For a general choice of the map φ_1 we expect $3 = 3 \cdot 7 - 6 \cdot 3$ linear syzygies. Choosing a 2-dimensional subspace among them gives ψ_1. The map $S(-1,-3)^{\oplus 2} \to S(0,-3)$ is just (x_0, x_1) up to a linear change of coordinates. Hence, it remains to find the differential $S(-1,-3)^{\oplus 2} \to S(0,-2)^{\oplus 4}$. We proceed as before and select a general matrix $P \in \mathrm{Hom}(R(-1,-2) \oplus R(0,-3), D^2)$ subject to the relation $\mathrm{Ann}_R(P) \subset \mathrm{im}(\psi_1, \psi_2)^{\mathrm{T}}$. The columns of the transpose of ψ_3 are chosen from $\mathrm{Ann}_R(P)_{(1,1)}$.

Corollary 3.24. *The Severi variety $\mathscr{V}_{d,g}$ of plane irreducible nodal curves of genus g and degree $d = \lceil \frac{2}{3}g + 2 \rceil$ is unirational for $g \leq 13$.*

Proof. The result is classical for $g \leq 10$, see for instance [ST02]. As an interim result, Chang and Ran [CR84] show that for $g = 12, 13$ there is a unirational component of the Hilbert scheme of curves in $\mathbf{P}^3$ of degree $d = g$ which dominates the moduli space. For these cases, the residual linear series are of degree $\deg(\omega_C \otimes \mathscr{O}_C(-H)) = 2g - 2 - g = g - 2 = \lceil \frac{2}{3}g + 2 \rceil$ As $h^1(\mathscr{O}_C(H)) = 3$ the bundle $\omega_C \otimes \mathscr{O}_C(-H)$ yields and a map to $\mathbf{P}^2$.

Finally, the unirational component of the Hilbert scheme in $\mathrm{Hilb}_{(7,10),11}(\mathbf{P})$ we obtain in the course of the unirationality proof above also dominates $\mathscr{V}_{10,11}$. $\square$

3.4.3 Classical Cases

For the sake of completeness we also want to show that , in principle, the unirationality of $\mathscr{H}_{g,k}$ for $3 \leq k \leq 5$ can also be covered by our approach.

Trigonal Curves. Note that a curve $C \in \mathscr{H}_{3,g}$ has no plane model as demanded in (G1). However, from the discussion in Section 3.1 it is clear that a generic projection from an embedding given by a divisor of degree $g + 3$ yields plane model of the same degree. With this being understood the construction is straightforward. For the twist $(s,t) = (\lceil (g+2)/3 \rceil, 2)$ the minimal free resolution of a good trigonal curve C is

$$0 \to R(-1,-2) \xrightarrow{(A|B)} R(0,-2) \oplus R(-1,-1)^{s+2+\varepsilon} \to$$

$$\to R^{2-\varepsilon} \oplus R(-1,0)^{1+\varepsilon} \oplus R(0,-1)^{s+\varepsilon} \to I_{C,\geq(s,t)} \to 0$$

where $\varepsilon \in \{-1,0,1\}$ with $\varepsilon \equiv g\ (3)$. The for a generic choice of $(A|B)$ the curve constructed via the syzygies is trigonal and smooth.

Tetragonal Curves. In this section we construct curves with a $\mathfrak{g}^1_4$. Along the lines of the proof of Proposition 3.9 we find that a good tetragonal curve C of genus g has a minimal bigraded free resolution

$$0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{I}_C \to 0$$

with

$$\begin{aligned} \mathscr{F} \quad &\cong \quad H^0(\mathscr{I}'_C) \otimes \mathscr{O}_{\mathbf{P}} \\ &\oplus H^1(\mathscr{I}'_C(-1,0)) \otimes \mathscr{O}_{\mathbf{P}}(-1,0) \\ &\oplus H^1(\mathscr{I}'_C \otimes \pi_2^*\Omega^1_{\mathbf{P}^2}(1)) \otimes \mathscr{O}_{\mathbf{P}}(0,-1) \\ &\oplus H^2(\mathscr{I}'_C(0,-1)) \otimes \mathscr{O}_{\mathbf{P}}(0,-2) \end{aligned}$$

$$\begin{aligned} \mathscr{G} \quad &\cong \quad H^2(\mathscr{I}'_C(-1,-1)) \otimes \mathscr{O}_{\mathbf{P}}(-1,-2) \\ &\oplus H^1(\mathscr{I}'_C(-1,0) \otimes \Omega^1_{\mathbf{P}^2}(1)) \otimes \mathscr{O}_{\mathbf{P}}(-1,-1) \end{aligned}$$

Thus, the minimal free resolution is described by the theorem of Hilbert-Burch [Eis95, Theorem 20.15].

Pentagonal Curves. Let C be a good curve in $\mathscr{H}_{g,5}$. The Hilbert function of the deficiency module $K' = K \otimes R(1,2)$ of C for $g \gg 0$ has nonzero values

$$\begin{array}{l|ccccccc} j & & & & & & & \\ & m+5b & \cdots & m+5 & m & & & \\ & n+2a & \cdots & & & \cdots\ n+2 & n & \\ \hline & & & & b & & a & \to i \end{array}$$

with $m \in \{1,2,3,4\}$ and $n \in \{0,1\}$. The construction is completely analogue to the 6-gonal case.

In conclusion, we obtain the following result:

Corollary 3.25. *Assume that for* $3 \leq k \leq 5$ *and* $g \geq k-1$ *there is a good curve in* $\mathscr{H}_{g,k}$. *Then* $\mathscr{H}_{g,k}$ *is unirational.*

Remark 3.26. As before, we can check computationally the existence of good curves for small values of g. A verification has been done for all cases in the range $3 \le k \le 5$ and $g \le 40$ and the author conjectures that there is a good curve for any (k, g) with $g \ge k - 1$, independently of the existence of a unirational description.

3.5 Outlook

Before turning to applications we want to make a few remarks. The presented result motivates to further investigate the birational geometry of Hurwitz spaces which, at the moment seems to be a far open problem. A conjectural picture of the birational type of $\mathscr{H}_{g,k}$ based on divisor class calculations due to Farkas (unpublished) asserts that $\mathscr{H}_{g,k}$ is of general type for $g \gg 0$ and $k \ge \kappa_g$ with $\frac{\kappa_g}{g} \to \frac{1}{3}$ as $g \to \infty$. But even with curves of large gonality understood, the cases of small gonality remain very interesting.

Problem 3.27. *Is $\mathscr{H}_{g,k}$ unirational for $\frac{g}{3} \gg k \ge 6$?*

The least one could hope for is that the "gaps" in our list are only due to our method. Hence, as a first step towards an extension of the presented result one might pursue the following

Conjecture 3.28. *$\mathscr{H}_{g,6}$ unirational for* all $g \le 45$.

The unirationality $\mathscr{H}_{g,k}$ for fixed gonality $k \ge 6$ and any $g \ge k - 1$ would also be an indicator for a structure theorem of Gorenstein ideals of codimension $k - 2$, as discussed in the next section.

4 Subvarieties of the Hurwitz Space of Hexagonal Covers

In this chapter we study 6-gonal curves in their canonical embedding. The union of the linear spans of the divisors in the special pencil of such a curve $C \subset \mathbf{P}^{g-1}$ forms a rational normal scroll X. We compile the background material in Section 1. In Section 2 we describe a resolution of the structure sheaf $\mathscr{O}_C$ by locally free sheaves on X. In the cases covered by the unirationality constructions, we compute the Betti numbers of the resolution of the general curve using *Macaulay2*. Motivated by recent results of Erman and Wood [Obe10] we examine the variety of 6-gonal curves whose resolution on the scroll is a Gulliksen-Negard complex.

4.1 Canonical Curves on Rational Normal Scrolls

We start by briefly resuming the necessary material on rational scrolls and their subvarieties. We follow in this section the lucid presentations in [Sch86] and [Har81].

Rational Normal Scrolls. Let $\mathscr{E} = \mathscr{O}_{\mathbf{P}^1}(e_1) \oplus \ldots \oplus \mathscr{O}_{\mathbf{P}^1}(e_d)$ with $e_1 \geq \ldots e_d \geq 0$ be a locally free sheaf of rank d on $\mathbf{P}^1$ and let

$$\pi : \mathbf{P}(\mathscr{E}) \to \mathbf{P}^1$$

be the corresponding $\mathbf{P}^{d-1}$-bundle. For $f = \sum_{i=1}^{d} e_i \geq 2$ consider the image of $\mathbf{P}(\mathscr{E})$ under the map associated to the tautological bundle $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$:

$$j : \mathbf{P}(\mathscr{E}) \to X \subset \mathbf{P}^r$$

with $r = f + d - 1$. The variety X is called a *rational normal scroll of type* $S(e_1, \ldots, e_d)$. X is a nondegenerate irreducible variety of minimal degree

$$\deg X = f = r - d + 1 = \mathrm{codim} X + 1.$$

If all $e_i > 0$ then X is smooth and $j : \mathbf{P}(\mathscr{E}) \to X$ is an isomorphism. Otherwise X is singular and $j : \mathbf{P}(\mathscr{E}) \to X$ is a resolution of singularities. The singularities of X are rational, i.e.

$$j_* \mathscr{O}_{\mathbf{P}(\mathscr{E})} = \mathscr{O}_X \text{ and } R^i j_* \mathscr{O}_{\mathbf{P}(\mathscr{E})} = 0 \text{ for } i > 0.$$

Therefore it is no problem to replace X by $\mathbf{P}(\mathscr{E})$ for most cohomological considerations, even if X is singular. In the following, let $H = [j^*\mathscr{O}_{\mathbf{P}^r}(1)]$ denote the hyperplane class and $R = [\pi^*\mathscr{O}_{\mathbf{P}^1}(1)]$ the class of the ruling on X.

Proposition 4.1. *For X as above the following holds:*

(i) $\operatorname{Pic}(X) = \mathbf{Z}H \oplus \mathbf{Z}R$ *with the relations*

$$H^d = f, \quad H^{d-1} \cdot R = 1, \quad R^2 = 0.$$

(ii) We have the identification

$$H^0(\mathbf{P}(\mathscr{E}), \mathscr{O}_{\mathbf{P}(\mathscr{E})}(aH + bR)) \cong H^0(\mathbf{P}^1, (\operatorname{Sym}_a \mathscr{E})(b))$$

where $\operatorname{Sym}_a \mathscr{E}$ denotes the symmetric product of $\mathscr{E}$.

(iii) For a, b such that $\sum_i \alpha_i e_i + b \geq 0$ for all α with $|\alpha| = a$ the number of global sections is given by

$$h^0(\mathbf{P}(\mathscr{E}), \mathscr{O}_{\mathbf{P}(\mathscr{E})}(aH + bR)) = f\binom{a+d-1}{d} + (b+1)\binom{a+d-1}{d-1}.$$

(iv) The dualizing sheaf of X is $\omega_X = \mathscr{O}_X(-dH + (f-2)R)$.

Proof. See [Sch86]. We note that (iii) was only shown for $b \geq -1$ but generalizes immediately to our situation. □

Remark 4.2. The identification in part (ii) can be written down explicitly. Let $\mathbf{k}[s,t]$ denote the homogeneous polynomial ring of $\mathbf{P}^1$. Then we can identify sections $\Psi \in H^0(\mathbf{P}(\mathscr{E}), \mathscr{O}_{\mathbf{P}(\mathscr{E})}(aH + bR))$ with homogeneous polynomials of the form

$$\Psi = \sum_\alpha P_\alpha(s,t) \varphi_1^{\alpha_1} \cdot \ldots \cdot \varphi_d^{\alpha_d}$$

of degree $a = \sum_i \alpha_i$ in the φ_i with coefficients P_α homogeneous polynomials in $\mathbf{k}[s,t]$ of degree $P_\alpha = \sum_i \alpha_i e_i + b$.

Rational normal scrolls are determinantal varieties.

Proposition 4.3. *For X as above, consider the basis*

$$z_{ij} = t^j s^{e_i - j} \varphi_i \text{ for } i = 1, \ldots, d,\ j = 0, \ldots, e_i$$

of $H^0(\mathscr{O}(H)) \cong H^0(\mathscr{O}_{\mathbf{P}^r}(1))$. Then X is the vanishing loci of the 2×2-minors of the matrix

$$\Phi = \left(\begin{array}{ccc|ccc|c|cc} z_{10} & \cdots & z_{1e_1-1} & z_{20} & \cdots & z_{2e_2-1} & \cdots & \cdots & z_{de_d-1} \\ z_{11} & \cdots & z_{1e_1} & z_{21} & \cdots & z_{2e_2} & \cdots & \cdots & z_{de_d} \end{array} \right)$$

consisting of d catalecticant blocks of size $2 \times e_1, \ldots, 2 \times e_d$.

Proof. [Sch86]. □

Proposition 4.4. *Let $X \cong \mathbf{P}(\mathscr{E})$ be a smooth rational normal scroll of dimension d and type $S(e_1, \ldots, e_d)$. Then the automorphism group $\mathrm{Aut}(\mathbf{P}(\mathscr{E}))$ fits in a short exact sequence*

$$0 \to \mathbf{P}\mathrm{Aut}(\mathscr{E}) \to \mathrm{Aut}(\mathbf{P}(\mathscr{E})) \to \mathrm{Aut}(\mathbf{P}^1) \to 0.$$

In particular,

$$\dim \mathrm{Aut}(\mathbf{P}(\mathscr{E})) = 2 + \binom{d+1}{2} + \sum_{j \geq i} (e_j - e_i) + \#\{(i,j) \,|i < j,\ e_i = e_j\}.$$

Proof. See [Har81]. The only divisors in $\mathbf{P}(\mathscr{E})$ which are isomorphic to $\mathbf{P}^{d-2}$ are the fibers of the projection $\pi : \mathbf{P}(\mathscr{E}) \to \mathbf{P}^1$ and thus, any automorphism of $\mathbf{P}(E)$ preserves these fibers. In this way we obtain a map $\mathrm{Aut}(\mathbf{P}(\mathscr{E})) \to \mathrm{Aut}(\mathbf{P}^1)$ which is actually onto. In fact, for every $\varphi \in \mathrm{Aut}(\mathbf{P}^1)$ we have $\varphi^*\mathscr{E} \cong \mathscr{E}$. Let F denote the kernel of this morphism so that we have the exact sequence

$$0 \to F \to \mathrm{Aut}(\mathbf{P}(\mathscr{E})) \to \mathrm{Aut}(\mathbf{P}^1) \to 0.$$

The subgroup F consists of those automorphisms which leave $\mathbf{P}^1$ fixed and carry every fiber to itself. Thus $F = \mathbf{P}\mathrm{Aut}(\mathscr{E})$. The automorphisms of $\mathscr{E}$ form the open set of invertible elements in $\mathrm{End}(\mathscr{E}) = H^0(\mathscr{E} \otimes \mathscr{E}^\vee)$. An element in the latter group is given by a collection of sections $\{\sigma_{ij} \in H^0(\mathscr{O}_{\mathbf{P}^1}(e_i - e_j))\}$. Hence,

$$\dim \mathrm{Aut}(\mathscr{E}) = h^0(\mathscr{E} \otimes \mathscr{E}^\vee) = \binom{d+1}{2} + \sum_{j \geq i} (e_j - e_i) + \#\{(i,j) \,|i < j,\ e_i = e_j\}.$$

□

Scrolls and Pencils. Let V be a smooth variety together with a linearly normal map

$$j : V \to \mathbf{P}^r = \mathbf{P}(H^0(V, \mathscr{O}_V(H))).$$

Let $X \subset \mathbf{P}^r$ be a scroll of degree f containing $j(V)$. Then the ruling R on X cuts out on V a pencil of divisors $(D_\lambda)_{\lambda \in \mathbf{P}^1} \subset |D|$. As the sections of $H^0(X, \mathscr{O}_X(H - R))$ restrict to V, we have $h^0(V, \mathscr{O}_V(H - D)) = f$.

Conversely, a pencil of divisors $(D_\lambda)_{\lambda \in \mathbf{P}^1}$ on V with $h^0(V, \mathscr{O}_V(H - D)) = f \geq 2$ yields a scroll of degree f as follows. Let $\overline{D_\lambda}$ denote the linear span of the image in $\mathbf{P}^r$ of D_λ under j. Then the scroll X is the variety swept out by these linear spaces:

$$X = \bigcup_{\lambda \in \mathbf{P}^1} \overline{D_\lambda} \subset \mathbf{P}^r$$

In more algebraic terms X can be described as follows: We denote with $G \subset H^0(V, \mathscr{O}_V(D))$ the 2-dimensional subspace which defines the pencil. Then the multiplication map

$$G \otimes H^0(V, \mathscr{O}_V(H-D)) \to H^0(V, \mathscr{O}_V(D))$$

yields a $2 \times f$ matrix Φ with linear entries whose 2×2 minors vanish on $j(V)$. The variety X defined by these minors contains $j(V)$ and is a scroll of degree f.

The type $S(e_1, \dots, e_d)$ of the scroll can be determined as follows. We can decompose the scroll in its fixed and moving part $D_\lambda = F + E_\lambda$ for $\lambda \in \mathbf{P}^1$ and consider the partition of $r+1$ defined by

$$\begin{array}{rcl} d_0 & := & h^0(\mathscr{O}_V(H)) - h^0(\mathscr{O}_V(H-D)) \\ d_1 & := & h^0(\mathscr{O}_V(H-D)) - h^0(\mathscr{O}_V(H-F-2E) \\ \vdots & & \\ d_i & := & h^0(\mathscr{O}_V(H-F-iE) - h^0(H-F-(i+1)E) \\ \vdots & & \end{array}$$

The dual partition (i.e. the partition which is obtained by reflecting the corresponding Young tableaux along the diagonal) defines then the numbers e_i:

$$e_i = \sharp\{j \mid d_j \geq 1\} - 1.$$

Theorem 4.5 (Harris, Bertini)**.** *With the notation from above, X is a scroll of dimension d_0 and type $S(e_1, \dots, e_{d_0})$.*

Proof. [Har81]. □

We recall from [Sch86] the family of complexes $\mathscr{C}^b$, $b \geq -1$, of locally free sheaves on V which resolve the b^{th}-symmetric power of the cokernel of a map $\Phi : F \to G$ of locally free sheaves of rank f and g, $f \geq g$, on V:

$$\mathscr{C}^b_j = \begin{cases} \bigwedge^j F \otimes \mathrm{Sym}_{b-j} G & \text{for } 0 \leq j \leq b, \\ \bigwedge^{j+g-1} F \otimes \mathrm{D}_{j-b-1} G^* \otimes \bigwedge^g G^* & \text{for } j \geq b+1. \end{cases}$$

The differentials $\mathscr{C}^b_j \to \mathscr{C}^b_{j-1}$ are induced by the multiplication with Φ resp. $\bigwedge^g \Phi$ in the appropriate term of the exterior, symmetric or divided power algebra.

Syzygies of Canonical Curves. We restrict to the case of curves. Let $C \subset \mathbf{P}^{g-1}$ be a canonical curve of genus g and $\{D_\lambda\}_{\lambda \in \mathbf{P}^1}$ a complete basepoint free pencil of divisors of degree $k \leq g-1$ on C. Then

$$X = \bigcup_{\lambda \in \mathbf{P}^1} \overline{D_\lambda} \subset \mathbf{P}^1$$

is a $(k-1)$-dimensional rational normal scroll of degree $f = g-k+1$. Let $\mathbf{P}(\mathscr{E})$ denote the corresponding $\mathbf{P}^1$ bundle.

Proposition 4.6. *With C as above the following holds:*

(i) *$\mathscr{O}_C$ has a resolution $F_\bullet$ as $\mathscr{O}_{\mathbf{P}(\mathscr{E})}$-module of the form*

$$0 \to \mathscr{O}_{\mathbf{P}(\mathscr{E})}(-kH+(f-2)R) \to \bigoplus_{j=1}^{\beta_{k-2}} \mathscr{O}_{\mathbf{P}(\mathscr{E})}(-(k+2)H+b^j_{k-2}R) \to \dots$$

$$\dots \to \bigoplus_{j=1}^{\beta_1} \mathscr{O}_{\mathbf{P}(\mathscr{E})}(-2H+b^j_1 R) \to \mathscr{O}_{\mathbf{P}(\mathscr{E})} \to \mathscr{O}_C \to 0$$

where $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$.

(ii) *$F_\bullet$ is self dual, i.e. $\mathscr{H}om(F_\bullet, \mathscr{O}_{\mathbf{P}(\mathscr{E})}(-kH+(f-2)R)) \cong F_\bullet$.*

(iii) *If all $b^j_i \geq -1$ then an iterated mapping cone*

$$\left[\dots \to \left[\mathscr{C}^{f-2}(-k) \to \bigoplus_{j=1}^{\beta_{k-3}} \mathscr{C}^{b^{(j)}_{k-3}}(-k+2)\right] \to \dots\right] \to \mathscr{C}^0$$

is a not necessarily minimal resolution of $\mathscr{O}_C$ as $\mathscr{O}_{\mathbf{P}^{g-1}}$-module.

Proof. [Sch86]. □

Theorem 4.7 (Ballico)**.** *Let C be a general k-gonal curve of genus g and let $\mathfrak{g}^1_k$ be the unique pencil of degree k on C. Then $\dim |r\mathfrak{g}^1_k| = r$ for $r \leq \left\lfloor \frac{g}{k-1} \right\rfloor$.*

Proof. [Bal89]. □

We say that a rational normal scroll X of dimension d and degree f of is of *generic type* if it is of type $S(e_1, \dots, e_d)$ with $e_1 = \dots = e_r = q+1$ and $e_{r+1} = \dots = e_d = q$ where $f = q \cdot d + r$ and $0 \leq r < d$.

Corollary 4.8. *Let $C \subset \mathbf{P}^{g-1}$ be a general k-gonal canonical curve and let X be the scroll swept out by the special pencil. Then X is of generic type.*

Proof. Let D be a divisor of the special pencil and let $s = \left\lfloor \frac{g}{k-1} \right\rfloor$. We apply Riemann-Roch to obtain

$$h^0(K-iD) = \begin{cases} g-i(k-1) & \text{for } i \leq s, \\ 0 & \text{else.} \end{cases}$$

Correspondly, the partition of g defined by $d_i = h^0(K-iD)-h^0(K-(i+1)D)$ is of the form $(d_1,\dots,d_s) = (k-1,\dots,k-1,g-s(k-1))$ and the dual partition is generic. $\square$

This motivates to fix a scroll X of generic type and to study the Hilbert scheme $\mathrm{Hilb}_{g,k}(X)$ of curves of genus g and degree $2g-2$ contained in X and its rational map to the Hurwitz scheme. Note that the ruling of X cuts out a $\mathfrak{g}^1_k$ on such a curve if smooth, hence the notation. We need the following lemma, see also [GV06].

Lemma 4.9. *There is a unique generically reduced component* $H \subset \mathrm{Hilb}_{g,k}(X)$ *of dimension* $\dim(H) = k^2+2g-2$ *which dominates* $\mathscr{H}_{g,k}$. *The fiber of the corresponding map* $H \dashrightarrow \mathscr{H}_{g,k}$ *over a point* $[C]$ *is precisely* $\mathrm{Aut}(X)/\mathrm{Aut}(C)$.

Proof. If $[C] \in \mathrm{Hilb}_{g,k}(X)$ is the point corresponding to a curve C then the Zariski tangent space to $\mathrm{Hilb}_{g,k}(X)$ at $[C]$ is $T_{[C]}(\mathrm{Hilb}_{g,k}(X)) = H^0(C,\mathscr{N}_{C/X})$. This allows to determine the following upper bound for the dimension of the Hilbert scheme. From the conormal exact sequence

$$0 \to \mathscr{N}^\vee_{C/X} \to \Omega_X \otimes \mathscr{O}_C \to \Omega_C \to 0$$

we obtain $\deg(\mathscr{N}_{C/X}) = (d+1)(2g-2)-(f-2)k$. Riemann-Roch yields

$$\chi(\mathscr{N}_{C/X}) = \deg(\mathscr{N}_{C/X}) - \mathrm{rk}(\mathscr{N}_{C/X})(g-1) = k^2+2g-2.$$

Let us now show that $H^1(\mathscr{N}_{C/X}) = H^0(\mathscr{N}^\vee_{C/X}\otimes\omega_C) = 0$ for a sufficiently general curve C. For this let $\pi: X \to \mathbf{P}^1$ the natural map to $\mathbf{P}^1$. Considering the conormal exact sequences of $\pi: X\to\mathbf{P}^1$ and $\pi: C \to \mathbf{P}^1$ we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\Omega_{\mathbf{P}^1}\otimes\omega_C \arrow[r] & \Omega^1_C\otimes\omega_C \arrow[r] & \Omega_{C/\mathbf{P}^1}\otimes\omega_C \arrow[r] & 0 \\
0 \arrow[r] & \pi^*\Omega_{\mathbf{P}^1}\otimes\omega_C \arrow[r] \arrow[u, equal] & \Omega^1_X\otimes\omega_C \arrow[r] \arrow[u] & \Omega_{X/\mathbf{P}^1}\otimes\omega_C \arrow[r] \arrow[u] & 0 \\
 & & \mathscr{N}^\vee_{C/X}\otimes\omega_C \arrow[u] & &
\end{tikzcd}$$

We show that $\varphi: H^0(C,\pi^*\Omega_{\mathbf{P}^1}\otimes\omega_C) \to H^0(\Omega^1_X\otimes\omega_C)$ is an isomorphism which implies $H^0(\mathscr{N}^\vee_{C/X}\otimes\omega_C) = 0$, as desired. First observe that $\pi^*\Omega_{\mathbf{P}^1}\otimes\omega_C \cong \mathscr{O}_C(K-2D)$, hence $h^0(C,\pi^*\Omega_{\mathbf{P}^1}) = -2k-1+g+3 = g-2k+2$, by Ballico's theorem. Moreover, by Riemann-Roch

$$h^0(\Omega_X\otimes\omega_C) - h^1(\Omega_X\otimes\omega_C) = (f-2)k - k(g-1) = -k^2+g-1.$$

Next, we note that $H^0(\mathcal{T}_X \otimes \mathcal{I}_{C/X}) = 0$ as this group can be intepreted as the group of vector fields of X which vanish along C but C has only finitely many automorphisms. Thus, we have

$$h^1(\Omega_X \otimes \omega_C) = h^0(\mathcal{T}_X \otimes \mathcal{O}_C) = h^0(\mathcal{T}_X) = \dim \operatorname{Aut}(X) = k^2 - 2k + 3$$

and hence $h^0(\Omega_X \otimes \omega_C) = g - 2k + 2$. We conclude φ is an injection of vector spaces of the same dimension and hence an isomorphism.

Let H denote the component of $\operatorname{Hilb}_{g,k}(X)$ containing C. The fiber of a general point, e.g. of C, under the projection $\pi : \operatorname{Hilb}_{g,k}(X) \to \mathscr{H}_{g,k}$ is precisely $\operatorname{Aut}(X)$. Counting dimensions we see that H dominates $\mathscr{H}_{g,k}$. As the general k-gonal curve has a unique pencil of degree k we see that H is unique. $\square$

Unirationality of k-gonal Curves for $k \leq 5$. For $3 \leq k \leq 5$ and any $g \geq k+1$ the resolution in Proposition 4.6 (i) is described by structure theorems of Gorenstein ideals of codimension $k-2$ and yields another proof of the unirationality of $\mathscr{H}_{g,k}$, see [Sch86, Section 6]. Let us briefly summarize the situation for these cases:

k=3. A trigonal canonical curve C lies on a two dimensional scroll X and from Proposition 4.6 one sees that C is a divisor on X defined by a section in $H^0(\mathcal{O}_{\mathbf{P}(\mathscr{E})}(3H - (f-2)R))$.

k=4. A tetragonal canonical curve C is contained in a three dimensional scroll and is a complete intersection defined by a pair of global sections in $H^0(2H - b_iR)$ with $b_1 + b_2 = g - 5$.

k=5. A pentagonal canonical curve C is contained in a four dimensional scroll and has a resolution

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbf{P}(\mathscr{E})}(-5H + (f-2)R) \to \bigoplus_{i=1}^{5} \mathcal{O}_{\mathbf{P}(\mathscr{E})}(-3H + b_iR) \\ \xrightarrow{\psi} \bigoplus_{i=1}^{5} \mathcal{O}_{\mathbf{P}(\mathscr{E})}(-2H + a_iR) \to \mathcal{O}_{\mathbf{P}(\mathscr{E})} \to \mathcal{O}_C \to 0 \end{aligned}$$

The Buchsbaum-Eisenbud structure theorem [BE77] states that ψ is skewsymmetric and its 5 Pfaffians generate the vanishing ideal of C in $\mathbf{P}(\mathscr{E})$. Hence, C can be described by a collection of sections $\psi_{ij} \in H^0(\mathcal{O}_{\mathbf{P}(\mathscr{E})}(H + (a_i - b_j)R)$.

4.2 Hexagonal curves...

Let C be a curve of genus g with a $\mathfrak{g}^1_6$. The pencil sweeps out a scroll

$$C \subset X = \bigcup_{D \in \mathfrak{g}^1_6} \overline{D} \subset \mathbf{P}^{g-1}$$

of type $S(e_1, \ldots, e_5)$ with $\frac{2g-2}{6} \geq e_1 \geq \ldots e_5 \geq 0$ and degree $f = \sum_i e_i = g-5$. If $\mathbf{P}(\mathscr{E})$ denotes the corresponding projective bundle over $\mathbf{P}^1$. From Proposition 4.6 we see that the resolution of $\mathscr{O}_C$ as $\mathscr{O}_{\mathbf{P}(\mathscr{E})}$-bundle is of the form

$$\begin{aligned} 0 &\to \mathscr{O}_{\mathbf{P}(\mathscr{E})}(-6H + (f-2)R) \to \textstyle\bigoplus_{i=1}^{9} \mathscr{O}_{\mathbf{P}(\mathscr{E})}(-4H + ((f-2) - \alpha_i)R) \\ &\to \textstyle\bigoplus_{i=1}^{16} \mathscr{O}_{\mathbf{P}(\mathscr{E})}(-3H + \beta_i R) \to \bigoplus_{i=1}^{9} \mathscr{O}_{\mathbf{P}(\mathscr{E})}(-2H + \alpha_i R) \\ &\to \mathscr{O}_{\mathbf{P}(\mathscr{E})} \to \mathscr{O}_C \to 0. \end{aligned} \tag{4.10}$$

Twisting with appropriate powers of $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(H)$ and considering Hilbert functions, we see that $\alpha_1, \ldots, \alpha_9$ and $\beta_1, \ldots, \beta_{16}$ are subject to the relations

$$\sum_{i=1}^{9} \alpha_i = 3g - 21$$

and

$$\sum_{i=1}^{16} \beta_i = 8g - 56.$$

Under the assumption that all maps are of maximal rank, we see that the degrees $(\alpha_1, \ldots, \alpha_9)$ and $(\beta_1, \ldots, \beta_{16})$ form generic partitions of $3g-21$ and $8g-56$, respectively. We can computationally verify this in the cases where $\mathscr{H}_{g,6}$ has a unirational parametrization.

Proposition 4.11. *For g among the values covered by our unirationality constructions, the general 6-gonal curve C lies on a scroll of generic type and has a resolution of the form (4.10) with generic degrees $(\alpha_1, \ldots, \alpha_9)$ and $(\beta_1, \ldots, \beta_{16})$.*

Proof. Having generic syzygy numbers is an open condition. It remains to verify the existence of a curve with these properties. We do this computationally using the *Macaulay2* code below. □

Code 4.12. We explain the computation along the example case $g = 15$. Initialization:

```
i1 : Fp=ZZ/32009; -- a finite field
     S=Fp[x_0,x1,y_0..y_2,Degrees=>{2:{1,0},3:{0,1}}];
        -- Cox-ring of P^1 x P^2
```

```
Sel=Fp[x_0,x_1,y_0..y_2,MonomialOrder=>Eliminate 2];
   -- elimination order
R2=Fp[y_0..y_2]; -- coordinate ring of P^2
m=ideal basis({1,1},S);
   -- irrelevant ideal
setRandomSeed("HurwitzSpaces");
   -- initialization of the random number generator
```

We compute the vanishing ideal of C in the rational normal scroll by using the plane model $\Gamma \subset \mathbf{P}^2$ and adjunction.

```
i2 : ICsat=construction6Gonal(g);
     IGammaC=sub(ideal selectInSubring(1,gens gb sub(ICsat,Sel)),R2);
     time IDelta=saturate ideal jacobian IGammaC + IGammaC;
        -- compute the singular locus
```

Let $d = \deg(\Gamma)$. Using the identification $H^0(K_C) \cong H^0(\mathscr{I}_\Delta(d-3))$ we compute a collection of polynomials $\omega_1, \ldots, \omega_g \in \mathbf{F}_p[y_0, y_1, y_2]$ which correspond to a basis of $H^0(K_C)$. To this end, we pick two points $P_1 = (1 : 0), P_2 = (0 : 1) \in \mathbf{P}^1$ and consider the associated divisors $D_i = \pi_1^{-1}(P_i)$ for $i = 1, 2$ on Γ which also correspond to the choice of a basis s_1, s_2 of $H^0(D)$.

```
i3 : P1=matrix{{1_Fp,0}};
     D1=ideal (mingens substitute(ICsat,P1|vars R2));
     P2=matrix{{0,1_Fp}};
     D2=ideal (mingens substitute(ICsat,P2|vars R2));
```

The following function computes a basis of $H^0(K - nD_1)$ considered as a subspace of $H^0(K)$.

```
i2 : HOK=(mingens IDelta)*
        random(source mingens IDelta, R2^{g:-(d-3)});
        -- the canonical system
     HOKnD=n->HOK*constantSyz(HOK
        % mingens saturate ideal mingens(D1^n+IGammaC))
```

We compute the type of the scroll by computing the differences in the numbers of global section $h^0(\mathscr{O}_C(K - iD_1)) - h^0(K - (i+1)D_1)$ as discussed in section 4.1.

```
i4 : eDual={}; -- the dual partition
     time for i from 0 do (
              Dual0:=rank source HOKnD(i) - rank source HOKnD(i+1);
              if eDual0==0 then break else (eDual=eDual|{eDual0}));
        -- used 9.66583 seconds
     e=apply(5,i->#select(eDual,e0->e0>=i+1)-1)

o4 = {2, 2, 2, 2, 2}
```

We now search for a $\mathbf{F}_p$-rational point on Γ and compute its value under the projection to $\mathbf{P}^1$. We will make us of this in the next step to rescale the basis elements in $H^0(K)$.

```
i5 : use R2
     time while (
            while (Q=random(Fp^1,Fp^3);
               Q==0 or sub(IGammaC,Q)!=0) do ();
               -- find a point Q on C
            st=sub(transpose syz contract(matrix{{x_0,x_1}},
               transpose mingens sub(ICsat,matrix{{x_0,x_1}}|Q)),Fp);
               -- the value under the map to PP^1
            st_(0,0)==0 or st_(0,1)==0) do ();
      r=st_(0,0)/st_(0,1)
         -- the value of the map to PP^1
```

We identify $H^0(K - nD_1 - mD_2)$ with the image of $H^0(K - (n + m)D)$ in $H^0(K)$ under the multiplication with $s_1^n s_2^m$, i.e. with the subspace of elements in $\langle\omega_1, \dots, \omega_g\rangle$ which vanish in D_1 with order at least n and in D_2 with order at least m. As we want to produce a basis $x_{ij} = s_1^{e_i-j} s_2^j \varphi_i$ we have to determine for an element $\varphi_i \in H^0(K - e_i D)$ its images under the multiplication maps $s_1^{e_i-j} s_2^j$ for $j = 0, \dots, e_i$. On Γ the element corresponding to $s_1^{e_i}\varphi_i$ cuts out the divisors 2Δ and nD_1 as well as a residual divisor E_i only depending on φ_i. Hence, $s_1^j s_2^{e_i-j}\varphi_i$ can be identified as the unique element (up to scalars) in $H^0(K - (e_i - j)D_1 - jD_2)$ that vanishes along E_i.

```
i6 : PHI0=apply(e,n->(
            H0KnD1:=H0KnD(n);
            phi:=H0KnD1*random(Fp^(rank source H0KnD1),Fp^1);
            E0:=saturate(ideal phi+IGammaC,IDelta);
            E:=saturate(E0,D1);
            b0:=apply(n+1,i->(
                  EDs:=intersect(E,D1^(n-i),D2^i);
                  rels:=mingens saturate ideal mingens(EDs+IGammaC);
                  (H0K*constantSyz(H0K%rels))_(0,0)));
                  apply(n+1,i->
                    (sub(b0_0,Q)/(r^i*sub(b0_i,Q)))*b0_i)));
     PHI=matrix{flatten PHI0};
```

In the next step we compute the vanishing ideal $I_{C,can}$ of the canonical embedding of C.

```
i7 : Z=Fp[z_0..z_(g-1)]; -- the ring for the canonical embedding
     phi=map(R2,Z,PHI);
     time ICcan=saturate(preimage_phi(IGammaC));
     (dim ICcan, degree ICcan, genus ICcan)

o7 = (2, 28, 15)
```

As pointed out in Remark 4.2, the Cox-Ring $R_X = \bigoplus_{a,b\in\mathbf{Z}} H^0(\mathscr{O}_X(aH + bR))$ of X is a subring of the polynomial ring $T = \mathbf{F}_p[s, t, \varphi_1, \dots, \varphi_5]$ equipped with the bigrading $\deg(s) = \deg(t) = (0, 1)$ and $\deg(\varphi_i) = (1, e_1 - e_i)$. In order to obtain the Betti numbers we compute the image $J \subset T$ of the ideal of the canonical curve $I_{C,can}$ under the natural map $\psi : Z/I_X \to T$.

```
i8 : degs={apply(0..#e-1, i->{1,max e-e_i}),2:{0,1}};
     T=Fp[p_0..p_4,s,t,Degrees=>degs];
     PSI=matrix{flatten
            apply(#e,i->flatten entries (basis({0,e_i},T)*p_i))};
     psi=map(T,Z,PSI); -- the parametrization of the scroll
     Scroll=preimage_psi(ideal 0_T);
     Z'=Z/Scroll;
     psi'=map(T,Z',PSI);
     J=psi'(ideal mingens sub(ICcan,Z'));
        -- the vanishing ideal of the scroll
     J1=saturate(J,ideal basis({0,1},T));
```

We obtain the generators of $I_{C/X}$ in J and all the Betti numbers by picking the degrees.

```
i9 : J2=ideal select(flatten entries gens J1,f->(degree f)_0 == 2);
     degsH={3,4,6};
     resX={gens J2};
     scan(#degsH,i->(
        M0=syz (resX_i);
        cols:=toList(0..rank source M0-1);
        M=M0_(select(cols,j->
        ((degrees source M0)_j)_0==degsH_i));
     resX=resX|{M}));
     betti chainComplex resX

            0 1  2 3 4
o9 = total: 1 9 16 9 1
         0: 1 .  . . .
         1: . .  . . .
         2: . 6  . . .
         3: . 3 16 3 .
         4: . .  . 6 .
         5: . .  . . .
         6: . .  . . 1
```

4.3 ...with a View Toward Gorenstein Ideals of Codimension 4

We examine in the following to which extent the hexagonal curves considered as subvarieties of the associated scroll have a determinantal structure.

The Gulliksen-Negard complex. Let X be a scheme, $\mathscr{F}$ and $\mathscr{G}$ be two vector bundles on X of rank m and $\alpha : \mathscr{F} \to \mathscr{G}$ a morphism. The degeneracy loci

$$X^r(\alpha) := \{x \in X \mid \mathrm{rk}(\alpha) \leq r\} \subset X$$

has a natural structure of a closed subscheme of X. In the case $r = 2$ we consider the line bundle $\mathscr{L} = \bigwedge^m \mathscr{E} \otimes \bigwedge^m \mathscr{F}^\vee$ and the Gulliksen-Negard complex $\mathrm{GN}(\alpha)$

$$0 \to \mathscr{L}^2 \to \mathscr{F} \otimes \mathscr{G}^\vee \otimes \mathscr{L} \to \det(\mathscr{F}) \otimes L_{e-1,1}\mathscr{G}^\vee \oplus \det \mathscr{G}^\vee \otimes L_{e-1,1}\mathscr{G}$$

$$\to \bigwedge\nolimits^{e-1} \mathscr{F} \otimes \bigwedge\nolimits^{e-1} \mathscr{G}^\vee \to 0$$

where $L_{e-1,1}$ is the Schur functor associated to the partition $(e-1,1)$. From the definitions of Schur functors, see [Wey03, Ch. 2], one sees that for a bundle $\mathscr{E}$ the resulting bundle $L_{e-1,1}\mathscr{E}$ is the cokernel of $\bigwedge^e \mathscr{E} \to \bigwedge^{e-1} \mathscr{E} \otimes \mathscr{E}$.

Theorem 4.13 (Gulliksen-Negard)**.** *In the situation as above the following holds:*

(i) If $X(\alpha)$ is of codimension 4 then $\mathrm{GN}(\alpha)$ is a resolution of $X^2(\alpha)$.

(ii) If in addition X is Gorenstein, then $X^2(\alpha)$ is subcanonical with

$$\omega_{X^2(\alpha)} \cong \omega_X \otimes \mathscr{L}^{-2}\big|_X.$$

Proof. This theorem was originally proved in [GN72]. See [Wey03, (6.1.8)] for a more conceptual proof. □

Let us turn to the situation of canonical curves on rational normal scrolls. Assume that there is a 6-gonal curve $C \subset X \subset \mathbf{P}^{g-1}$ of genus g and a map $\alpha : \mathscr{F} \to \mathscr{G}$ of vector bundles of rank 3 on X such that $C = X^2(\alpha)$. Then by Theorem 4.13 the complex $\mathrm{GN}(\alpha)$ resolves the curve. The equality

$$\bigwedge^2 \mathscr{F} \otimes \bigwedge^2 \mathscr{G}^\vee = \bigoplus_{i=1}^{9} \mathscr{O}_X(-2H + \alpha_i R)$$

implies $-18H + (3g-21)R = 6(c_1(\mathscr{F}) - c_1(\mathscr{G}))$ and we see that g must be odd independently of the bundles $\mathscr{F}$ and $\mathscr{G}$. In this case, say $g = 2m+1$, we have

$$c_1(\mathscr{F}) - c_1(\mathscr{G}) = -3H + \frac{g-7}{2}R = -3H + (m-3)R.$$

Moreover, the determinant $\det(\alpha)$ which is a section of the line bundle

$$\bigwedge^3 \mathscr{F}^\vee \otimes \bigwedge^3 \mathscr{G} = \mathscr{O}_X(3H - \frac{g-7}{2}R)$$

vanishes with multiplicity at least two along C. However, the following calculation shows that, in general, we do not expect to have such a form.

Lemma 4.14. *Let $g = 2m+1$ and assume C is a sufficiently general 6-gonal curve of genus g such that* *(i)* C *is of maximal rank in the associated scroll* X *and*

(ii) $H^0(\mathscr{I}_{C/X}(3H-(m-3)R)) \to H^0(\mathscr{N}^\vee_{C/X}(3H-(m-3)R))$ *is of maximal rank.*

Then $H^0(\mathscr{I}^2_{C/X}(3H-(m-3)R)) = 0$.

Proof. We abbreviate $E = 3H-(m-3)R$. Consider standard exact sequence

$$0 \to \mathscr{I}_{C/X} \to \mathscr{O}_X \to \mathscr{O}_C \to 0$$

to compute the number of global sections of the twisted ideal sheaf. First note that $\deg(\mathscr{O}_C(E) = 3\cdot(2g-2) - 6\cdot(m-3) = 6m+18 > 2g-2$. Hence, $\chi(\mathscr{O}_C(E)) = h^0(\mathscr{O}_C(E)) = 4m+18$ by Riemann-Roch. By Ballico's theorem the scroll is balanced and we can use Proposition 4.1 (iii) to compute $h^0(\mathscr{O}_X(E)) = 7m+56$. Under the assumption of maximal rank, we obtain $h^0(\mathscr{I}_C(E)) = 3m+38$.

Again, by Riemann-Roch, we obtain $\chi(\mathscr{N}^\vee_{C/X}(E)) = 4m+36$ and hence $h^0(\mathscr{N}^\vee_{C/X}(E) = \chi(\mathscr{N}^\vee_{C/X}(E)) + h^1(\mathscr{N}^\vee_{C/X}(E)) \geq 4m+36$. Hence, considering the exact sequence

$$0 \to \mathscr{I}^2_{C/X} \to \mathscr{I}_{C/X} \to \mathscr{N}^\vee_{C/X} \to 0$$

and using (ii), we see $H^0(\mathscr{I}^2_{C/X}(E)) = 0$. □

In the cases covered by the unirationality constructions we can compute the saturated vanishing ideal in the Cox ring of the scroll to check whether there is a determinant or not.

Code 4.15. Continuing the computation for $g = 15$ from Code 4.12 we compute the saturated square of the vanishing ideal

```
i121 : time J2sat=saturate(J2^2,ideal basis({1,1},T));
     -- used 1316.76 seconds

o121 : Ideal of T

i122 : tally degrees J2sat

o122 = Tally{{3, 4} => 5 }
             {4, 2} => 21
             {4, 3} => 18
             {5, 1} => 12
             {6, 0} => 1
             {6, 1} => 40
             {7, 0} => 41
```

A section in $H^0(\mathscr{I}^2_{C/X}(3H-4R))$ would have been represented by a polynomial of bidegree $(3, 2\cdot 3 - 4) = (3,2)$.

Unirational Subvarieties of $\mathscr{H}_{g,6}$. As shown above the resolution of a general 6-gonal curve of genus g is not described by a Gulliksen-Negard complex for small g (and conjecturally not for any g).

However, if we consider for $g = 2m+1$ a fixed rational normal scroll $X \subset \mathbf{P}^{g-1}$ of generic type $S(e_1, \dots, e_5)$ and restrict to the case that $\mathscr{F}$ and $\mathscr{G}$ split as a direct sum of line bundles, we can determine the dimension of the obtained unirational subvariety.

Theorem 4.16. *For $g = 2m+1$, $m \geq 2$, there is a unirational subvariety $V \subset \mathscr{H}_{g,6}$ of dimension $3m+9 = 2/3(g+5)$ (and hence codimension $m = (g-1)/2$).*

Proof. After twisting with an appropriate power of $\mathscr{O}_X(H)$ we can assume that

$$\mathscr{F} = \bigoplus_{i=1}^{3} \mathscr{O}_{\mathbf{P}(\mathscr{E})}(-H + s_i R) \quad \text{and} \quad \mathscr{G} = \bigoplus_{j=1}^{3} \mathscr{O}_{\mathbf{P}(\mathscr{E})}(t_j R).$$

Let $r_{ij} = s_i - t_j$ and consider an arbitrary but fixed ordering of the 2×2 minors. We then have a system of equations of the form

$$r_{i_1,j_1} + r_{i_2,j_2} = r_{i_1,j_2} + r_{i_2,j_1} = \alpha_k, \quad k = 1, \dots, 9$$

with $\sum \alpha_i = 3g - 21$. It is easy to see that this implies $\sum_{ij} r_{ij} = 3m - 9$ and for any solution we have most two different values among the r_{ij} depending on the congruence class of m modulo 3. In particular, for $m = 3n$ or $m = 3n+2$ the only solution for α_k is the generic partition. For $m = 3n+1$ we have two solutions, one leading to a unirational variety of strictly smaller dimension which we omit.

We discuss the case $m = 3n$ (leaving the other cases to the reader). Then $r_{ij} = n - 1$. A morphism of bundles $\psi : \mathscr{F} \to \mathscr{G}$ is given by a collection of global sections $\psi_{ij} \in H^0(\mathscr{O}_X(H - r_{ij}R))$. Thus we have a rational map

$$\Psi : \mathbf{G}(3, H^0(\mathscr{O}_X(H - (n-1)R))^{\oplus 3})/\mathrm{SL}(3) \dashrightarrow \mathrm{Hilb}_{g,6}(X), \ [\psi] \to \left[X^2(\psi)\right].$$

From $\min\{e_1, \dots, e_5\} = \left\lfloor \frac{f}{k-1} \right\rfloor \geq n-1$ we conclude that the sheaf

$$\mathscr{H}om(\mathscr{F}, \mathscr{G}) \cong \mathscr{O}_{\mathbf{P}(\mathscr{E})}(H - (n-1)R)^{\oplus 9}$$

is generated by global sections. Thus, by a Bertini-type theorem, see e.g. [Ott95, Teorema 2.8], it follows that for the general morphism $\psi : \mathscr{F} \to \mathscr{G}$ the associated scheme $X^2(\psi)$ is a smooth curve. Hence, Ψ factors over the component H of $\mathrm{Hilb}_{g,6}(X)$. Note that for any automorphism $\varphi \in \mathrm{Aut}(X)$ the induced map $\varphi^*\mathscr{F} \to \varphi^*\mathscr{G}$ defines a curve isomorphic to $C = X^2(\psi)$. From this we see that the image of Ψ already contains the fiber of C under the projection $H \dashrightarrow \mathscr{H}_{g,6}$. We count dimensions

$$\begin{aligned}
\dim V &= \dim \mathbf{G}(3, H^0(\mathscr{O}_X(H-(n-1)R))^{\oplus 3}) - \dim \mathrm{SL}(3) - \dim \mathrm{Aut}(X) \\
&= 3 \cdot (3(n+6)-3) - 9 - (36 - 2 \cdot 6 + 3) \\
&= 9n + 9 = 2/3(g+5).
\end{aligned}$$

$\square$

5 Ulrich Bundles on Cubic Threefolds

In this chapter we apply the unirationality construction of the preceeding part to prove the existence of stable ACM bundles of any rank on the cubic threefold. This is joint work with M. Casanellas, R. Hartshorne and F.-O. Schreyer which is published in [CH11b]. The main result of the author are Propositions 5.35 and 5.45.

We start by collecting the necessary informations about ACM sheaves and Ulrich sheaves and their moduli from [CH11a], [CH11b] and [HL10] in Section 1 of this chapter. Section 2 is devoted to the proof of the central theorem of this chapter stating that the general cubic threefold has stable Ulrich bundles of every possible rank.

5.1 ACM Sheaves and Ulrich Sheaves

ACM Sheaves. Let $\mathscr{F}$ be a coherent sheaf of rank r on a projective variety $X \subset \mathbf{P}^n$ of dimension m and degree d and let $S = \mathbf{k}[x_0, \dots, x_n]$ be the homogeneous coordinate ring of $\mathbf{P}^n$. The sheaf $\mathscr{F}$ is called *locally Cohen-Macaulay* if for every point $x \in X$ the stalk $\mathscr{F}_x$ is a Cohen-Macaulay module over the local ring $\mathscr{O}_{X,x}$, i.e. $\operatorname{depth}(\mathscr{F}_x) = \dim(\mathscr{F}_x)$.

Definition 5.1. *$\mathscr{F}$ is called* arithmetically Cohen-Macaulay *or* ACM *if the module of global sections $\Gamma_*(\mathscr{F}) = \bigoplus_{i\in\mathbf{Z}} H^0(\mathscr{F}(i))$ is a Cohen-Macaulay Module over S.*

We can also characterize ACM sheaves as follows.

Proposition 5.2. *$\mathscr{F}$ is* ACM *if and only if $\mathscr{F}$ is locally Cohen-Macaulay and all intermediate cohomology groups of $\mathscr{F}$ vanish, i.e. $H^i(\mathscr{F}(k)) = 0$ for all $1 \le i \le m-1$ and all k.*

Proof. We consider the sheafification of a minimal free resolution of $F := \Gamma_*(\mathscr{F})$ by free S-Modules

$$0 \to \mathscr{F}_r \xrightarrow{\varphi_r} \mathscr{F}_{r-1} \to \dots \to \mathscr{F}_1 \xrightarrow{\varphi_1} \mathscr{F}_0 \xrightarrow{\varphi_0} \mathscr{F} \to 0 \tag{5.3}$$

and set $\mathscr{E}_j = \ker\varphi_j$ for $j = 0, \dots, r-1$. Then $H^{i+1}(\mathscr{E}_j(k)) \cong H^i(\mathscr{E}_{j-1}(k))$ for $1 \le i \le n-2$ and $0 \le i \le r$ and any k. Hence $H^i(\mathscr{F}(k)) \cong H^{i+r}(\mathscr{F}_r(k))$ for $1 \le i \le n-r-1$.
Now assume that $\mathscr{F}$ is ACM. Then for every point $x \in X$ the module $\mathscr{F}_x$ is also Cohen-Macaulay. By the Auslander-Buchsbaum formula [Eis95, p. 479],

the minimal free resolution (5.3) has length $r = n - m$ and the vanishing of the cohomology groups follows.

Conversly, assume that the minimal free resolution (5.3) has length $n-m+1$. As $\mathscr{F}$ is locally Cohen-Macaulay, we see that the ideal of maximal minors $I(\varphi_{n-m+1})$ can be only supported on $\mathfrak{m} = (x_0, \dots, x_n)$. Hence, it remains to show that $F_\mathfrak{m}$ is Cohen-Macaulay. But this follows from the identification of local cohomology with sheaf cohomology, see [Eis95, Theorem A 4.1]. Indeed, from $H^i_\mathfrak{m}(F) \cong \bigoplus_{k\in\mathbf{Z}} H^{i-1}(\mathscr{F}(k))$ for $i \geq 2$ one deduces that the local cohomology groups $H^i_{S_\mathfrak{m}\mathfrak{m}}(F_\mathfrak{m}) = 0$ for $i \leq m$. But for $\delta = \operatorname{depth}(S_\mathfrak{m}\mathfrak{m}, F_\mathfrak{m})$ we have $H^d_{S_\mathfrak{m}\mathfrak{m}}(F_\mathfrak{m}) \neq 0$ by [Eis95, Theorem A 4.3] and thus $\delta = m + 1 = \dim F_\mathfrak{m}$. We conclude that the support of $I(\varphi_{n-m+1})$ is empty in contradiction to the minimality of the resolution of $\mathscr{F}$. □

Corollary 5.4. *If X is nonsingular then any ACM sheaf is locally free.*

Proof. Since X is smooth for any point $x \in X$ the local ring $\mathscr{O}_{X,x}$ is regular, thus $\operatorname{gl}\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{X,x} = \operatorname{depth} \mathscr{O}_{X,x}$. For an ACM sheaf $\mathscr{F}$ the stalk $\mathscr{F}_x$ is a Cohen-Macaulay module over $\mathscr{O}_{X,x}$ and by the Auslander-Buchsbaum formula we see that $\operatorname{pr}\dim \mathscr{F}_x = 0$. Thus, $\mathscr{F}_x$ is a projective module over a local ring and hence free. □

Corollary 5.5 (Horrocks Theorem)**.** *A locally free sheaf $\mathscr{F}$ on $\mathbf{P}^n$ with vanishing intermediate cohomology splits as a direct sum of line bundles.*

Proof. The module $\Gamma_*(\mathscr{F})$ is ACM of dimension n and hence has projective dimension 0. It follows from the graded version of the Auslander-Buchsbaum formula [Eis95, Exercise 19.8] that the minimal free resolution of $\Gamma_*(\mathscr{F})$ has only one term F_0. Thus $\Gamma_*(\mathscr{F})$ is free. □

Horrocks theorem indicates that ACM bundles should be in some sense the simplest bundles on a projective variety.

Proposition 5.6. *The number of minimal generators $\mu(\mathscr{F})$ of the S-Module $\Gamma_*(\mathscr{F})$ for an ACM sheaf $\mathscr{F}$ on X is bounded by*

$$h^0(\mathscr{F}) \leq \mu(\mathscr{F}) \leq dr. \tag{5.7}$$

Proof. This is [CH11a, Theorem 3.1]. Consider a noether normalization $T = \mathbf{k}[y_0, \dots, y_m] \subset S/I_X$ and the corresponding finite projection $\pi : X \to \mathbf{P}^m$. Then $H^i(\pi_*\mathscr{F}(k)) = H^i(\mathscr{F}(k))$ as π is finite. Hence $\pi_*\mathscr{F}$ is locally free of rank dr and has no intermediate cohomology. By Horrocks theorem $\pi_*\mathscr{F}$ splits into a direct sum of line bundles: $\pi_*\mathscr{F} = \mathscr{O}_{\mathbf{P}^k}(e_1) \oplus \dots \oplus \mathscr{O}_{\mathbf{P}^k}(e_{dr})$. Hence $\Gamma_*(\mathscr{F}) = \Gamma_*(\pi_*(\mathscr{F}))$ is minimally generated by dr elements over T. $\Gamma_*(\mathscr{F})$ is also generated by these elements as S-Module, but they do not necessarily form a minimal system of generators any more. □

Ulrich Sheaves. We call a sheaf $\mathscr{F}$ on a projective variety *normalized* if the cohomology groups $H^0(\mathscr{F}) \neq 0$ and $H^0(\mathscr{F}(-1)) = 0$. If not otherwise mentioned, we will consider all sheaves as normalized.

Definition 5.8. *An ACM sheaf $\mathscr{F}$ is called* Ulrich sheaf *or* maximally ACM sheaf *if $\mu(\mathscr{F}) = dr$.*

Proposition 5.9. *Let $\mathscr{F}$ be an sheaf on $X \subset \mathbf{P}^n$. $\mathscr{F}$ is Ulrich if and only if $\Gamma_*(\mathscr{F})$ is a Cohen-Macaulay S-module with a linear free resolution*

$$0 \to \mathscr{O}_{\mathbf{P}^n}(n-k)^{\oplus dr} \to \ldots \to \mathscr{O}_{\mathbf{P}^n}(-i)^{\oplus dr\binom{n-k}{i}} \to \ldots \to \mathscr{O}_{\mathbf{P}^n}^{\oplus dr} \to \mathscr{F} \to 0. \tag{5.10}$$

Proof. This is [BHU87, Proposition 1.5]. □

Corollary 5.11. *If $\mathscr{F}$ is an Ulrich bundle of rank r on a nonsingular projective variety X of degree d and dimension N then its Hilbert polynomial is*

$$P_{\mathscr{E}}(n) = rd\binom{n+N}{N}. \tag{5.12}$$

Existence of Ulrich Sheaves. In this paragraph we briefly survey known results on the existence of Ulrich sheaves. In general, it seems to be a delicate question to determine whether a variety has an Ulrich sheaf.

Conjecture 5.13 (Eisenbud, Schreyer)**.** *Every variety has an Ulrich sheaf.*

Remark 5.14. The conjecture was originally stated in [ESW03] and a strong motivation for this conjecture arises from Boij-Soederberg theory [ES11]. The existence of an Ulrich sheaf $\mathscr{F}$ on a variety X of dimension k implies that the cone of cohomology tables of coherent sheaves

$$\operatorname{Pos}\big(\gamma(\mathscr{E}) = \{h^i(\mathscr{E}(j))\}_{ij} \mid \mathscr{E} \text{ coherent sheaf on } X\big) \subset \bigoplus_{i=0}^{k} \prod_{j\in\mathbf{Z}} \mathbf{Q}$$

of X equals the corresponding cone of $\mathbf{P}^k$. Indeed, for a finite projection $\pi : X \to \mathbf{P}^n$ we have $\gamma(\mathscr{G}) = \gamma(\pi_*(\mathscr{G}))$ for any sheaf $\mathscr{G}$ on X. Conversely, for any sheaf $\mathscr{E}$ on $\mathbf{P}^n$ we consider the sheaf $\mathscr{G} = \mathscr{F} \otimes \pi^*(\mathscr{E})$. The projection formula implies $\gamma(\mathscr{G}) = \gamma(\pi_*\mathscr{G}) = dr\gamma(\mathscr{E})$ since $\pi_*\mathscr{E} = \mathscr{O}_{\mathbf{P}^k}^{dr}$.

Remark 5.15. We collect a number of existence results.

1. On a smooth embedded projective curve $C \subset \mathbf{P}^n$ of genus g a line bundle $\mathscr{L}$ is Ulrich if and only if $\mathscr{L}(-1)$ has degree $g-1$ and no global sections. Eisenbud and Schreyer show in [ESW03, Corollary 4.5] that for arbitrary curves over arbitrary fields it is still possible to find higher rank Ulrich sheaves.

2. The existence of Ulrich bundles on hypersurfaces is a classical topic: From Proposition 5.9 we see immediately that a smooth hypersurface $X = V(F) \subset \mathbf{P}^n$ has an Ulrich line bundle if and only if F can be expressed as determinant of matrix with linear entries. This generally possible only for plane curves and surfaces of degree ≤ 3.
   A fact of which we will make use later is that X has a rank 2 Ulrich bundle if F can be expressed as the pfaffian of skew-symmetric matrix with linear entries. Beauville [Bea00] gives a complete treatment of this.

3. Eisenbud [Eis80] showed that Ulrich bundles on hypersurfaces correspond to linear matrix factorizations. This leads to the fact that every smooth hypersurface has an Ulrich bundle of rank $r \gg 0$ due to Backelin, Herzog and Sanders [BHS88].

Serre Correspondence. Our main tool for the construction of the Ulrich bundles will be the following adaption of the well-known Serre correspondence [HL10, Section 5.1] between locally free sheaves and certain classes of subschemes.

Lemma 5.16. *Let $X \subset \mathbf{P}^4$ be a smooth hypersurface of degree d and let $\mathscr{F}$ be an Ulrich bundle of rank r on X. Then $\deg(\mathscr{F}) = r\binom{d}{2}$.*

Proof. See [CH11b, Lemma 2.4]. The minimal free resolution of $\mathscr{F}$ is given by

$$0 \to \mathscr{O}_{\mathbf{P}^4}^{dr}(-1) \to \mathscr{O}_{\mathbf{P}^4}^{dr} \to \mathscr{F} \to 0 \tag{5.17}$$

from which we see that $\chi(\mathscr{F}) = dr$ and $\chi(\mathscr{F}(1)) = 4dr$. By Bertinis theorem the general hyperplane section of X is a smooth surface S of degree d and we have a short exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}(1) \to \mathscr{F}_H(1) \to 0 \tag{5.18}$$

From the long exact sequence in cohomology we see that the bundle $\mathscr{F}_H$ is ACM and $h^0(\mathscr{F}_H) = h^0(\mathscr{F}) = dr$, hence $\mathscr{F}_H$ is Ulrich or rank r.

The general hyperplane section of S is a smooth plane curve C of genus $g = \binom{d-1}{2}$. Repeating the argument we obtain an Ulrich bundle $\mathscr{F}_{H^2}$ on C and by Riemann-Roch $\chi(\mathscr{F}_{H^2}(1)) = \deg(\mathscr{F}_{H^2}(1)) + r(1-g)$. As $\deg(\mathscr{F}_{H^2}(1)) = \deg(\mathscr{F}_{H^2}) + rH$ and $\deg(\mathscr{F}_{H^2}) = \deg(\mathscr{F})$ we deduce $\deg(\mathscr{F}) = r(d+g-1)$. $\square$

A coherent sheaf $\mathscr{F}$ on a scheme X is called *torsion-free* if $\mathscr{F}_x$ is a torsion-free $\mathscr{O}_{X,x}$-module for all $x \in X$. If $\mathrm{rk}(\mathscr{F}) = r$ then $\mathscr{F}$ locally embeds into $\mathscr{O}_X^r$.

Lemma 5.19. *Let $X \subset \mathbf{P}^4$ be a smooth hypersurface of degree d.*

(i) Let $\mathscr{F}$ be an Ulrich bundle of rank r on X. Then, there is an exact sequence

$$0 \to \mathscr{O}_X^{r-1} \to \mathscr{F} \to \mathscr{I}_{C/X}(r(d-1)/2) \to 0 \tag{5.20}$$

with C a smooth ACM curve of genus

$$g(C) = 1 + r \cdot \binom{d}{2} \cdot \frac{2r^2(d-1)^2 + r(d-11)(d-1) - 2(d-3)^2}{24} \tag{5.21}$$

and degree

$$\deg(C) = r \cdot \binom{d}{2} \cdot \frac{3r(d-1) - 2(d-2)}{12} \tag{5.22}$$

with the additional property that the canonical module $H^0_*(\omega_C)$ is generated by exactly $r-1$ sections in degree $\alpha = \frac{10-(2+r)d+r}{2}$.

(ii) *Conversely, let C be an ACM curve of genus $g(C)$ and degree $d(C)$ as in (i) such that $H^0_*(\omega_C)$ is generated by $r-1$ sections. Then, there is an exact sequence*

$$0 \to \mathscr{O}_X^{r-1} \to \mathscr{F} \to \mathscr{I}_{C/X}(r(d-1)/2) \to 0$$

where $\mathscr{F}$ is an Ulrich bundle of rank r on X.

Proof. The sheaf $\mathscr{F}$ is generated by global sections and a generic collection of $r-1$ sections gives rise to a map $\mathscr{F}^\vee \to \mathscr{O}_X^{r-1}$. The Eagon-Northcott complex associated to this the map is of the form

$$0 \to \mathscr{O}^{r-1} \otimes \mathscr{L} \to \mathscr{F} \otimes \mathscr{L} \to \mathscr{I}_C \to 0$$

with $\mathscr{L} = \bigwedge^r \mathscr{F} = \mathscr{O}(D)$ for a divisor D of degree $\deg D = \deg c_1(\mathscr{F})$. The codimension 2 subscheme C represents $c_2(\mathscr{F})$ and is smooth and irreducible, by Bertini. We compute the Hilbert polynomial of C to obtain the formulas for the degree and genus:

$$\begin{aligned} \chi(\mathscr{O}_C(t)) &= \chi(\mathscr{O}_X(t)) - \chi(\mathscr{I}_{C/X}(t)) \\ &= \chi(\mathscr{O}_X(t)) - \chi(\mathscr{F}(t-r)) + \chi(\mathscr{O}_X^{r-1}(t-r)). \end{aligned}$$

Evaluating this expression yields the claimed formulas. As we assumed X to be general, we have $\mathrm{Pic}(X) = \mathbf{Z}H$ and we see that $D = mH$ with $m = r(d-1)/2$. Finally, the dual of the sequence (5.20) is

$$0 \to \mathscr{O}_X(-m) \to \mathscr{F}^\vee \to \mathscr{O}_X^{r-1} \to \omega_C(\alpha) \to 0 \tag{5.23}$$

and taking global sections shows that $H^0_*(\omega_C)$ is generated by $r-1$ sections in degree α.

Conversely, the generators of $H^0(\omega_C(\alpha)) \cong \mathrm{Ext}^1(\mathscr{I}_C(m), \mathscr{O}_X)$ give rise to an extension

$$0 \to \mathscr{O}_X^{r-1} \to \mathscr{F} \to \mathscr{I}_{C/X}(m) \to 0. \tag{5.24}$$

As C is ACM, we have $H^1_*(\mathscr{F}) \cong H^1_*(\mathscr{I}_C) = 0$. From the dual sequence of

(5.24) which is again (5.23) we obtain

$$H^1_*(\mathscr{F}^\vee) = H^2(\mathscr{F} \otimes \omega_X) = H^2_*(\mathscr{F}(-d)) \cong 0.$$

Hence, $\mathscr{F}$ is an ACM bundle. It remains to show that $\mathscr{F}$ is a normalized Ulrich bundle. Moreover, $h^0\mathscr{I}_{C/X}(m-1) = h^0\mathscr{O}_X(m-1)$ which implies that $h^0(\mathscr{I}_{C/X}(m-1)) = 0$ because C is ACM. This shows that $h^0(\mathscr{F}(-1)) = 0$. Similarly, we obtain $h^0(\mathscr{I}_{C/X}(m)) = dr - r + 1$ and hence $h^0(\mathscr{F}) = dr$. Thus, $\mathscr{F}$ is Ulrich. □

Remark 5.25. From the above computation we see that due to numerical conditions, on a smooth threefold X in $\mathbf{P}^4$ of even degree there can only exist Ulrich bundles of even rank.

Moduli of Ulrich Bundles. For the construction of the moduli space of vectorbundles with given Hilbertpolynomial on a given projective variety X as GIT-Quotient of the Quot-scheme we need to restrict to the class of semistable vectorbundles. We recall from [HL10] the definition of stability for vector bundles.

Definition 5.26 (Stability of Vectorbundles)**.** *Let X be a smooth polarized projective variety and let $\mathscr{F}$ be a vector bundle on X.*

(i) *$\mathscr{F}$ is* semistable *if for every subsheaf $\mathscr{E}$ of $\mathscr{F}$*

$$\frac{p_{\mathscr{E}}}{\mathrm{rk}(\mathscr{E})} \leq \frac{p_{\mathscr{F}}}{\mathrm{rk}(\mathscr{F})} \tag{5.27}$$

$\mathscr{F}$ is called stable *if the inequality is strict.*

(ii) *For a sheaf $\mathscr{E}$ we define the* slope $\mu(\mathscr{E}) := \deg(c_1\mathscr{E})/\mathrm{rk}(\mathscr{E})$. *$\mathscr{F}$ is μ-*semistable *if for every subsheaf $\mathscr{E}$ of $\mathscr{F}$ with $0 < \mathrm{rk}(\mathscr{E}) < \mathrm{rk}(\mathscr{F})$ we have*

$$\mu(\mathscr{E}) \leq \mu(\mathscr{F}). \tag{5.28}$$

$\mathscr{F}$ is called μ-stable *if the above inequality is strict.*

Theorem 5.29. *Let $X \subset \mathbf{P}^n$ be a nonsingular projective variety and let $\mathscr{F}$ be an Ulrich bundle on X. Then*

(i) $\mathscr{F}$ is semistable and μ-semistable.

(ii) If $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ is an exact sequence of coherent sheaves where $\mathscr{G}$ torsion-free and $\mu(\mathscr{E}) = \mu(\mathscr{F})$ then $\mathscr{F}$ and $\mathscr{G}$ are both Ulrich bundles.

(iii) If $\mathscr{F}$ is stable then it is also μ-stable.

Proof. [CH11b, Theorem 2.9]. □

Let $\mathscr{M}^{ss}(P, X)$ denote the moduli space of semistable vectorbundles on X with Hilbertpolynomial P. Recall that the Zariski tangent space to $\mathscr{M}^{ss}(P, X)$ at the isomorphism class $[\mathscr{E}]$ is given by

$$T_{[\mathscr{E}]}\mathscr{M}^{ss}(P, X) = \operatorname{Ext}^1(\mathscr{E}, \mathscr{E}).$$

We also want to very briefly resume the concept of Mumford's modular families which can be considered as a remedy to the lack of a tautological family of a coarse moduli space.

Definition 5.30 (Modular family)**.** *A* modular family *of bundles is a flat family $\mathscr{E}$ on $X \times S/S$ with S a scheme of finite type such that*

(i) Each isomorphism class of bundles occurs at least once, and at most finitely many times in the family,

(ii) For each $s \in S$ the local ring $\mathscr{O}_{S,s}$ together with the induced family pro represents the local deformation functor,

(iii) For any other flat family $\mathscr{E}'$ on $X \times S'/S'$ of such bundles there exists a surjective etale map $S'' \to S'$ for some scheme S'', and a morphism $S'' \to S$ such that $\mathscr{E}' \times_{S'} S''/X \times S'' \cong \mathscr{E} \times_S S''/X \times S''$.

Under mild additional assumptions on can show that modular families exist, see [Har10, §28] and [CH11b].

Proposition 5.31. *On a nonsingular projective variety X any bounded family of simple bundles $\mathscr{E}$ with given rank and Chern classes and $H^2(\mathscr{E} \otimes \mathscr{E}^\vee) = 0$ has a smooth modular family.*

Proof. This is [CH11b, Proposition 2.10]. □

5.2 Ulrich Bundles on the Cubic Threefold

Ulrich Bundles of Rank 2. We start by examining the smallest possible rank $r = 2$. The main result of this paragraph is the following.

Proposition 5.32. *Let $E \subset X \subset \mathbf{P}^4$ be a general pair of an elliptic normal curve on a general cubic threefold over an algebraically closed field of characteristic 0. Then the twisted normal bundle of E in X splits as*

$$\mathscr{N}_{E/X}(-1) = \mathscr{L} \oplus \mathscr{L}^{-1}$$

with $\mathscr{L} \in \operatorname{Pic}^0(E)$, $\mathscr{L} \ncong \mathscr{O}_E$. In particular, $H^1(\mathscr{N}_{E/X}(-1)) = 0$.

Proof. [CH11b, Proposition A.2]. First, we check the corresponding statement for a general pair $E \subset X \subset \mathbf{P}^4$ defined over a finite field $\mathbf{F}_p$ by computation in *Macaulay2*. Initialization:

```
i1: p=101 -- a fairly small prime number
    Fp=ZZ/p -- a finite ground field
    R=Fp[x_0..x_4] -- coordinate ring of P^4
    setRandomSeed("beta")
```

We start by randomly choosing a smooth cubic threefold X and a smooth quintic elliptic curve E on it.

```
i2 : m1=random(R^6,R^{6:-1});
     m=m1-transpose m1;
        -- a random skew symmetric 6x6 matrix of linear forms
     I=pfaffians(4,m_{0..4}^{0..4});
        -- the ideal of an elliptic normal curve E
     singE=minors(codim I,jacobian I)+I;
     (codim I==3, degree I==5, genus I==1, codim singE==5)

o2 = (true, true, true, true)

i3 : f=pfaffians(6,m) -- ideal of X
     singf=ideal jacobian f;
     (codim f==1, degree f==3, codim singf == 5)

o3 = (true, true, true)
```

Next, we compute the normal bundle and the first values of its Hilbert function:

```
i4 : I2=saturate(I^2+f);
     coN=prune (image( gens I)/ image gens I2);
         -- a module whose sheafification is the conormal sheaf
         -- of E in X
      N=Hom(coN,R^1/I); -- the module of global sections
                          -- of the normal bundle
      apply(toList(-1..2),i->hilbertFunction(i,N))

 o4 = {0, 10, 20, 30}
```

Hence, $\mathscr{N}_{E/X}(-1)$ has no sections, and since $\det \mathscr{N}_{E/X}(-1) \cong \mathscr{O}_E$ has degree 0, we have $H^1(\mathscr{N}_{E/X}(-1)) = 0$ as well. There are two possibilities for the rank 2 vector bundle $\mathscr{N}_{E/X}(-1)$ according to the Atiyah classification [Ati57]. Either

$$\mathscr{N}_{E/X}(-1) \cong \mathscr{L}_1 \oplus \mathscr{L}_2$$

with $\mathscr{L}_2 \cong \mathscr{L}_1^{-1} \in \mathrm{Pic}^0(E)$ or $\mathscr{N}_{E/X}(-1)$ is an extension

$$0 \to \mathscr{L} \to \mathscr{N}_{E/X}(-1) \to \mathscr{L} \to 0$$

with $\mathscr{L} \in \mathrm{Pic}^0(E)$ is 2-torsion. We check that we are in the first case:

```
i5 : Nminus1 = N**R^{-1};
     time betti(EndN=Hom(Nminus1,Nminus1))

             0  1
o5 = total: 12 40
         0:  2  .
         1: 10 40
```

Thus, $H^0(\mathscr{E}nd(\mathscr{N}_{E/X}(-1)))$ is two-dimensional. We compute the characteristic polynomial and the eigenvalues of this pencil of endomorphisms. The command SetRandomSeed("beta") above was chosen such that the characteristic polynomial decomposes completely over $\mathbf{F}_p$ in this step of the computation.

```
i6 : h0=homomorphism EndN_{0};
     h0a=map(R^10,R^10,h0)
     h1=homomorphism EndN_{1};
     h1a=map(R^10,R^10,h1) -- the corresponding matrices

i7 : T=Fp[t] -- an extra ring
     chiA=det(sub(h0a,T)-t*sub(h1a,T));
         -- the characteristic polynomial
     chiAFactors = factor chiA

             5        5
o7 = (t - 47) (t - 14)

i8 : -- We compute the eigenvalues and eigenspaces
     eigenValues=apply(2,c-> -((chiAFactors#c)#0)%ideal t)
     betti (N1=syz(h0a-eigenValues_0*h1a)
     betti (N2=syz(h0a-eigenValues_1*h1a)) -- the eigenspaces
     betti N
     L1=prune coker(presentation N|N1)**R^{-1};
     L2=prune coker(presentation N|N2)**R^{-1};
         -- the corresponding line bundles
     betti res L1 -- L1 (and L2) has a linear resolution

            0  1  2 3
o8 = total: 5 15 15 5
         1: 5 15 15 5
```

Finally, we check that $\mathscr{L}_1 \oplus \mathscr{L}_2 \cong \mathscr{N}_{E/X}(-1)$.

```
i9 : time betti (homL1L2=Hom(L1**L2,R^1/I))  -- used 32.25 seconds
         -- => L1 tensor L2 = O_E
      annihilator homL1L2==I -- check
```

```
o9 = true

i10 : time betti(iso=Hom(L1++L2,N))  -- used 9.22 seconds
      iso0=homomorphism iso_{0}
      iso1=homomorphism iso_{1}

o10 = | 0 0 0 0 0 10  42  31  7   -9  |
      | 0 0 0 0 0 16  -27 -30 -21 -35 |
      | 0 0 0 0 0 6   -13 -19 -5  -29 |
      | 0 0 0 0 0 38  9   41  22  -30 |
      | 0 0 0 0 0 -3  -9  34  -31 1   |
      | 0 0 0 0 0 20  -4  -19 -5  6   |
      | 0 0 0 0 0 17  -2  -37 -6  -19 |
      | 0 0 0 0 0 -46 -18 -31 -26 -20 |
      | 0 0 0 0 0 43  -23 -47 -33 -43 |
      | 0 0 0 0 0 34  41  -35 -13 1   |

i11 : det map(R^10,R^10,iso0+iso1)=!=0
         -- N(-1) is isomorphic to L1++L2

o11 = true

i12 : prune ker(iso0+iso1)==0 and prune coker(iso0+iso1)==0
          -- kernel and cokernel are zero

o12 = true
```

Since $\mathscr{L}_1 \in \mathrm{Pic}^0(E)(\mathbf{F}_p)$ it has finite order. We compute the order, just for fun, in the most naive way. If the prime p is larger a better method is necessary.

```
i13 : time betti(twoL1=prune Hom(L2,L1))
      k=2;
      kL1=twoL1;
      time while (rank target gens kL1=!=1) do (k=k+1;
          kL1=prune Hom(L2,kL1)); -- used 29 seconds
                                  -- in a case where the order k=52
      k -- the order of L1 in Pic E.

o13 = 52

i15 : betti kL1;
      kL1==R^1/I

o15 = true
```

To conclude from these computations the desired result in characteristic zero, we argue that the computation above can be seen as the reduction mod p of computation over $\mathbf{Z}$. By semi-continuity the vanishing

$$H^0(\mathscr{N}_{E_{\mathbf{Q}}/X_{\mathbf{Q}}}(-1)) = H^1(\mathscr{N}_{E_{\mathbf{Q}}/X_{\mathbf{Q}}}(-1)) = 0$$

holds for the corresponding pair $(E_{\mathbf{Q}}, X_{\mathbf{Q}})$ defined over $\mathbf{Q}$ as well. The splitting into line bundles will be defined over a quadratic extension field K of $\mathbf{Q}$ and the line bundle most likely will have infinite order in $\mathrm{Pic}^0(E_{\mathbf{Q}})(K)$. $\square$

Theorem 5.33. *On the general cubic threefold X there exist stable rank 2 Ulrich bundles with first Chern class $c_1 = 2H$, where H is the hyperplane class, and $c_2 = 5$. The moduli space of these bundles is smooth of dimension 5.*

Proof. We make use of the Serre correspondence (5.19) to to obtain an Urlich bundle $\mathscr{F}$ as extension

$$0 \to \mathscr{O}_X \to \mathscr{F} \to \mathscr{I}_C(2) \to 0 \tag{5.34}$$

where C is an elliptic normal curve in X as (5.32). $\mathscr{F}$ is stable as there are no bundles of rank 1 on X. $\square$

Ulrich Bundles of Rank 3. In this section we prove the following

Proposition 5.35. *The space of pairs $C \subset X \subset \mathbf{P}^4$ of smooth arithmetically Cohen-Macaulay curves C of degree 12 and genus 10 on a cubic threefold X has a component which dominates the moduli space $\mathscr{M}_{10}$. This component is defined over $\mathbf{Q}$ and unirational (over $\mathbf{Q}$) and dominates the Hilbert scheme of cubic threefolds in $\mathbf{P}^4$ as well. Moreover, for a general pair $C \subset X$ in this component the following holds:*

(i) The line bundle $\mathscr{O}_C(1)$ is a smooth isolated point of the Brill-Noether space $W^4_{12}(C) \subset \mathrm{Pic}^{14}(C)$.

(ii) The module of global sections $H^0_*(\omega_C(n))$ of the dualizing sheaf ω_C is generated by its two sections in degree -1 as an $S = \sum_{n\in\mathbf{Z}} H^0(\mathbf{P}^4, \mathscr{O}(n))$-module.

(iii) The twisted normal bundle of C in X satisfies $H^1(\mathscr{N}_{C/X}(-1)) = 0$.

As in the preceding paragraph, we will prove the result by a computation over a finite field and semi-continuity. We utilize the unirational parametrization of 6-gonal curves as depicted in Chapter 3. We want to describe the construction in detail for this case: Suppose C is a smooth projective curve of genus 10 defined over a field k together with line bundles $\mathscr{L}_1$, $\mathscr{L}_2$ with $|\mathscr{L}_1|$ a $\mathfrak{g}^1_6$ and $|\mathscr{L}_2|$ a $\mathfrak{g}^2_9$. Let C' denote the image under the map

$$C \xrightarrow{|\mathscr{L}_1|,|\mathscr{L}_2|} \mathbf{P}H^0(C,\mathscr{L}_1) \times \mathbf{P}H^0(C,\mathscr{L}_2) = \mathbf{P}^1 \times \mathbf{P}^2.$$

We say that C is of maximal rank if the map $H^0\mathscr{O}_{\mathbf{P}^2}(n,m) \to H^0(L_1^{\otimes n} \otimes L_2^{\otimes m})$ is of maximal rank for all $n, m \geq 1$. Under the assumption of maximal rank of C the image C' is isomorphic to C and the Hilbert series of the truncated vanishing ideal

$$I_{\text{trunc}} = \bigoplus_{n\geq 3, m\geq 3} H^0(\mathscr{I}_{C'}(n,m))$$

in the Cox-Ring $S = k[x_0, x_1, y_0, y_1, y_2]$ of $\mathbf{P}^1 \times \mathbf{P}^2$ is

$$H_{I_{\text{trunc}}}(s,t) = \frac{3s^4t^5 - 6s^4t^4 - 3s^3t^5 + 3s^3t^4 + 4s^3t^3}{(1-s)^2(1-t)^3}.$$

In other words, we expect a bigraded free resolution of type

$$0 \to F_2 \to F_1 \to F_0 \to I_{\text{trunc}} \to 0$$

with modules $F_0 = S(-3,-3)^4 \oplus S(-3,-4)^3$, $F_1 = S(-3,-5)^3 \oplus S(-4,-4)^6$ and $F_2 = S(-4,-5)^3$.
Turning things around, we find the following unirational construction for such curves: For a general map $M : F_2 \to F_1$ let K be the cokernel of the dual map $M^* : F_1^* \to F_2^*$. For the first terms of a minimal free resolution of K we expect

$$\ldots \to G \xrightarrow{N'} F_1^* \to F_2^* \to K \to 0$$

with $G = S(2,4)^3 \oplus S(3,3)^9 \oplus S(3,4)^3 \oplus S(4,2)^6$. Composing N' with a general map $F_0^* \to G$ and dualizing again yields a map $N : F_1 \to F_0$. Finally, $\ker(F_0^* \xrightarrow{N^*} F_1^*) \cong S$ and the entries of the matrix $S \to F_0^*$ generate I_{trunc}. The following Code for *Macaulay2* [GS] realizes this construction over an arbitrary field, here in particular for random choices over a finite field $\mathbf{F}_p$:

```
i1 : setRandomSeed"I am feeling lucky"; -- initiate random generator
     p=32009; -- a prime number
     Fp=ZZ/p; -- a prime field
     S=Fp[x_0,x_1,y_0..y_2, Degrees=>{2:{1,0},3:{0,1}}];
         -- Cox ring of P^1 x P^2
     m=ideal basis({1,1},S); -- irrelevant ideal

i2 : randomCurveGenus10Withg16=(S)->(
     M:=random(S^{6:{-4,-4},3:{-3,-5}},S^{3:{-4,-5}});
         -- a random map F1 <--M-- F2
```

```
    N':=syz transpose M; -- syzygy-matrix of the dual of M
    N:=transpose(N'*random(source N',S^{3:{3,4},4:{3,3}}));
    ideal syz transpose N) -- the vanishing ideal of the curve

i3 : IC'=saturate(randomCurveGenus10Withg16(S),m);
```

As being of maximal rank is an open condition this computation proves the existence of a nonempty unirational component H in the Hilbert scheme $\mathrm{Hilb}_{(6,9),10}(\mathbf{P}^1 \times \mathbf{P}^2)$ of curves of bidegree $(6,9)$ and genus 10.

By semi-continuity we get the first half of the following proposition.

Proposition 5.36. *The Hilbert scheme* $\mathrm{Hilb}_{(6,9),10}(\mathbf{P}^1 \times \mathbf{P}^2)$ *has a unirational component* H *over* $\mathbf{Q}$ *that dominates the moduli space* $\mathscr{M}_{10}$.

Proof. The main missing ingredient is to prove that in our example above the line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ will be behave like general line bundles in $W^1_6(C)$ and $W^2_9(C)$ for a general curve C. According to Theorem 2.6 for a general smooth curve C of genus g the Brill-Noether loci $W^r_d(C)$ are non-empty and smooth away from $W^{r+1}_d(C)$ of dimension ρ if and only if $\rho = \rho(g,r,d) = g-(r+1)(g-d+r) \geq 0$. Moreover, $W^r_d(C)$ is connected if $\rho > 0$ and the tangent space at a linear series $\mathscr{L} \in W^r_d(C) \smallsetminus W^{r+1}_d(C)$ is the dual of the cokernel of the Petri-map

$$H^0(C,\mathscr{L}) \otimes H^0(C,\omega_C \otimes \mathscr{L}^{-1}) \to H^0(C,\omega_C).$$

Now let $\eta : C \to C'$ be a normalization of our given point $C' \in H$. η will be an isomorphism, but we do not know this yet. We can check computationally that the linear systems $\mathscr{L}_1 = \eta^*\mathscr{O}_{\mathbf{P}^1}(1)$ and $\mathscr{L}_2 = \eta^*\mathscr{O}_{\mathbf{P}^2}(1)$ are smooth points in the respective $W^{r_i}_{d_i}(C)$ for $i = 1,2$:
In order to check $\mathscr{L}_2$, we start by computing the plane model $\Gamma \subset \mathbf{P}^2$ of C':

```
i4 : Sel=Fp[x_0,x_1,y_0..y_2,MonomialOrder=>Eliminate 2];
         -- eliminination order
     R=Fp[y_0..y_2]; -- coordinate ring of P^2
     IGammaC=sub(ideal selectInSubring(1,gens gb sub(IC',Sel)),R);
         -- ideal of the plane model
```

We check that Γ is a curve of desired degree and genus and its singular locus Δ consists only of ordinary double points:

```
i5 : distinctPoints=(J)->(
          singJ:=minors(2,jacobian J)+J;
          codim singJ==3)
```

```
i6 : IDelta=ideal jacobian IGammaC + IGammaC; -- singular locus
     distinctPoints(IDelta)

o6 = true

i7 : delta=degree IDelta;
     d=degree IGammaC;
     g=binomial(d-1,2)-delta;
     (d,g,delta)==(9,10,18)

o7 = true
```

We compute the free resolution of I_Δ:

```
i8 : IDelta=saturate IDelta;
     betti res IDelta

            0 1 2
o8 = total: 1 4 3
         0: 1 . .
         1: . . .
         2: . . .
         3: . . .
         4: . 3 .
         5: . 1 3
```

From Proposition 2.10 we conclude that C is irreducible. Proposition 2.11 then shows that $\mathscr{L}_2 \in W^3_9(C)$ is a smooth point of dimension $\rho_2 = 1$. Thus C the normalization of Γ is isomorphic to a smooth irreducible curve of genus $g = 10$, and C' is smooth because $10 = g \le p_a C' \le 10$.

Turning to $\mathscr{L}_1$, we compute the embedding $C \to \mathbf{P}H^0(C, \omega_C \otimes L_1^{-1}) = \mathbf{P}^4$ as follows

```
i9 : LK=(mingens IDelta)*random(source mingens IDelta, R^{10:{-6}});
         -- compute a basis of the Riemann-Roch space L(Omega_C)
     Pt=random(Fp^1,Fp^2); -- random point in P^1
     L1=substitute(IC',Pt|vars R); -- L1 is the fiber over Pt
     KD=LK*(syz(LK % gens L1))_{0..4};
         -- compute a basis of those elements in L(Omega_C) that
         -- vanish in L1
     T=Fp[z_0..z_4]; -- coordinate ring of P^4
     phiKD=map(R,T,KD); -- embedding
     IC=preimage_phiKD(IGammaC);
     degree IC==12 and genus IC==10
o9 = true
```

```
i10 : betti(FC=res IC)

             0 1 2 3
o10 = total: 1 8 9 2
          0: 1 . . .
          1: . . . .
          2: . 8 9 .
          3: . . . 2
```

From the length of the resolution F_C we see that the image of C in $\mathbf{P}^4$ is arithmetically Cohen-Macaulay. The dual complex $\mathrm{Hom}_S(F_C, S(-5))$ is a resolution of $\bigoplus_{n\in\mathbf{Z}} H^0(\omega_C(n))$. Thus this module is generated by its two sections in degree -1 and $h^0(L_1) = h^0(C, \omega_C(-1)) = 2$. The Petri map for $\mathscr{L}_2$ can be identified with

$$H^0(C, \omega_C(-1)) \otimes H^0(\mathbf{P}^4, \mathscr{O}_{\mathbf{P}^4}(1)) \to H^0(C, \omega_C).$$

Here, this map is an isomorphism, because there is no linear relation among the two generators, and $\mathscr{L}_1$ is a smooth isolated point in $W^1_6(C)$. Thus our random example over the finite field is as expected, and semi-continuity proves that the same is true for the triple $(C, \mathscr{L}_1, \mathscr{L}_2)$ defined over an open part of $\mathrm{Spec}\,\mathbf{Z}$ whose reduction mod p is the given randomly selected curve.

The map $H \to \mathscr{M}_{10}$ factors over $Z = \mathscr{W}^1_6 \times_{\mathscr{M}_{10}} \mathscr{W}^2_9$ and the fiber of $H \to Z$ for a triple $(C, \mathscr{L}_1, \mathscr{L}_2)$ (without automorphisms) with $h^0(C, \mathscr{L}_1) = 2$ and $h^0(C, \mathscr{L}_2) = 3$ is $\mathrm{PGL}(2) \times \mathrm{PGL}(3)$. The fiber dimension of $Z \to \mathscr{M}_g$ is $\rho_1 + \rho_2 = 0 + 1 = 1$, as expected. □

Proof of Proposition 5.35. We are nearly done. The embedding of

$$C \hookrightarrow \mathbf{P}H^0(C, \omega_C \otimes \mathscr{L}_1^{-1}) \cong \mathbf{P}^4$$

is a curve which satisfies (i) and (ii). Since $\mathscr{L}_1$ and equivalently $\mathscr{O}_C(1) \in W^4_{12}(C)$ is Petri general this proves the existence of a unirational component

$$H' \subset \mathrm{Hilb}_{12t+1-10}(\mathbf{P}^4).$$

Since the Hurwitz scheme $H_{6,10}$ is irreducible, we can conclude that the induced rational map $H'//\mathrm{PGL}(5) \to \mathscr{M}_{10}$ is generically finite of degree

$$g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!} = 42 = \deg W^1_6(C),$$

see Theorem 2.7. Choosing a cubic threefold containing C is the same as choosing a point in the projective space $\mathbf{P}H^0(\mathbf{P}^4, \mathscr{I}_C(3))$. Hence,

$$V = \{(C, X) \mid C \in H' \text{ ACM and } X \in \mathbf{P}H^0(\mathbf{P}^4, \mathscr{I}_C(3)) \text{ smooth } \}$$

is unirational as well. For a random pair $(C, X) \in V$ we compute the normal sheaf $\mathscr{N}_{C/X}$ of C in X and check that $H^i(\mathscr{N}_{C/X}(-1)) = 0$ for $i = 0, 1$:

```
i11 : IX=ideal((mingens IC)*random(source mingens IC,T^{1:-3}));
      IC2=saturate(IC^2+X);
      cNCX=image gens IC/ image gens IC2; -- the conormal sheaf in X
      NCX=sheaf Hom(cNCX,T^1/IC); -- the normal sheaf in X

i12 : HH^0 NCX(-1)==0 and HH^1 NCX(-1)==0

o12 = true

i14 : HH^0 NCX==Fp^24 and HH^1 NCX==0

o14 = true
```

With a similar computation for $\mathscr{N}_{C/\mathbf{P}^4}$ we check that H' is a generically smooth component of the Hilbert scheme $\mathrm{Hilb}_{12t+1-10}(\mathbf{P}^4)$ of expected dimension 51 and C is a smooth point in H'.

```
i15 : cNCP= prune(image (gens IC)/ image gens saturate(IC^2));
      NCP=sheaf Hom(cNCP,T^1/IC);
      HH^1 (NCP)==0 and HH^0 (NCP)==Fp^51

o15 = true
```

Consider the maps

$$\begin{tikzcd} V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathbf{P}H^0(\mathbf{P}^4, \mathscr{O}_{\mathbf{P}^4}(3)) \cong \mathbf{P}^{34} \\ H' & \end{tikzcd}$$

The fibre of π_1 over a point C is exactly $\mathbf{P}H^0(\mathbf{P}^4, \mathscr{I}_C(3)) \cong \mathbf{P}^7$, hence V is irreducible of dimension 58. The map π_2 is smooth of dimension $h^0(C, \mathscr{N}_{C/X}) = 24$ at (C, X). Thus π_2 is surjective. By semicontinuity the desired vanishing holds for the general curve on a general cubic. $\square$

Proposition 5.37. *On a general cubic threefold $X \subset \mathbf{P}^4$, there exist stable Ulrich bundles of rank 3.*

Proof. We apply the Serre correspondence (5.19) on a curve C as in (5.35) to establish the existence of a rank 3 Ulrich bundle $\mathscr{F}$ on X. $\mathscr{F}$ is necessarily stable as there are no rank 1 Ulrich bundles on X. $\square$

Ulrich Bundles of Higher Rank. We can now formulate the main theorem of this chapter.

Theorem 5.38. *For any $r \geq 2$ the moduli space of stable rank r Ulrich bundles on a general cubic threefold $Y \subset \mathbf{P}^4$ is non-empty and smooth of dimension r^2+1.*

Proof. We recall the proof from [CH11b, Theorem 5.7]. To prove the existence we proceed by induction on r. For $r = 2$ and $r = 3$ this is already shown. For $r \geq 4$ let $\mathscr{E}'$ and $\mathscr{E}''$ be stable Ulrich bundles of rank 2 and $r-2$, respectively (we choose $\mathscr{E}'$ different from $\mathscr{E}''$ in the case $r = 4$). Then $\dim_{\mathbf{k}} \operatorname{Ext}^1(\mathscr{E}'', \mathscr{E}') = h^1(\mathscr{E}' \otimes \mathscr{E}''^{\vee}) = 2(r-2)$. Hence, there exist non split extensions

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0 \tag{5.39}$$

and $\mathscr{E}$ will be a simple Ulrich bundle of rank r. It remains to show that there are stable bundles of rank r. If the general element of the modular family of simple Ulrich bundles of rank r on X is not stable then it must have the same splitting type as any non-split extension (5.39). However, the family of these bundles has dimension

$$\begin{aligned} & \dim\{\mathscr{E}'\} + \dim\{\mathscr{E}''\} + \dim_{\mathbf{k}} \operatorname{Ext}^1(\mathscr{E}'', \mathscr{E}') - 1 \\ = \ & 2^2 + 1 + (r-2)^2 + 1 + 2(r-2) - 1 \\ = \ & r^2 - 2r + 5. \end{aligned}$$

which is strictly less than $r^2 + 1$ for $r \geq 4$. Hence, there are stable Ulrich bundles or rank r. $\square$

Remark 5.40. In order to apply this method to prove the existence of Ulrich sheaves on higher degree 3-folds we would need to establish the existence of ACM curves with very high genus, e.g. to a rank 3 Ulrich bundle on a quintic threefold corresponds a curve C of degree $d(C) = 75$ and genus $g(C) = 261$. At the moment we do not have a method to prove existence of such curves.

Cohomology of Extensions and Restriction to Hyperplane Sections. The goal of the remainder of the chapter is prove the following

Theorem 5.41. *For each $r \geq 2$ there is a nonempty open set of a modular family of stable rank r Ulrich bundles on the general cubic threefold Y restricting by an étale dominant map to a modular family of stable rank r bundles on a hyperplane section H.*

Some propositions are in order.

Proposition 5.42. *Suppose that $\mathscr{E}$ is a stable Ulrich bundle of rank r on Y with the property that $H^i(\mathscr{E} \otimes \mathscr{E}^\vee(-1)) = 0$ for all i (in which case we say $\mathscr{E} \otimes \mathscr{E}^\vee(-1)$ has no cohomology). Then the restriction map from bundles on Y to bundles on the general hyperplane section H induces an étale dominant map from an open subset of a modular family of stable rank r Ulrich bundles on Y to a modular family of stable rank r Ulrich bundles on H.*

Proof. [CH11b, Proposition 5.8]. □

Lemma 5.43 (Lemma 5.9)**.** *Let $\mathscr{E}$ be a rank r Ulrich bundle on Y corresponding to a nonsingular curve C via the exact sequence*

$$0 \to \mathscr{O}_Y^{r-1} \to \mathscr{E} \to \mathscr{I}_C(r) \to 0$$

Then

(i) $H^i(\mathscr{E} \otimes \mathscr{E}^\vee(-1)) = 0$ *for* $i = 0, 3$*,*

(ii) $H^i(\mathscr{E} \otimes \mathscr{E}^\vee) \cong H^{i-1}(\mathscr{N}_{C/Y}(-1))$ *for* $i = 1, 2$.

Proof. [CH11b, Lemma 5.9]. □

Corollary 5.44. *There exist rank 2 and rank 3 stable Ulrich bundles $\mathscr{E}$ on a general cubic threefold Y such that $\mathscr{E} \otimes \mathscr{E}^\vee(-1)$ has no cohomology.*

Proof. Using Lemma 5.43 this follows immediately from Proposition 5.32 and 5.35. □

To extend this result to bundles of rank $r \geq 4$ the following computational result is needed.

Proposition 5.45. *Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. There is an open subset U of the space of triples $C, E \subset X$ with C and ACM curve of genus 10 and degree 12, E an elliptic normal curve of degree 5 not meeting C and X a smooth cubic threefold over $\mathbf{k}$ with the following properties:*

(i) *U dominates the space $\mathbf{P}H^0(\mathbf{P}^4, \mathscr{O}_{\mathbf{P}^4}(3))$ of cubic threefolds and the spaces of pairs $E \subset X$ and $C \subset X$. In particular the pair $E \subset X$ and the pair $C \subset X$ satisfy all assertions of Proposition 5.32 and 5.35 respectively.*

(ii) *For every triple $C, E \subset X$ in U the extension group $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}_{E/X}(2), \mathscr{O}_X)$ is 1-dimensional and for the non-trivial extension*

$$0 \to \mathscr{O}_X \to \mathscr{F} \to \mathscr{I}_{E/X}(2) \to 0$$

we have the vanishing $H^1(\mathscr{F} \otimes \mathscr{I}_{C/X}) = H^2(\mathscr{F} \otimes \mathscr{I}_{C/X}) = 0$.

Proof. Again, our strategy is to construct a triple $C, E \subset X$ over a finite field with the help of *Macaulay2* and then establishing the theorem in characteristic 0 with semi-continuity.

The bottleneck of this approach is to construct E and C such that there is a cubic threefold which contains both curves. Since $H^0(\mathscr{O}_{\mathbf{P}^4}(3))$ is 35-dimensional, for a general pair (E, C) the 20-dimensional subspace $W_E = H^0(\mathscr{I}_{E/\mathbf{P}^4}(3))$ and the 8-dimensional subspace $W_C = H^0(\mathscr{I}_{E/\mathbf{P}^4}(3))$ will have a trivial intersection. More precisely, the locus M of pairs (E, C) with $W_E \cap W_C \neq 0$ has expected codimension 8 in $H = H_1 \times H_2 \subset \mathrm{Hilb}_{5t}(\mathbf{P}^4) \times \mathrm{Hilb}_{12t-9}(\mathbf{P}^4)$ where H_1 is the subscheme whose points correspond to smooth elliptic normal curves and H_2 the subscheme whose points correspond to smooth ACM curves.

One way to find points in M is by searching over a small finite field. Heuristically, the probability for a random point $(E, C) \in H(\mathbf{F}_p)$ to lie in $M(\mathbf{F}_p)$ is

$$\frac{\#M(\mathbf{F}_p)}{\#H(\mathbf{F}_p)} \approx \frac{1}{p^8}.$$

From the Weil formula [Har77, Appendix C] we see that this approximation is asymptotically correct as $p \to \infty$. Practice shows that it is a reasonable heuristic even for small p in many cases. However, over very small fields ($p = 2, 3$) most hypersurfaces are singular, see [vBS05]. Hence we must not choose p too small in order to minimize the total runtime. Empirically, $p = 5$ is a good choice. Turning to the construction, we start with a random smooth arithmetically Cohen-Macaulay curve C of genus 10 and degree 12 in $\mathbf{P}^4$. To keep things clear we capsulated the construction of the preceding section in a function that returns the vanishing ideal of such a curve:

```
i1 : load"UlrichBundlesOnCubicThreefolds.m2";
     Fp=ZZ/5;
     T=Fp[z_0..z_4];
     setRandomSeed("gamma");

i2 : time IC=randomCurveGenus10Degree12(T);
         -- used 32.4096 seconds
```

For the sake of replicability we write down the curve used in our example. To do this is in a space-saving way, we write down the 9×8 matrix m_C with linear entries in the free resolution of I_C. From m_C the curve can easily be regained:

```
i3 : mC= transpose((res IC).dd_2);

i4 : IC==ideal syz mC
     -- regain the curve from mC

o4 : true
```

In our example, we have

$$m_C = \left(\begin{matrix} -2z_3 & 2z_1 - 2z_3 - z_4 & -2z_0 - z_1 - 2z_3 + 2z_4 & z_0 + z_3 - 2z_4 \\ 2z_1 - z_2 + z_3 - 2z_4 & -2z_0 + 2z_1 + 2z_4 & -2z_0 + 2z_4 & z_3 - z_4 \\ -2z_1 + z_3 & 2z_0 - z_1 - z_2 - 2z_3 & z_0 + 2z_3 + z_4 & 2z_3 - z_4 \\ -z_1 - z_3 - z_4 & z_0 - z_1 - 2z_3 & z_0 - z_3 - z_4 & -2z_3 + 2z_4 \\ 2z_3 & 2z_1 + z_3 + z_4 & -2z_0 - z_2 - 2z_4 & z_3 - z_4 \\ -z_1 - z_3 + 2z_4 & z_0 + z_3 + 2z_4 & 2z_3 & -2z_3 \\ z_1 + 2z_3 & -z_0 - 2z_1 + 2z_3 + z_4 & 2z_0 - 2z_3 - 2z_4 & -z_2 \\ 2z_1 + 2z_3 - z_4 & -2z_0 - 2z_1 - 2z_4 & 2z_0 + 2z_3 - z_4 & z_4 \\ z_1 - 2z_3 & -z_0 - z_1 + z_3 - z_4 & z_0 - z_3 - 2z_4 & -z_3 + 2z_4 \end{matrix}\right.$$

$$\left.\begin{matrix} -2z_3 & z_2 - 2z_3 - z_4 & -2z_4 & -z_2 + z_3 - z_4 \\ z_0 - z_3 & 2z_2 + 2z_3 - z_4 & 2z_2 - z_3 + 2z_4 & -z_2 + z_3 - 2z_4 \\ z_1 + 2z_4 & -2z_2 - 2z_3 - z_4 & -z_2 - z_3 - 2z_4 & 2z_2 + 2z_3 \\ -z_1 - z_3 & z_0 + 2z_2 + z_3 + 2z_4 & z_3 + 2z_4 & z_2 + 2z_3 + 2z_4 \\ z_3 & z_1 + 2z_2 + z_3 & -2z_3 - z_4 & z_3 + z_4 \\ -z_3 + 2z_4 & -z_1 - 2z_2 + 2z_3 - z_4 & z_0 - z_4 & 2z_3 - 2z_4 \\ 0 & z_2 + z_3 & z_1 + z_2 + 2z_3 - z_4 & -z_2 - z_3 + 2z_4 \\ -z_2 - z_3 - 2z_4 & 2z_2 & -z_2 + z_3 + z_4 & z_0 - 2z_2 + 2z_3 + 2z_4 \\ -z_3 - z_4 & z_2 - z_3 & z_3 - 2z_4 & z_1 - z_2 + z_4 \end{matrix}\right).$$

In the next step we search for an elliptic normal curve E such that C and E lie on a common cubic threefold X. Picking E at random and checking whether there is a relation between the generators of $H^0(\mathscr{I}_{C/\mathbf{P}^4}(3))$ and $H^0(\mathscr{I}_{E/\mathbf{P}^4}(3))$ takes about 0.01 seconds a time on a 2.4 GHz processor. Hence we expect to find a such an E within a span of about one hour.

```
i5 : getEllipticWithCommonThreefold=(IC)->(
        max3:=ideal basis(3,T);
           -- third power of the maximal ideal
        for attemptsHS from 1 do (
           mEtmp:=random(T^5,T^{5:-1});
           mE:=mEtmp-transpose mEtmp;
              -- the 5x5 skew-symmetric matrix
           IE:=pfaffians(4,mE);
              -- the elliptic curve E
           if rank source gens intersect(IE+IC,max3)<28 then (
              rltn:=(syz(gens IC|gens intersect(IE,max3)))_{0};
                 -- the relation between the generators
              X:=ideal (gens IC*rltn^{0..7});
                 -- the cubic threefold
              <<"attempts hypersurface = "<<attemptsHS;
                 -- print number of attempts
              return(mE,X))))
```

We also have to check that the cubic hypersurface X is smooth and that the twisted normal bundles $\mathscr{N}_{E/X}(-1)$ has no global sections, as expected.

```
i6 : normalSheaf=(I,X)->(
        I2:=saturate(I^2+X);
        cNIX:=image gens I/ image gens I2;
        sheaf Hom(cNIX,(ring I)^1/I))

i7 : sectionsTwistedNormalBundle=(mE,X)->(
        IE:=pfaffians(4,mE);
        NEX:=normalSheaf(IE,X);
        HH^0(NEX(-1)))
```

Recall from [Eis80], that $\mathscr{O}_E$ has an eventually 3-periodic free resolution as an $\mathscr{O}_X$-module

$$\dots \xrightarrow{q} \mathscr{O}_X(-6)^6 \xrightarrow{m} \mathscr{O}_X(-5)^6 \xrightarrow{q} \mathscr{O}_X(-3)^6 \to \mathscr{O}_X(-2)^5 \to \mathscr{O}_X \to \mathscr{O}_E \to 0$$

whose higher syzygy modules are independent of choice of the section $s \in H^0(\mathscr{F})$ defining E. Thus the number of sections

$$\mathscr{N}_{E/X}(-1) \cong \mathscr{I}_{E/X}/\mathscr{I}^2_{E/X}(1) \cong \mathscr{F} \otimes \mathscr{O}_E(-1)$$

depends only on $\mathscr{F}$ but not on $s \in H^0(\mathscr{F})$: Tensoring the perodic resolution with $\mathscr{F}(-1)$ and the fact that $\mathscr{F}$ has no intermediate cohomology yields

$$H^0(\mathscr{N}_{E/X}(-1)) \cong \ker(H^3(\mathscr{F}(-1)) \otimes \mathscr{K}_3) \to H^3(\mathscr{F}^6(-4)),$$

and $\mathscr{K}_3 = \ker(\mathscr{O}_X(-3)^6 \to \mathscr{O}_X(-2)^5) \cong \mathrm{im}(\mathscr{O}_X(-4)^6 \xrightarrow{q} \mathscr{O}_X(-3)^6)$ is independent of E. So if $H^0(\mathscr{N}_{E/X}(-1)) = 0$ then for any other elliptic curve E' corresponding to a global section of $\mathscr{F}$ the cohomology $H^0(\mathscr{N}_{E'/X}(-1))$ is also vanishing. Putting everything together, we have the following search routine:

```
i8 : time for attemptsN from 1 do (
        time for attemptsS from 1 do (
           time (mE,X)=getEllipticWithCommonThreefold(IC);
           if isSmooth X then (
              <<"attempts smooth = "<<attemptsS;
              break));
        if sectionsTwistedNormalBundle(mE,X)==0 then (
           <<"attempts normalbundle = "<<attemptsN;
           break));

    -- the output:
    attempts hypersurface = 25831     -- used 221.619 seconds
    attempts hypersurface = 206719     -- used 1825.24 seconds
    attempts hypersurface = 132506     -- used 1154.79 seconds
    attempts smooth = 3     -- used 3201.66 seconds
    attempts normalbundle = 1     -- used 3202.02 seconds
```

The extension is given as the cokernel of m which is accessible through the resolution of $\mathscr{O}_E$:

```
i9 : m0=sub((res sub(pfaffians(4,mE),T/X)).dd_4,T);
       -- the matrix in the resolution
     baseChange=Hom(coker m0,coker transpose m0);
     b=map(T^6,T^6,homomorphism baseChange_{0});
       -- we compute a skewsymmetrization of m0
     m=b*m0;
```

```
      pfaffians(6,m)==X

o9 = true
```

In our example, we have

$$m = \left(\begin{matrix} 0 & -2z_1 - z_2 + 2z_4 & -2z_3 - z_4 \\ 2z_1 + z_2 - 2z_4 & 0 & z_0 + 2z_2 - 2z_3 + z_4 \\ 2z_3 + z_4 & -z_0 - 2z_2 + 2z_3 - z_4 & 0 \\ 2z_2 - 2z_3 - 2z_4 & -2z_0 + 2z_1 - z_2 + 2z_3 - 2z_4 & -2z_2 + 2z_3 + z_4 \\ -z_2 + z_3 + z_4 & 2z_0 - z_2 - z_3 - 2z_4 & z_0 + 2z_2 - z_4 \\ -2z_0 + z_2 - z_4 & z_0 + 2z_1 + z_2 + 2z_3 + 2z_4 & -z_1 - z_2 + 2z_3 + 2z_4 \end{matrix} \right.$$

$$\left. \begin{matrix} -2z_2 + 2z_3 + 2z_4 & z_2 - z_3 - z_4 & 2z_0 - z_2 + z_4 \\ 2z_0 - 2z_1 + z_2 - 2z_3 + 2z_4 & -2z_0 + z_2 + z_3 + 2z_4 & -z_0 - 2z_1 - z_2 - 2z_3 - 2z_4 \\ 2z_2 - 2z_3 - z_4 & -z_0 - 2z_2 + z_4 & z_1 + z_2 - 2z_3 - 2z_4 \\ 0 & 2z_1 - z_2 - z_3 - 2z_4 & -z_0 + z_1 + z_2 + 2z_4 \\ -2z_1 + z_2 + z_3 + 2z_4 & 0 & -z_0 + z_1 - 2z_2 \\ z_0 - z_1 - z_2 - 2z_4 & z_0 - z_1 + 2z_2 & 0 \end{matrix} \right).$$

A smooth random section E' of the bundle $\mathscr{F}$ can also be obtained very easily:

```
i10 : IE'=for i from 1 do (
              b:=random(T^6,T^6);
              m':=b*m*transpose b;
              IE':=pfaffians(4,m'_{0..4}^{0..4});
              if isSmooth IE' and dim(IC+IE')==0 then break(IE'));
```

In order to check that C and E' are smooth points in $\mathrm{Hilb}_{12t-9}(X)$ and $\mathrm{Hilb}_{5t}(X)$, respectively, we compute the cohomology groups of the normal sheaves:

```
i11 : NE'X=normalSheaf(IE',X);
      HH^0(NE'X)==Fp^10 and HH^1(NE'X)==0

o11 = true

i12 : NCX=normalSheaf(IC,X);
      HH^0(NCX)==Fp^24 and HH^1(NCX)==0

o12 = true
```

Finally, we compute the cohomology groups of $\mathscr{F} \otimes \mathscr{I}_{C/X}$:

```
i13 : M=coker sub(m,T/X);
         -- this is a module whose sheafification is an extension
      sheafMIC=sheaf(M)**sheaf(module sub(IC,T/X));
      HH^1 sheafMIC==0 and HH^2 sheafMIC==0

o13 = true
```

□

Proposition 5.46. *For each $r \geq 2$ there is a stable rank r Ulrich bundle $\mathscr{E}$ on the general cubic threefold Y such that $\mathscr{E} \otimes \mathscr{E}^\vee(-1)$ has no cohomology.*

Proof. We recall the proof from [CH11b, Proposition 5.11]. Let $\mathscr{E}_0$ be a stable rank 2 Ulrich bundle such that $\mathscr{E}_0 \otimes \mathscr{E}_0^\vee(-1)$ has no cohomology. We will prove by induction the following statement

(*) For each $r \geq 2$ there is a stable rank r Ulrich bundle $\mathscr{F}$ on Y, $\mathscr{F} \not\cong \mathscr{E}_0$ such that $\mathscr{F} \otimes \mathscr{F}^\vee(-1)$ and $\mathscr{F} \otimes \mathscr{E}_0^\vee(-1)$ have no cohomology.

The condition of having no cohomology is an open condition, so for $r = 2$ we can take $\mathscr{F}$ to be a deformation of $\mathscr{E}_0$. Then by semicontinuity, both $\mathscr{F} \otimes \mathscr{F}^\vee(-1)$ and $\mathscr{F} \otimes \mathscr{E}_0^\vee(-1)$ will have no cohomology.

For $r = 3$, we make use of Proposition 5.45 which shows that on a general cubic threefold Y, there are curves E and C as in the earlier Propositions 5.32 and 5.35 respectively, such that if $\mathscr{E}_0$ (changing notation) is the rank 2 bundle corresponding to E:

$$0 \to \mathscr{O}_Y \to \mathscr{E}_0 \to \mathscr{I}_{E/Y}(2) \to 0$$

then we have also the additional property that $H^i(\mathscr{E}_0 \otimes \mathscr{I}_{C/Y}) = 0$ for $i = 1, 2$. Let F be a stable rank 3 bundle corresponding to C (as in Proposition 5.35):

$$0 \to \mathscr{O}_Y^2 \to \mathscr{F} \to \mathscr{I}_{C/Y}(3) \to 0$$

Tensoring with $\mathscr{E}_0^\vee(-1)$ we have

$$0 \to \mathscr{E}_0^\vee(-1) \oplus 2 \to \mathscr{F} \otimes \mathscr{E}_0^\vee(-1) \to \mathscr{E}_0^\vee(-1) \otimes \mathscr{I}_{C/Y}(3) \to 0.$$

Now $\mathscr{E}_0$ has rank 2, so $\mathscr{E}_0^\vee \cong \mathscr{E}_0(-2)$. Thus $\mathscr{E}_0^\vee(-1) \cong \mathscr{E}_0(-3)$, which has no cohomology. Furthermore, since $\mathscr{F}$ and $\mathscr{E}_0$ are distinct stable bundles, already $H^0(\mathscr{F} \otimes \mathscr{E}_0^\vee) = 0$, so also $H^0(\mathscr{F} \otimes \mathscr{E}_0^\vee(-1)) = 0$, and by duality also $H^3(\mathscr{F} \otimes \mathscr{E}^\vee(-1)) = 0$. To show therefore that $\mathscr{F} \otimes \mathscr{E}_0^\vee$ has no cohomology, we have only to check the vanishing of H^i for $i = 1, 2$. Since $E_0^\vee$ has no cohomology, the groups on question are isomorphic to $H^i(\mathscr{E}_0(-1) \otimes \mathscr{I}_{C/Y}(3)) = H^i(\mathscr{E}_0 \otimes \mathscr{I}_{C/Y}(3))$, and these are zero by Proposition 5.45. We have shown that $\mathscr{F} \otimes \mathscr{F}^\vee(-1)$ has no cohomology earlier in Corollary 5.44. (Note that at this step we have redefined the rank 2 bundle $\mathscr{E}_0$ chosen before, but we can just as well use this one from the beginning.) For $r \geq 4$, choose by the induction hypothesis a stable bundle $\mathscr{F}_0$ of rank $r-2$, different from $\mathscr{E}_0$, such that $\mathscr{F}_0 \otimes \mathscr{F}_0^\vee(-1)$ and $\mathscr{F}_0 \otimes \mathscr{E}_0^\vee(-1)$ have no cohomology. As in the proof of existence of stable bundles, consider an extension

$$0 \to \mathscr{E}_0 \to \mathscr{G} \to \mathscr{F}_0 \to 0$$

Then $\mathscr{G}$ will be simple of rank r. Tensoring with $\mathscr{E}_0^\vee(-1)$ and using our hypotheses on $\mathscr{E}_0$ and $\mathscr{F}_0$, we see that $\mathscr{G} \otimes \mathscr{E}_0^\vee(-1)$ has no cohomology. Similarly tensoring with $\mathscr{F}_0^\vee(-1)$ we find that $\mathscr{G} \otimes \mathscr{F}_0^\vee(-1)$ has no cohomology. (Note that $\mathscr{E}_0 \otimes \mathscr{F}_0^\vee(-1) = (\mathscr{F}_0 \otimes \mathscr{E}_0^\vee(-1))^\vee \otimes \omega_Y$ so by Serre duality it has also no cohomology.) Now tensor $\mathscr{G}(-1)$ with the dual sequence

$$0 \to \mathscr{F}_0^\vee \to \mathscr{G}^\vee \to \mathscr{E}_0^\vee \to 0$$

to see that $\mathscr{G} \otimes \mathscr{G}^\vee(-1)$ has no cohomology. Finally, as in Theorem 5.38 we can deform $\mathscr{G}$ into a stable bundle, call it $\mathscr{F}$, and by semicontinuity it will satisfy $\mathscr{F} \otimes \mathscr{F}^\vee(-1)$ and $\mathscr{F} \otimes \mathscr{E}_0^\vee(-1)$ have no cohomology. $\square$

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