k-Divisible Partitions in Free Probability

Dissertation
zur Erlangung des Grades
des Doktors der Naturwissenschaften
der Naturwissenschaftlich-Technischen Fakultäten
der Universität des Saarlandes

von
Octavio Arizmendi Echegaray

Saarbrücken, 2012

Tag des Kolloquiums: 18. Oktober 2012

Dekan:
Prof. Dr. Mark Groves

<u>Prüfungsausschuss</u>

Vorsitzender:
Prof. Dr. Hannah Markwig

Gutachter:
Prof. Dr. Roland Speicher
Prof. Dr. Jörg Eschmeier
Prof. Dr. Uwe Franz

Akademischer Mitarbeiter:
Dr. Moritz Weber

Hiermit versichere ich an Eides statt, dass ich die vorliegende Arbeit selbstständig und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe. Die aus anderen Quellen oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet. Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form in einem Verfahren zur Erlangung eines akademischen Grades vorgelegt.

Saarbrücken, July 2012

Zusammenfassung

In dieser Arbeit untersuchen wir die Rolle der k-teilbaren nicht-kreuzenden Partitionen in der Freien Wahrscheinlichkeitstheorie.

Wir betrachten zunächst die kombinatorische Faltung $*$ auf den Gittern der nicht-kreuzenden Partitionen NC und der k-teilbaren nicht-kreuzenden Partitionen NC^k. Wir zeigen, dass die k-fache Faltung mit der Zetafunktion in NC äquivalent ist zur einfachen Faltung mit der Zetafunktion in NC^k. Dies eröffnet neue Wege, um Objekte wie "k-equal" Partitionen, k-teilbare Partitionen oder k-Multiketten zu zählen – sowohl in NC wie auch in NC^k. Darüber hinaus analysieren wir die Statistik der Größe von Blöcken in k-teilbaren nicht-kreuzenden Partitionen.

Des Weiteren führen wir den Begriff der k-teilbaren Elemente in einem nicht-kommutativen Wahrscheinlichkeitsraum ein und untersuchen diese. Ein k-teilbares Element ist eine (nicht-kommutative) Zufallsvariable, deren n-te Momente null sind, für alle n, die kein Vielfaches von k sind. Für solch k-teilbare Elemente x leiten wir eine Formel für die freien Kumulanten von x^k her, die sich auf die freien Kumulanten von x zurückführen lässt. Hierbei werden k-teilbare nicht-kreuzende Partitionen verwendet sowie unsere entsprechenden kombinatorischen Resultate. Wir beweisen, dass sps und a frei sind, falls a und s frei sind, s k-teilbar ist und p ein Polynom (in a und s) ist, dessen Grad in s gerade $k-2$ ist. Anschließend definieren wir den Begriff R-diagonaler k-Tupel und erhalten ähnliche Aussagen.

Ein weiteres Ergebnis dieser Arbeit ist, dass die freie additive Faltung eines Maßes auf der positiven reellen Achse mit einem k-symmetrischen Wahrscheinlichkeitsmaß wohldefiniert ist. Analytische Methoden um diese Faltung zu berechnen werden entwickelt.

Wir konzentrieren uns dann auf Potenzen der freien additiven Faltung k-symmetrischer Verteilungen und zeigen, dass $\mu^{\boxplus t}$ ein wohldefiniertes Wahrscheinlichkeitsmaß ist, für alle $t > 1$. Wir leiten zentrale Grenzwertsätze her und solche vom Typ Poissons. Etwas allgemeiner untersuchen wir frei unbegrenzt teilbare Maße und beweisen, dass die frei unbegrenzte Teilbarkeit unter der Abbildung $\mu \to \mu^k$ erhalten bleibt.

Einige Beziehungen zwischen Potenzen der freien multiplikativen Faltung und k-teilbaren nicht-kreuzenden Partitionen werden herausgearbeitet und auf das beliebige Produkt freier Zufallsvariablen verallgemeinert.

Schlussendlich nehmen wir (k-symmetrische) frei stabile Verteilungen in den Blick, für die wir eine Eigenschaft der Reproduktion beweisen. Diese verallgemeinert die für einseitige und reell-symmetrische frei stabile Gesetze bekannte.

Abstract

In this thesis we study the role of k-divisible non-crossing partitions in Free Probability.

First, we consider the combinatorial convolution $*$ in the lattices NC of non-crossing partitions and NC^k of k-divisible non-crossing partitions. We show that convolving k times with the zeta-function in NC is equivalent to convolving once with the zeta-function in NC^k. This gives new ways of counting objects like k-equal partitions, k-divisible partitions and k-multichains both in NC and NC^k. We also consider some statistics of block sizes in k-divisible non-crossing partitions.

Second, we introduce and study the notion of k-divisible elements in a non-commutative probability space. A k-divisible element is a (non-commutative) random variable whose n-th moment vanishes whenever n is not a multiple of k. For such k-divisible element x, we derive a formula for the free cumulants of x^k in terms of the free cumulants of x. For this we use our combinatorial results on the lattice of k-divisible non-crossing partitions.

We prove that if a and s are free and s is k-divisible then sps and a are free, where p is any polynomial (in a and s) of degree $k-2$ in s. Moreover, we define a notion of R-diagonal k-tuples and prove similar results.

Next, we show that free multiplicative convolution between a measure concentrated on the positive real line and a probability measure with k-symmetry is well defined. Analytic tools to calculate this convolution are developed.

We then concentrate on free additive powers of k-symmetric distributions and prove that $\mu^{\boxplus t}$ is a well defined probability measure, for all $t > 1$. We derive central limit theorems and Poisson type ones. More generally, we consider freely infinitely divisible measures and prove that free infinite divisibility is maintained under the mapping $\mu \to \mu^k$.

Relations between free multiplicative powers and k-divisible non-crossing partitions are also found and generalized to any product of free random variables.

We conclude by focusing on (k-symmetric) free stable distributions, for which we prove a reproducing property generalizing the ones known for one sided and real symmetric free stable laws.

Acknowledgements

This thesis would not have been possible without the support of many people.

First and foremost, I would like to thank Professor Roland Speicher, from whom I have learned a lot. His comments and ideas have been very enlightening and motivating. I really appreciated his friendly, flexible and supporting guidance and his willingness to hear my, often erratic, mathematical thoughts.

I thank Professor Jörg Eschmeier and Professor Uwe Franz for accepting being examiners for my PHD defense.

Special thanks are due to my friend Carlos Vargas for his patience and tolerance during many, many hours we have spent together.

I am deeply grateful to Professor Victor Perez Abreu for his constant support during all these years.

I would also like to thank Dr. Noriyoshi Sakuma and Dr. Takahiro Hasebe for allowing me to include part of our joint work in this thesis. It is a pleasure to work with them.

I am grateful to John Treilhard and Dr. Moritz Weber for many comments that helped improving the presentation of this work.

I thank the support from funds of R. Speicher from the Alfried Krupp von Bohlen und Halbach Stiftung and the support of DFG (Deutsche Forschungsgemeinschaft Project SP419/8-1).

Finally, I thank my parents and my brothers, to whom this thesis is dedicated.

Contents

Introduction

This thesis is part of the theory of Free Probability, where the objects of interest are not classical random variables but free (non-commutative) random variables and tensor products are replaced by free products.

Free Probability started in the 80s with the work of Voiculescu [79, 80, 82] linked to some questions in the context of operator algebras. The free additive convolution $\boxplus$ and free multiplicative convolution $\boxtimes$ of measures supported on the real line (explained in Chapter 1) were introduced by Voiculescu [82] to describe the sum and the product of free (non-commuting) random variables. These operations have many applications in the theory of large dimensional random matrices, since they allow to compute the asymptotic spectrum of the sum and the product of two independent random matrices from the individual asymptotic spectra [36], [83].

Later, in the early 90s, Speicher [72] introduced purely combinatorial tools, relying on the notion of free cumulants, to study Free Probability. A central object in this combinatorial approach is the lattice of non-crossing partitions, since free cumulants are related to the lattice of non-crossing partitions in the same way as classical cumulants are related to the lattice of all partitions. The present work is based on this combinatorial approach.

Nowadays, Free Probability has also significant relations with other branches of mathematics such as combinatorics, classical probability, representations of symmetric groups, as well as some mathematical models in physics, communications and information theory.

Until recently, k-divisible non-crossing partitions have been overlooked in Free Probability and have barely appeared in the literature. However, their structure is very rich and there are, for instance, quite natural bijections between k-divisible non-crossing partitions and $(k+1)$-equal non-crossing partitions which preserve a lot of structure, (see e.g. [3]). In this work we explore k-divisible non-crossing partitions in connection to various aspects of Free Probability.

This thesis is mainly based on the results of [3] where we introduce the notion of k-divisible elements in a non-commutative probability space and study the combinatorial structure of k-divisible non-crossing partitions. Statistical properties of the block structure of k-divisible partitions were considered in [2]. Connections to free multiplicative convolution have been further explored in a joint work with Vargas [10]. The case $k = 2$, that we briefly explain, was studied in more detail as part of a joint work with Hasebe and Sakuma [8].

Before passing to a detailed description of the results, let us briefly explain their setting. Let $\mathcal{M}$ and $\mathcal{M}_{\mathbb{C}}$ be the classes of all Borel probability measures on the real line $\mathbb{R}$ and on the complex plane, respectively. Moreover, let $\mathcal{M}_b$ and $\mathcal{M}^+$ be the subclasses of $\mathcal{M}$ consisting of probability measures with bounded support and of probability measures having support on $\mathbb{R}_+ = [0, \infty)$, respectively.

For q a primitive k-th root of unity, consider the k-semiaxes $A_k := \{x \in \mathbb{C} \mid x = tq^s$ for some $t > 0$ and $s \in \mathbb{N}\}$ and denote by $\mathcal{M}_k$ the subclass of $\mathcal{M}_{\mathbb{C}}$ of probability measures supported on A_k such that $\mu(B) = \mu(qB)$, for all Borel sets B. A measure in $\mathcal{M}_k$ will be called k-symmetric. We say that a measure in $\mathcal{M}_{\mathbb{C}}$ has all moments if $m_k(\mu) := \int_{\mathbb{C}} |t|^n \mu(\mathrm{d}t) < \infty$, for each integer $n \geq 1$.

In this thesis we will be interested in random variables whose distribution is k-symmetric, which we will call *k-divisible*. We give a framework to these k-divisible random variables from the free probabilistic point of view. We consider various aspects of k-symmetric distributions including combinatorial, algebraic and probabilistic ones. It will turn out that, as for the cases of even elements and even partitions, k-divisible non-crossing partitions are exactly the objects involved in the combinatorics of k-divisible elements.

These k-divisible (non-commutative) random variables appear naturally in Free Probability. A typical example of a k-divisible random variable is the so called k-Haar unitary with distribution $\mu = \frac{1}{k}\sum_{j=1}^{k} \delta_{q^j}$. k-divisible free random variables appear not only in the abstract setting but also in applications to random matrices. For instance, in [54] it is shown that an independent family $U_1, U_2, ..., U_s$ of random $N \times N$ permutation matrices with cycle lengths of size k converges in $*$-distribution to a $*$-free family $u_1, u_2, ..., u_s$ of k-Haar unitaries.

Other interesting examples of k-divisible free random variables come from the context of quantum groups. In Banica et al. [13], where free Bessel laws are studied in detail, a modified k-symmetric version appears as the asymptotic law of the truncated characters of certain quantum groups. Similarly, from their studies of the law of characters of quantum isometry groups, Banica and Skalski [14] found k-symmetric measures which are the analog of free compound Poissons, see Theorem 4.4 and Remark 4.5 in [14].

As we have mentioned the free additive convolution $\boxplus$ and free multiplicative convolution $\boxtimes$ of measures are two of the main operations in Free Probability. Even though some work has been done in the physics literature (see e.g. [31]) until now, this machinery could only be used for selfadjoint random variables and, in general, k-divisible random variables are not selfadjoint whenever $k > 2$. Let us mention that k-symmetric distributions were considered by Goodman [35] in the framework of graded independence.

The Main Theorem (stated below) enables to define free multiplicative convolution between a measure concentrated on the positive real axis and a probability measure with k-symmetry. We extend the definition of Voiculescu's S-transform to any k-symmetric measure μ to calculate effectively the free multiplicative convolution $\mu \boxtimes \nu$, between a k symmetric measure μ and a measure ν supported on $\mathbb{R}^+$.

The Main Theorem also permits to define free additive powers for k-divisible measures leading to central limit theorems and Poisson type ones. Once we have free additive powers, the concept of free infinite divisibility arises naturally. We prove that for a k-symmetric measure μ, free infinite divisibility is maintained under the mapping $\mu \to \mu^k$.

Moreover, interesting combinatorial implications regarding the combinatorial convolution in NC^k (the poset of k-divisible non-crossing partitions) are derived from the Main Theorem. This gives new ways of counting objects like k-equal partitions, k-divisible partitions and k-multichains, both in NC and NC^k.

From the combinatorial results on the poset of k-divisible non-crossing partitions we derive a formula for the free cumulants of x^k in terms of the free cumulants of x involving k-divisible non-crossing partitions. Moreover, we define a notion of R-diagonal k-tuples and prove similar results.

The thesis is organized as follows. The preliminaries on Free Probability needed in this thesis are explained in Chapter 1. We study k-divisible partitions in Chapter 2. The concept of k-divisible elements and more generally R-diagonal k-tuples is introduced in Chapter 3, where we discuss some of the combinatorial aspects of their cumulants. In Chapter 4, we present the Main Theorem of the thesis and its direct consequences, including free multiplicative convolution and free additive powers. Chapter 5 is dedicated to limit theorems: free central limit theorems, free compound Poisson, free infinite divisibility and connections to limit theorems in free multiplicative convolution are made. The role of k-divisible partitions in free multiplicative convolution has been further explored in a joint work with Vargas [10]; this is the content of Chapter 6. Finally, Chapter 7 deals with the case of unbounded measures, the S-transform of any k-symmetric probability measure is defined and the free multiplicative convolution of distributions in $\mathcal{M}_k$ with distributions in $\mathcal{M}^+$ is considered. We end by focusing on free stable distributions.

Statement of Results

Chapter 2 is devoted to combinatorics and statistics of k-divisible non-crossing partitions. First, we study the poset $NC^k(n)$ and its associated combinatorial convolution $*$ and translate the combinatorial convolution in $NC^k(n)$ to the convolution in $NC(n)$ of dilated sequences. Basically, we show that convolving k times with the zeta-function in NC is equivalent to convolving once with the zeta-function in NC^k.

Theorem 1. *The following statements are equivalent.*

(1) *The multiplicative family $f := (f_n)_{n>0}$ is the result of applying the zeta-function k times to $g := (g_n)_{n>0}$, that is,*

$$f = g * \underbrace{\zeta * \cdots * \zeta}_{k \text{ times}}.$$

(2) *The multiplicative family $f^{(k)} := (f_n^{(k)})_{n>0}$ is the result of applying the zeta-function once to $g^{(k)} := (g_n^{(k)})_{n>0}$, that is,*

$$f^{(k)} = g^{(k)} * \zeta,$$

where, for a sequence $(a_n)_{n>0}$, the sequence $(a_n^{(k)})_{n>0}$ denotes the dilated sequence given by $a_{kn}^{(k)} = a_n$ and $a_n^{(k)} = 0$ if n is not a multiple of k.

Then we study some statistics of the block structure of non-crossing partitions. In this direction, a recent paper by Ortmann [61] studies the asymptotic behavior of the sizes of the blocks of a uniformly chosen random partition. This lead him to a formula for the right-edge of the support of a measure in terms of the free cumulants, when these are positive. He noticed a very simple picture of this statistic as $n \to \infty$. Roughly speaking, in average, out of the $\frac{n+1}{2}$ blocks of this random partition, half of them are singletons, one fourth of the blocks are pairings, one eighth of the blocks have size 3, and so on.

In trying to get a better understanding of this asymptotic behavior, the question of the exact calculation of this statistic arose. We answer this question and refine these results by considering the number of blocks given. Moreover, we generalize to k-divisible partitions, as follows.

Theorem 2. *The sum of the number of blocks of size tk over all the k-divisible non-crossing partitions of $\{1, 2, .., kn\}$ is given by*

$$\binom{n(k+1)-t-1}{nk-1}.$$

In particular, asymptotically, we have a similar phenomenon as for the case $k = 1$; about a $\frac{k}{k+1}$ portion of all the blocks have size k, then a $\frac{k}{(k+1)^2}$ portion have size $2k$, then $\frac{k}{(k+1)^3}$ are of size $3k$, etc.

In Chapter 3, we introduce the concept of k-divisible random variables. Noticing that, when x is k-divisible, the moments of x are nothing other than the dilation of the moments of x^k and using the so called moment-cumulant formula of Speicher (see e.g. [60]) which relates the moments and the free cumulants via the combinatorial convolution in $NC(n)$ we give a relation between the free cumulants of x and x^k which generalizes results in [57].

Theorem 3. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and let x be a k-divisible element with k-determining sequence $(\alpha_n = \kappa_{kn}(x, ..., x))_{n\geq 1}$. Then the following formula holds for the free cumulants of x^k.*

$$\kappa_n(x^k, x^k, ..., x^k) = [\alpha * \underbrace{\zeta * \cdots * \zeta}_{k \text{ times}}]_n.$$

Second, we consider how freeness behaves when conjugating with k-divisible elements in a non-commutative probability space. More precisely, if a and s are free and s is k-divisible then a is also free from sps, where p is any polynomial in a and s of degree $k-2$ on s.

Moreover, we generalize the concept of diagonally balanced pairs from Nica and Speicher [57], which contains three of the most frequently used examples in Free Probability, that is, semicircular, circular and Haar unitaries, and prove similar results for what we call diagonally balanced k-tuples.

Theorem 4. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, and let $(s_1, ..., s_k)$ be a diagonally balanced k-tuple free from a. Moreover, let $h = s_1a_2s_2a_3s_3 \cdots s_{k-1}a_{k-1}s_k$, where for all $i = 1, ..., n$ the element a_i is free from $\{s_1, \cdots, s_k\}$. Then h and a are free.*

Furthermore, we realize k-divisible random variables as R-cyclic matrices [55] with diagonally balanced k-tuples as entries. Implications of these results to the theory of Random Matrices is explained at the end of this chapter.

Chapter 4 deals with probability measures with k-symmetry and free convolutions $\boxtimes$ and $\boxplus$. Given a k-symmetric probability measure μ on $\mathcal{M}_k$, let μ^k be the probability measure in $\mathcal{M}^+$ induced by the map $t \to t^k$. In other words if x is a k-divisible element with distribution μ, then μ^k is the distribution of x^k.

One of the main results of this thesis shows that it is possible to define a free multiplicative convolution $\mu \boxtimes \nu$ between a probability measure μ in $\mathcal{M}^+$ and k-symmetric distribution ν.

The Main Theorem, which enables to define this free multiplicative convolution is the following.

Main Theorem. Let $x, y \in (\mathcal{A}, \phi)$ with x positive and y a k-divisible element free from x. Consider $x_1, ..., x_k$ free positive elements with the same moments as x. Then $(xy)^k$ and $y^k x_1 \cdots x_k$ have the same moments, i.e.

$$\phi((xy)^{kn}) = \phi((y^k x_1 \cdots x_k)^n) \quad \forall n \in \mathbb{N}.$$

As a byproduct we show that this free multiplicative convolution gives a k-symmetric distribution satisfying the relation $(\mu \boxtimes \nu)^k = \mu^{\boxtimes k} \boxtimes \nu^k$. Using this identity we give a formula for the moments of $\mu^{\boxtimes k}$ in terms of k-divisible partitions.

An important analytic tool for computing the free multiplicative convolution of two probability measures is Voiculescu's S-transform. It was introduced in [82] for non-zero mean distributions with bounded support and further studied by Bercovici and Voiculescu [22] in the case of probability measures in $\mathcal{M}^+$ with unbounded support, see also [21].

Raj Rao and Speicher [64] extended the S-transform to the case of random variables having zero mean and all moments. Their main tools are combinatorial arguments based on moment calculations.

We use the approach of [64] to extend the S-transform to random variables with first k moments vanishing. After this, we specialize to the case of k-divisible random variables where simple relations between the S-transforms of x and x^k are found.

Another remarkable consequence of the Main Theorem is that we can define free additive powers $\mu^{\boxplus t}$ for $t > 1$ when μ is a k-symmetric distribution. This opens the possibility to new limit theorems.

In Chapter 5, we prove new limit theorems on k-divisible elements and k-symmetric measures: free central limit theorems, free compound Poisson and connections to limit theorems in free multiplicative convolution are made.

Theorem 5 (Free central limit theorem for k-symmetric measures)**.** *Let μ be a k-symmetric measure with finite moments and $\kappa_k(\mu) = 1$ then, as N goes to infinity,*

$$D_{N^{-1/k}}(\mu^{\boxplus N}) \to s_k,$$

where s_k is the only k-symmetric measure with free cumulant sequence $\kappa_n(s_k) = 0$ for all $n \neq k$ and $\kappa_k(s_k) = 1$. Moreover,

$$(s_k)^k = \pi^{\boxtimes(k-1)},$$

where π is a free Poisson measure with parameter 1.

Free compound Poisson distributions exist in $\mathcal{M}_k$ and Poisson limit theorems also hold. We give a framework to the results in [13] and, in particular, generalize Theorem 7.3 in [13], where $\nu = \frac{1}{k}\sum_{j=1}^{k} \delta_{q^j}$ was considered in connection with free Bessel laws.

Theorem 6. *Let ν be a k-symmetric distribution, then the Poisson type limit convergence holds*

$$((1 - \frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\nu)^{\boxplus N} \to \pi(\lambda, \nu).$$

We also address questions of free infinite divisibility. A measure $\mu \in \mathcal{M}_k$ is said to be infinitely divisible if $\mu^{\boxplus t} \in \mathcal{M}_k$ for all $t > 0$. For these measures, it is also shown that free additive convolution is well defined. Moreover we show that μ^k is also freely infinitely divisible.

Theorem 7. *If μ is k-symmetric and $\boxplus$-infinitely divisible, then μ^k is also $\boxplus$-infinitely divisible.*

We end Chapter 5 by specializing to the case $k = 2$. The results for this case are part of the joint work with Hasebe and Sakuma[8].

Chapter 6 explains results of the paper with Vargas [10]. As mentioned before, by choosing ν to be a k-Haar measure in the relation $(\mu \boxtimes \nu)^k = \mu^{\boxtimes k} \boxtimes \nu^k$ one can give a formula for the cumulants and moments of the free multiplicative convolution $\mu^{\boxtimes k}$ in terms of k-divisible partitions. We generalize this formula for non-identically distributed random variables in [10] where it was used to give new proofs of results in Kargin [40, 42] and Sakuma and Yoshida [68] regarding the asymptotic behaviors of $\mu^{\boxtimes k}$ and $(\mu^{\boxtimes k})^{\boxplus k}$, respectively.

Theorem 8. *Let $a_1, \ldots, a_k \in (\mathcal{A}, \tau)$ be free random variables. Then the free cumulants and the moments of $a := a_1 \ldots a_k$ are given by*

$$\begin{aligned} \kappa_n(a) &= \sum_{\pi \in NC_k(n)} \kappa_{Kr(\pi)}(a_1, \ldots, a_k) \\ \tau(a^n) &= \sum_{\pi \in NC^k(n)} \kappa_{Kr(\pi)}(a_1, \ldots, a_k) \end{aligned}$$

where $NC_k(n)$ and $NC^k(n)$ denote, respectively the k-equal and k-divisible partitions of $[kn]$. Here $Kr(\pi)$ denotes the Kreweres complement.

Finally, in Chapter 7, we consider the unbounded case. We are able to extend the S-transform for the case of k-symmetric probability measures even if we have no moments. To do this, we follow an analytic approach similar to [9] and show that this S-transform allows the computation of the desired free multiplicative convolution between probability measures on $[0, \infty)$ and general k-symmetric measures. As an important example of distributions without finite moments and unbounded supports we consider free stable laws and show reproducing properties similar to the ones found in [9] and [24].

Theorem 9. *For any $s, r > 0$, let $\sigma^k_{1/(1+r)}$ be a k-symmetric strictly stable distribution of index $1/(1+r)$ and $\nu_{1/(1+s)}$ be a positive strictly stable distribution of index $1/(1+s)$. Then*

$$\sigma^k_{1/(1+t)} \boxtimes \nu_{1/(1+s)} = \sigma^k_{1/(1+t+s)}.$$

Chapter 1

Preliminaries on Free Probability

In this chapter we give some basic definitions and results on Free Probability. We mainly follow the monograph [60]. First, we recall basic concepts on probability. The reason for this is that we believe the problems addressed in this thesis could be of interest to readers not familiar with probability. Next, we introduce a structure known as a non-commutative probability space, the appropriate framework for Free Probability. Later, the notions of free independence and its associated additive and multiplicative convolutions are reviewed. Free cumulants, an important object in the combinatorial approach to Free Probability are explained in detail. We end by reviewing some facts about free infinite divisibility that are mainly taken from [22]. For an introduction to Free Probability the reader is advised to check the monograph by Voiculescu, Dykema and Nica [83] and the book by Nica and Speicher [60]. The latter explains clearly the combinatorial approach to Free Probability.

1.1 Classical Probability Spaces

Let us start recalling the basic notions of classical probability.

Definition 1.1.1. A probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a non-empty set, $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a probability measure. That is, $\mathcal{F}$ satisfies the following properties:

(1) $\Omega \in \mathcal{F}$.
(2) If $A_1, A_2, ... \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.
(3) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
While $\mathbb{P} : \mathcal{F} \to [0, 1]$ is such that
(4) $\mathbb{P}(\Omega) = 1$.
(5) $\mathbb{P}$ is σ-additive: If $A_1, A_2, ... \in \mathcal{F}$ and $A_n \cap A_m = \phi$ whenever $n \neq m$, then

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

In other words, the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is nothing other than a measure space with $P(\Omega) = 1$. One may think of Ω as the set of possible event and for a set $B \in \mathcal{F}$ we think of $\mathbb{P}(B)$ as the probability that this event happens. The collection of all Borel sets in $\mathbb{R}$, that we will denote by $\mathcal{B}(\mathbb{R})$, is the σ-algebra generated by the open intervals in $\mathbb{R}$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is measurable, if it is $\mathcal{B}(\mathbb{R})-$measurable.

Definition 1.1.2 (Classic Random Variable)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a real random variable, if it is $\mathcal{F}$-measurable in $\mathbb{R}$, that is, $\{w : X(w) \in B\} \in \mathcal{F}$, whenever $B \in \mathcal{B}(\mathbb{R})$. We say that the measure μ in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is the distribution of X if

$$\mathbb{P}(X \in A) = \mu(A) = \int_A \mu(dt) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}).$$

Definition 1.1.3 (Expectation)**.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, then the expected value of X, denoted by $E[X]$ is defined as the Lebesgue integral

$$E[X] := \int_\Omega X(\omega)\mathbb{P}(d\omega) = \int_\mathbb{R} x\mu(dx) \ .$$

whenever it exists.

The expectation may be understood as the “average” value of the random variable X. Not all random variables have a finite expected value, since the integral may not converge absolutely; furthermore, it may be not defined at all (e.g., Cauchy distribution).

Remark 1.1.4. Note that the expectation E satisfies the following properties:
(i)$E(1) = 1$ (normalization).
(ii) $E(f) \geq 0$ if $f \geq 0$ (positivity).
(iii) $E(X + \lambda Y) = E(X) + \lambda E(Y)$ (linearity).

More generally, for a bounded measurable function $f : \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is given by

$$E[f(X)] := \int_\Omega f(X(\omega))\mathbb{P}(d\omega) = \int_\mathbb{R} f(x)\mu(dx) \ . \tag{1.1.1}$$

In particular we define the moment of order n of X (with distribution μ on $\mathbb{R}$) as

$$m_n(\mu) = E(X^n) = \int_{-\infty}^{\infty} x^n \mu(dx).$$

Notice that, alternatively, we can define the distribution μ as the only measure that satisfies (1.1.1) for all bounded measurable functions.

Let X and Y be random variables in a probability space and let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bivariate function. Under certain conditions, $f(X, Y)$ is also a random variable. In this case the expectation $E[f(X, Y)]$ is well defined and calculated as

$$E[f(X, Y)] = \int_\Omega f(X(\omega), Y(\omega))\mathbb{P}(d\omega)$$

if the integral exists.

It will be very useful to look at the notion of independence of random variables in terms of the expectation, seen as a linear functional. In order to do this we first, recall the concept of classical independence.

Definition 1.1.5. Two classical random variables X and Y are said to be independent if

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B) \quad \text{for all } A, B \in \mathcal{B}(\mathbb{R}).$$

More generally, we say that the random variables $X_1, ..., X_n$ are independent if for all $A_1, A_2, ..., A_n \in \mathcal{B}(\mathbb{R})$ we have that

$$\mathbb{P}\left(\bigcap_{i=1}^{n}(X_i \in A_i)\right) = \prod_{i=1}^{n} \mathbb{P}(X_i \in A_i).$$

An infinite collection of random variables is said to be independent if any finite sub-collection of them is.

If we assume that the random variables X and Y have distributions with bounded support, then all moments of X and Y exist and determine their distributions. In addition, the condition that X and Y are independent is equivalent to the following

$$E[X^{n_1}Y^{m_1}...X^{n_k}Y^{m_k}] = E[X^{n_1+...+n_k}]E[Y^{m_1+...+m_k}], \tag{1.1.2}$$

for each m_i, $n_i \in \mathbb{N}$.

More formally, independence of X and Y is equivalent to the uncorrelation

$$E[(f(X) - E[f(X)]) \cdot (g(Y) - E[g(Y)])] = 0, \tag{1.1.3}$$

for any bounded Borel functions $f, g : \mathbb{R} \to \mathbb{R}$.

In this way, one can see from (1.1.2) that if X and Y are independent random variables with compact support, the moments of the random variable $X + Y$ can be obtained from the moments of X and the moments of Y. Equivalently, we have that the distribution μ_{X+Y} is determined by the distributions μ_X and μ_Y. This is the classical convolution

$$\mu_X * \mu_Y = \mu_{X+Y}.$$

The main tool to handle this convolution is the so called **characteristic function** or Fourier transform of a random variable.

Definition 1.1.6. Let μ be a probability measure in $\mathbb{R}$ and X be a random variable with distribution μ. We define $\widehat{\mu_X}$, the **characteristic function** of X, by

$$\widehat{\mu_X}(t) := \int e^{itx}\mu(dx) = E[e^{itX}].$$

In the case that the distribution of X has bounded support we have the expansion

$$\widehat{\mu}_X(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} E[X^n],$$

that is, the characteristic function is simply the exponential series of moments of X. The importance of the characteristic function in probability theory is the fact that its logarithm (when it exists) linearizes classical convolution. That is, if μ_X and μ_Y are probability distributions of the independent random variables X and Y, then $\widehat{\mu_{X+Y}}(t) = \widehat{\mu_X}(t)\widehat{\mu_Y}(t)$ and then

$$\log \widehat{\mu_{X+Y}}(t) = \log \widehat{\mu_X}(t) + \log \widehat{\mu_Y}(t).$$

Finally let us recall that a sequence $(\mu_n)_{n\geq 1}$ of measures converges weakly to the measure μ, if for every continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\mu_n(dx) = \int_{\mathbb{R}} f(x)\mu(dx).$$

In this case we will use the notation $\mu_n \to \mu$. When $(\mu_n)_{n\geq 1}$ are probability measures, this convergence is also known as convergence in distribution. The convergence $\mu_n \to \mu$ is equivalent to the pointwise convergence of $\widehat{\mu}_n \to \widehat{\mu}$. Moreover, when μ is determined by moments the convergence of all the moments implies the convergence $\mu_n \to \mu$. As an example we recall the so called Central Limit Theorem.

Example 1.1.7 (Classical CLT)**.** Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of centered random variables, which are independent and identically (i.i.d), with variance σ^2 and mean 0.

$$\lim_{N\to\infty} E\left(\left(\frac{X_1 + \ldots + X_N}{\sqrt{N}}\right)^n\right) = \begin{cases} \sigma^{2k}\frac{(2k)!}{2^k k!}, & \text{if } n = 2k, \\ 0, & \text{otherwise.} \end{cases}$$

In other words if $\mu = \mu_{X_1} = \mu_{X_i}$ then

$$\lim_{N\to\infty} D_{N^{-1/2}}(\underbrace{\mu * \mu \cdots * \mu}_{N\ times}) = \mathcal{N}(0, \sigma^2),$$

where, for a probability measure ν, the measure $D_t(\nu)$ denotes the dilation by t, such that $D_t(\nu)(B) = \nu(tB)$ for any Borel set B.

1.2 Non-Commutative Probability Spaces

In order to give a non-commutative analog of the triplet $(\Omega, \mathcal{F}, P)$ we need to rephrase the notion of classical probability space. The main observation is that the knowledge of the triplet $(\Omega, \mathcal{F}, P)$ is equivalent to the knowledge of $L^\infty(\Omega, \mathbb{P})$ (the algebra of random variables) and E. Thus we can think of a classical probability space as a commutative algebra with a linear functional. With this definition, from the operator algebraic point of view, it is natural to consider a non-commutative algebra instead of $L^\infty(\Omega, \mathbb{P})$.

Non-Commutative Probability Spaces

Definition 1.2.1. (1) A **non-commutative probability space** is a pair $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is a unital complex algebra and ϕ is a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(\mathbf{1}_{\mathcal{A}}) = 1$.

(2)A **non-commutative random variable** (or simply random variable) is just an element $a \in \mathcal{A}$.

We will assume that $\mathcal{A}$ is a $*$**-algebra**. That is, $\mathcal{A}$ is endowed with an antilinear involution *, an operation such that $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$, $\forall\ \alpha, \beta \in \mathbb{C}$, $a, b \in \mathcal{A}$,

$(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. If we have that $\phi(a^*a) \geq 0$, for all $a \in \mathcal{A}$, we say that ϕ is positive and we will call $(\mathcal{A}, \phi)$ a **$*$-probability space**. Moreover, if for all $a \neq 0$ we have that $\phi(a^*a) > 0$, we say that ϕ is **faithful**.

In the frame of a $*$-probability space we say that a random variable $a \in \mathcal{A}$ is **normal** if $aa^* = a^*a$. A normal random variable $a \in \mathcal{A}$ is called **selfadjoint**, if $a = a^*$. Moreover if $u \in \mathcal{A}$ is such that $u^*u = uu^* = 1$ we call u a **unitary**.

Example 1.2.2. ($*$-probability spaces)

(1) Classical Probability Space. Consider the set $L^\infty_{\mathbb{R}}(\Omega, \mathbb{P})$ of real valued random variables, then the set

$$L^\infty(\Omega, \mathbb{P}) = \{X + iY : X, Y \in L^\infty_{\mathbb{R}}(\Omega, \mathbb{P})\}$$

is a $*$-algebra with the involution $(X + iY)^* = X - iY$. The pair $(\mathcal{A}, E)$, where E is the usual expectation extended by linearity to $L^\infty(\Omega, \mathbb{P}$, is a $*$-probability space.

(2) Classical Matrices. Let $M_d(\mathbb{C})$ be the space of complex $d \times d$ matrices with the canonical involution given by transposing and conjugating entrywise. Then $M_d(\mathbb{C})$ is a $*$-algebra and the linear functional $tr : M_d(\mathbb{C}) \to \mathbb{C}$ defined by $tr(A) = \frac{1}{d} Trace(A)$ makes $(M_d(\mathbb{C}), tr)$ a $*$-probability space.

(3) Random Matrices. Let $(\mathcal{A}, \phi)$ be a $*$-probability space and let d be a positive integer. Let $M_d(\mathcal{A})$ be the space of $d \times d$ matrices over $\mathcal{A}$ with the canonical involution given by transposing and applying the involution * entrywise. $M_d(\mathcal{A})$ is a $*$-algebra and the linear functional $\phi_d : M_d(\mathcal{A} \to \mathbb{C}$ defined by

$$\phi_d(A) = \frac{1}{d} \sum_{i=1}^{d} \phi(a_{ii}), \quad A = (a_{ij})_{i,j=1}^d \in M_d(\mathcal{A}).$$

makes $(M_d(\mathcal{A}), \phi_d)$ a $*$-probability space.

A particularly important example of this kind of $*$-probability spaces comes when taking $\mathcal{A} = L^\infty(\Omega, \mathbb{P})$ since this corresponds to the algebra of random matrices. Many applications have been encountered in this setting. We briefly touch these applications in Section 1.5 and 3.5.

(4) Compressed space. Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and $p \in \mathcal{A}$ a projection ($p^2 = p$) such that $\phi(p) \neq 0$, then we can consider the compressed space $(p\mathcal{A}p, \phi^{p\mathcal{A}p})$, where

$$p\mathcal{A}p := \{pap | a \in \mathcal{A}\}$$

and for an element $b = pap \in p\mathcal{A}p$ we define $\phi^{p\mathcal{A}p}(b) = \frac{1}{\phi(p)}\phi(b)$. This makes the pair $(p\mathcal{A}p, \phi^{p\mathcal{A}p})$ a non-commutative probability space with unit element $p = p \cdot 1 \cdot p$.

(5) Group algebra. Let G be a discrete group with identity 1_G and $\mathbb{C}G$ the group algebra

$$\mathbb{C}G := \{\sum_{g \in G} \alpha_g g \mid \alpha_g \in \mathbb{C}, \alpha_g \neq 0 \text{ finitely many times}\}$$

with the canonical involution and multiplication. We consider the group trace ϕ, that is, for $g \in G \subset \mathbb{C}G$, $\phi(g) = 0$ if $g \neq 1_G$ and $\phi(1_G) = 1$ and extend linearly. Then $(\mathbb{C}G, \phi)$ is a $*$-probability space and ϕ is a faithful trace. The same construction works with $\mathbb{C}G$ replaced by the reduced group C^*-algebra of G, or the von Neumann algebra of G.

(6) Let $\mathcal{H}$ be a Hilbert space and let $B(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. This is a $*$-algebra with the usual involution given by the adjoint. More explicitly, for an element $A \in B(\mathcal{H})$, the adjoint of A is the unique operator A^* determined by the property that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in \mathcal{H}$.

If we take $\mathcal{A}$ a unital $*$-subalgebra of $B(\mathcal{H})$ and a vector $x_0 \in \mathcal{H}$ of norm one we can form the $*$-probability space (A, ϕ), where $\phi : \mathcal{A} \to \mathbb{C}$ is defined by $\phi(A) = \langle Ax_0, x_0\rangle$. The linear functional ϕ is usually called a vector-state.

Distributions and $*$-distributions

In the frame of a $*$-probability space, the notion of distribution of a random variable will be given through its moments. More precisely, we will be interested in knowing the $*$-moments of a random variable a in $(\mathcal{A}, \phi)$, that is, the values of

$$\phi(a^{m_1}(a^*)^{n_1}...a^{m_k}(a^*)^{n_k})$$

for m_i, $n_i \in \mathbb{N}$, and in order to keep track of them we define the **$*$-distribution** of a. Denote by $\mathbb{C}\langle x, y\rangle$ the algebra of polynomials in two variables (no-commutative) with complex coefficients

Definition 1.2.3. Given a random variable a in $(\mathcal{A}, \phi)$, the **$*$-distribution (in the algebraic sense)** of a is the linear functional $\mu_a : \mathbb{C}\langle x, y\rangle \to \mathbb{C}$ defined by

$$\mu_a(x^{m_1}(y)^{n_1}...x^{m_t}(y)^{n_k}) = \phi(a^{m_1}(a^*)^{n_1}...a^{m_k}(a^*)^{n_k})$$

for each m_i, $n_i \in \mathbb{N}$.

In the case where $a \in \mathcal{A}$ is normal, alternatively, we define the $*$-distribution of $a \in \mathcal{A}$ in the following way.

Definition 1.2.4. Let $(\mathcal{A}, \phi)$ be a $*$-probability space and suppose that $a \in \mathcal{A}$ is a normal element. If there is a measure μ in $\mathbb{C}$ with compact support, such that

$$\int_{\mathbb{C}} z^k \overline{z}^l \mu(dz) = \phi(a^k(a^*)^l), \text{ for all } k, l \in \mathbb{N}, \tag{1.2.1}$$

we say that μ is a **$*$-distribution (in the analytical sense**) of a and denote it by μ_a.

Remark 1.2.5. (1) When μ satisfies the condition (1.2.1) then μ is unique due to Stone-Weierstrass Theorem.

(2) Since ϕ is unital then (1.2.1) ensures that μ is a probability measure.

(3) When a is selfadjoint the $*$-distribution of a is supported on $\mathbb{R}$ and Equation (1.2.1) takes the form

$$\int_{\mathbb{R}} t^p \mu(dt) = \phi(a^p), \text{ for all } p \in \mathbb{N}.$$

In this case we call μ the *distribution* of a.

Since our notion of distribution is defined in terms of moments, we must phrase convergence in distribution in terms of convergence of all the moments. As previously mentioned, for distributions determined by their moments (e.g. random variables with compact support), this type of convergence is stronger than weak convergence.

Definition 1.2.6 (Convergence in distribution)**.** Let $(\mathcal{A}_N, \phi_N)_{N\in\mathbb{N}}$ and $(\mathcal{A}, \phi)$ be non-commutative probability spaces and consider random variables $a_N \in \mathcal{A}_N$ for each $N \in \mathbb{N}$ and $a \in \mathcal{A}$. We say that a_N **converges in distribution** towards a, as $N \to \infty$ if

$$\lim_{N\to\infty} \phi_N(a_N^n) = \phi(a^n), \qquad \forall n \in \mathbb{N}.$$

In this case we write $a_N \to a$.

Let us give some typical examples of normal random variables and their distributions considered in the literature of non-commutative probability.

Example 1.2.7. (1) Projection. Let p be a projection (i.e $p = p^2 = p^*$) such that $\phi(p) = t$. Then clearly all the moments of p satisfy the $\phi(p^n) = \phi(p) = t$. The measure $(1-t)\delta_0 + t\delta_1$ clearly has this moments and thus is the distribution of p.

2) Haar Unitaries. A unitary such that $\phi(u^n) = \phi((u^*)^n) = 0$ for all $n \in \mathbb{N}$ is called Haar unitary. The $*$-distribution is given by the Lebesgue Measure in the unit circle $\mathbb{T}$, also called Haar measure. Indeed since $u^k(u^*)^l = u^{k-l}$ for every $k, l \in \mathbb{N}$ we have that $\phi(u^k(u^*)^l) = \delta_{kl}$ which can be easily seen to coincide with the integral

$$\int_{\mathbb{T}} z^k \bar{z}^l dz.$$

3) Arcsine. Let u be a Haar unitary and consider the element $a = u+u^*$. The moments of a are given by

$$\phi(a^n) = \begin{cases} \binom{2k}{k}, & \text{if } n = 2k, \\ 0, & \text{otherwise.} \end{cases}$$

These are the moments of the arcsine distribution supported on $(-2, 2)$ with density

$$\frac{1}{\pi\sqrt{4-t^2}}, \qquad |t| < 2.$$

4) Semicircle. In the framework of Example 1.2.2 consider the Hilbert space $\mathcal{H} = l^2(\mathbb{N} \cup 0)$ with orthonormal basis formed by elements of the form

$$x_i = (0, ..., 0, 1, 0, 0, ...).$$

The one-sided shift operator $L \in B(\mathcal{H})$ given by $L(x_i) = x_{i+1}$ has adjoint given by $L^*(x_{i+1}) = x_i$, for $i = 0, 1, ...$ and $L^*(x_0) = 0$. One can see that for the vector-state defined by $\phi(T) := \langle Tx_0, x_0\rangle$, we have that

$$\phi(s^n) = \begin{cases} \frac{1}{k+1}\binom{2k}{k}, & \text{if } n = 2k, \\ 0, & \text{otherwise.} \end{cases}$$

These are the moments of the semicircle distribution supported on $(-2, 2)$ with density

$$\frac{1}{2\pi}\sqrt{4-t^2} \qquad |t| < 2.$$

5) Empirical distribution. Consider a selfadjoint $N \times N$ matrix A with eigenvalues $\lambda_1, ..., \lambda_N$, counted with multiplicity. Then the distribution (with respect of tr) of A is given by probability measure that assigns mass $1/N$ to each eigenvalue. Indeed

$$tr(A^k) = \frac{1}{N}(\lambda_1^k + \cdots + \lambda_N^k) = E(\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i})$$

C^*-probability spaces and W^*-probability spaces

The most frequently used non-commutative probability spaces $(\mathcal{A}, \phi)$ belong either to the class of C^*-probability spaces or to the class of W^*-probability spaces. These are $*$-probability spaces where $\mathcal{A}$ is given a structure of C^*-algebra (resp. von Neumann algebra).

Working with C^*-algebras ensures the existence of $*$-distributions in the analytical sense for any normal element. Moreover, the notion of affiliated operator to W^*-algebras permits to extend some of the operations in Free Probability to unbounded measures.

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the algebra of bounded linear operators of $\mathcal{H}$. Recall that a $*$-subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is called a C^* algebra if it is closed in the operator norm and taking adjoints. If, moreover, $\mathcal{A}$ is closed in the weak operator topology and contains the identity operator it is called a W^*-algebra or Von Neumann algebra. A C^*-algebra is a W^*-algebra if and only if it is equal to its bicommutant, that is, $\mathcal{A} = \mathcal{A}''$.

The corresponding $*$-probability space is defined as follows.

Definition 1.2.8. (1) A $*$-probability space $(\mathcal{A}, \phi)$ is called C^*-*probability space* if $\mathcal{A}$ is a unital C^*-algebra.

(2) A $*$-probability space $(\mathcal{A}, \phi)$ is called a W^*-*probability space* if $\mathcal{A}$ is a non-commutative von Neumann algebra and ϕ is a normal faithful trace.

Let us review some of basic results in the theory of C^*-algebras. Recall that the spectrum of a is the set

$$Sp(a) = \{z \in \mathbb{C} : z1_{\mathcal{A}} - a \text{ is not invertible}\}.$$

We denote by $C(Sp(a))$ the algebra of continuous functions $f : Sp(a) \to \mathbb{C}$.

Theorem 1.2.9. *Let $\mathcal{A}$ be a unital C^*-algebra.*

(1) For any $a \in \mathcal{A}$, $Sp(a)$ is a non-empty set contained in the disc $\{z \in \mathbb{C} : |z| \leq |a|\}$.

(2) Let $a \in \mathcal{A}$ be a normal element. There is a mapping $\Phi_a : C(Sp(a)) \to \mathcal{A}$ with the following properties:

(i) Φ_a is a homomorphism.

(ii) $|\Phi(f)| = |f|_\infty$ for all $f \in C(Sp(a))$.

(iii) If $id : Sp(a) \to \mathbb{C}$ denotes the identity $id(z) = z$ in $C(Sp(a))$, then $\Phi_a(id) = a$.

Φ_a is known as *functional calculus with continuous functions* for the element a. From these properties we can obtain, for a normal element a and any function $f \in C(Sp(a))$ that

$$Sp(f(a)) = f(Sp(a))$$

where $f(a)$ is defined by the functional calculus.

The importance of C^*-probability spaces lies in the existence of $*$-distributions in the analytic sense.

Theorem 1.2.10. *Let $(\mathcal{A},\phi)$ be a C^*-probability space such that ϕ is faithful and let $a \in \mathcal{A}$ be a normal element. Then a has a $*$-distribution in the analytic sense. Moreover, if μ is the $*$-distribution of a we have that*

(i) The support of μ is contained in the spectrum of a.

(ii) For $f \in C(Sp(a))$ we have the formula

$$\int f d\mu = \phi(f(a)).$$

As seen from Theorem 1.2.10, so far we could only talk about measures with bounded support. However, W^*-algebras allow to consider random variables (and probability measures) with unbounded support. So let $\mathcal{A}$ be a W^*-algebra. A self-adjoint operator X *is said to be affiliated with* $\mathcal{A}$ if $f(X) \in \mathcal{A}$ for any bounded Borel function f on $\mathbb{R}$. Given a self-adjoint operator X affiliated with $\mathcal{A}$, the *distribution* of X is the unique measure μ_X in $\mathcal{M}$ satisfying

$$\phi(f(X)) = \int_{\mathbb{R}} f(x)\mu_X(\mathrm{d}x)$$

for every Borel bounded function f on $\mathbb{R}$.

1.3 Free independence

The notion of **free independence** or **freeness** between non-commutative random variables was introduced in 1985 by Voiculescu [79], who noticed that freeness behaves in an analogous way to the concept of classical independence, but replacing tensor products with free products. In order to see this resemblance recall the notion of independence of random variables in terms of expectation E.

Two random variables X and Y with bounded support, and thus with all moments, are independent if and only if

$$E[X^{n_1}Y^{m_1}...X^{n_k}Y^{m_k}] = E[X^{n_1+...+n_k}]E[Y^{m_1+...+m_k}], \ \forall m_i, n_i \in \mathbb{N}. \tag{1.3.1}$$

That is if X and Y are independent and have bounded support, we can calculate the mixed moments of X and Y. The general idea is that independence may be understood as a "universal rule" for calculating moments.

We will be interested in so-called free independence, which is defined in a similar way to formula (1.3.1) for the classical case of independence. However, in this case we will use the linear functional ϕ and we deal with not necessarily classical random variables, but rather non-commutative random variables, that is, elements of $(\mathcal{A},\phi)$.

In the context of a $*$- probability space $(\mathcal{A},\phi)$ there are other notions of independence or "rules for calculating moments". Of course, these rules must satisfy some properties so that they give rise to an interesting theory. There is an axiomatized concept of independence leading to 5 notions of independence, namely, tensor (or classical), free, Boolean, monotone and antimonotone. The interested reader is referred to [17], [53] and [73] for this axiomatization.

In this frame we may define classical independence in the following way.

Definition 1.3.1. The non-commutative random variables $a, b \in \mathcal{A}$ are said **tensor independent** (**with respect to** ϕ) if $ab = ba$ and

$$\phi(a^n b^m) = \phi(a^n)\phi(b^m) \qquad \forall n, m \in \mathbb{N}.$$

Again, by linearity of ϕ (resp. E), the moments of the random variable $a + b$ can be obtained from the moments of a and the moments of b. In particular, we have that μ_{a+b} is determined by the distributions μ_a and μ_a. This is the classical convolution

$$\mu_a * \mu_b := \mu_{a+b}.$$

Remark 1.3.2. If $X = X_1 \times X_2$ and $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$ (the product measure), then any functions X, Y so that X depends only on the first coordinate and Y only on the second coordinate are tensor independent. In other words, the random variables $x_1 \otimes 1$ and $1 \otimes y_1$ are (tensor) independent.

Just like the notion of a tensor product can be used to recover the notion of independence and classical convolution, free products lead to the notion of free independence and free convolution.

Definition 1.3.3. A family of subalgebras A_i, $\mathbf{1}_{\mathcal{A}} \in A_i$, $i \in I$ in a non-commutative probability space $(\mathcal{A}, \phi)$ is said to be **free** if

$$\phi(a_1 a_2 ... a_n) = 0$$

whenever $\phi(a_j) = 0$, $a_j \in A_{i(j)}$, and $i(1) \neq i(2) \neq ... \neq i(n)$. More generally, a family of subsets $\Omega_i \subset A, i \in I$ is free if the algebras generated by $\Omega_i \cup \{\mathbf{1}\}$ are free.

Let us illustrate how the notion of free products is related to free independence. The following example is taken from [70].

Example 1.3.4 (Free products of groups)**.** In the context of Example 1.2.2 , let G_1 and G_2 be two discrete groups. We regard the group algebra of the free product $\mathbb{C}(G_1 * G_2)$ as a non-commutative probability space by letting ϕ be the group trace; for $g \in G = G_1 * G_2$, $\phi(g) = 0$ unless $g = 1_G$.

Now, let $w \in G_1 * G_2$ be a word. Thus $w = g_1 \cdots g_n$ with $g_j \in G_{i(j)}$. We may assume, by reducing the word, that consecutive letters lie in different groups; i.e., $i(1) \neq i(2), i(2) \neq i(3)$ and so on,. The resulting word is non-trivial if all $g_1, ..., g_n$ are non-trivial.

Remember that $\phi(g) = 0$ unless $g = 1_G$, thus, rephrasing in terms of ϕ, we have that

$$\phi(g_1 g_2 ... g_n) = 0$$

whenever $g_j \in G_{i(j)}$, $i(1) \neq i(2), i(2) \neq i(3)$ and $\phi(g_1) = \phi(g_2) = ... = 0$ (i.e. $g_i \neq 1_G$).

By linearity we get that if $a \in \mathbb{C}(G_1 * G_2)$ has the form $a = a_1 \cdots a_n$, then $\phi(a) = 0$ whenever $a_j \in \mathbb{C}(G_{i(j)})$, $i(1) \neq i(2), i(2) \neq i(3)$ and $\phi(a_1) = \phi(a_2) = ... = 0$, which is nothing other than free independence between the algebras $\mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$.

To be more concrete, for two random variables $a, b \in \mathcal{A}$ free independence (with respect to ϕ) is equivalent to the property that

$$\phi(p_1(a)q_1(b)p_2(a)q_2(b)...p_n(a)q_n(b)) = 0 \quad \forall n \in \mathbb{N},$$

whenever p_i and q_j are polynomials such that $\phi(p_i(a)) = \phi(q_j(b)) = 0$. As for the case of classical random variables, this new relation of independence may be understood as a "rule" for calculating mixed moments of a and b, this allows us to compute the moments of $a+b$ and ab in terms of the moments of a and b. Hence, if $\{a, b\}$ is a free pair of random variables with all moments, then the $*$-distributions μ_{a+b} of $a+b$ and μ_{ab} of ab depend only on the $*$-distribution μ_a of a and the $*$-distribution μ_b of b. The free sum and free product will be studied in the following sections.

As an example let us state the free version of the Central Limit Theorem.

Theorem 1.3.5 (Free CLT)**.** *Let $(\mathcal{A}, \phi)$ be a C^*-probability space and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of centered self-adjoint random variables, which are identically distributed and independent in the free sense, with common variance* 1 *Then*

$$\lim_{N\to\infty} \phi\left(\left(\frac{a_1 + \ldots + a_N}{\sqrt{N}}\right)^n\right) = \begin{cases} \frac{1}{k+1}\binom{2k}{k}, & \text{if } n = 2k, \\ 0, & \text{otherwise.} \end{cases}$$

In other words we have the convergence

$$\frac{a_1 + \ldots + a_N}{\sqrt{N}} \to s$$

where s is the semicircle element on Example 1.2.7

1.4 Free Additive Convolution

If μ and ν are probability measures on $\mathbb{R}$ with compact support, we can find non-commutative selfadjoint random variables a and b in a C^*-probability space such that a has a $*$-distribution μ and b has $*$-distribution ν. If we ask a and b to be free, then the $*$-distribution of $a+b$ is called the free convolution of μ and ν, which is denoted by $\mu \boxplus \nu$. For example, we can take a and b as multiplication operators with the identity in the Hilbert spaces $L^2(\mu)$ and $L^2(\nu)$, respectively, and then take the free product of this C^*-probability spaces to make a and b free; this can be seen in detail in the book by Nica and Speicher [60, Lec. 6].

The fact that $\mu \boxplus \nu$ does not depend on the choice of a and b follows from the fact that the $*$-distribution of $a+b$ only depends on the moments of a and b, which are determined by μ and ν. Moreover it is not hard to see that $\boxplus$ is associative and commutative.

Given a finite measure μ on $\mathbb{R}$ (with the Borel σ-field $B(\mathbb{R})$), its **Cauchy transform** G_μ is defined as

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t}\mu(dt), \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{1.4.1}$$

It is well known that G_μ is an analytic function in $\mathbb{C}\backslash\mathbb{R}$, $G_\mu : \mathbb{C}_+ \to \mathbb{C}_-$.

The Cauchy and the Fourier transform are related by the expression

$$G_\mu(z) = \begin{cases} i\int_{-\infty}^{0} e^{-itz}\widehat{\mu}(t)dt, & \mathrm{Im}(z) > 0 \\ -i\int_{0}^{\infty} e^{-itz}\widehat{\mu}(t)dt, & \mathrm{Im}(z) < 0, \end{cases} \tag{1.4.2}$$

which already tells that that G_μ determines uniquely the measure μ.

Moreover, we can recover the measure μ via the Stieltjes inversion formula:

$$\mu((t_0,t_1]) = -\frac{1}{\pi}\lim_{\delta\to 0+}\lim_{y\to 0+}\int_{t_0+\delta}^{t_1+\delta} \Im(G_\mu(x+iy))dx, \quad t_0 < t_1. \tag{1.4.3}$$

In particular, if μ is absolutely continuous with respect to the Lebesgue measure with density f_μ,

$$f_\mu(x) = -\frac{1}{\pi}\lim_{y\to 0+}\Im G_\mu(x+iy). \tag{1.4.4}$$

From the above considerations we easily obtain the following properties.

Proposition 1.4.1. *Let μ be a finite measure on $\mathbb{R}$. Then*

i) $G_\mu(\mathbb{C}^\pm) \subset \mathbb{C}^\mp$ *and* $G_\mu(\overline{z}) = \overline{G_\mu(z)}$.

ii) $|G_\mu(z)| \le \dfrac{\mu(\mathbb{R})}{|\Im(z)|}$.

iv) $\lim\limits_{y\to\infty} y\,|G_\mu(iy)| < \infty$.

v) $\lim\limits_{y\to\infty} iyG_\mu(iy) = \mu(\mathbb{R})$. *In particular, if μ is a probability measure*

$$\lim_{y\to\infty} iyG_\mu(iy) = 1.$$

The *reciprocal of the Cauchy transform* is the function $F_\mu(z) : \mathbb{C}_+ \to \mathbb{C}_+$ defined by $F_\mu(z) = 1/G_\mu(z)$. It was proved in [22] that there are positive numbers η and M such that F_μ has a right inverse F_μ^{-1} defined on the region

$$\Gamma_{\eta,M} := \{z \in \mathbb{C}; |Re(z)| < \eta Im(z), \quad Im(z) > M\}. \tag{1.4.5}$$

The *Voiculescu transform of* μ is defined by

$$\phi_\mu(z) = F_\mu^{-1}(z) - z \tag{1.4.6}$$

on any region of the form $\Gamma_{\eta,M}$, where F_μ^{-1} is defined, see [20], [22]. The *free cumulant transform* is a variant of ϕ_μ defined as

$$\mathcal{C}_\mu^{\boxplus}(z) = z\phi_\mu(\frac{1}{z}) = zF_\mu^{-1}\left(\frac{1}{z}\right) - 1, \tag{1.4.7}$$

for z in a domain $D_\mu \subset \mathbb{C}_-$ such that $1/z \in \Gamma_{\eta,M}$ where F_μ^{-1} is defined, see [12].

The *free additive convolution* of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 \boxplus \mu_2$ on $\mathbb{R}$ such that $\phi_{\mu_1\boxplus\mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z)$ or equivalently

$$\mathcal{C}_{\mu_1\boxplus\mu_2}^{\boxplus}(z) = \mathcal{C}_{\mu_1}^{\boxplus}(z) + \mathcal{C}_{\mu_2}^{\boxplus}(z) \tag{1.4.8}$$

for $z \in D_{\mu_1} \cap D_{\mu_2}$. It turns out that $\mu_1 \boxplus \mu_2$ is the distribution of the sum $a + b$ of two free random variables a and b with distributions μ_1 and μ_2 respectively.

1.5 Free Multiplicative Convolution

Free multiplicative convolution is a little bit more subtle that free additive convolution. The reason is that, in general, if a, b are selfadjoint random variables in a C^*-probability space (A, ϕ), then it is not true that ab is also selfadjoint. To fix this problem we need to assume that a is positive and then, if a, b are free, the random variable ab has the same moments as the selfadjoint random variable $a^{1/2}ba^{1/2}$. Indeed,

$$\phi((a^{1/2}ba^{1/2})^n) = \phi((a^{1/2}b(ab)^{n-1}a^{1/2}) = \phi((a^{1/2}a^{1/2}b(ab)^{n-1}) = \phi((ab)^n),$$

where we used in the second equality the fact that a and b are free. Since μ is supported on $\mathbb{R}_+$, a is a positive self-adjoint operator and $\mu_{a^{1/2}}$ is uniquely determined by μ. Hence the distribution $\mu_{a^{1/2}ba^{1/2}}$ of the self-adjoint operator $a^{1/2}ba^{1/2}$ is determined by μ and ν.

So *free multiplicative convolution* $\boxtimes$ on $\mathcal{M}$ is defined as follows, see [22].

Definition 1.5.1. Let μ, ν be probability measures on $\mathbb{R}$, with $\mu \in \mathcal{M}^+$ and let a, b be free random variables such that $\mu_a = \mu$ and $\mu_b = \nu$. The **free multiplicative convolution** of μ and ν is the distribution of $\mu_{a^{1/2}ba^{1/2}}$ and it is denoted by $\mu \boxtimes \nu$.

If we restrict to the case where both $\mu, \nu \in \mathcal{M}^+$, the operation $\boxtimes$ is commutative since the moments of $a^{1/2}ba^{1/2}$ equal the moments of $b^{1/2}ab^{1/2}$.

On the other hand it is clear that as for the free additive convolution we can find free variables a, b in some C^* -probability space (A, ϕ) with distributions μ and ν and that $\mu \boxtimes \nu$ does not depend on the choice of a and b.

Finally note that $a^{1/2}ba^{1/2}$ is only used to ensure that we are dealing with moments of a probability measure so we can define for $\mu, \nu \in \mathcal{M}^+$, $\mu \boxtimes \nu$ as the probability measure whose moments are $\phi((ab)^n)$. This measure is unique and has support in $\mathbb{R}^+$.

Free multiplicative convolution behaves nicely with respect to weak convergence.

Proposition 1.5.2 ([22]). *Let $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ be sequences of probability measures in $\mathcal{M}^+$ converging to probability measures μ and ν in $\mathcal{M}^+$, respectively, in the weak* topology and such that $\mu \neq \delta_0 \neq \nu$. Then, the sequences $\{\mu_n \boxtimes \nu_n\}_{n=1}^{\infty}$ converges to $\mu \boxtimes \nu$ in the weak* topology.*

An important analytic tool for computing the free multiplicative convolution of two probability measures is Voiculescu's S-transform. It was introduced in [82] for non-zero mean distributions in $\mathcal{M}_b$ and further studied by Bercovici and Voiculescu [22] in the case of probability measures in $\mathcal{M}^+$ with unbounded support, see also [21].

The next result was proved in [22] for probability measures in $\mathcal{M}^+$ with unbounded support.

Proposition 1.5.3. *Let $\mu \in \mathcal{M}^+$ such that $\mu(\{0\}) < 1$. The function*

$$\Psi_\mu(z) = \int_0^\infty \frac{zx}{1 - zx}\mu(\mathrm{d}x), \quad z \in \mathbb{C}\backslash\mathbb{R}_+ \tag{1.5.1}$$

is univalent in the left-plane $i\mathbb{C}_+$ and $\Psi_\mu(i\mathbb{C}_+)$ is a region contained in the circle with diameter $(\mu(\{0\}) - 1, 0)$. Moreover, $\Psi_\mu(i\mathbb{C}_+) \cap \mathbb{R} = (\mu(\{0\}) - 1, 0)$.

A sometimes useful relation between Ψ and the Cauchy transform is the following

$$\Psi_\mu(z) = \int_{\mathbb{C}} \frac{zt}{1-zt}\mu(\mathrm{d}t) = \frac{1}{z}G_\mu\left(\frac{1}{z}\right) - 1, \quad z \in \mathbb{C}\backslash\mathbb{R}_+. \tag{1.5.2}$$

Let $\chi_\mu : \Psi_\mu(i\mathbb{C}_+) \to i\mathbb{C}_+$ be the inverse function of Ψ_μ. The *S-transform* of μ is the function

$$S_\mu(z) = \chi(z)\frac{1+z}{z}.$$

The following result shows the role of the S-transform as an analytic tool for computing free multiplicative convolutions. It was shown in [80] for measures in $\mathcal{M}^+$ with bounded support and in [22] for measures in $\mathcal{M}^+$ with unbounded support.

Proposition 1.5.4. *Let μ_1 and μ_2 be probability measures in $\mathcal{M}^+$ with $\mu_i \neq \delta_0$, $i = 1, 2$. Then $\mu_1 \boxtimes \mu_2 \neq \delta_0$ and*

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z)S_{\mu_2}(z)$$

in that component of the common domain which contains $(-\varepsilon, 0)$ for small $\varepsilon > 0$. Moreover, $(\mu_1 \boxtimes \mu_2)(\{0\}) = \max\{\mu_1(\{0\}), \mu_2(\{0\})\}$.

Recently, Raj Rao and Speicher [64] extended the S-transform to the case of measures in $\mathcal{M}$ having zero mean and all moments. Their main tools are combinatorial arguments based on moment calculations. This allows them to compute interesting free multiplicative convolutions of measures with bounded support, like the Marchenko-Pastur distribution with the semicircle distribution.

The next proposition is a particular case of a recent result proved in [64] for probability measures μ_1, μ_2 on $\mathbb{R}$ with all moments, when μ_1 has zero mean and $\mu_2 \in \mathcal{M}^+$.

Proposition 1.5.5. *Let μ_1 be a compactly supported symmetric probability measure on $\mathbb{R}$ and let $\mu_2 \in \mathcal{M}^+$ have compact support, with $\mu_i \neq \delta_0$, $i = 1, 2$. Then, $\mu_1 \boxtimes \mu_2 \neq \delta_0$ and*

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z)S_{\mu_2}(z).$$

As we proved in the joint paper with Perez-Abreu [9] the definition of S-transform can be extended to symmetric probability measures μ on $\mathbb{R}$ (even without moments) as follows. Consider the cones

$$H = \left\{z \in \mathbb{C}^-; \quad |\mathrm{Re}(z)| < |\mathrm{Im}(z)|\right\} \qquad \widetilde{H} = \left\{z \in \mathbb{C}^+; \quad |\mathrm{Re}(z)| < \mathrm{Im}(z)\right\}.$$

When $\mu(\{0\}) < 1$, the transform Ψ_μ has a unique inverse on H, $\chi_\mu : \Psi_\mu(H) \to H$ and a unique inverse on $\widetilde{H}$, $\widetilde{\chi}_\mu : \Psi_\mu(\widetilde{H}) \to \widetilde{H}$. In this case there are two S-transforms for μ given by

$$S_\mu(z) = \chi_\mu(z)\frac{1+z}{z} \text{ and } \widetilde{S}_\mu(z) = \widetilde{\chi}_\mu(z)\frac{1+z}{z} \tag{1.5.3}$$

and these are such that

$$S^2_\mu(z) = \frac{1+z}{z}S_{\mu^{(2)}}(z) \text{ and } \widetilde{S}^2_\mu(z) = \frac{1+z}{z}S_{\mu^{(2)}}(z) \tag{1.5.4}$$

for z in $\Psi_\mu(H)$ and $\Psi_\mu(\widetilde{H})$, respectively.

Thus, the free multiplicative convolution of a probability measure μ_1 supported on $\mathbb{R}_+$ with a symmetric probability measure μ_2 on $\mathbb{R}$ may be defined as the symmetric probability measure $\mu_1 \boxtimes \mu_2$ on $\mathbb{R}$ such that

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z) S_{\mu_2}(z). \tag{1.5.5}$$

It is easily shown then that

$$(\mu_1 \boxtimes \mu_2)^2 = \mu_1 \boxtimes \mu_1 \boxtimes \mu_2^2. \tag{1.5.6}$$

This expression and its generalization to k-symmetric distributions (that we prove in Chapter 5) has many implications and will play a crucial role when considering multiplicative convolution between k-symmetric distributions and distribution supported on the positive real line.

From (1.4.7) and the fact that $\Psi_\mu(z) = \frac{1}{z} G_\mu(\frac{1}{z}) - 1$, one obtains the following relation observed in [58] between the free cumulant transform and the S-transform

$$z = \mathcal{C}_\mu^{\boxplus}(z S_\mu(z)). \tag{1.5.7}$$

This equation holds for measures in $\mathcal{M}^+$ or in $\mathcal{M}_b$ with zero mean. It was suggested in [64] that (1.5.7) may be used to define S-transforms of general probability measures on $\mathbb{R}$.

As is readily seen from Equation (1.5.7), free additive powers may also be described by the S-transform in the following way

$$S_{\mu^{\boxplus t}}(z) = \frac{1}{t} S_\mu(z/t), \tag{1.5.8}$$

while the S transform of a dilation is given by

$$S_{D_t(\mu)}(z) = \frac{1}{t} S_\mu(z), \tag{1.5.9}$$

from where we can deduce the following equality (see [15])

$$(\mu \boxtimes \nu)^{\boxplus t} = D_t(\mu^{\boxplus t} \boxtimes \nu^{\boxplus t}) \qquad t > 1. \tag{1.5.10}$$

1.6 Free cumulants

In this section we will define free cumulants and see the relation with free convolution. The **free cumulants** (k_n) were introduced by Roland Speicher in [72], in his combinatorial approach to Voiculescu's free probability theory. We refer the reader to the book of Nica and Speicher [60] for an introduction to this combinatorial approach.

Free cumulants of a probability measure

Let us start by defining free cumulants for probability measures. If we only care about probability measures and their convolutions, free cumulants can be defined in a very simple way and for many purposes this definition is enough. Later, we will define the free cumulants of random variables as a way to encode their joint distribution.

We say that a measure μ has *all moments* if $m_k(\mu) = \int_{\mathbb{R}} t^k \mu(\mathrm{d}t) < \infty$, for each integer $k \geq 1$. Probability measures with compact support have all moments.

Let $\mu \in \mathcal{M}$ be a probability measure with all moments. The free cumulants are the coefficients $k_n = k_n(\mu)$ in the series expansion

$$\mathcal{C}_\mu^{\boxplus}(z) = \sum_{n=1}^{\infty} k_n(\mu) z^n.$$

Since, by definition the free cumulant transform $\mathcal{C}_\mu^{\boxplus}$ linearizes additive free convolution, then free cumulants also additive with respect to the free convolution $\mu_1 \boxplus \mu_2$

$$k_n(\mu_1 \boxplus \mu_2) = k_n(\mu_1) + k_n(\mu_2)$$

and

$$k_n(\mu^{\boxplus t}) = t k_n(\mu).$$

The main object to describe the relation between the free cumulants and the moments is the set of non-crossing partitions of $\{1, \ldots, n\}$, denoted by $NC(n)$ and described in Chapter 2. A partition π is an equivalence relation on the set $\{1, \ldots, n\}$. We say that a partition π is **non-crossing** if $a \sim_\pi c$, $b \sim_\pi d \Rightarrow a \sim_\pi b \sim_\pi c \sim_\pi d$, for all $1 \leq a < b < c < d \leq n$. So, let us state the so-called moment-cumulant formula of Speicher [72] which gives a relation between moments and free cumulants.

$$m_n(\mu) = \sum_{\pi \in NC(n)} k_\pi(\mu), \tag{1.6.1}$$

where $\pi \to k_\pi$ is the multiplicative extension of the free cumulants to non-crossing partitions, that is

$$k_\pi := k_{|V_1|} \cdots k_{|V_r|} \qquad \text{for} \qquad \pi = \{V_1, ..., V_r\} \in NC(n).$$

We can calculate easily the first terms using (1.6.1)

$$\begin{aligned}
m_1 &= k_1 \\
m_2 &= k_2 + k_1^2 \\
m_3 &= k_3 + 3k_2k_1 + k_1^3 \\
m_4 &= k_4 + 4k_3k_1 + 2k_2^2 + 6k_2k_1^2 + k_1^4 \\
m_5 &= k_5 + 5k_4k_1 + 5k_2k_3 + 10k_3k_1^2 + 10k_2^2k_1 + 10k_2k^3 + k_1^5.
\end{aligned}$$

On the other hand, free multiplicative convolution may be described in terms of free cumulants as follows.

$$k_{\mu_1 \boxtimes \mu_2}(z) = \sum_{\pi \in NC(n)} k_\pi(\mu_1) k_{K(\pi)}(\mu_2) \tag{1.6.2}$$

where $K(\pi)$ is the Kreweras complement defined in the Remark 2.1.3. We also show the first cumulants of $\mu_1 \boxtimes \mu_2$:

$$\begin{aligned}
k_1(\mu_1 \boxtimes \mu_2) &= k_1(\mu_1)k_1(\mu_2) \\
k_2(\mu_1 \boxtimes \mu_2) &= k_2(\mu_1)k_1^2(\mu_2) + k_1^2(\mu_1)k_2(\mu_2) \\
k_3(\mu_1 \boxtimes \mu_2) &= k_3(\mu_1)k_1^3(\mu_2) + 3k_2(\mu_1)k_1(\mu_1)k_2(\mu_2)k_1(\mu_2) + k_3(\mu_1)k_1^3(\mu_2).
\end{aligned}$$

Free cumulants for random variables

We want to extend the definition of free cumulants to random variables. We will use some facts and definitions concerning non-crossing partitions.

Remark 1.6.1. The set $NC(n)$ can be equipped with the partial order $\leq$ of reverse refinement: for $\pi, \sigma \in \mathcal{P}(n)$, $\pi \leq \sigma$ iff every block of π is completely contained in a block of σ. This partial order turns $NC(n)$ into a lattice. This allows us to consider multiplicative functions and combinatorial convolutions on these families of partitions, and the corresponding Möbius inversions. We refer to Chapter 2 for these definitions.

We adopt the general definition of cumulants described by Lehner [48].

Definition 1.6.2. Given a notion of independence in a non-commutative probability space $(\mathcal{A}, \phi)$ we say that a sequence of applications $t_n : \mathcal{A} \to \mathbb{C}$, which sends $a \to t_n(a)$, $n = 1, 2, 3...$ is called the sequence of **cumulants** (with respect to some independence) if the following properties hold

a) $t_n(a)$ is a polynomial in the first n moments of a with greatest term $m_n(a)$.

b) Homogeneity of degree n: $t_n(\lambda a) = \lambda^n t_n(a)$.

c) Additivity with respect to independence: if a and b are independent random variables, then $t_n(a + b) = t_n(a) + t_n(b)$.

Definition 1.6.3. Let $\mathcal{A}$ be a unital algebra and let $\tau : \mathcal{A} \to \mathbb{C}$ be a unitary linear functional. Given a sequence of multilinear functionals $(p_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$,

$$\begin{array}{rcl} p_n & : & \mathcal{A}^n \to \mathbb{C}, \\ (a_1, ..., a_n) & \longmapsto & p_\pi[a_1, ..., a_n] \end{array}$$

we extend this sequence to a family $(p_\pi)_{n\in\mathbb{N},\pi\in NC(n)}$ of multilinear functionals by the formula

$$p_\pi[a_1, ..., a_n] := \prod_{V\in\pi} p(V)[a_1, ..., a_s] \quad \text{for } a_1, ..., a_n \in \mathcal{A}$$

where

$$p(V)[a_1, ..., a_s] := p_s(a_{i_1}, ..., a_{i_s})$$

for $V = \{i_1, i_2..., i_s\}$ and $i_1 < i_2... < i_s$.

The family $(p_\pi)_{n\in\mathbb{N},\pi\in NC(n)}$ is called the *multiplicative family of functionals* in $NC(n)$ determined by the sequence $(p_n)_{n\in\mathbb{N}}$. The multiplicativity of the family $(p_\pi)_{n\in\mathbb{N},\pi\in NC(n)}$ means that we have a factorization according to the block structure of $NC(n)$.

Notation 1.6.4. Let $\mathcal{A}$ be a unitary algebra and let $\phi : \mathcal{A} \to \mathbb{C}$ be a unitary linear functional. Define the multilinear functionals $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ by the formula

$$\phi_n(a_1, ..., a_n) := \phi(a_1 \cdots a_n).$$

We extend this notation for the corresponding multiplicative functionals in the non-crossing partitions via the formula

$$\phi_\pi[a_1, ..., a_n] := \prod_{V\in\pi} \phi(V)[a_1, ..., a_n] \text{ for } a_1, ..., a_n \in \mathcal{A}.$$

Now we define the free cumulants through the Möbius Inversion.

Definition 1.6.5. Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. The corresponding **free cumulants** $(k_\pi)_{\pi \in NC(n)}$ are, for each $n \in N$, $\pi \in NC(n)$, multilinear functionals

$$\begin{array}{rcl} k_\pi & : & \mathcal{A}^n \to \mathbb{C}, \\ (a_1, ..., a_n) & \longmapsto & k_\pi[a_1, ..., a_n] \end{array}$$

where

$$k_\pi[a_1, ..., a_n] := \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi}} \phi_\sigma[a_1, ..., a_n]\mu(\sigma, \pi),$$

and μ is the Möbius function in $NC(n)$. (See Section 2.1 below)

Because of the canonical factorization in intervals of $NC(n)$ free cumulants may also be described as follows.

Proposition 1.6.6. *The mapping $\pi \longmapsto k_\pi$ is a multiplicative family of functionals, that is*

$$k_\pi[a_1, ..., a_n] := \prod_{V \in \pi} k(V)[a_1, ..., a_n].$$

Moreover, the Definition 1.6.5 is equivalent to the following statements

i) $\pi \longmapsto k_\pi$ is a multiplicative family of functionals and for each $n \in \mathbb{N}$ and all the $a_1, ..., a_n \in \mathcal{A}$ we have that

$$k_\pi(a_1, ..., a_n) = \sum_{\sigma \in NC(n)} \phi_\sigma[a_1, ..., a_n]\mu(\sigma, 1_n). \tag{1.6.3}$$

ii) $\pi \longmapsto k_\pi$ is a multiplicative family of functionals and for each $n \in \mathbb{N}$ and all $a_1, ..., a_n \in \mathcal{A}$, we have

$$\phi(a_1, ..., a_n) = \sum_{\sigma \in NC(n)} k_\sigma[a_1, ..., a_n]. \tag{1.6.4}$$

The importance of free cumulants is given by the next theorem by Speicher [72], which says that free independence is equivalent to the vanishing of mixed free cumulants.

Theorem 1.6.7. *Let $\mathcal{A}, \phi)$ be a C^*-non commutative probability space and let $(k_n)_{n \in \mathbb{N}}$ be the free cumulants. Consider a family $(\mathcal{A}_i)_{i \in I}$ of unital subalgebras of $\mathcal{A}$. Then the following statements are equivalent.*

i) $(\mathcal{A}_i)_{i \in I}$ are in free relation.

ii) For any $n \geq 2$ and for any $a_j \in \mathcal{A}_{i(j)}$, $(j = 1, ..., n)$ with $i(1), ..., i(n) \in I$ we have that $k_n(a_1, ..., a_n) = 0$ whenever there are $1 \leq l, k \leq n$ with $i(l) \neq i(k)$.

Recall that we were interested in the moments and distribution of a random variable. Now, we will be more interested in the free cumulants of these random variables. So let us fix some notation.

Notation 1.6.8. Let $(a_i)_{i\in I}$ be random variables in a non-commutative probability space $(\mathcal{A},\phi)$ and let $(k_n)_{n\in\mathbb{N}}$ be their corresponding free cumulant functionals

(1) The **free cumulants** of $(\mathbf{a}_i)_{i\in I}$ are all the expressions of the form $k_n(a_{i(1)},...,a_{i(n)})$ for $n\in\mathbb{N}$ and $i(1),...,i(n)\in I$.

(2)If $(\mathcal{A},\phi)$ is a $*$-probability space, then the $*$- free cumulants of $(a_i)_{i\in I}$ are the free cumulants of $(a_i,a_i^*)_{i\in I}$.

(3) If we have only one random variable a we use the notation $k_n^a := k_n(a,a,...,a)$.

(4) When the probability measure μ is the $*$-distribution of a we say that its cumulants are $(k_n)_{n\in\mathbb{N}} := k_n^a$.

The fact that (4) agrees with the definition given before is a consequence of the following theorem which shows that free cumulants linearize free additive convolution.

Corollary 1.6.9. *Lat a and b be free random variables in some non-commutative probability space. Then we have*

$$k_n^{a+b} = k_n^a + k_n^b \qquad \text{for all } n\geq 1.$$

Proof. From Theorem 1.6.7 we have that all the cumulants that have both a and b as arguments must vanish and then

$$k_n^{a+b} \;=\; k_n(a+b,...,a+b) = k_n(a,...,a) + k_n(b,...,b) = k_n^a + k_n^b.$$

□

We will often use the **formula for product as arguments**, first proved by Krawczyk and Speicher [44]. For a proof see Theorem 11.12 in Nica and Speicher [60]

Theorem 1.6.10 (Formula for products as arguments)**.** *Let $(\mathcal{A},\phi)$ be a non-commutative probability space and let $(k_\pi)_{\pi\in\mathbb{N}}$ be the corresponding free cumulants. Let $m,n\in\mathbb{N}$ and $1\leq i(1)<i(2)\cdots<i(m)=n$ be given and consider the partition*

$$\hat{0}_m = \{\{1,...,i(1)\},...,\{i(m-1)+1,...,i(m)\}\}\in NC(n)$$

and the random variables $a_1,...,a_n\in\mathcal{A}$ then the following equation holds:

$$k_m(a_1\cdots a_{i(1)},...,a_{i(m-1)+1}\cdots a_{i(m)}) = \sum_{\substack{\pi\in NC(n)\\ \pi\vee\hat{0}_m=1_n}} k_\pi(a_1,...,a_n). \tag{1.6.5}$$

Let a,b be two random variables we want to be able to calculate the free cumulants of ab in terms of the free cumulants. This is the content of next theorem.

Theorem 1.6.11. *Let (A,ϕ) be a non-commutative probability space and consider random variables $a_1,...,a_n,b_1,...b_n\in A$ such that $\{a_1,a_2,...,a_n\}$ and $\{b_1,...,b_n\}$ are freely independent. Then we have*

$$\phi(a_1b_1a_2b_2...a_nb_n) = \sum k_\pi[a_1,a_2,...,a_n]\phi_{K(\pi)}[b_1,b_2,...,b_n]$$

and

$$k_n(a_1b_1,a_2b_2,...,a_nb_n) = \sum k_\pi[a_1,a_2,...,a_n]k_{K(\pi)}[b_1,b_2,...,b_n].$$

In particular, one should note that in the case when $a = a_1 = a_2 = ... = a_n$ and $b := b_1 = b_2 = ... = b_n$ the last formula gives

$$k_n^{ab} = \sum_{\pi \in NC(n)} k_\pi^a k_{K(\pi)}^b$$

which is exactly the formula (1.6.2) for the free multiplicative convolution.

1.7 Free Infinite divisibility

In this section we recall the notion of infinite divisibility and give some basic properties. In classical probability it is a well known fact that the infinitely divisible laws are characterized based on a Lévy-Khintchine representation for the classical cumulant function (the logarithm of its Fourier transform), see Sato [69] or Steutel and Van Harn [75]. Similarly, we can find a Lévy-Khintchine type characterization for free infinitely divisible distributions in terms of the Voiculescu transform. This was proved in 1993 by Bercovici and Voiculescu [22]. More recently Barndorff-Nielsen and Thorbjønsen [12] proposed a variant that is more similar to the classical case.

Definition 1.7.1. Let μ be a probability measure in $\mathbb{R}$. We say that μ is **freely infinitely divisible**, if for all n, there exists a probability measure μ_n such that

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus .... \boxplus \mu_n}_{n \text{ times}}. \tag{1.7.1}$$

We denote by $ID(\boxplus)$ the class of freely infinitely divisible measures . If $\mu \in ID(\boxplus)$ we also say that μ is **$\boxplus$-infinitely divisible** or infinitely divisible with respect to the convolution $\boxplus$.

Any freely infinitely divisible distribution defines a continuous $\boxplus$-semigroup of measures $(\mu_t)_{t\geq 0}$ in the space of probability measures on $\mathbb{R}$ with the weak topology of probability measures.

Proposition 1.7.2 ([22])**.** *Let μ be a $\boxplus$-infinitely divisible probability measure. There is a family $(\mu_t)_{t\geq 0}$ of probability measures on $\mathbb{R}$ such that*
i) $\mu_0 = \delta_0, \mu_1 = \mu$.
ii) $\mu_{t+s} = \mu_t \boxplus \mu_s$ for $s, t \geq 0$
iii) The mapping $t \to \mu_t$ is continuous with respect to the weak topology.

From the Voiculescu Transform and its variants it is possible to characterize the measures $\mu \in ID(\boxplus)$, with a representation analogous to Lévy-Kintchine´s. The first characterization for general measures (even with unbounded support) was given by Bercovici and Voiculescu [22] in terms of the characteristic pair (γ, σ).

Theorem 1.7.3 ([22, Th. 5.10])**.** *Let μ be a probability measure in $\mathbb{R}$. The following statements are equivalent*
i) μ is $\boxplus$-infinitely divisible.
ii) ϕ_μ has an analytic extension defined in $\mathbb{C}^+$ with values in $\mathbb{C}^- \cup \mathbb{R}$.

iii) There a finite measure σ on $\mathbb{R}$ and a real constant γ such that

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\sigma(dt), \qquad z \in \mathbb{C}^+.$$

In case that (i), (ii) and (iii) is satisfied, the pair (γ, σ) is called generating pair of μ.

Recall that a probability measure μ is infinitely divisible in the classical sense if and only if its classical cumulant transform $\log \widehat{\mu}$ has the Lévy-Khintchine representation

$$\log \widehat{\mu}(u) = i\eta u - \frac{1}{2}au^2 + \int_{\mathbb{R}} (e^{iut} - 1 - iut1_{[-1,1]}(t))\nu(dt), \quad u \in \mathbb{R}, \tag{1.7.2}$$

where $\eta \in \mathbb{R}$, $a \geq 0$ and ν is a Lévy measure in $\mathbb{R}$, that is $\int_{\mathbb{R}} \min(1, t^2)\nu(dt) < \infty$ and $\nu(\{0\}) = 0$. If this representation exists, the triplet (η, a, ν) is uniquely determined and is called the characteristic triplet of μ.

The corresponding Lévy-Khintchine representation of μ in terms of a triplet (η, a, ν) for a measure μ which is infinitely divisible was suggested by Barndorff-Nielsen and Thorbjørnsen [12].

Theorem 1.7.4 ([12])**.** *A probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if there are $\eta \in \mathbb{R}$, $a \geq 0$ and a Lévy measure ν on $\mathbb{R}$ such that*

$$\mathcal{C}_\mu(z) = \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1-zt} - 1 - tz1_{[-1,1]}(t) \right) \nu(dt), \quad z \in \mathbb{C}^-. \tag{1.7.3}$$

In this case the triplet (η, a, ν) is determined in a unique way and is called the free characteristic triplet of μ.

An important class of freely infinitely divisible measures is the class of free compound Poisson distributions, since any freely infinitely divisible measure on $\mathbb{R}$ can be approximated by free compound Poissons. This fact is often used since sometimes proving properties for free compound Poisson is easy and then these properties are extended to all $ID(\boxplus)$ by approximation arguments.

Definition 1.7.5. A probability measure μ whose free cumulants are of the form

$$k_n(\mu) = \lambda m_n(\nu).$$

for some $\lambda > 0$ and some distribution ν is called a free compound Poisson with rate λ and jump distribution ν.

Remark 1.7.6. The case when $\nu = \delta_\alpha$, $\alpha \in \mathbb{R}$, corresponds to the distribution

$$\mu(A) = \begin{cases} (1 - \frac{1}{\lambda})\mathbf{1}_{0 \in A} + \nu(A), & \text{if } \lambda > 1 \\ \nu(A), & \text{if } 0 \leq \lambda \leq 1, \end{cases}$$

with

$$d\nu(x) = \frac{1}{2\pi\sigma^2} \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{\lambda x} \mathbf{1}_{[\lambda_-, \lambda_+]}\, dx$$

and

$$\lambda_\pm = \sigma^2(1 \pm \sqrt{\lambda})^2.$$

The justification for the name of free Compound Poisson is the following limit theorem.

Proposition 1.7.7. *Let $\lambda \geq 0$ and ν a probability measure on $\mathbb{R}$ with compact support. The limit in distribution as $N \to \infty$ of*

$$((1-\frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\nu)^{\boxplus N}$$

has free cumulants $(k_n)_{n\geq 1}$ which are given by

$$k_n = \lambda m_n(\nu), \qquad (n \geq 1).$$

It is remarkable that if π is the distribution of a free Poisson (1) and ν a measure with all moments. Then the free cumulants of $\pi \boxtimes \nu$ are given by

$$k_n(\pi \boxtimes \nu) = m_n(\nu).$$

Indeed, since the cumulants of a free Poisson are just 1 then by the formula for the free cumulants of a product we get that if b has a free Poisson distribution then

$$k_n^{ab} = \sum_{\pi \in NC(n)} k_\pi^a k_{K(\pi)}^b = \sum_{\pi \in NC(n)} k_\pi^a = m_n(a).$$

Thus we have the following relations between the three concepts.

Proposition 1.7.8. *Let π be a free Poisson and let ν be a probability measure with all moments. Then the following statements are equivalent*
i) $\mu = \pi \boxtimes \nu$
ii) μ is a free compound Poisson with rate 1 and jump distribution ν.
iii) $m_n(\mu) = \sum_{\sigma \in NC(n)} m_\sigma(\nu)$.

Notice from relation 1.5.6 that

$$(\pi \boxtimes \nu)^2 = \pi \boxtimes \pi \boxtimes \nu^2.$$

Thus the square of compound Poisson with rate 1 is also a compound Poisson with rate 1. This has some consequences in free infinite divisibility of squares of symmetric measures which were explored in further detail in a paper with Hasebe and Sakuma which we explain in Chapter 5.

As in classical probability, free infinite divisibility is preserved under convergence in distribution.

Proposition 1.7.9. *Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of $\boxplus$-infinitely divisible probability measures on $\mathbb{R}$ and suppose that the sequence μ_n converges in distribution to a probability measure μ. Then μ is $\boxplus$-infinitely divisible.*

Freely infinitely divisible distributions are typically very different to classical ones. For instance if μ is $\boxplus$-infinitely divisible then μ has at most one atom. In particular, there is no non-trivial discrete distribution which is $\boxplus$-infinitely divisible, as opposed to the classical case where there are a lot of them. Also, there are many examples of distributions with compact support that are $\boxplus$-infinitely divisible, while in the classical case only the Dirac measures are. For this reason, one may have the intuition that their intersection is very small. One of the consequences of the results in the paper [8] is that we find new examples of probability distribution which are both classically and freely infinitely divisible. More recently in the joint work with Hasebe [7] we exhibit an infinite family of distributions with this property.

1.8 Random Matrices

Random Matrix Theory started in the 20s, with the work of Wishart [77]. Later, in the 50s random matrices gained some interest in the physics literature due to the work of Wigner [78] . Wigner's idea was to replace the Hamiltonian, an operator in a Hilbert space of infinite dimension which describes the energy levels of a system, by a symmetric random matrix of large size. This idea was the beginning of the study of large random matrices.

Two of the most celebrated results in random matrix theory of large size are, on one hand, Wigner's Semicircle Law, which states that the eigenvalue distribution of random matrices with independent entries converges in the limit as $N \to \infty$ to the semicircle distribution s of Example 1.2.7, and on the other hand, the Marchenko-Pastur Law, which says that the limit of the empirical spectral measure of Wishart matrices coincides with the free Poisson law having rate λ and jump size α. Figure 1.1 shows a simulation of this convergence.

Now, since the semicircle distribution appears from the free central limit theorem (Theorem 1.3.5) and the Marchenko-Pastur law corresponds to the free version of the law of small numbers it is natural to ask if this is just a coincidence or if there is a deeper connection. Voiculescu discovered, in 1991, that certain random matrices are *asymptotically* free. This observation established a connection between what apparently seem to be quite different fields and lead to important implications on operator algebras. We shall briefly describe this connection between Free Probability and Random Matrices.

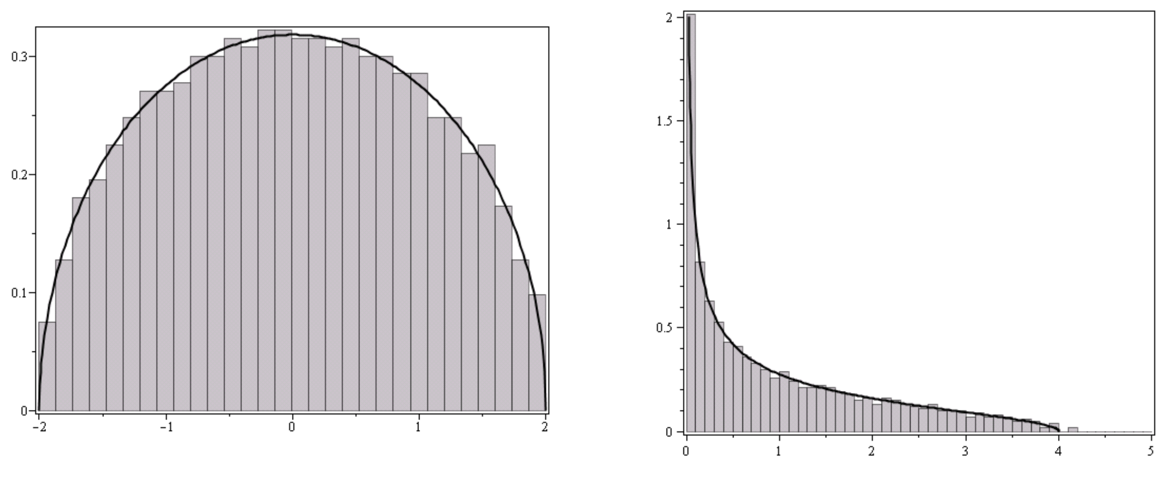


Figure 1.1: Histograms of the eigenvalues of 1 realization of a 1200×1200 sized random matrices of the form W (left) and WW^*(right) where W is a Wigner matrix with Gaussian independent entries. The black lines show the semicircle and free Poisson densities.

By "random matrices" we mean matrices whose entries are classical random variables. We want to consider families of random matrices whose size is growing. We will call a family $\{A^{(N)}\}_{N\in\mathbb{N}}$ an *ensemble* if for each $N \in \mathbb{N}$, $A^{(N)}$ is a random matrix of size $N \times N$.

Before passing to asymptotic freeness let us review the basic results mentioned above for Large Random Matrices.

An important example in the theory random matrices is the class of Wigner matrices.

Definition 1.8.1. A real Wigner matrix is an $N \times N$ random matrix $W^{(N)} = (W_{ij})_{i,j=1}^{N}$ with $W = W^*$ and such that the entries $(W_{ij})_{i,j=1}^{N}$ are independent and identically distributed with mean zero. The family $\{W^{(N)}\}_{N\in\mathbb{N}}$ is called a Wigner ensemble.

Wigner's Theorem is stated as follows.

Theorem 1.8.2 (Wigner's Semicircle Law). *Let $W_{N\in\mathbb{N}}^{(N)}$ be a Wigner ensemble whose entries have variance 1. Then W_N converges in distribution as $N \to \infty$, towards a semicircle,*

$$W^{(N)} \to s.$$

Probably the most important class of random matrices are the Gaussian random matrices whose entries consist of Gaussian classical random variables.

Definition 1.8.3. A selfadjoint Gaussian random matrix is an $N \times N$ random matrix $G^{(N)}$ with $G^{(N)} = (G^{(N)})^*$ and such that the entries $a_{ij} = G_{ij}^{(N)}$ $(i, j = 1, ..., N)$ form a Gaussian Family which is determined by the covariance

$$E[a_{ij}a_{kl}] = \frac{1}{N}\delta_{i,l}\delta_{j,k} \quad (i, j, k, l = 1, ..., N)$$

An ensemble consisting of selfadjoint Gaussian random matrices is called a GOE(Gaussian Orthogonal Ensemble).

Wigner's semicircle law is also valid for Gaussian random matrices.

Theorem 1.8.4. *Let $\{G^{(N)}\}_{N\in\mathbb{N}}$ be GOE. Then $G^{(N)}$ converges in distribution as $N \to \infty$, towards a semicircle,*

$$G^{(N)} \to s.$$

Definition 1.8.5. A Wishart matrix is an $N \times N$ random matrices of the form $M_N = XX^*$, where X is an $M \times N$ random matrix with independent entries.

The special case considered by Wishart assumes the entries to be identically distributed Gaussian random variables. The limiting distribution, now known as the Marchenko-Pastur law, was calculated by Marchenko and Pastur and coincides with the free Poisson.

Theorem 1.8.6 (Marchenko-Pastur law). *If X denotes a $M \times N$ random matrix whose entries are independent identically distributed random variables with mean 0 and variance $\sigma^2 < \infty$, let $Y_N = XX^*$ Assume that M, $N \to \infty$ so that the ratio $M/N \to \lambda \in (0, +\infty)$. Then $\mu_M \to \mu$ (in weak* topology in distribution), where*

$$\mu(A) = \begin{cases} (1 - \frac{1}{\lambda})\mathbf{1}_{0\in A} + \nu(A), & \text{if } \lambda > 1 \\ \nu(A), & \text{if } 0 \leq \lambda \leq 1, \end{cases}$$

and

$$d\nu(x) = \frac{1}{2\pi\sigma^2}\frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{\lambda x}\mathbf{1}_{[\lambda_-,\lambda_+]}\,dx$$

with

$$\lambda_\pm = \sigma^2(1 \pm \sqrt{\lambda})^2.$$

Before stating the main results of Voiculescu we shall define convergence in distribution for a family of random variables.

Definition 1.8.7 (Convergence in distribution)**.** Let $(\mathcal{A}_N, \phi_N)$ $(N \in \mathbb{N})$ and $(\mathcal{A}, \phi)$ be non-commutative probability spaces and consider families of random variables $a_1^{(N)}, ..., a_k^{(N)} \in \mathcal{A}_N$ for each $N \in \mathbb{N}$ and $a_1, ...a_k \in \mathcal{A}$. We say that a_N **converges in distribution** towards $a_1, ...a_k$, as $N \to \infty$ if we have

$$\lim_{N\to\infty} \phi_N(a_{i(1)}^{(N)} \cdots a_{i(m)}^{(N)}) = \phi(a_{i(1)} \cdots a_{i(m)})$$

and denote this by $a_1^{(N)}, ..., a_k^{(N)} \to a_1, ..., a_k$.

Definition 1.8.8 (Asymptotic Freeness)**.** Let $(\mathcal{A}_N, \phi_N)_{N\in\mathbb{N}}$ be non-commutative probability spaces and consider families of random variables $a_1^{(N)}, ..., a_k^{(N)} \in \mathcal{A}_N$ for each $N \in \mathbb{N}$. The random variables $a_1^{(N)}, ..., a_k^{(N)}$ are said to be asymptotically free if $a_1^{(N)}, ..., a_k^{(N)} \to a_1, ..., a_k$ for some free variables $a_1, ..., a_k$ in a non-commutative probability space $(\mathcal{A}, \phi)$ $a_N \to a$.

Theorem 1.8.9. *Let X_n be a selfadjoint Wigner matrix, such that the distribution of the entries is centered and has all moments, and let A_N be a random matrix which is independent from X_n. If A_N has almost surely an asymptotic eigenvalue distribution and if we have*

$$\sup_{\in\mathbb{N}} \|A_N\| < \infty.$$

Then A_N and X_N are almost surely asymptotically free.

Theorem 1.8.10. *For each $N \in \mathbb{N}$, Let A_N and B_N $N\times N$ independent random matrices, such that both A_N and B_N almost surely have an asymptotic eigenvalue distribution, as $N \to \infty$ and B_N is a unitarily invariant ensemble. Then A_N and B_N are almost surely asymptotically free.*

In particular if A_N and B_N are diagonal matrices with asymptotic eigenvalue distributions, as $N \to \infty$, and U is a Haar unitary matrix then A and UBU^* are asymptotically free. By Haar Unitary random matrices we mean the compact group $\mathcal{U}(N)$ of unitary $N \times N$ matrices with its Haar Measure. This observation has been used repeatedly in order to go from free probability to random matrices and viceversa.

Chapter 2

Combinatorics in k-divisible Non-Crossing Partitions

In this chapter we recall the partially ordered set (poset) of the k-divisible non-crossing partitions and study the combinatorial convolution of the associated incidence algebra. The poset of k-divisible non-crossing partitions was introduced by Edelman [33] and reduces to the poset of all non-crossing partitions for $k = 1$. A first systematic study of non-crossing partitions was done by Kreweras [47]. More recently, much more attention has been paid to non-crossing partitions because, among other reasons, they play a central role in the combinatorial approach of Speicher to Voiculescu's free probability as we explained in the previous chapter.

2.1 Preliminaries on non-crossing partitions

Basic properties and definitions

Definition 2.1.1. (1) We call $\pi = \{V_1, ..., V_r\}$ a **partition** of the set $[n] := \{1, 2, .., n\}$ if and only if V_i $(1 \leq i \leq r)$ are pairwise disjoint, non-void subsets of S, such that $V_1 \cup V_2 ... \cup V_r = \{1, 2, .., n\}$. We call $V_1, V_2, .., V_r$ the **blocks** of π. The number of blocks of π is denoted by $|\pi|$.

(2) A partition $\pi = \{V_1, ..., V_r\}$ is called **non-crossing** if for all $1 \leq a < b < c < d \leq n$ if $a, c \in V_i$ then there is no other subset V_j with $j \neq i$ containing b and d. We denote the set of non-crossing partitions of $[n]$ by $NC(n)$, t

Remark 2.1.2. The following characterization of non-crossing partitions is sometimes useful: for any $\pi \in NC(n)$, one can always find a block $V = \{r+1, \ldots, r+s\}$ containing consecutive numbers. If one removes this block from π, the partition $\pi \setminus V \in NC(n-s)$ remains non-crossing.

There is a graphical representation of a partition π which makes clear the property of being crossing or non-crossing, usually called the circular representation. We think of $[n]$ as labelling the vertices of a regular n-gon, clockwise. If we identify each block of π with the convex hull of its corresponding vertices, then we see that π is non-crossing precisely when its blocks are pairwise disjoint (that is, they do not cross).

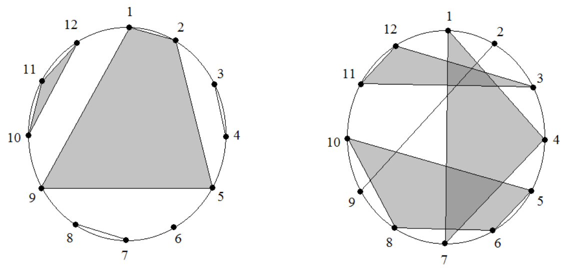


Figure 2.1: Non-Crossing and Crossing Partitions

Figure 2.1 shows the non-crossing partition $\{\{1,2,5,9\},\{3,4\},\{6\},\{7,8\},\{10,11,12\}\}$ of the set $[12]$, and the crossing partition $\{\{1,4,7\},\{2,9\},\{3,11,12\},\{5,6,8,10\}\}$ of $[12]$ in their circular representation.

The set $NC(n)$ can be equipped with the partial order $\leq$ of reverse refinement ($\pi \leq \sigma$ if and only if every block of π is completely contained in a block of σ), making it a lattice. With this order, the poset $(NC(n), \leq)$ is self-dual (see Figure 2.2).

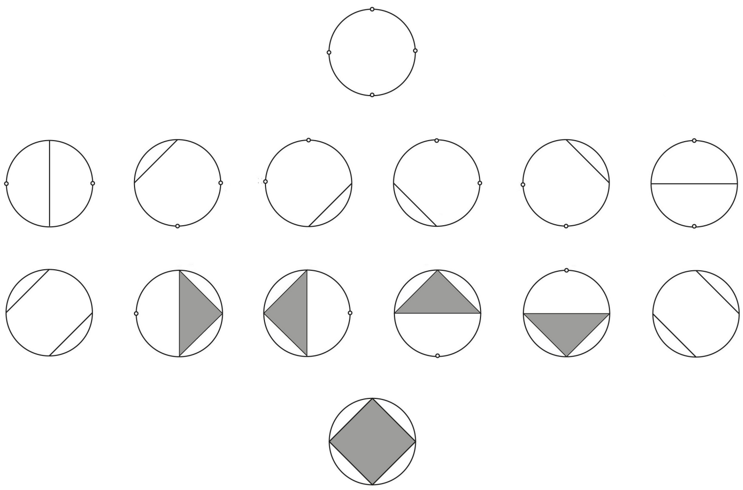


Figure 2.2: The poset $NC(4)$

Moreover, there exists a very natural order reversing isomorphism, called the Kreweras complement.

Remark 2.1.3 (Definition of Kreweras complement)**.** Let π be a partition in $NC(n)$. Then the Kreweras complement $K(\pi)$ is characterized in the following way. It is the only element $\sigma \in NC(\overline{1}, 2, ...\overline{n})$ with the properties that $\pi \cup \sigma \in NC(1, \overline{1}, 2, \overline{2}, ..., n, \overline{n}\} \backsimeq NC(2n)$ is non-crossing and that

$$\pi \cup \sigma \vee \{(1,\overline{1}),(2,\overline{2}),...,(n,\overline{n})\} = 1_{2n}.$$

The map $Kr : NC(n) \to NC(n)$ is an order reversing isomorphism. Furthermore, for all $\pi \in NC(n)$ we have that $|\pi| + |Kr(\pi)| = n + 1$, see [60] for details.

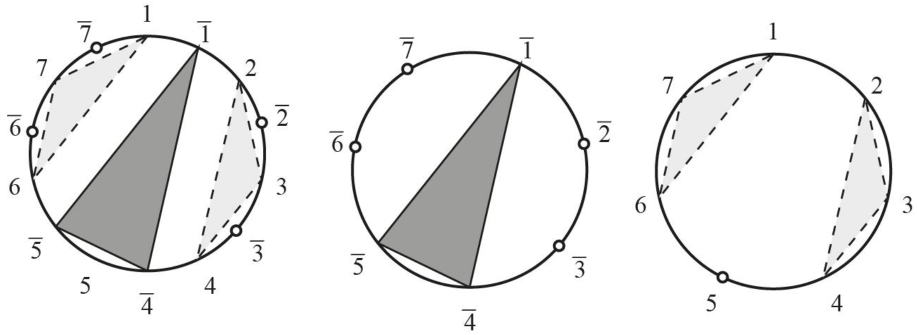


Figure 2.3: Kreweras complementation map

We recall the following result which gives a formula for the number of partitions with a given type [47].

Proposition 2.1.4. *Let $r_1, r_2, ..., r_n$ be nonnegative integers such that $r_1+2r_2...+nr_n = n$. Then the number of partitions of π in $NC(n)$ with r_1 blocks of size 1, r_2 blocks of size 2, ..., r_n blocks of size n equals*

$$\frac{n!}{p_r(n-m+1)!}, \tag{2.1.1}$$

where $p_r = r_1!r_2!\cdots r_n!$ and $r_1 + r_2... + r_n = m$.

We say that a partition π is **k-divisible** if the size of all the blocks is a multiple of k. If all the blocks are exactly of size k we say that π is **k-equal**.

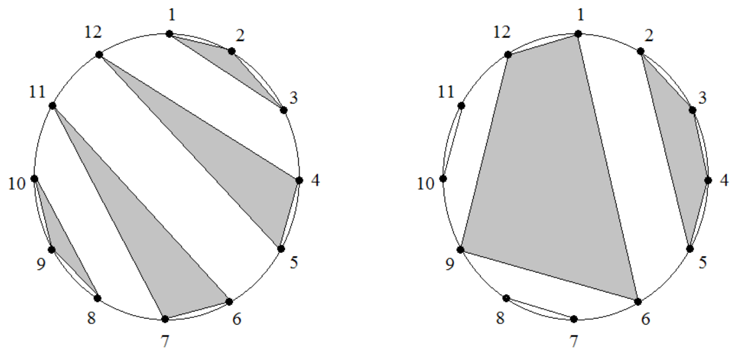


Figure 2.4: 3-equal and 2-divisible non-crossing partitions

The set of k-divisible non-crossing partitions of $[kn]$ is denoted by $NC^k(n)$ and the set of k-equal non-crossing partitions of $[kn]$ by $NC_k(n)$[1].

It is well known that the number of non-crossing partition is given by the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. More generally we can count k-divisible partitions, see [33].

[1]The notation that we follow is the one of Armstrong [11] which does not coincide with Nica and Speicher [60] for 2-equal partitions.

Proposition 2.1.5. *Let $NC^k(n)$ be the set of non-crossing partitions of $[nk]$ whose sizes of blocks are multiples of k. Then*

$$\#NC^k(n) = \frac{\binom{(k+1)n}{n}}{kn+1}.$$

On the other hand, from Proposition 2.1.4, one can easily count k-equal partitions.

Corollary 2.1.6. *Let $NC_k(n)$ be the set of non-crossing partitions of nk whose blocks are of size of k. Then*

$$\#NC_k(n) = \frac{\binom{kn}{n}}{(k-1)n+1}.$$

Definition 2.1.7. Given a partially ordered set, a k-multichain (or multichain of length $k-1$) is a sequence $x_0 \leq x_1 \leq \cdots \leq x_{k-1}$ of elements of P. We denote by $NC^{[k]}(n)$ the set of k-multichains in $NC(n)$.

The number of k-multichains in $NC(n)$ was given by Edelman in [33].

Proposition 2.1.8. *Let $NC^{[k]}(n)$ be the set of k-multichains in $NC(n)$. Then*

$$\#NC^{[k]}(n) = \frac{\binom{(k+1)n}{n}}{kn+1}.$$

The reader may have noticed from Proposition 2.1.5 and Corollary 2.1.6 that the number of $(k+1)$-equal non-crossing partitions of $n(k+1)$ and the number of k-divisible non-crossing partitions of nk coincide with the number of k-multichains on $NC(n)$. This will be of relevance for this work, and we will give a proof in Example 2.3.1 as an application on how the zeta-function in $NC^k(n)$ is related to ζ^{*k} in $NC(n)$. We derive a bijective proof of this fact and study further consequences in Section 2.6.

Similar to Proposition 2.1.4, one can count the number of partitions π, such that π and $Kr(\pi)$ have certain block structures. Let $(r_i)_{1\leq i\leq n}$, $(b_j)_{1\leq j\leq n}$ be tuples satisfying

$$1r_1 + 2r_2 + \cdots + nr_n = n = 1b_1 + 2b_2 + \cdots + nb_n, \tag{2.1.2}$$

$$|\pi| + |Kr(\pi)| = r_1 + \cdots + r_n + b_1 + \cdots + b_n = n+1. \tag{2.1.3}$$

Then the number of partitions such that π has r_i blocks of size i and $Kr(\pi)$ has b_j blocks of size j is given by the formula

$$n\frac{(|\pi|-1)!(|Kr(\pi)|-1)!}{r_1!\dots r_n! b_1!\dots b_n!}. \tag{2.1.4}$$

When π is k-equal, Equation (2.1.4), reduces to

$$k\frac{((k-1)n)!}{b_1!\cdots b_n!}.$$

As a consequence, we can show that for large k, the Kreweras complements of k-equal partitions have "typically" small blocks. More precisely, for $n, k \geq 1$ let

$$NC(k,n)_{2,1} := \{\pi \in NC_k(n) : Kr(\pi) \text{ contains only pairings and singletons}\} \subseteq NC_k(n).$$

In this case, the only possibility is that $b_1 = n(k-2)+2$, $b_2 = n-1$ and $b_i = 0$ for $i > 2$. So

$$|NC(k,n)_{2,1}| = k\frac{((k-1)n)!}{((n(k-2)+2)!(n-1)!}. \tag{2.1.5}$$

An easy application of Stirling's approximation formula shows that

$$\lim_{k\to\infty}\frac{|NC(k,n)_{2,1}|}{|NC_k(n)|} = 1. \tag{2.1.6}$$

Incidence algebra in NC

Let us recall the main concepts about posets and incidence algebras first introduced by Rota et al. [32]. The incidence algebra $I(P) = I(P,\mathbb{C})$ of a finite poset $(P,\leq)$ consists of all functions $f : P^{(2)} \to \mathbb{C}$ such that $f(\pi;\sigma) = 0$ whenever $\pi \nleq \sigma$. We can also consider functions of one variable; these are restrictions of functions of two variables as above to the case where the first argument is equal to 0, i.e. $f(\pi) = f(0,\pi)$ for $\pi \in P$.

We endow $I(P,\mathbb{C})$ with the usual structure of vector space over $\mathbb{C}$. On this incidence algebra we have a canonical multiplication or (combinatorial) convolution[2] defined by

$$(F*G)(\pi,\sigma) := \sum_{\substack{\rho\in P\\ \pi\leq\rho\leq\sigma}} F(\sigma,\rho)G(\rho,\sigma).$$

Moreover, for functions $f : P \to \mathbb{C}$ and $G : P^{(2)} \to \mathbb{C}$ we consider the convolution $f*G : P \to$ defined by

$$(f*G)(\sigma) := \sum_{\substack{\rho\in P\\ \rho\leq\sigma}} f(\rho)G(\rho,\sigma).$$

The convolutions defined above are associative and distributive with respect to taking linear combinations of functions in $P^{(2)}$ or in P. It is easy to verify that the function $\delta : P^{(2)} \to \mathbb{C}$ defined as

$$\delta(\pi,\sigma) = \begin{cases} 1 & \pi = \sigma \\ 0 & \pi \neq \sigma \end{cases}$$

is the unity with respect to the convolutions, making $I(P,\mathbb{C})$ a unital algebra. Two other prominent functions in the in incidence algebra $I(P,\mathbb{C})$ are the zeta-function and its inverse the Möbius function.

Definition 2.1.9. Let $(P,\leq)$ be a finite partially ordered set. The **zeta function** of P, $\zeta : P^{(2)} \to \mathbb{C}$ is defined by

$$\zeta(\pi,\sigma) = 1, \qquad \text{for all } \pi \leq \sigma \in P.$$

The inverse of ζ under the convolution is called the **Möbius function** of P, which will be denoted by μ.

[2]Not to be confused with the concept of convolution of measures.

Remark 2.1.10. Note that

$$\zeta * \zeta(\pi,\sigma) = \sum_{\pi \leq \rho \leq \sigma} 1 = card[\pi,\sigma],$$

and, more generally,

$$\underbrace{\zeta * \zeta * \cdots * \zeta}_{k \text{ times}}(\pi,\sigma) = \sum_{\pi=\rho_0 \leq \rho_1 \cdots \leq \rho_k = \sigma} 1$$

counts the number of $(k+1)$-multichains from π to σ.

Definition 2.1.11. Let $(\alpha_n)_{n\geq 1}$ be a sequence of complex numbers. Define a family of functions $f_n : NC(n) \to \mathbb{C}$, $n \geq 1$, by the following formula: if $\pi = \{V_1, ..., V_r\} \in NC(n)$ then

$$f_n(\pi) = \alpha_{|V_1|} \cdots \alpha_{|V_r|}.$$

Then (f_n) is called the **multiplicative family of functions** on NC determined by $(\alpha_n)_{n\geq 1}$.

To emphasize the fact that the α_n encode the information of the multiplicative family of functions f_n we will use the following notation.

Notation 2.1.12. Let $(\alpha_n)_{n\geq 1}$ be a sequence of complex numbers, and let (f_n) be the multiplicative family of functions on NC determined by $(\alpha_n)_{n\geq 1}$. Then we will use the notation

$$\alpha_\pi := f_n(\pi) \quad \text{for } \pi \in NC(n),$$

and we will call the family of numbers $(\alpha_\pi)_{n\in\mathbb{N}, \pi\in NC(n)}$ the multiplicative extension of $(\alpha_n)_{n\in\mathbb{N}}$.

Finally, for $g := (g_n)_{n\geq 1}$ and $f := (f_n)_{n\geq 1}$ multiplicative families in the lattice of non-crossing partitions we can define the combinatorial convolution $f * g := ((f * g)_n)_{n\geq 1}$ in NC by the following formula:

$$(f * g)_n := \sum_{\pi \in NC(n)} f_n(\pi) g_n(K(\pi)).$$

The importance of this combinatorial convolution is that the multiplicative family $((f * g)_n)_{n\geq 1}$ can be used to describe free multiplicative convolution, in the following sense, see Equation (1.6.2):

$$\kappa_n(ab) = \sum_{\pi \in NC(n)} k_\pi(a) k_{K(\pi)}(b).$$

Moreover, the so-called moment-cumulant formula (see Equation (1.6.1)) may be stated as follows:

$$m_n(x) = \sum_{\pi \in NC(n)} \kappa_\pi(a) \tag{2.1.7}$$

which in our notation (if $m := m_n(x)$ and $\kappa : \kappa_n(x)$) is nothing other than $m = k * \zeta$ or $k = m * \mu$. There is a functional equation for the power series two multiplicative families $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ on NC, related by

$$g = f * \mu \text{ (or equivalently: } f = g * \zeta). \tag{2.1.8}$$

This is the content of next proposition.

Proposition 2.1.13 (Speicher [72]). *Let $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ be two multiplicative families on NC, which are related as in Equation (2.1.8). Let $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ be the sequences of numbers that determine the multiplicative families; that is, we denote $\alpha_n := f_n(\mathbf{1}_n)$ and $\beta_n := g_n(\mathbf{1}_n)$, $n \geq 1$. If we consider the power series*

$$A(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n \quad \text{and} \quad B(z) = 1 + \sum_{n=1}^{\infty} \beta_n z^n.$$

Then A and B satisfy the functional equation

$$A(z) = B(zA(z)) \quad \text{and} \quad B(z) = A(\frac{z}{B(z)})$$

2.2 The poset $NC^k(n)$

In this section we study the poset $NC^{(k)}(n)$ of k-divisible non-crossing partitions and the combinatorial convolution associated with this poset.

Recall that a partition $\pi \in NC(nk)$ is called k-divisible if the size of each block in π is divisible by k. As we have done for non-crossing partitions, we can regard the set $NC^{(k)}(n)$ as a subposet of $NC(nk)$.

Definition 2.2.1. We denote by $(NC^k(n), \leq)$ the induced subposet of $NC(kn)$ consisting of partitions in which each block has cardinality divisible by k.

This poset was introduced by Edelman [33], who calculated many of its enumerative invariants. Observe that coarsening of partitions preserves the property of k-divisibility, hence the set of k-divisible non-crossing partitions form a join-semilattice. However $NC^k(n)$ is not a lattice for $k > 1$ since, in general, some elements $\pi, \sigma \in NC^{(k)}(n)$ do not have a meet in $NC^k(n)$ (for instance, two different elements of the type $\lambda = (k, k, ..., k)$).

Since $NC^{(k)}(n)$ is a finite poset we can define the incidence algebra $I(NC^{(k)}(n), \mathbb{C})$. Recall that for a poset P and functions $f : P \to \mathbb{C}$ and $G : P^{(2)} \to \mathbb{C}$ the convolution $f * G : P \to \mathbb{C}$ is defined as

$$(f * G)(\sigma) := \sum_{\substack{\rho \in P \\ \rho \leq \sigma}} f(\rho) G(\rho, \sigma).$$

In particular, when $P = NC^k(n)$ and G is the zeta function ζ (in $NC^k(n)$) we have that

$$f * \zeta(\sigma) = \sum_{\substack{\pi \in NC^{(k)}(n) \\ \pi \leq \sigma}} f(\pi).$$

We will be interested in the case when $f(\pi)$ is part of a multiplicative family on NC^k. So let us define a multiplicative family on NC^k in analogy to the case of NC.

Definition 2.2.2. Let $(\alpha_n)_{n\geq 1}$ be a sequence of complex numbers. Define a family of functions $f_n^{[k]} : NC^k(n) \to \mathbb{C}$, $n \geq 1$, by the following formula: if $\pi = \{V_1, ..., V_r\} \in NC^{(k)}(n)$ then

$$f_n^{[k]}(\pi) = \alpha_{|V_1|/k} \cdots \alpha_{|V_r|/k}.$$

Then $(f_n^{[k]})_{n\geq 1}$ is called the **multiplicative family of functions** on $NC^{(k)}(n)$ determined by $(\alpha_n)_{n\geq 1}$.

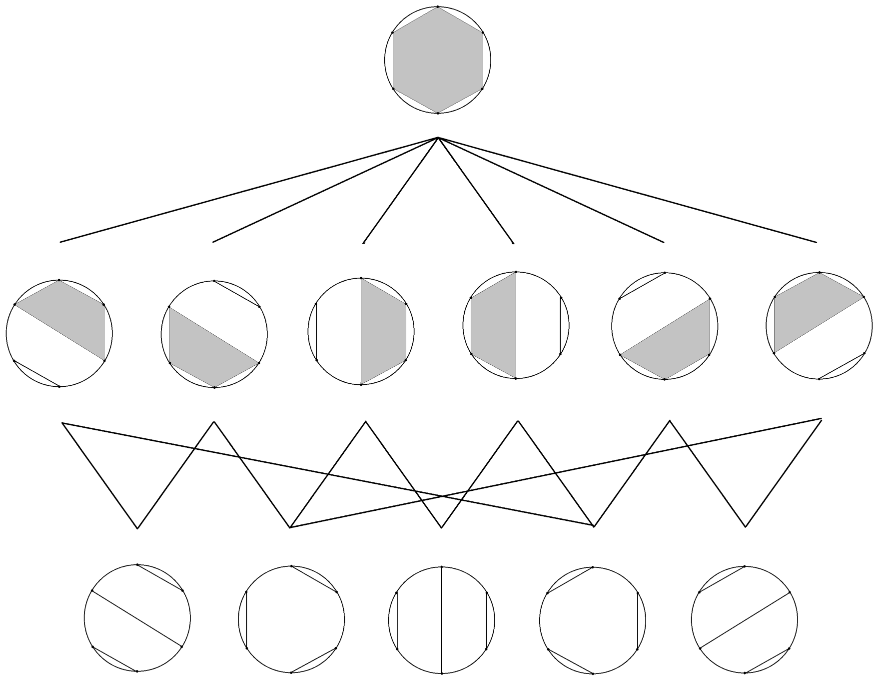


Figure 2.5: The Hasse Diagram of the poset $NC^2(3)$

Observe, on one hand, that if $\pi = \{V_1, ..., V_r\}$ is a k-divisible partition then the value

$$f_{nk}(\pi) = \alpha_{|V_1|} \cdots \alpha_{|V_r|}$$

only depends on the α_i's such that k divides i and thus the values of α_i for i not divisible by k can be chosen arbitrarily. In particular, we can choose them to be 0.

On the other hand, if $(f_n)_{n\geq 1}$ is the multiplicative family on $NC(n)$ determined by a sequence $(\alpha_n)_{n\geq 1}$ such that $\alpha_i = 0$ when i is not divisible by k then for $\pi \notin NC^k(n)$ we have that $\alpha_{|V_1|} \cdots \alpha_{|V_r|} = 0$ and thus, in $I(NC(kn), \mathbb{C})$, we have

$$(f * \zeta)_{nk}(\sigma) = \sum_{\substack{\pi \in NC(kn) \\ \pi \leq \sigma}} f(\pi) = \sum_{\substack{\pi \in NC^k(n) \\ \pi \leq \sigma}} f_{nk}(\pi).$$

and

$$(f^{[k]} * \zeta)_{nk}(\sigma) = \sum_{\substack{\pi \in NC^k(n) \\ \pi \leq \sigma}} f_n^{[k]}(\pi) = \sum_{\substack{\pi \in NC^k(n) \\ \pi \leq \sigma}} f_{nk}(\pi).$$

So, for multiplicative families on NC determined by sequences such that $\alpha_i = 0$ whenever i is not divisible by k, the convolution with the zeta function ζ is exactly the same in $I(NC(kn), \mathbb{C})$ as in $I(NC^{(k)}(n), \mathbb{C})$.

Let us fix some notation to encode the information in sequences of this type.

Notation 2.2.3. We call a sequence $\alpha_n^{(k)}$ the k-dilation of α_n if $\alpha_{kn}^{(k)} = \alpha_n$ and $\alpha_n^{(k)} = 0$ if n is not a multiple of k.

By the arguments given above we can deal with the convolution between $(f_n^{[k]})_{n\geq 1}$ (a multiplicative family on $NC^{(k)}(n)$) and $\zeta \in I(NC^{(k)}(n), \mathbb{C})$ by just considering the k-dilations of the original sequence and work with the usual convolution in $NC(n)$. In particular, we can use the functional equation in Proposition 2.1.13 to get a functional equation for multiplicative families in $NC^{(k)}(n)$.

Proposition 2.2.4. *Let $g_n^{[k]}$ be a multiplicative family in $NC^{(k)}(n)$ determined by the sequence $(\beta_n)_{n\geq 1}$ and $f_n^{[k]}$ be a multiplicative family in $NC^{(k)}(n)$ determined by the sequence $(\alpha_n)_{n\geq 1}$. Suppose that $f^{[k]} = g^{[k]} * \zeta$. If we consider the power series*

$$A(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n \quad \textit{and} \quad B(z) = 1 + \sum_{n=1}^{\infty} \beta_n z^n.$$

then

$$A(z) = B(zA(z)^k).$$

Proof. Since $f^{[k]} = g^{[k]} * \zeta$ is equivalent to $f^{(k)} = g^{(k)} * \zeta$ then, by Proposition 2.1.13, the power series $A_k(z) = 1 + \sum_{n=1}^{\infty} \alpha_n^{(k)} z^n$ and $B_k(z) = 1 + \sum_{n=1}^{\infty} \beta_n^{(k)} z^n$ are related by the functional equation

$$A_k(z) = B_k(zA_k(z)).$$

Note that $A_k(z) = A(z^k)$ and $B_k(z) = B(z^k)$, hence

$$A(z^k) = B(z^k A(z^k)^k).$$

Making the change of variable $z^k = y$ we get

$$A(y) = B(yA(y)^k).$$

as desired. □

2.3 Motivating example

Consider the following three objects.

(i) $NC_{k+1}(n)$: Non-crossing partitions in $NC((k+1)n)$ with each block of size $k+1$.

(ii) $NC^k(n)$: Non-crossing partitions in $NC(kn)$ with blocks of size a multiple of k.

(iii) $NC^{[k]}(n)$: Multichains of order $k+1$ in $NC(n)$.

It is well known that the Fuss-Catalan numbers count all three of them. Different ways to count them are now known. The first ones were counted by Kreweras [47]. Also bijections between them have been given in [2] and [33] . Moreover in [11] an order has been given to (ii) makings the objects in (ii) and (iii) isomorphic as posets and generalized to other Coxeter groups.

We want to show we can use Proposition 2.2.4 to derive the same functional equation for the three of them without counting them explicitly.

Example 2.3.1. Denote the cardinality of $NC_k(n)$ by the number Z_n^k. Let $(\beta_n = 0)_{n\geq 2}$, $\beta_1 = 1$ and $(\alpha_n)_{n\geq 1}$ be two sequences with respective multiplicative families $(g_n^{[k]})_{n>0}$ and $(f_n^{[k]})_{n>0}$ on NC^k related by the formula $f^{[k]} = g^{[k]} * \zeta$. Then α_n equals Z_n^k and $A(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n$ satisfies

$$A(z) = 1 + zA(z)^k.$$

Indeed,

$$\alpha_n = f_n^{(k)}(1_{nk}) = g_n^{(k)} * \zeta(1_{nk}) = \sum_{\substack{\pi\in NC(nk)\\ \pi\leq 1_{nk}}} g_n^{(k)}(\pi) = \sum_{\substack{\pi\in NC_k(n)\\ \pi\leq 1_{nk}}} 1 = Z_n^{(k)}.$$

Then, by Proposition 2.2.4 the power series $A(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n$ and $B(z) = 1 + \sum_{n=1}^{\infty} \beta_n z^n$ are related by

$$A(z) = B(zA(z)^k).$$

The power series for the sequence $(\beta_n)_{n\geq 1}$ is $B(z) = 1 + z$ and then

$$A(z) = 1 + zA(z)^k.$$

Example 2.3.2. Denote the cardinality of $NC^k(n)$ by the number $C_n^{(k)}$. Let $(\beta_n = 1)_{n\geq 1}$ and $(\alpha_n)_{n>1}$ be two sequences with respective multiplicative families on NC^k related by the formula $f^{[k]} = g^{[k]} * \zeta$. Then $f_n^{(k)}$ equals $C_n^{(k)}$ and

$$A(z) = 1 + zA(z)^{k+1}.$$

Indeed,

$$\alpha_n = f_n^{(k)}(1_{nk}) = g_n^{(k)} * \zeta(1_{nk}) = \sum_{\substack{\pi\in NC^k(n)\\ \pi\leq 1_{nk}}} g_n^k(\pi) = \sum_{\substack{\pi\in NC^{(k)}(n)\\ \pi\leq 1_{nk}}} 1 = C_n^{(k)}.$$

Again, by Proposition 2.2.4 the power series $A(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n$ and $B(z) = 1 + \sum_{n=1}^{\infty} \beta_n z^n$ are related by

$$A(z) = B(zA(z)^k).$$

The power series for the sequence $(\beta_n = 1)_{n\geq 1}$ is

$$B(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

and then

$$A(z) = \frac{1}{1 - zA(z)^k}$$

or equivalently

$$A(z) = 1 + zA(z)^{k+1}.$$

Finally, for k-multichains we have the following.

Example 2.3.3. Let c_n^k $(n, k \geq 1)$ denote the number of k-multichains in $NC(n)$. For every $k \geq 1$ let $(f_{n,k})_{n\geq 1}$ be the multiplicative family of functions on NC determined by the sequence c_n^k. As we have noticed in Remark 2.1.10, for every poset P, the number of $(k+2)$-multichains from $\pi \in P$ to $\sigma \in P$ is given by

$$\underbrace{(\zeta * \zeta * \cdots * \zeta)}_{k+1 \text{ times}}(\pi, \sigma). \quad \text{for all } k \geq 1$$

In particular, for $NC(n)$, if we plug $\pi = \mathbf{0}_n$ and $\pi = \mathbf{1}_n$ we get the that the number of $(k+1)$-multichains is given by

$$\underbrace{(\zeta_n * \zeta_n * \cdots * \zeta_n)}_{k+1 \text{ times}}(\mathbf{0}_n, \mathbf{1}_n) \quad \text{for all } n, k \geq 1.$$

In other words

$$f_{n,k} = \underbrace{\zeta_n * \zeta_n * \cdots * \zeta_n}_{k+1 \text{ times}} \quad \text{for all } n, k \geq 1,$$

or equivalently

$$f_{n,k+1} = f_{n,k} * \zeta_n \quad \text{for all } n, k \geq 1.$$

Now, consider for each $k \geq 1$, the power series

$$A_k(z) := 1 + \sum_{n=1}^{\infty} c_n^k z^n.$$

From the Proposition 2.1.13, the power series $A_k(z)$ and $A_{k+1}(z)$ must satisfy the functional equation

$$A_{k+1}(z) = A_k(zA_{k+1}(z)).$$

It is easy to see that power series of c_n^2 (the Catalan numbers) satisfy the relation

$$A_1(z) = 1 + zA_1(z)^2.$$

By induction we see that A_k satisfies the functional equation

$$A_k(z) = 1 + zA_k(z)^{k+1}.$$

Indeed, since $A_k(y) = 1 + yA_k(y)^{k+1}$ plugging $y = zA_{k+1}$ we get

$$\begin{aligned} A_{k+1}(z) &= A_k(zA_{k+1}(z)) = 1 + zA_{k+1}(z)A_k(zA_{k+1}))^{k+1} \\ &= 1 + zA_{k+1}(z)(A_{k+1}(z))^{k+1} \\ &= 1 + zA_{k+1}(z)^{k+2}. \end{aligned}$$

We have seen that all of the three objects satisfy the same functional equation and then must be counted by the same sequence. So the multichains of length $k+1$ in $NC(n)$ are in bijection with the k-divisible non-crossing partitions in $NC(nk)$ and with the $(k+1)$-equal partitions in $NC(n(k+1))$. This result is known and was already in [33] but we emphasize that our derivation never used the explicit calculation of the cardinality of these objects but rather relies on deriving a functional equation. These ideas will be used later in this thesis.

Remark 2.3.4. This bijection goes further. In fact, one can give an explicit order to k-multichains so that $NC^k(n) \backsimeq NC^{(k)}(n)$ as ordered sets. We will not give details about this but rather refer the reader to Chapters 3 and 4 of [11]. The point here is that we may think of both object as the same.

2.4 The convolution of k-dilated sequences in NC

Since the convolution with ζ in $NC^{(k)}(n)$ is equivalent to convolution with ζ in $NC(n)$ for sequences dilated by k we can forget about the former and focus on how convolution with k-dilated sequences behave in $NC(n)$. From now on, we will prefer to use the notation $\alpha_\pi = \alpha_{|V_1|}\cdots\alpha_{|V_r|}$ instead of $f(\pi)$ since there is no confusion.

The first result gives a relation between the formal power series of the k-dilation of the sequence $(m_n)_{n\geq 1}$ and the $(k+1)$-dilation of the sequence $(\beta_n)_{n\geq 1}$, when the two sequences related $m_n = \beta_n * \zeta$, namely,

$$m_n = \sum_{\pi\in NC(n)} \beta_\pi$$

Proposition 2.4.1. *Let k be a positive integer and let*

$$\begin{aligned} A(z) &= 1 + \sum \alpha_n z^n \\ B(z) &= 1 + \sum \beta_n z^n \\ M(z) &= 1 + \sum m_n z^n. \end{aligned}$$

Then any two of the following three statements imply the third

(i) $M(z) = B(zM(z))$.

(ii) $M(z) = A(zM(z)^k)$.

(iii) $B(z) = A(zB(z)^{k-1})$.

Equivalently, each two of the following three statement imply the third.

(i) *The sequences m_n and β_n are related by the formula*

$$m_n = \sum_{\pi\in NC(n)} \beta_\pi.$$

(ii) *The sequences α_n and m_n are related by the formula*

$$m_n^{(k)} = \sum_{\pi\in NC(kn)} \alpha_\pi^{(k)}.$$

(iii) *The sequences α_n and β_n are related by the formula*

$$\beta_n^{(k-1)} = \sum_{\pi\in NC((k-1)n)} \alpha_\pi^{(k-1)}.$$

Proof. (i) & (ii) $\Rightarrow$ (iii). Evaluating in B in $zM(z)$ we get

$$B(zM(z)) = M(z) = A(zM(z)^k) = A(zM(z)M(z)^{k-1}) = A(zM(z)B(zM(z))^{k-1}),$$

making the change of variable $y = zM(z)$ the result holds.

(i) & (iii) ⇒ (ii). The relation (i) is equivalent to $B(z) = M(z/B(z))$ so

$$M(z/B(z)) = B(z) = A(zB(z)^k) = A(\frac{z}{B(z)}B(z)^{s+1}) = A(\frac{z}{B(z)}M(z/B(z))^k),$$

making the change of variable $y = z/B(z)$ we get the result.

The last equality follows along the same lines. The equivalence of the next three statements in terms of sums on non-crossing partitions follows from Proposition 2.2.4.. □

We can use the last result recursively to get a formula for the k-fold convolution with the zeta function ζ.

Corollary 2.4.2. *Let $M(z), A(z), B_i(z)$ formal power series and such that*

(i) $M(z) = A(zM(z)^k)$

(ii) $M(z) = B_1(z(M(z))$

(iii) $B_i(z) = B_{i+1}(zB_i(z))$, *for* $i = 1, ..., k-1$.

Then $B_i(z) = A(zB_i(z)^{k-i})$, *in particular* $B_n(z) = A(z)$.

Proof. We will use induction on i.

For $i = 1$, we use i) and ii) and Proposition 2.4.1 to get

$$B_1(z) = A(zB_1(z)^{k-1}).$$

Now suppose that the statement is true for $i = n$. Then

$$B_n(z) = A(zB_n(z)^{k-n})$$

and by iii) $B_n(z) = B_{n+1}(zB_n(z))$, so again by Proposition 2.4.1 we get

$$B_{n+1}(z) = A(zB_{n+1}(z)^{k-n-1}).$$

□

The last proposition may look rather artificial. But it explains how the successive convolution with the zeta-function in $NC(n)$ is equivalent to the convolution with the zeta-function in $NC^{(k)}(n)$, as we state more precisely in the following theorem.

Theorem 2.4.3. *The following statements are equivalent.*

(1) *The multiplicative family $f := (f_n)_{n>0}$ is the result of applying the zeta-function k times to $g := (g_n)_{n>0}$, that is*

$$f = g * \underbrace{\zeta * \cdots * \zeta}_{k \ times}.$$

(2) *The multiplicative family $f^{(k)} := (f_n^{(k)})_{n>0}$ is the result of applying the zeta-function once to $g^{(k)} := (g_n^{(k)})_{n>0}$, that is*

$$f^{(k)} = g^{(k)} * \zeta.$$

Proof. This is just a reformulation of Corollary 2.4.2 in terms of combinatorial convolution. □

Example 2.4.4. To make the previous theorem more clear let us calculate the first elements of a sequence $g * \zeta * \zeta$, first directly and then by applying the last theorem for $k = 2$. So, let $h = g * \zeta$, and $f = g * \zeta * \zeta = h * \zeta$. That is

$$h_n = \sum_{\pi \in NC(n)} g_\pi \text{ and } f_n = \sum_{\pi \in NC(n)} b_\pi.$$

explicitly

$$\begin{aligned} h_1 &= g_1 \\ h_2 &= g_2 + g_1^2 \\ h_3 &= g_3 + 3g_1g_2 + g_1^3 \\ h_4 &= g_4 + 4g_3g_1 + 2g_2^2 + 6g_2g_1^2 + g_1^4 \end{aligned}$$

and

$$\begin{aligned} f_1 &= h_1 \\ f_2 &= h_2 + h_1^2 \\ f_3 &= h_3 + 3h_1h_2 + h_1^3 \\ f_4 &= h_4 + 4h_3h_1 + 2h_2^2 + 6h_2b_1^2 + h_1^4 \end{aligned}$$

combining this equations we get

$$\begin{aligned} f_1 &= g_1 && (2.4.1) \\ f_2 &= g_2 + 2g_1^2 && (2.4.2) \\ f_3 &= g_3 + 6g_2g_1 + 5g_2^3. && (2.4.3) \\ f_4 &= g_4 + 8g_3g_1 + 4g_2^2 + 28g_1^2g_2 + 14g_1^4 \end{aligned}$$

On the other hand let $\alpha_n = a_n^{(2)}$ and let $\gamma_n = \alpha_n * \zeta_n$. That is, $\gamma_1 = \gamma_3 = \gamma_5 = \gamma_7 = 0$. While

$$\gamma_{2n} = \sum_{\pi \in NC(2n)} \alpha_\pi = \sum_{\pi \in NC_2(n)} \alpha_\pi.$$

which explicitly is written, (for γ_6 it is instructive to look at Fig 2.1 , but the coefficients are easily calculated from Theorem 2.1.4)

$$\begin{aligned} \gamma_2 &= \alpha_2 \\ \gamma_4 &= \alpha_4 + 2\alpha_2^2 \\ \gamma_6 &= \alpha_6 + 6\alpha_4\alpha_2 + 6\alpha_2^3 \\ \gamma_8 &= \alpha_8 + 8\alpha_6\alpha_1 + 4\alpha_2^2 + 28\alpha_1^2\alpha_2 + 14\alpha_1^4 \end{aligned}$$

verifying that indeed $\gamma_n = f_n^{(2)}$, since this agrees with equality (2.4.3).

Remark 2.4.5. This phenomenon is very specific for the non-crossing partitions. For instance, it does not occur if we change $NC(n)$ by $P(n)$ the lattice of all partition nor $IN(n)$ the lattice of interval partitions.

To finish this section let us see how Theorem 2.4.3 may be applied to our motivating example to give a shorter proof of the fact that $(k+1)$-multichains, k-divisible non-crossing partitions and $(k+1)$-equal non-crossing partitions have the same cardinality.

Example 2.4.6. Let a_n be the sequence determined by $a_1 = 1$ and $a_n = 0$ for $n > 1$ (notice that this is just the sequence associated with the delta-function δ). Next, consider

$$c = a * \underbrace{\zeta * \cdots * \zeta}_{k+1 \ times}.$$

By Remark 2.1.10, c_n counts the number of $(k+1)$-multichains of $NC(n)$. Now, by Theorem 2.4.3 applied to a_n

$$c_n = c^{(k+1)}_{(k+1)n} = \sum_{\pi \in NC((k+1)n)} a^{(k+1)}_{\pi} = \sum_{\pi \in NC_{k+1}(n)} 1 = \#NC_{k+1}(n)$$

and we get the number of $(k+1)$-equal noncrossing partitions. Finally, for k-divisible partitions, consider $b = a * \zeta$. Then

$$c = b * \underbrace{\zeta * \cdots * \zeta}_{k \ times}$$

and

$$b_n = \sum_{\pi \in NC(n)} a_{\pi} = 1.$$

So, again by Theorem 2.4.3, applied to b_n,

$$c_n = c^{(k)}_{kn} = \sum_{\pi \in NC(nk)} b^{(k)}_{\pi} = \sum_{\pi \in NC^k(n)} 1 = \#NC^k(n).$$

Thus we have proved that c_n counts k-divisible non-crossing partitions of $[kn]$, $(k+1)$-equal non crossing partitions of $[(k+1)n]$ and $(k+1)$-multichains on $NC(n)$. We will give a bijective proof in the next section.

We can push more this example to also recover Theorem 3.6.9 of Armstrong [11] for the case of classical k-divisible non-crossing partitions. The proof is left to the reader.

Corollary 2.4.7. *The number of l-multichains of k-divisible noncrossing partitions equals the number of lk multichains of $NC(n)$ and is given by the Fuss-Catalan number $C_{kl,n}$.*

It would be very interesting to see if similar arguments can be used to count invariants for non-crossing partitions in the different Coxeter groups. To the knowledge of the author this is not known.

2.5 Statistics of blocks in k-divisible non-crossing partitions

In this section we present part of the paper [3], concerning statistics of the block structure of non-crossing partitions. As can be seen in [3] and [10], k-divisible non-crossing partitions play an important role in the calculation of the free cumulants and moments of products of k free random variables. Moreover, in the approach given in [10] for studying asymptotic behavior of the size of the support when $k \to \infty$ understanding the asymptotic behavior of the sizes of blocks was a crucial step.

In this direction, a recent paper by Ortmann [61] studies the asymptotic behavior of the sizes of the blocks of a uniformly chosen random partition. This lead him to a formula for the right-edge of the support of a measure in terms of the free cumulants, when these are positive. He noticed a very simple picture of this statistic as $n \to \infty$. Roughly speaking, in average, out of the $\frac{n+1}{2}$ blocks of this random partition, half of them are singletons, one fourth of the blocks are pairings, one eighth of the blocks have size 3, and so on.

Trying to get a better understanding of this asymptotic behavior, the question of the exact calculation of this statistic arose. In this section, we answer this question and refine these results by considering the number of blocks given. Moreover, we generalize to k-divisible partitions, as follows.

Theorem 2.5.1. *The sum of the number of blocks of size tk over all the k-divisible non-crossing partitions of $\{1, 2, .., kn\}$ is given by*

$$\binom{n(k+1)-t-1}{nk-1}. \tag{2.5.1}$$

In particular, asymptotically, we have a similar phenomena as for the case $k = 1$; about a $\frac{k}{k+1}$ portion of all the blocks have size k, then a $\frac{k}{(k+1)^2}$ portion have size $2k$, then $\frac{k}{(k+1)^3}$ are of size $3k$, etc.

Another consequence is that the expected number of blocks of a k-divisible non-crossing partition is given by $\frac{kn+1}{k+1}$. An equivalent formulation of this result was also observed by Armstrong [11, Theorem 3.9] for any Coxeter group. It is then a natural question if this simple formula can be derived in a bijective way. We end with a bijective proof of this fact, for type A and B k-divisible non-crossing partitions.

Let us finally mention that there exists a type B free probability. Free probability of type B was introduced by Biane, Goodman and Nica [27] and was later developed by Belinschi and Shlyakhtenko [16], Nica and Février [34] and Popa [63].

Some combinatorial lemmas

The following two summation lemmas will enable us to use Proposition 2.1.4 to get the number of blocks of size t subject to the restriction of having a fixed number m of blocks.

Lemma 2.5.2. *The following identity holds*

$$\sum_{\substack{r_1+r_2+\cdots r_n=m \\ r_1+2r_2+\cdots(n)r_n=n}} \frac{m!}{r_1!\cdots r_n!} = \binom{n-1}{m-1}. \tag{2.5.2}$$

Proof. This is proved easily by counting in two ways the number of paths from $(0, 0)$ to $(n-1, m-1)$ using the steps $(a, b) \to (a, b+1)$ or $(a, b) \to (a+1, b)$ by observing that

$$\frac{(m+1)!}{r_1!\cdots r_n!} = \binom{m+1}{r_1}\binom{m+1-r_1}{r_2}\cdots\binom{m+1-(r_1+\cdots r_{n-1})}{r_n}.$$

□

Lemma 2.5.3. *The following identity holds*

$$\sum_{\substack{r_1+r_2+\cdots r_n=m\\ r_1+2r_2+\cdots nr_n=n}} \frac{(m-1)!r_t}{r_1!\cdots r_n!} = \binom{n-t-1}{m-2}. \tag{2.5.3}$$

Proof. We make the change of variable $\tilde{r}_t = r_t - 1$ and $\tilde{r}_i = r_i$ for $i \neq t$. Then

$$\begin{aligned}\sum_{\substack{r_1+r_2+\cdots r_n=m\\ r_1+2r_2+\cdots nr_n=n}} \frac{(m-1)!r_t}{r_1!\cdots r_n!} &= \sum_{\substack{\tilde{r}_1+\tilde{r}_2+\cdots \tilde{r}_n=m-1\\ \tilde{r}_1+2\tilde{r}_2+\cdots(n-t)\tilde{r}_{n-t}=n-t}} \frac{(m-1)!}{\tilde{r}_1!\cdots \tilde{r}_n!}\\ &= \binom{n-t-1}{m-2},\end{aligned}$$

where we used the Lemma 2.5.2 in the last equality.

□

We remind the so-called Chu-Vandermonde's identity which will enable us to remove the restriction of having a number of blocks given.

$$\sum_{m=0}^{s} \binom{y}{m}\binom{x}{s-m} = \binom{x+y}{s}. \tag{2.5.4}$$

Number of blocks in k-divisible non-crossing partitions.

First we calculate the expected number of blocks of a given size t, subject to the restriction of having m blocks from which the main result will follow.

Proposition 2.5.4. *The sum of the number of blocks of size tk of all non-crossing partitions in $NC^k(n)$ with m blocks*

$$\binom{nk}{m-1}\binom{n-t-1}{m-2}. \tag{2.5.5}$$

Proof. First we treat the case $k=1$. In order to count the number of blocks of size t of a given partition π with r_1 blocks with size 1, r_2 blocks of size 2, … , r_n blocks of size n, we need to multiply by r_t. So we want to calculate the following sum

$$\begin{aligned}\sum_{\substack{r_1+r_2+\cdots r_n=m\\ r_1+2r_2+\cdots nr_n=n}} \frac{n!r_t}{(n+1-m)!p_r} &= \binom{n}{m-1} \sum_{\substack{r_1+r_2+\cdots r_n=m\\ r_1+2r_2+\cdots nr_n=n}} \frac{(m-1)!r_t}{p_r}\\ &= \binom{n}{m-1}\binom{n-t-1}{m-2}.\end{aligned}$$

We used Lemma 2.5.3 in the last equality. This solves the case $k=1$.

For the general case we follow the same strategy. In this case we need $(r_1,\ldots,r_n)$ such that $r_i=0$ if k does not divide i. So the condition $r_1+r_2+\cdots+r_n=m$ is really $r_k + r_{2k} + \cdots + r_{nk} = m$ and the condition $r_1 + 2r_2 + \cdots + nr_n = nk$ is really $kr_k + 2kr_{2k} + \cdots + (nk)r_{nk} = nk$, or equivalently $r_k + 2r_{2k} + \cdots + nr_{nk} = n$. Making the change of variable $r_{ik} = s_i$ we get.

$$\sum_{\substack{r_k+r_{2k}+\cdots+r_{nk}=m \\ kr_k+2kr_{2k}+\cdots+(nk)r_{nk}=nk}} \frac{(nk)!r_{tk}}{(nk+1-m)!r_k!r_{2k}!\ldots r_{nk}!} = \sum_{\substack{s_1+s_2+\cdots+s_n=m \\ s_1+2s_2+\cdots+ns_n=n}} \frac{(nk)!s_t}{(nk+1-m)!\prod_{i=0}^{n} s_i!}. \tag{2.5.6}$$

Now, the last sum can be treated exactly as for the case $k = 1$, yielding the result. This reduction to the case $k = 1$ will be obviated for types B and D. □

Now we can prove Theorem 2.5.1, which we state again for the convenience of the reader.

Theorem 2.5.5. *The sum of the number of blocks of size tk over all the k-divisible non-crossing partitions of $\{1, ..., kn\}$ is given by*

$$\binom{n(k+1)-t-1}{nk-1}. \tag{2.5.7}$$

Proof. We use Proposition 2.5.4 and sum over all possible number of blocks. Letting $\tilde{m} = m - 1$, we get

$$\sum_{m=1}^{nk} \binom{nk}{m-1}\binom{n-t-1}{m-2} = \sum_{\tilde{m}=0}^{nk-1} \binom{nk}{nk-\tilde{m}}\binom{n-t-1}{\tilde{m}-1}.$$

Now, using the Chu-Vandermonde's identity for $s = nk - 1$, $x = n(k+1) - t - 1$ and $y = nk - 1$ we obtain the result. □

Corollary 2.5.6. *The expected number of blocks of size tk of a non-crossing partition chosen uniformly at random in $NC^k(n)$ is given by*

$$\frac{(nk+1)\binom{n(k+1)-t-1}{nk-1}}{\binom{(k+1)n}{n}}. \tag{2.5.8}$$

Moreover, similar to the case $k = 1$, asymptotically the picture is very simple, about a $\frac{k}{k+1}$ portion of all the blocks have size k, then $\frac{k}{k+1}$ of the remaining blocks are of size $2k$, and so on. This is easily seen using (2.5.8).

Corollary 2.5.7. *When $n \to \infty$ the expected number of blocks of size tk of a non-crossing partition chosen uniformly at random in $NC^k(n)$ is asymptotically $\frac{nk}{(k+1)^{t+1}}$.*

The following is a direct consequence of Theorem 1.

Corollary 2.5.8. *The sum of the number of blocks of all the k-divisible non-crossing partitions in $NC^k(n)$ is*

$$\binom{n(k+1)-1}{nk}.$$

Proof. Summing over t, in (2.5.7), we easily get the result. □

Finally, from Corollary 2.5.8 one can calculate the expected number of block of k-divisible non-crossing partition.

Corollary 2.5.9. *The expected number of blocks of a k-divisible partition of $[kn]$ chosen uniformly at random is given by* $\frac{kn+1}{k+1}$.

Remark 2.5.10. 1) Corollary 2.5.9 was proved by Armstrong [11] for any Coxeter group.

2) When $k = 1$ there is a nice proof of Corollary 2.5.9. Recall from Remark 2.1.3 that the Kreweras complement $Kr : NC(n) \to NC(n)$ is a bijection such that $|Kr(\pi)| + |\pi| = n+1$. Then summing over all $\pi \in NC(n)$ and dividing by 2 we obtain that the expected value is just $\frac{n+1}{2}$. This suggests that there should be a bijective proof of Corollary 2.5.9. This is done in Section 2.6.

2.6 The bijection

In this section we give a bijective proof of the fact that $NC^k(n) = NC_{k+1}(n)$. From this bijection we derive Corollary 2.5.9.

Lemma 2.6.1. *For each n and each k let $f : NC_{k+1}(n) \to NC^k(n)$ be the map induced by the identification of the pairs $\{k+1, k+2\}, \{2(k+1), 2(k+1)+1\}, \dots, \{n(k+1), 1\}$. Then f is a bijection.*

Proof. First, we see that the image of this map is in $NC^k(n)$. So, let π be a $(k+1)$-equal partition.

(i) Every block has one element on each congruence *mod* $k+1$. Indeed, because of the characterization of non-crossing partitions on Remark 2.1.2, there is at least one interval, which has of course this property. Removing this interval does not affect the congruence in the elements of other blocks. So by induction on n every block has one element of each congruence *mod* $k+1$.

(ii) Note that for each two elements identified we reduce 1 point. So suppose that m blocks (of size k) are identified in this bijection to form a big block V. Then the number of vertices in this big block equals $m(k+1) - \#(\text{ identified vertices})/2$. Now, by (i), there are exactly two elements in each block to be identified with another element, that is $2m$. So

$$\begin{aligned} |V| &= m(k+1) - \#(\text{identified vertices})/2 \\ &= m(k+1) - (2m)/2 = mk. \end{aligned}$$

this proves that $f(\pi) \in NC^k(n)$.

Now, it is not hard to see that by splitting the points $1, k+2, ..., nk+1$ of $\pi \in NC^k(n)$ we get a unique inverse $f^{-1}(\pi) \in NC_{k+1}(n)$. More specifically, f^{-1} is defined as follows. Let $\pi = \{V_i, ...V_t\}$ be a k-divisible partition and $V_i = \{e_{1,1}, ..., e_{1,k}, e_{2,1}, ...e_{s_i,1}, ..., e_{s_i,k}\}$, a block with $k(s_i)$ elements arranged clockwise and $e_{j,l} = ka_j + l$ for some $a_j \in \mathbb{N}$ (which is possible by a similar argument as (i) above). Then $f^{-1}(\pi)$ has, for each V_i, s_i blocks $V_i^1, V_i^2, ...V_i^{s_i}$ with $V_i^j = \{\hat{e}_{j,1}, ..., \hat{e}_{j,k}, \hat{e}_{j,k+1}\}$ consisting of the elements $\hat{e}_{j,l} = (k+1)a_j+l$ for $l = 1, ..., k$ and $\hat{e}_{j,k+1}$ comes from the splitting of $e_{j+1,k}$, that is, $\hat{e}_{j,k+1} = (k+1)a_j+l-1$. □

Now we can prove Corollary 2.5.9.

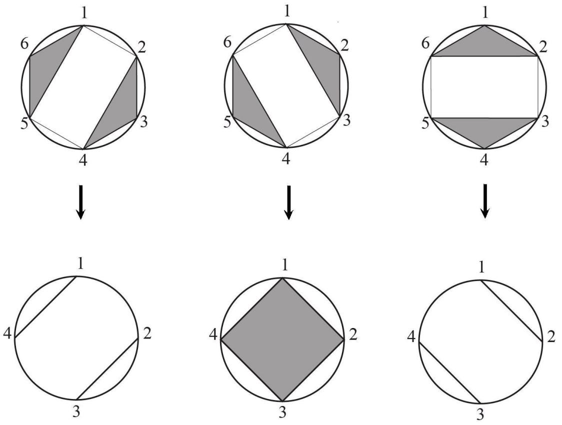


Figure 2.6: Bijection f between 3-equal and 2-divisible non-crossing partitions

Proof of Corollary 2.5.9. For each n and each k and each $0 < i \leq k+1$ let $f_i : NC_{k+1}(n) \to NC^k(n)$ the map induced by the identification of the pairs $\{k+1+i, k+1+i+1\}, \ldots \{2(k+1)+i, 2(k+1)+i+1\}, \ldots \{n(k+1)+i, n(k+1)+i+1\}$ (we consider elements *mod nk*). Then by the proof of the previous lemma, each f_i is a bijection. So, let π be a fixed $(k+1)$-equal partition. Considering all the bijections f_i on this fixed partition, we see that every point j is identified twice (one with f_{j-1} and one with f_j). Note that each block obtained by the identification corresponds to a block in the Kreweras complement. So, for each partition π in $NC_{k+1}(n)$, the collection $(f_i(\pi))_{i=1}^k$ consists of $k+1$ partitions in $NC^k(n)$ whose number of blocks add $kn+1$. $\square$

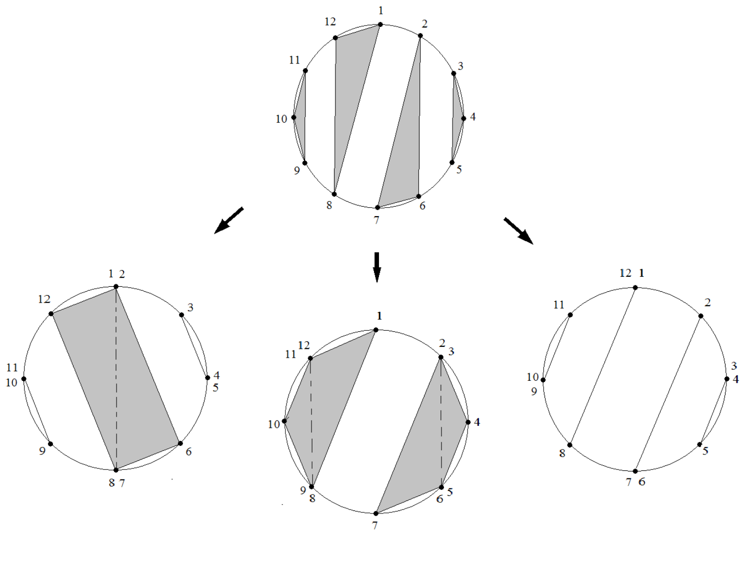


Figure 2.7: Bijections f_1,f_2 and f_3 applied to $\pi = \{\{1, 8, 9\}, \{2, 6, 7\}, \{3, 4, 5\}, \{9, 10, 11\}\}$

Remark 2.6.2. Note that for a k-divisible partition on $[2kn]$ points, the property of being centrally symmetric is preserved under the bijections f_i (see e. g. Fig.6), and then the arguments given here also work for the partitions of type B. We expect that a similar argument works for type D.

In the following example we want to illustrate how the bijection given by Lemma 2.6.1 allows us to count k-divisible partitions with some restrictions by counting the preimage under f.

Example 2.6.3. Let $NC^k_{1\to 2}(n)$ be the set of k-divisible non-crossing partitions of $[kn]$ such that 1 and 2 are in the same block. It is clear that $\pi \in NC^k_{1\to 2}(n)$ if and only if $f^{-1}(\pi)$ satisfies that 1 and 2 are in the same block.

Now, counting the $(k+1)$-equal non-crossing partitions of $[(k+1)n]$ such that 1 and 2 are in the same blocks is the same as counting non-crossing partitions of $[(k+1)n-1]$ with $n-1$ blocks of size $k+1$ and 1 block of size k containing the element 1, since 1 and 2 can be identified. From Proposition 2.1.4, the size of this set is easily seen to be

$$\frac{k}{(k+1)n-1}\binom{(k+1)n-1}{n-1} = \frac{k}{n-1}\binom{(k+1)n-2}{n-2}$$

where the first factor of the LHS is the probability that the block of size k contains the element 1.

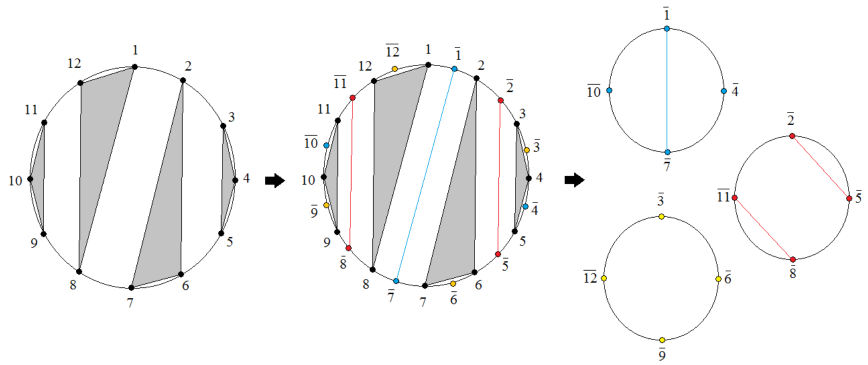


Figure 2.8: A 3-equal and its Kreweras complement divided mod 3.

Let us finally mention that the bijections f_i are closely related to the Kreweras complement of a $(k+1)$-equal non-crossing partitions, which was considered in [10]. Indeed $Kr(\pi)$ can be divided in a canonical way into $k+1$ partitions of [n], $\pi_1, ..., \pi_{k+1}$, such that $|\pi_i| = |f_i(\pi)|$. Fig. 6 shows the bijections f_1, f_2 and f_3 for $k = 3$, $n = 4$ and $\pi = \{\{1,8,9\},\{2,6,7\},\{3,4,5\},\{9,10,11\}\}$, while Fig. 7 shows the same partition as Fig. 6 with its Kreweras complement divided into the partitions π_1, π_2 and π_3.

Chapter 3

k-divisible elements

We introduce the concept of k-divisible elements and study some of the combinatorial aspects of their cumulants. The main result in this section describes the cumulants of the k-th power of a k-divisible element.

3.1 Basic properties and definitions

Let $(\mathcal{A}, \phi)$ be a non-commutative probability space.

Notation 3.1.1. 1) An element $x \in \mathcal{A}$ is called k-divisible if the only non vanishing moments of x are multiples of k. That is

$$\phi(x^n) = 0 \text{ if } k \nmid n$$

2) Let $x \in \mathcal{A}$ be k-divisible and let $\alpha_n := \kappa_{kn}(x, ..., x)$. We call $(\alpha_n)_{n \geq 1}$ the k-determining sequence of x.

It is clear that $x \in \mathcal{A}$ is k-divisible if and only if its non-vanishing free cumulants are multiples of k.

Example 3.1.2. 1) As a first example of a k-divisible element we consider the k-Haar unitary. An element $u \in \mathcal{A}$ is said to be a k-Haar Unitary if it is a unitary, if $u^p = 1$, and if

$$\phi(u^k) = 0 \quad \text{unless } k|n.$$

k-Haar unitaries appear naturally in the framework of Example 1.2.2 since they correspond to elements of order k. Also superdiagonal matrices of the form

$$A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & & & 0 & 1 \\ 1 & 0 & & \cdots & 0 \end{pmatrix}$$

are k-Haar unitaries. We will come back to this realization of k-Haar unitaries later. It is easily checked that a k-Haar unitary has $*$-distribution

$$\mu = \sum_{i=1}^{k} \delta_{w_i}$$

where $w_i = w^i$ and w is a k-th primitive root of unity.

2) Consider $X_1, X_2, ..., X_k$ free positive random variables identically distributed and for w a k-th primitive root of unity, denote $w_i = w^i$ $i = 1, ...k$. Then

$$X = \sum_{i=1}^{k} w_i X_i$$

is a k-divisible element. Indeed let suppose the n does not divide k. Then

$$\begin{aligned} k_n(X, ...X) &= k_n(\sum_{i=1}^{k} w_i X_i, ..., \sum_{i=1}^{k} w_i X_i) = \sum_{i=1}^{k} \kappa_n(w_i X_i, ..., w_i X_i) && (3.1.1) \\ &= \sum_{i=1}^{k} w_i^n \kappa_n(X_i, ..., X_i) = \kappa_n(X_1, ..., X_1) \sum_{i=1}^{k} w^{in} = 0 && (3.1.2) \end{aligned}$$

Note that in the case when X_i are Poisson distributed with parameter $1/k$ the non-vanishing cumulants of X are 1, thus we may think of X as a free compound Poisson with jump distribution uniformly distributed in the roots of unity. Similar argument applies if we take classical random variables. We will come back to this example in Chapter 6.

The following is a generalization of Theorem 11.25 in Nica and Speicher [60] where, for an even element x, the free cumulants of x^2 are given in terms of the moments of x.

Theorem 3.1.3 (Free cumulants of x^k, First formula)**.** *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and let x be a k-divisible element with k-determining sequence $(\alpha_n)_{n\geq 1}$. Then the following formula holds for the cumulants of x^k.*

$$\kappa_n(x^k, x^k, ..., x^k) = \sum_{\pi \in NC((k-1)n)} \alpha_\pi^{(k-1)}. \qquad (3.1.3)$$

First proof. Set $\alpha_n = \kappa_{kn}(x)$, $\beta_n = \kappa_n(x^k, ..., x^k)$, $m_n = m_n(x^k) = m_{kn}(x)$ and let

$$\begin{aligned} A(z) &= 1 + \sum \alpha_n z^n \\ B(z) &= 1 + \sum \beta_n z^n \\ M(z) &= 1 + \sum m_n z^n \end{aligned}$$

The moment-cumulant formula for x^k gives

$$M(z) = B_1(z(M(z))$$

and the moment-cumulant formula for x says

$$M(z) = A(zM(z)^k)$$

so by Proposition 2.4.1 we get

$$B(z) = A(zB(z)^{k-1})$$

or equivalently,

$$\kappa_n(x^k, x^k, ..., x^k) = \sum_{\pi \in NC((k-1)n)} \alpha_\pi^{(k-1)}.$$

□

Second proof. This proof is more involved but gives a better insight into the combinatorics of k-divisible elements and works for the more general setting of diagonally balanced k-tuples (defined later). The argument is very similar as in the proof in [60] for $k = 2$. The formula for products as arguments Eq. (1.6.5) yields

$$\kappa_n(x^k, x^k, ..., x^k) = \sum_{\substack{\pi \in NC(kn) \\ \pi \vee \sigma = 1_{kn}}} \kappa_\pi(x, x, ..., x, x)$$

with $\sigma = \{(1, 2, 3, ..., k), (k + 1, k + 2, ..., 2k), ..., (kn - n + 1, ..., kn)\}$.

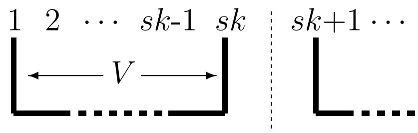
Observe that since x is k-divisible, we have

$$\sum_{\substack{\pi \in NC(kn) \\ \pi \vee \sigma = 1_{kn}}} \kappa_\pi(x, x, ..., x, x) = \sum_{\substack{\pi \in NC(kn),\ \pi \text{ k-divisible} \\ \pi \vee \sigma = 1_{kn}}} \kappa_\pi[x, x, ..., x, x].$$

The basic observation is the following

$$\begin{aligned} \{\pi &\in NC(kn) \mid \pi\ k\text{-divisible}, \pi \vee \sigma = 1_{kn}\} = \\ \{\pi &\in NC(kn) \mid \pi\ k\text{-divisible},\ 1 \sim_\pi kn, sk \sim_\pi sk + 1\ \forall s = 1, ..., n - 1\} \end{aligned}$$

Let V be the block of π which contains the element 1. Since π is k-divisible, in order that the size of all the blocks of π to be a multiple of k the last element of V must be sk for some $s \in \{1, ...n\}$. But if $k \neq n$ then sk would not be connected to $sk + 1$ in π and neither in σ.



This of course means that $\pi \vee \sigma \neq 1_{kn}$. Therefore $1 \sim_\pi kn$. Relabelling the elements in $\{1, \ldots, kn\}$ by a rotation of k does not affect the properties of π being k-divisible or $\pi \vee \sigma = 1_{kn}$, so the same argument implies that $sk \sim_\pi sk + 1$, $\forall k = 1, ..., n - 1$.

Now, the set $\{\pi \in NC(kn) \mid \pi\ k\text{-divisible},\ 1 \sim_\pi kn, sk \sim_\pi sk + 1\ \forall s = 1, ..., n - 1\}$ is in canonical bijection with $\{\widetilde{\pi} \in NC((k - 1)n) \mid \widetilde{\pi} \text{ is } (k - 1)\text{-divisible }\}$ induced by the identification $sk \equiv sk + 1$, for $s = 1, ..., n - 1$, and $1 \equiv kn$.

And since

$$\begin{aligned} \alpha_{kn}^{(k)} &= \kappa_{kn}^x \\ \alpha_n^{(k)} &= 0 \text{ if } k \nmid n \end{aligned}$$

Then $\kappa_\pi(x, x, ..., x, x) \to \alpha_{\widetilde{\pi}}^{(k-1)}$. So

$$\kappa_n(x^k, x^k, ..., x^k) = \sum_{\substack{\widetilde{\pi} \in NC(kn) \\ \widetilde{\pi}\ k\text{-divisible}}} \alpha_{\widetilde{\pi}}^{(k-1)} = \sum_{\pi \in NC((k-1)n)} \alpha_\pi^{(k-1)}$$

as desired. □

Proposition 3.1.4 (Free cumulants of x^k, Second formula). *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and let x be a k-divisible element with k-determining sequence $(\alpha_n)_{n\geq 1}$. Then the following formula holds for the cumulants of x^k.*

$$\kappa_n(x^k, x^k, ..., x^k) = \sum_{\pi \in NC(n)} \beta_\pi$$

where

$$\beta_k = \sum_{\pi \in NC((k-1)n)} \alpha_\pi. \tag{3.1.4}$$

Proof. This follows from Proposition 2.4.1 and Theorem 3.1.3. □

The last theorem gives a moment-cumulant formula between β_n and $\kappa_n(x^k..., x^k)$ which says, for example, that when β_n is a cumulant sequence then x^{k+1} is a free compound Poisson and thus $\boxplus$-infinitely divisible. This will be explained in detail in Section 5.

Proposition 3.1.5 (Free cumulants of x^k, Third formula). *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and let x be a k-divisible element with k-determining sequence $(\alpha_n)_{n\geq 1}$. Then the following formula holds for the cumulants of x^k.*

$$\kappa_n(x^k, x^k, ..., x^k) = [\alpha * \underbrace{\zeta * \cdots * \zeta}_{k \text{ times}}]_n. \tag{3.1.5}$$

Proof. This follows from Corollary 2.4.2 and Proposition 3.1.4. □

3.2 Freeness and k-divisible elements

Recall the definition of diagonally balanced pairs from Nica and Speicher [57].

Definition 3.2.1. Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, and let a_1, a_2 be in $\mathcal{A}$. We will say that (a_1, a_2) is a diagonally balanced pair if

$$\phi(\underbrace{a_1a_2 \cdots a_1a_2a_1}_{2n+1}) = \phi(\underbrace{a_2a_1...a_2a_1a_2}_{2n+1}) = 0. \tag{3.2.1}$$

Two prominent examples of diagonally balanced pairs are (u, u^*) where u is a Haar unitary and (s, s) where s is even. It is well known in free probability that if a is free from $\{u, u^*\}$ then uau^* is free from a, and similarly if a is free from s then it is also free from sas.

More generally, it was proved in [57] that if (b_1, b_2) is a diagonally balanced pair and a is free from $\{b_1, b_2\}$ then b_1ab_2 is free from a. Now, notice that if s is k-divisible then the pair (s^i, s^{k-i}) is diagonally balanced and then $sas^{k-1}, s^2as^{k-2}, ..., s^{k-1}as$ and a are free. Instead of s^ias^{k-i}, we can consider any monomial on a and s of degree k on s and freeness will still hold. Furthermore, we see that if a and s are free and s is k-divisible then shs and a are free, where p is any polynomial in a and s of degree k on s. This is the content of the next proposition for monomials. The general case follows trivially from it.

Proposition 3.2.2. *Let s be k-divisible and a be free of s. And let $h = sa_1sa_2sa_3s...sa_{k-1}s$, where for all $i = 1, ..., n$ the element a_i is free from s. Then h and a are free.*

Proof. Consider a mixed cumulant of h and a and use the formula for cumulants with products as arguments.

$$\kappa_n(..., h, ..., a, ...) = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \sigma = 1_n}} \kappa_n(..., \underbrace{s, a_1, s, a_2, s, ...s, a_{k-1}, s}_{h}, ..., a, ...). \tag{3.2.2}$$

Let us analyze the summands of the RHS and show that they must vanish. In order to satisfy the minimum condition a must be joined with some element on h. Now, for this $h = sa_1s, a_2s...sa_{k-1}s$, a can not be joined with some s, since they are free. So it must join with some a_i as follows.

$$\kappa_n(..., \overbrace{s, a_1, s, ...s, a_i, \underbrace{s, ...s, a_{k-1}, s, ..., s}_{km-i \text{ number of } s}}^{h}, a, ...)$$

In this case there must be a block of size not a multiple of k containing only s´s (since s is free from $\{a, a_1, ...a_n\}$) and then $\kappa_n(..., s, a_1, s, a_2, s, ...s, a_{k-1}, s, ..., a, ...)$ must vanish for all summands in RHS. So any mixed cumulant of h and a vanishes and hence a and h are free. □

3.3 Diagonally balanced k-tuples

We may generalize the concept of diagonally balanced pairs to k-tuples.

Definition 3.3.1. Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, and let $a_1, \dots, a_k$ be in $\mathcal{A}$. We will say that $(a_1, \dots, a_k)$ is a *diagonally balanced k-tuple* if every ordered sequence of size not a multiple of k vanishes with ϕ, i.e.

$$\phi(a_j a_{j+1} \cdots a_k a_1 \cdots a_k a_1 \cdots a_{i-1} a_i) = 0 \tag{3.3.1}$$

whenever $a_{j-1} \neq a_i$ (the indices are taken modulo k).

The proof of Proposition 3.2.2 can be easily modified for diagonally balanced k-tuples, and is left to the reader. We have a more general result.

Theorem 3.3.2. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, and let $(s_1, \dots, s_k)$ be a diagonally balanced k-tuple free from a. And let $h = s_1 a_2 s_2 a_3 s_3 \cdots s_{k-1} a_{k-1} s_k$, where for all $i = 1, ..., n$ the element a_i is free from $\{s_1, \dots, s_k\}$. Then h and a are free.*

A special kind of diagonally balanced pair which is very important in the free probability literature is the one of R-diagonal pair, introduced in [57]. There is a lot of structure in these elements and their relation to even elements is well known [60]. Moreover a big class of invariant subspaces have been studied by Speicher and Sniady [71] and their relation to R-cyclic matrices was pointed out in [55].

Definition 3.3.3. Let $(\mathcal{A}, \phi)$ be a non-commutative probability space, and let $a_1, \cdots, a_k$ be in $\mathcal{A}$. We will say that $(a_1, \cdots, a_k)$ is an R-diagonal k-tuple if the only non-vanishing free cumulants have increasing order, i.e. they are of the form

$$\kappa_{kn}(a_1, a_2, ..., a_k, a_1, a_2, ..., a_k, ..., a_1, a_2, ..., a_k) = \kappa_{kn}(a_i, a_{i+1}, ..., a_k, a_1, ..., a_k, a_1, ..., a_{k-i+1}).$$

Remark 3.3.4. The case $k = 2$ was well studied in [57]. An element a is R-diagonal if and only if the pair (a, a^*) is R-diagonal.

Theorem 3.3.5 (cumulants of R-diagonal tuples)**.** *Let $(a_1, \dots, a_s)$ be an R-diagonal k-tuple in a tracial state and denote by*

$$\alpha_n := \kappa_{kn}(a_1, \dots, a_k, \dots, a_1 \dots a_k). \tag{3.3.2}$$

Then, if $a = a_1 a_2 \cdots a_k$, we have

$$\kappa_n(a, \dots, a) = \sum_{\pi \in NC(n)} \alpha_\pi^{(k-1)}. \tag{3.3.3}$$

Proof. Again, the formula for products as arguments yields

$$\kappa_n(a, a, \dots, a) = \sum_{\substack{\pi \in NC(kn) \\ \pi \vee \sigma = 1_{kn}}} \kappa_\pi(a_1, a_2, ..., a_{k-1}, a_k)$$

with $\sigma = \{(1, 2, 3, ..., k), (k+1, k+2, ..., 2k), ..., (k(n-1)+1, ..., nk)\}$.

Observe that by the fact that $(a_1, \dots, a_k)$ is an R-diagonal k-tuple

$$\sum_{\substack{\pi \in NC(kn) \\ \pi \vee \sigma = 1_{kn}}} \kappa_\pi(a_1, a_2, ..., a_{k-1}, a_k) = \sum_{\substack{\pi \in NC(kn),\ \pi\ k\text{-divisible} \\ \pi \vee \sigma = 1_{kn}}} \kappa_\pi(a_1, a_2, ..., a_{k-1}, a_k).$$

From this point, the argument is identical as in the second proof of Theorem 3.1.3. □

Similar formulas as in Theorems 3.1.4 and 3.1.5 hold for R-diagonal tuples.

Proposition 3.3.6. *Let $(a_1, \dots, a_k)$ be an R-diagonal k-tuple in a tracial state and denote by*

$$\alpha_n := \kappa_{kn}(a_1, \dots, a_k, \dots, a_1, \dots, a_k). \tag{3.3.4}$$

The following formulas hold for the cumulants of $a = a_1 a_2 \cdots a_k$

$$\kappa_n(a, \dots, a) = [\alpha * \underbrace{\zeta * \cdots * \zeta}_{k\ times}]_n$$

and

$$\kappa_n(a, \dots, a) = \sum_{\pi \in NC(n)} \beta_\pi$$

where

$$\beta_k = \sum_{\pi \in NC((k-1)n)} \alpha_\pi^{(k-1)}.$$

Remark 3.3.7. (1) Theorem 3.3.5 and Proposition 3.3.6 are also true for diagonally balanced k-tuples. One can easily modify the proofs by using Remark 2.1.2.

(2) Notice that the determining sequence of a diagonally balanced k-tuple is determined by the moments of $a = a_1 a_2 \cdots a_k$ but the same is not true for the whole distribution of $(a_1, a_2, \dots, a_k)$.

3.4 R-cyclic matrices and R-diagonal tuples

Let $(\mathcal{A},\phi)$ be a non-commutative probability space, and let d be a positive integer. Consider the algebra $M_d(\mathcal{A})$ of $d\times d$ matrices over $\mathcal{A}$ and the linear functional ϕ_d on $M_d(\mathcal{A})$ defined by the formula

$$\phi((a_{i,j})_{i,j=1}^n)=\frac{1}{d}\sum_{i=1}^d \phi(a_{ii}). \tag{3.4.1}$$

As explained in Example 1.2.2 the pair $(M_d(\mathcal{A}),\phi_d)$ is itself a non-commutative probability space.

Definition 3.4.1. Let $(\mathcal{A},\phi)$ and let $A\in(M_d(\mathcal{A}),\phi_d)$. A is said to be R-cyclic if the following conditions holds

$$\kappa_n(a_{i_1,j_1},\dots,a_{i_n,j_n})=0 \tag{3.4.2}$$

for every $n>0$ and every $1\le i_1,j_1,...\le d$ for which it is not true that $j_1=i_2,\dots,j_{n-1}=i_n, j_n=i_1$.

We can realize k-divisible elements as R-cyclic matrices with R-diagonal k-tuples as entries. A formula for the distribution of an R-cyclic matrix in terms of its entries was given in [55]. However, in the case treated here, this formula will not be needed in full generality and we will rather use the special information we know to obtain the desired distribution.

Proposition 3.4.2. *Let $(a_1,a_2,...a_k)$ be a tracial diagonally balanced k-tuple in $(\mathcal{A},\phi)$ and consider the superdiagonal matrix*

$$A:=\begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ \vdots & \ddots & a_2 & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & & & 0 & a_{k-1} \\ a_k & 0 & & \cdots & 0 \end{pmatrix}$$

as an element in $(M_k(\mathcal{A}),\phi_k)$.

(1) A is k-divisible.

(2) A^k has the same moments as $a:=a_1\cdots a_k$. In particular, if a is positive A has moments as a k-symmetric distribution.

(3) A has the same determining sequence as $(a_1,a_2,...a_k)$.

(4) A is R-cyclic if and only if $(a_1,\dots,a_d)$ is an R-diagonal tuple.

Proof. (1) A is k-divisible since the powers of A which are not a multiple of k have zero entries on the diagonal.

(2) This is clear since

$$A^k:=\begin{pmatrix} a_1\cdots a_k & & & 0 \\ & a_2\cdots a_k a_1 & & \\ & & \ddots & \\ 0 & & & a_k\cdots a_{k-1} \end{pmatrix}$$

which by traciality has moments $\phi((a_1...a_k)^n) = \phi(a^n)$.

(3) By Theorems 3.1.3 and 3.3.5, the determining sequence of A depends on the moments of A^k in the same way as $(a_1, a_2, ..., a_k)$ so by (2) the determining sequences must coincide.

(4) The definition of R-cyclicity says that $\kappa_n(a_{i_1}, a_{1_2}, \ldots a_{i_n}) = 0$ whenever is not true that $i_2 = i_1 + 1$, $i_3 = i_2 + 1$,...,$i_1 = i_n + 1$. This is equivalent to the fact that n is a multiple of k and the indices are increasing, which is exactly the definition of R-diagonal tuples. □

Example 3.4.3 (Free k-Haar unitaries)**.** The simplest example of the last theorem is given by taking $a_i = 1$.

$$A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & & & 0 & 1 \\ 1 & 0 & & \cdots & 0 \end{pmatrix}$$

Clearly, this matrix is a k-Haar unitary, with distribution $\mu_A = \frac{1}{k}\sum_{j=1}^{k} \delta_{q^j}$ as an element in $(M_k(\mathcal{A}), \phi_k)$. Notice that, instead of the upperdiagonal matrix, we can choose any permutation matrix of size $nk \times nk$ in which any cycle has length k. Of course, if we choose one of them at random, we still get a k-Haar unitary. Moreover, Neagu [54] proved that if we let $N \to \infty$ we get asymptotic freeness in the following sense.

Theorem 3.4.4. *Let $\{U_1^N, U_2^N, ..., U_r^N\}_{N>0}$ be a family of $Nk \times Nk$ independent random permutation matrices with cycle lengths of size k. Then as N goes to infinity, $\{U_1^N, U_2^N, ..., U_r^N\}$ converges in $*$-distribution to a $*$-free family $u_1, u_2, ..., u_r$ of random variables with each u_i k-Haar unitary.*

This gives a matrix model for $u_1, ..., u_r$ free k-Haar unitaries. Moreover, Neagu showed asymptotic freeness with Gaussian Ensembles.

Theorem 3.4.5. *Let $\{U_1^N, U_2^N, ..., U_r^N\}_{N>0}$ be a family of $Nk \times Nk$ independent random permutation matrices with cycle lengths of size k and $\{G_1^N, G_2^N, ..., G_l^N\}_{N>0}$ be a family of Gaussian Matrices. Then as N goes to infinity, $\{U_1^N, U_2^N, ..., U_r^N\}$ and $\{G_1^N, G_2^N, ..., G_l^N\}_{N>0}$ are asymptotically free.*

3.5 Random Matrices

In this section we want to show how our results combined with the asymptotic freeness results of Neagu can be used in the theory of Random Matrices. One important realization of freeness by random matrices is done using the so-called Haar unitary random matrices. Recall from Section 1.8 that if A_N and B_N are two sequences of constant matrices, each of which has a limit distribution with respect to the normalized trace, and if U_N is a Haar Unitary $N \times N$ random matrix, then A_N and $U_N B_N U_N^*$ are asymptotically free as $N \to \infty$. This observation has been used repeatedly in order to go from Free Probability to Random Matrices and vice versa.

Now, in practice, it is not so easy to generate Haar unitary random matrices; the most frequent method relies on the polar decomposition of Gaussian Matrices. We want to show that in certain cases, simpler methods may be used. The main observation is that we can replace a Haar unitary by a permutation matrix with cycles of size k (corresponding to k-Haar unitaries), which are fairly easy to generate.

Selfadjoint random variables

First, we consider some typical examples of selfadjoint random variables.

Example 3.5.1. Consider two free Bernoulli random variables b_1 and b_2 with distribution:

$$\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1.$$

The free additive convolution $\mu \boxplus \mu$ may be easily shown to be an arcsine law with density given by

$$\frac{1}{\pi\sqrt{4-t^2}} \quad |t| < 2$$

By considerations of the previous section, one may realize this as $b_1 = v_2$ and $b_2 = u_k v_2 u_k{}^*$, where v_2 is a 2-Haar unitary (i.e a Bernoulli) and u_k is a k-Haar unitary free from b (conjugating with unitaries does not change the distribution). This means that the distribution of $b_1 + b_2 = v_2 + u_k v_2 u_k^*$ is $\mu \boxplus \mu$.

Moreover, Theorem 3.4.4 gives us a random matrix approximation for this convolution by considering the ensemble $U_k^{(N)} V_2^{(N)} U_k^{(N)} + V_2^{(N)}$ where $U_k^{(N)}$ and $V_2^{(N)}$ are independent random $Nk \times Nk$ permutation matrices with cycle lengths of size k and 2, respectively. Figure 3.1 shows the accuracy of this approximation, for different values of k.

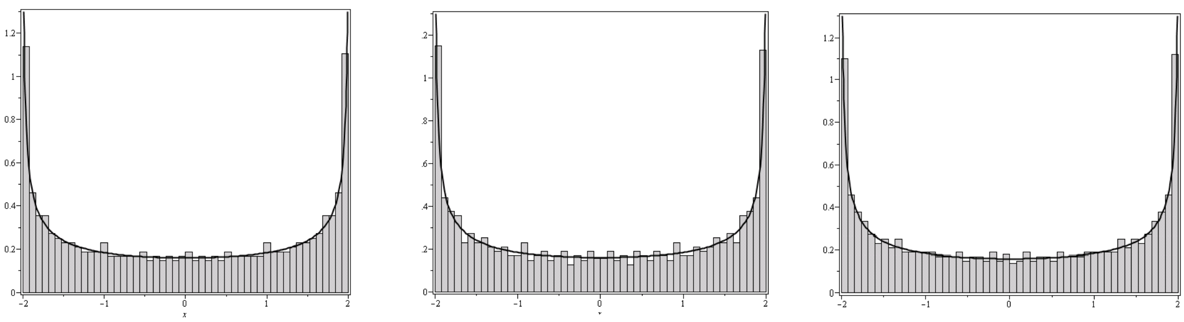


Figure 3.1: Histograms of the eigenvalues of a 1200×1200 sized random matrices of the form $U_k^{(N)} V_2^{(N)} U_k^{(N)} + V_2^{(N)}$ where $U_k^{(N)}$ and $V_2^{(N)}$ are independent random permutation matrices with cycle lengths of size k and 2, respectively. From left to right we show samples for $k = 3, 5, 12$ compared to the arcsine density.

Example 3.5.2 (multiplication with a projection)**.** Consider a Bernoulli random variable b with distribution:

$$\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1.$$

and a free projection p with distribution

$$\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1.$$

The free multiplicative convolution $\mu \boxtimes \nu$ may be easily shown to be $1/2(\delta_0 + a)$ where a arcsine law with density given by

$$\frac{1}{\pi\sqrt{4-t^2}} \quad |t| < 2$$

One may realizebp with $b = v_2$ and $p = u_k(v_2+1){u_k}^*$, where v_2 is a 2-Haar unitary (i.e a Bernoulli) and u_k is a k-Haar unitary free from b.

Theorem 3.4.4 gives us a random matrix approximation for this convolution by considering the ensemble $U_k^{(N)}(V_2^{(N)} + I^{(N)})U_k^{(N)} + V_2^N$ where $U_k^{(N)}$ and $V_2^{(N)}$ are independent random $Nk \times Nk$ permutation matrices with cycle lengths of size k and 2, respectively and $I^{(N)}$ is the identity matrix. Figure 3.2 shows the accuracy of this approximation, for $k = 5$.

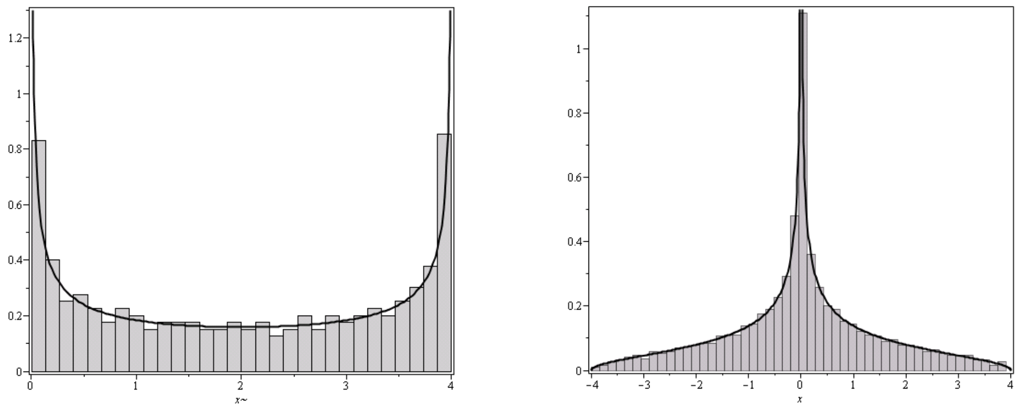


Figure 3.2: Arcisne distribution(left) and symmetric Beta distribution (right) compared with histograms of the eigenvalues of their respective random matrix models. The histograms correspond to a realization 1200×1200 sized random matrix.

One may want to combine additive and multiplicative convolution. For instance, in the setting of free compound Poissons we can multiply the arcsine distribution from the previous example with a free Poisson of parameter 1.

Example 3.5.3 (Symmetric Beta distribution)**.** Let π be a free Poisson distribution and $\mathbf{a}$ be an arcsine distribution on $(-2, 2)$, respectively. The free multiplicative convolution $\pi \boxtimes \mathbf{a}$ has been shown in [4] to have a symmetric distribution with density

$$\frac{1}{2\pi}\,|x|^{-1/2}\,(2-|x|)^{1/2}\mathrm{d}x, \quad |x| < 2.$$

Since Gaussian random matrices converge to a Wigner semicircle and a free Poisson is the square of a semicircle then we can model $\pi \boxtimes \mathbf{a}$ with $A^{(N)}(G^{(N)})^2$ where $A^{(N)} =$

$U_k^{(N)} V_2^{(N)} U_k^{(N)} + V_2$ from the last example and $G^{(N)}$ is a Gaussian Orthogonal Ensemble independent of $A^{(N)}$. Figure 3.2 compares theory with simulation.

In a similar fashion we can model other distributions coming from free multiplicative convolutions.

Example 3.5.4. Now we give examples that can be considered in our framework using k-Haar unitaries to obtain freeness. Figure 3.3 shows their approximations with random matrices.

1) Powers of free Poisson. The free multiplicative powers of a free Poisson $\pi^{\boxtimes k}$ are determined by the equation $S_{\pi_k}(z) = S_\pi^k(z)$. The case $k=2$ has an explicit density given by

$$\frac{2^{1/3}\sqrt{3}(2^{1/3}(27+3\sqrt{(81-12x)})^{2/3}-6x^{1/3})}{((12\pi)x^{2/3}27+3\sqrt{(81-12x)})^{1/3})}$$

2) Product of free Poisson with a centered free Poisson. The free multiplicative convolution $\mu = \pi \boxtimes \hat{\pi}_\lambda$ between a free Poisson of parameter λ shifted by λ with a free Poisson with parameter 1 is determined in terms of the Cauchy transform $g = G_\mu$ by the equation

$$1 + g^4 \lambda z^2 - (2\lambda z - z^2 - 2\lambda^2 z)g^3 + (\lambda^3 + \lambda z + \lambda - 2\lambda^2)g^2 - (\lambda + z - 1)g = 0$$

3) Wigner with Poisson. The free multiplicative convolution $\mu = w \boxtimes \pi_\lambda$ between a Wigner semicircle with a free Poisson with parameter λ is determined in terms of the Cauchy transform $g = G_\mu$ by the equation

$$1 + g^4 z^2 - (2z - 2\lambda)g^3 + (-2\lambda + \lambda^2 + 1)g^2 - gt = 0$$

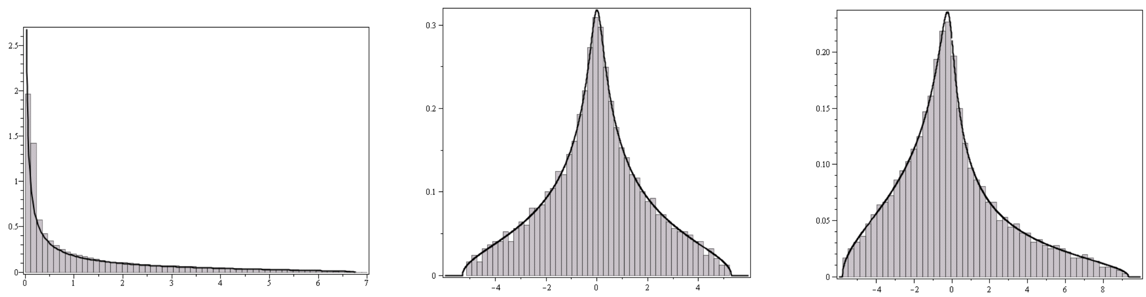


Figure 3.3: Histograms of the eigenvalues of random matrices of size 1200×1200 modeling the free multiplicative convolution $\pi \boxtimes \pi$(left), $w \boxtimes \pi_\lambda$ (center) and $\pi \boxtimes \hat{\pi}_\lambda$ (right) for $\lambda = 2$ compared with their density.

In the next example we consider different ensembles that converge to the same distribution.

Example 3.5.5. The three following examples have the same distribution, namely the probability measure with density

$$f(t) = \frac{\sqrt{3}}{2\pi \mid t \mid}\left(\frac{3t^2+1}{9h(t)} - h(t)\right), \qquad \mid t \mid \leq \sqrt{(11+5\sqrt{5})/2}, \tag{3.5.1}$$

where

$$h(t) = \sqrt{\frac{18t^2+1}{27} + \sqrt{\frac{t^2(1+11t^2-t^4)}{27}}}.$$

1) Free Sum of Compound Poisson. Let π be a free Poisson distribution and b be a Bernoulli distribution. Then the density of $(\pi \boxtimes b) \boxplus (\pi \boxtimes b)$ is given by (3.5.1) . We may approximate this distribution by matrices of the form $U_k G V_2 G U_k^* + G V_2 G$.

2) Free difference of free Poissons. Let P_1 and P_2 be two free random variables with Marchenko-Pastur distribution (or free Poisson). The difference $P_1 - P_2$ has a distribution $\pi(b, 2)$ a free compound Poisson with a jump distribution a Bernoulli and rate 2. The density of this measure is also given by (3.5.1). We can model this distribution by matrices of the form $U_k G^2 {U_k}^* - G^2$.

3) Free commutator. Let w be the Wigner semicircle distribution, and denote by $w \Box w$ the free commutator of w, that is, the distribution of $i(S_1S_2 - S_2S_1)$ where S_1 and S_2 are free semicircle variables; $w \Box w$ is given in [59] and coincides with the distribution (3.5.1). We may model $w \Box w$ with matrices of the form $U_k G U_k^* G + G U_k G U_k^*$.

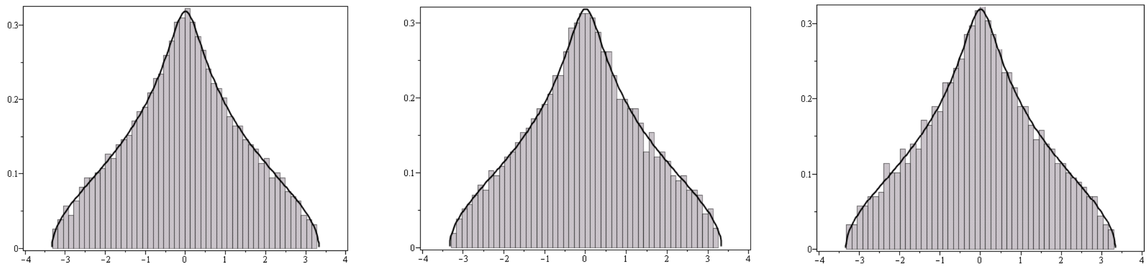


Figure 3.4: Histograms of the eigenvalues of random matrices of size 1200×1200 of the form $U_k G V_2 G {U_k}^* + G V_2 G$(left), $U_k G^2 {U_k}^* - G^2$(center) and $U_k G {U_k}^* G + G U_k G {U_k}^*$ (right) compared with the density of the free commutator $w \Box w$.

Non selfadjoint random variables

Now, let us consider some non selfadjoint operators, more specifically R-diagonal ones. For non-normal operators the right notion of spectral distribution is given by the Brown Measure introduced by Brown in 1893 [30]. The Brown Measure of R-diagonal operators in a von Neumann algebra was computed by Haagerup and Larsen [38]. We will use their results without further comments.

Example 3.5.6 (Product of centered Bernoullis)**.** By a well known result of Nica and Speicher ([59]), the product of two free Bernoullis $b_1 b_2$ is R-diagonal and moreover it is distributed as a Haar unitary, that is, the Brown Measure of $b_1 b_2$ is the uniform distribution on the circle. So let U_2^N, V_2^N be a pair of ensembles of random permutations with cycles of size 2. Then for all N the distribution of U_2^N and V_2^N is a Bernoulli $b = \frac{1}{2}(\delta_{-1} + \delta_1)$. Since U_2^N and V_2^N are asymptotically free, one expects that the asymptotic distribution of $U_2^N V_2^N$ is also the Haar measure on the unit circle. Figure 3.5 shows a simulation of this using random matrices.

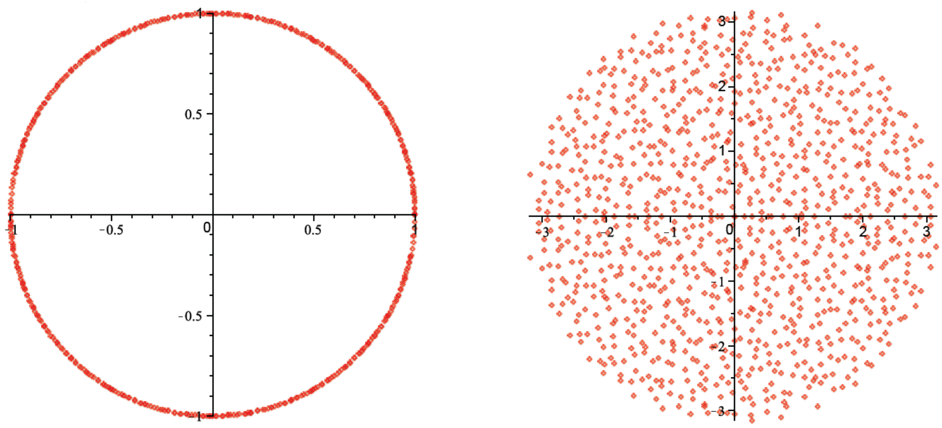


Figure 3.5: Eigenvalues of a 1200×1200 random matrix of the type $B_1 B_2$ with B_1 and B_2 independent permutation matrices with cycles of size 2 (left) and of the type $U_5 G$ with U_5 a permutation matrix with cycles of size 5 and G is an independent Gaussian matrix (right).

Theorem 3.5.7. *Let b_1 be a k_1-Haar unitary and b_2 a k_2-Haar unitary with $\{b_1, b_1^*\}$ free from $\{b_2, b_2^*\}$ and $k_1, k_2 > 1$. Then $b_1 b_2$ is a Haar unitary.*

Proof. The proof is identical as the one of Nica and Speicher [60, Theorem 15.17] where they prove this for $k_1 = k_2 = 2$ and general even elements a and b. For $x = b_1 b_2$ we have $x^* = b_2^{k_1-1} b_1^{k_2-1}$. The cumulants of odd length clearly vanish because of divisibility. Then to show that the free cumulants of $\kappa_n(..., ab, ab, ...) = 0$, using the formula for products as arguments, we can write these cumulants as

$$\kappa_n(..., b_1 b_2, b_1 b_2, ...) = \sum_{\pi \vee \sigma = 1_n} \kappa_\pi[..., b_1, b_2, b_1, b_2, ...],$$

where $\sigma = \{(1,2), (3,4), ..., (2n-1, 2n)\}$. We conclude by an argument of divisibility, similar as in the proof of Theorem 3.2.2 $\square$

Example 3.5.8 (Product of semicircle and Bernoulli)**.** Let U_2^N be a random permutation matrix with cycles of size 2 (a 2-Haar unitary) and G^N a Wigner Matrix with Gaussian entries. Then for all N the distribution of U_2^N is a Bernoulli $b = \frac{1}{2}(\delta_{-1} + \delta_1)$ and the distribution of G^N is asymptotically a semicircle. The product of free random variables b and s, b being a Bernoulli and s a semicircle, is also R-diagonal and moreover a circular operator with Brown measure a uniform distribution on the unit disk. Again, one expects that the empirical distribution of $U_2^N G^N$ approximates the uniform distribution on the unit circle, as $N \to \infty$. Figure 3.5 shows a simulation using random matrices.

Similar arguments as in Theorem 3.5.7 show the following result which generalizes example 3.5.8.

Theorem 3.5.9. *Let u_k be k-Haar unitary and s be an even operator free from $\{u_k, u_k^*\}$. Then $u_k s$ is an R-diagonal with determining sequence as s.*

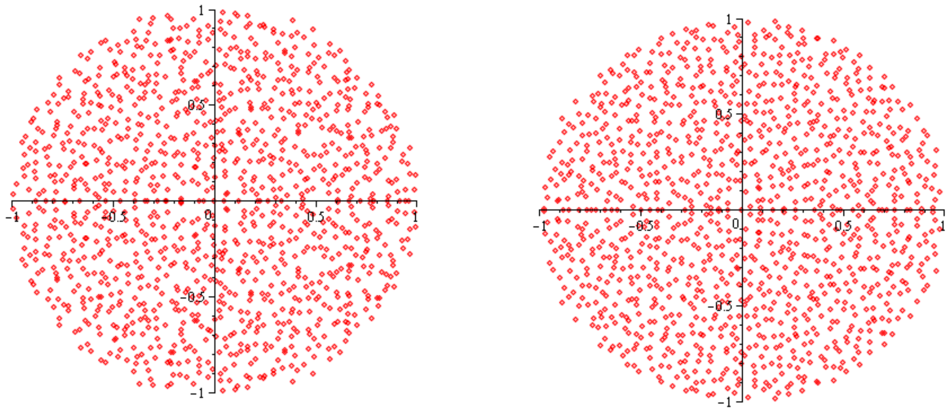


Figure 3.6: Eigenvalues of a 1200×1200 random matrix of the type $U_k G$ with G a selfadjoint Gaussian matrix and U_k an independent permutation matrix with cycles of size $k = 5$ (left) and $k = 12$ (right).

As a final example let us consider sums of free k-Haar unitaries.

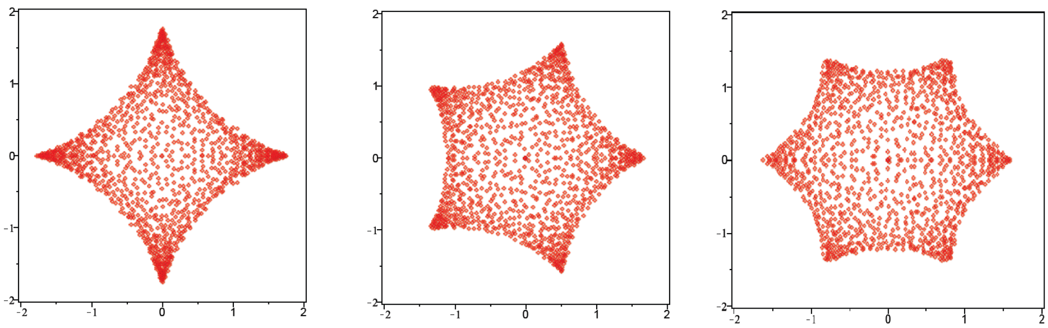


Figure 3.7: Eigenvalues of a sum of two independent random permutation matrices of size 1200×1200 with cycles of length 4,5 and 6.

Example 3.5.10 (Sum of u_k Haar unitaries)**.** Let $U^{(N)}, V^{(N)}$ be a pair of ensembles of random permutations of size $Nk \times Nk$ with cycles of size k. Then for all N the distribution of $U^{(N)}$ and $V^{(N)}$ is a k-Haar. One may ask what is the asymptotic distribution of $U^{(N)} + V^{(N)}$. By asymptotic freeness one expects that, as $N \to \infty$, the eigenvalues concentrate in the spectrum of $u_n + v_m$, for u_n and v_m free n- and m-Haar unitaries.

As shown by Lehner [49], for $n, m \in \mathbb{N}$ if u_n and v_m are free m- and n-Haar unitaries the spectrum of $u_n + v_m$ may be calculated as follows. Consider the equations

$$s(1-t^m) - t(1-s^n) = 0 \quad (3.5.2)$$

$$\lambda t - 1 - s^{n-1}t = 0 \quad (3.5.3)$$

$$(|s|^2 + \cdots + |s|^{2n-2})(|t|^2 + \cdots + |t|^{2m-2}) < 1 \quad (3.5.4)$$

Then λ is not in the spectrum if there is a solution for the system above. Figure 3.8 below shows the case $k = 3$ compared with a random matrix approximation.

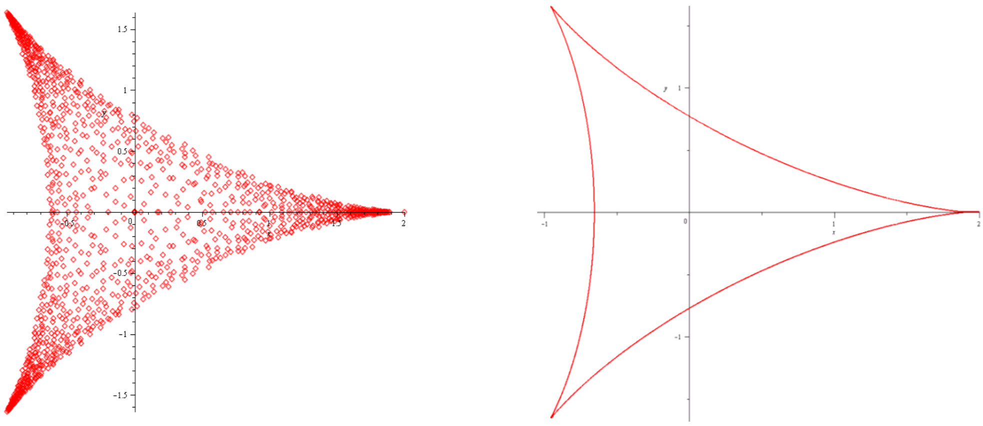


Figure 3.8: Eigenvalues of the sum of two independent random permutation matrices of size 1200×1200 with cycles of length 3 (left) compared with the curve describing the spectrum of the sum of two free 3-Haar unitaries

Recall that for q a primitive k-th root of unity, we consider the k-semiaxes $A_k := \{x \in \mathbb{C} \mid x = tq^s \text{ for some } t > 0 \text{ and } s \in \mathbb{N}\}$ and denote by $\mathcal{M}_k$ the subclass of $\mathcal{M}_{\mathbb{C}}$ of probability measures supported on A_k such that $\mu(B) = \mu(qB)$, for all Borel sets B. A measure in $\mathcal{M}_k$ will be called k-symmetric.

From these last examples one may have the impression that trying to define a free convolution on the k-symmetric distributions might not make sense, since the spectrum of the sum of two operator with k-symmetric distribution in general leaves the k semiaxes A_k. However, as we will see in the next chapter, if we only care about moments (and not $*$-moments) it **does** make sense to define free powers and we can define free additive convolution for a big class of measures.

One may also ask about free multiplicative convolution. As can be seen above, even if we care only about moments it makes no sense to consider a free multiplicative convolution between a k_1 and k_2 symmetric distribution, whenever k_1 and k_2 are greater than 1. This is not a surprise since even in the real case (as pointed in [64]) when both random variables have mean zero the moments of the product are all 0. However if we consider one of the variables to be positive and the other k-symmetric we will define the free multiplicative convolution between them. Figure 3.9 shows the matrix approximation for the product of a 5-Haar unitary with free Poisson. Again, as for the case of the sum, we will only deal with moments and not $*$-moments.

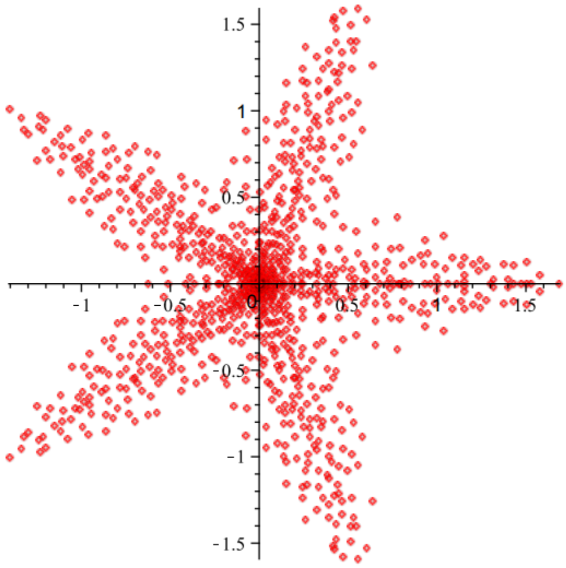


Figure 3.9: Eigenvalues of the product U_5P of two independent random matrices of size 1200×1200 with U_5 a permutation with cycles of size 5 and P the square of a Gaussian Matrix

Chapter 4

Main Theorem and first consequences

In this, the main chapter of the thesis, we will prove the Main Theorem. This theorem will not only allow us to define free multiplicative convolution between k-symmetric distributions and probability measures in $\mathcal{M}^+$ but, moreover, will permit us to define free additive convolution powers for k-symmetric distributions. Also, on the combinatorial side, we generalize Theorem 2.4.3 to any multiplicative family.

The main tool that we will use is the S-transform. This S-transform is not defined uniquely for k-divisible random variables, the principal problem is choosing an inverse for the transform ψ.

4.1 The S-transform for random variables with k vanishing moments

We will start in the general setting of an algebraic non-commutative probability space $(\mathcal{A}, \phi)$ and define an S-transform for random variables such the first $k-1$ moments vanish.

Recall the definition of the S-transform for positive measures. For a probability measure μ on $\mathbb{R}$, we let $\psi_\mu(z) := \int_{\mathbb{R}} \frac{zx}{1-zx}\mu(dx)$. ψ_μ coincides with a moment generating function if μ has finite moments of all orders. Denoting by χ_μ the inverse under composition of ψ_μ, the S-transform is defined as

$$S_\mu(z) := \frac{1+z}{z}\chi_\mu(z), \quad z \in \psi_\mu(i\mathbb{C}_+). \tag{4.1.1}$$

In general, when x is a selfadjoint random variable with non-vanishing mean the S-transform can be defined as follows.

Definition 4.1.1. Let x be a random variable with $\phi(x) \neq 0$. Then its S-transform is defined as follows. Let χ denote the inverse under composition of the series

$$\psi(z) := \sum_{n=1}^{\infty} \phi(x^n) z^n, \tag{4.1.2}$$

then

$$S_x(z) := \chi(z)\frac{1+z}{z}. \tag{4.1.3}$$

Here, $\phi(x) \neq 0$ ensures that the inverse of ψ exists as a formal power series. The importance of the S-transform is the fact that $S_{xy} = S_x S_y$ whenever x, y are free random variables with $\phi(x) \neq 0$ and $\phi(y) \neq 0$.

We want to consider the case when $\phi(x) = 0$. The case when x is selfadjoint and $\phi(x^2) > 0$ was treated in Raj Rao and Speicher in [64]. The main observation is that although ψ cannot be inverted by a power series in z it can be inverted by a power series in $\sqrt{z}$. This inverse is not unique, but there are exactly two choices.

The more general case where $\phi(x^n) = 0$ for $n = 1, 2, ..., k-1$ and $\phi(x^k) \neq 0$ can be treated in a similar fashion. In this case there are k possible choices to invert the function ψ. We include the proof for the convenience of the reader.

Proposition 4.1.2. *Let $\psi(z)$ be a formal power series of the form*

$$\psi(z) = \sum_{n=k}^{\infty} \alpha_n z^n \tag{4.1.4}$$

with $\alpha_k > 0$. There exist exactly k power series in $z^{1/k}$ which satisfy

$$\psi(\chi(z)) = z. \tag{4.1.5}$$

Proof. Let

$$\chi(z) = \sum_{i=1}^{\infty} \beta_i z^{i/k} \tag{4.1.6}$$

The equation $\psi(\chi(z)) = z$ is equivalent to

$$\sum_{n=k}^{\infty} \alpha_n (\sum_{n=1}^{\infty} \beta_i z^{i/k})^n = z. \tag{4.1.7}$$

This yields the system of equations

$$1 = \alpha_k \beta_1^k$$

and

$$0 = \sum_{n=k}^{r} \sum_{i_1+\cdots+i_n=r}^{r} \alpha_n \beta_{i_n} \ldots \beta_{i_n}$$

for all $r > 2$. Clearly the solutions of the first equation are

$$\beta_1 = \alpha_k^{1/k}$$

while the other equations ensure that β_n is determined by β_1 and the α's. □

Now, we can define the S-transform for random variables having vanishing moments up to order $k-1$.

Definition 4.1.3. Let x be a random variable with $\phi(x^n) = 0$ for $n = 1, 2, \ldots, k-1$ and $\phi(x^k) > 0$. Then its S-transform is defined as follows. Let $\chi(z) = \sum_{i=1}^{\infty} \beta_i z^{i/k}$ be the inverse under composition of the series

$$\psi(z) = \sum_{n=k}^{\infty} \phi(x^n) z^n \tag{4.1.8}$$

with leading coefficient $\beta_1 > 0$. Then

$$S_x(z) = \chi(z) \frac{1+z}{z}. \tag{4.1.9}$$

The following theorem is a generalization of Theorem 2.5 in [64] and shows the role of the S-transform with respect to multiplication of free random variables.

Theorem 4.1.4. *Let $x \in (\mathcal{A}, \phi)$ such that $\phi(x^n) = 0$ for $n = 1, 2, ..., k-1$ and $\phi(x^k) > 0$ and let $y \in (\mathcal{A}, \phi)$ be such that $\phi(y) \neq 0$. If S_x and S_y denote their respective S-transforms, then*

$$S_{xy}(z) = S_x S_y(z),$$

where S_{xy} is the S-transform of xy.

Proof. The proof is exactly the same as in [64]. The only observation to be made is that xy also satisfies the conditions in Definition 4.1.3. Indeed, by freeness $\phi((xy)^n) = 0$ for $n = 1, 2, ..., k-1$ and $\phi((xy)^k) = \phi(x^k)\phi(y)^k > 0$ and then all the manipulations are valid for the case when $k > 2$. The key point is to verify that $C_{xy}(S_{xy}(z)) = z$. □

Remark 4.1.5. We cannot drop the assumption $\phi(y) \neq 0$ in Theorem 4.1.4. As pointed out by Rao and Speicher [64], freeness would yield $\phi((yx)^n) = 0$, for all $n \in \mathbb{N}$.

4.2 Free Multiplicative convolution of k-symmetric distributions

Recall the notion of free multiplicative convolution of two measures μ in $\mathcal{M}$ and ν in $\mathcal{M}^+$. The idea is to consider a selfadjoint random variable x and a positive random variable y (free from x) with distributions μ and ν, respectively, and call $\mu \boxtimes \nu$ the distribution of $y^{1/2}xy^{1/2}$. This element is selfadjoint so we can be sure that $\mu \boxtimes \nu$ is a well defined probability measure on $\mathcal{M}$, but moreover $y^{1/2}xy^{1/2}$ and xy have the same moments. In other words, $\mu \boxtimes \nu$ can be defined as the only distribution in $\mathcal{M}$ whose moments equal the moments of xy.

Following these ideas, the strategy for defining a free multiplicative convolution $\mu \boxtimes \nu$, for μ k-symmetric and ν with positive support, is clear. We consider a k-divisible random variable x and a positive element y (free from x) with distributions μ and ν, respectively. Given a k-divisible random variable x and a positive element y, it is clear that xy is a also k-divisible in the algebraic sense. The interesting question is how to find an element with a k-symmetric distribution with the same moments as xy. In this section we prove that this element does exist. Observe that in this case taking the random variable $y^{1/2}xy^{1/2}$ does not work since it is not necessarily normal.

Recall that given a k-symmetric probability measure μ in $\mathcal{M}_k$, we denote by μ^k the probability measure in $\mathcal{M}^+$ induced by the map $t \to t^k$.

Example 4.2.1. 1) Let X be a k-divisible random variable with X^k positive and Y a positive operator. Moreover suppose that X and Y are classically independent (in particular they commute). Now, XY is also k-divisible since $\phi((XY)^n) = \phi(X^n)\phi(Y^n)$ and $(XY)^k = X^k Y^k$ is also positive. So the property of X of being k-divisible with X^k positive is maintained when multiplying with a (tensor) independent positive operator.

2) Let μ be a k-symmetric distribution. We see that μ can be realized as the distribution of XY, where X is a k-Haar element and Y is an independent positive operator.

Indeed consider the pushforward μ^k of μ, then there exist some positive operator Z distributed as μ^k. Now, let Y be the positive operator with $Z = Y^k$. By (1) if Y is a k-Haar unitary independent from X then XY has also a k-symmetric distribution. Moreover $(XY)^k = X^k$. So the distribution of $(XY)^k$ is μ^k.

Remark 4.2.2. 1) In view of Example 4.2.1 we can easily define the classical multiplicative convolution between a k-symmetric distribution and a positive one. In the free probability setting this requires much more work, since free random variables do not commute.

We start by stating a relation between the S-transform of a k-divisible element x and the S-transform of x^k.

Lemma 4.2.3. *Let $x \in (\mathcal{A}, \phi)$ be a k-divisible element. Then the S-transforms of x and x^k are related by the formula*

$$S_{x^k}(z) = S_x(z)^k (\frac{z}{1+z})^{k-1}.$$

Proof. By definition $m_n(x^k) = m_{nk}(x)$ and $m_s(x) = 0$ if $k \nmid s$. So

$$\psi_x(z) = \sum_{n=1}^{\infty} m_n(x) = \sum_{n=1}^{\infty} m_{nk}(x) z^n k$$

and

$$\psi_{x^k}(z) = \sum_{n=1}^{\infty} m_n(x^k) = \sum_{n=1}^{\infty} m_{nk}(x) z^n.$$

Thus $\psi_x(z) = \psi_{x^k}(z^k)$, or equivalently, $\chi_{x^k}(z) = \chi_x(z)^k$ and then

$$S_x(z)^k = (\frac{1+z}{z})^k \chi_x(z)^k = (\frac{1+z}{z})^k \chi_{x^k}(z) = (\frac{1+z}{z})^{k-1} S_{x^k}(z).$$

So

$$S_{x^k}(z) = S_x(z)^k (\frac{z}{1+z})^{k-1}.$$

□

Now we are in position to prove the Main Theorem.

Main Theorem. Let $x, y \in (\mathcal{A}, \phi)$ with x positive and y a k-divisible element free from x. Consider $x_1, ..., x_k$ free positive elements with the same moments as x. Then $(xy)^k$ and $y^k x_1 \cdots x_k$ have the same moments, i.e.

$$\phi((xy)^{kn}) = \phi((y^k x_1 \cdots x_k)^n) \quad \forall n \in \mathbb{N}. \tag{4.2.1}$$

Proof. It is enough to check that the S- transforms of $(xy)^k$ and $y^k x_1 \cdots x_k$ coincide. Now

$$\begin{aligned} S_{(xy)^k}(z) &= S_{xy}(z)^k (\frac{z}{1+z})^{k-1} = S_x(z)^k S_y(z)^k (\frac{z}{1+z})^{k-1} \\ &= S_x(z)^k \cdot S_{y^k}(z) = S_{(x_1)}(z) \cdots S_{x_k}(z) \cdot S_{y^k}(z) \\ &= S_{y^k x_1 \cdots x_k}(z) \end{aligned}$$

Note that if x positive and y is k-divisible element free from x, then xy is also a k-divisible element. □

Remark 4.2.4. In the tracial case, Theorem 3.2.2 gives another proof of Main Theorem. Indeed, consider the moments of $sas...sasa$ when s is k-divisible; since $sas...sas$, and a are free, by Theorem 3.2.2, then these moments coincide with the moments of $sas..sasa_1$ where a_1 is free from s and a. Now by, traciality the moments of $sas...sas$ coincide with the moments of $s^2as...sa$, which again, by Theorem 3.2.2 coincide with the moments of $s^2as...sa_2$ where a_2 is free from s and a. So the moments of $sas...sasa$ coincide with the moments of $s^2as...sa_2a_1$ with a_1, a_2, a and s free between them. Continuing with this procedure we see that the moments of $sas...sasa = (sa)^k$ coincide with the moments of $s^k a_1 a_2 \cdots a_k$, with a_i's and s free between them.

Corollary 4.2.5. *Let x be k-divisible with x^k positive and let y be positive. For $Z = (xy)^k$ there is a positive element $\hat{Z}$ with $\phi(Z^n) = \phi(\hat{Z}^n)$ for all $n \in \mathbb{N}$.*

Proof. This is a straightforward consequence of the Main Theorem. □

The previous corollary allows us to define free multiplicative convolution between a k-symmetric distribution and probability measure in $\mathcal{M}^+$.

Definition 4.2.6. Let $\mu \in \mathcal{M}^+$ and let $\nu \in \mathcal{M}^k$ be a k-symmetric probability measure. Suppose that μ and ν are the distributions of X and Y, free elements in some probability space $(\mathcal{A}, \phi)$, respectively. We define $\mu \boxtimes \nu = \nu \boxtimes \mu$ to be the unique k-symmetric probability measure with the same moments as XY.

Remark 4.2.7. Notice that the last definition does not depend on the choice of X and Y since the distribution of X and Y (by freeness) determine the moments of XY moments uniquely.

Finally, we obtain the mentioned relation.

Corollary 4.2.8. *Let $\mu \in \mathcal{M}^+$ and let $\nu \in \mathcal{M}^k$. The following formula holds:*

$$(\mu \boxtimes \nu)^k = \mu^{\boxtimes k} \boxtimes \nu^k. \tag{4.2.2}$$

Remark 4.2.9. One may ask if any k-divisible symmetric can be represented as the free multiplicative convolution of a k-Haar $\nu_k = \frac{1}{k}\sum_{j=1}^{k} \delta_{q^j}$ and a positive measure. However, Corollary 4.2.8 shows that this is not the case since

$$(\mu \boxtimes \nu_k)^k = \mu^{\boxtimes k}.$$

4.3 Free additive powers

Just as in the multiplicative case, it is not straightforward to show that free additive convolution of k-symmetric distributions is well defined. In fact, at this point this is an open problem.

Open Question. Can we define free additive convolution of k-symmetric probability measures?

We will give a partial answer in the next section, see Theorem 5.3.9. However, another important consequence of the Main Theorem is the existence of free additive powers $\mu^{\boxtimes t}$, when μ is a probability measure with k-symmetry and $t > 1$.

The framework of a compressed space from Example 1.2.2 allows to define a convolution semigroups for probability measures on $\mathbb{R}$, see [56] .

Recall that given a non-commutative probability space $(\mathcal{A}, \phi)$ and a projection $p \in \mathcal{A}$ such that $\phi(p) = t \neq 0$, we can form the space $(p\mathcal{A}p, \phi^{p\mathcal{A}p})$. When p is free from x, we can obtain the distribution of pxp from the distribution of x.

More explicitly Speicher and Nica [56] proved the following:

Theorem 4.3.1. *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and $p \in \mathcal{A}$ a projection ($p^2 = p$) such that $\phi(p) = \lambda > 0$ then we have that*

$$k_n^{p\mathcal{A}p}(pxp, \ldots, pxp) = \frac{1}{\lambda} k_n(\lambda x, ..., \lambda x)$$

We will use the same ideas used in [56] combined with the Main Theorem to prove the existence of additive powers for $t > 1$.

Theorem 4.3.2. *Let $\mu \in \mathcal{M}_k$ be a k-symmetric distribution. Then for each $t > 1$ there exists a k-symmetric measure $\mu^{\boxplus t}$ with $\kappa_n(\mu^{\boxplus t}) = t\kappa_n(\mu)$.*

Proof. Let $x \in (\mathcal{A}, \phi)$ be a tracial C^*-probability space and let $x \in (\mathcal{A}, \phi)$ be such that x^k is positive and with distribution μ and let $p \in (\mathcal{A}, \phi)$ be a projection such that $\phi(p) = \frac{1}{t}$, with x and p free. Now consider the compressed space $(p\mathcal{A}p, \phi^{p\mathcal{A}p})$ and the element $x_t := pXp \in (p\mathcal{A}p, \phi^{p\mathcal{A}p})$, with $X = tx$. By Theorem 14.10 in [60] the cumulants of x_t (with respect to $\phi^{p\mathcal{A}p}$) are

$$\kappa_n^{p\mathcal{A}p}(x_t, \ldots, x_t) = t\kappa_n(\frac{1}{t}X, \ldots, \frac{1}{t}X) = t\kappa_n(x, \ldots, x). \tag{4.3.1}$$

Now, X is k-divisible, X^k is positive and p is positive. Thus, by the Main Theorem, the moments of Xp also define a k-symmetric distribution. Also, since ϕ is tracial we have

$$\phi(pXppXp \cdots pXp) = \phi(pXpXp \cdots pXpXp) = \phi(XpX \cdots Xp). \tag{4.3.2}$$

This means that the moments of $(pXp)^k$ define a measure ν on $\mathbb{R}_+$ with an atom at 0 of size at least $(1 - 1/t)$. Now consider the compressed $(pXp)^k$. Then

$$\phi^{p\mathcal{A}p}(pXppXp \cdots pXp) = t\phi(pXppXp \cdots pXppXp). \tag{4.3.3}$$

Writing the measure $\nu = (1-1/t)\delta_0 + 1/t\mu_t$ we see that μ_t has moments $m_n(\mu_t) = tm_n(\nu)$ which are the moments of x_t in the compressed space, as desired. $\square$

Although we are not able to define free additive convolution for all k-symmetric measures, having free additive powers is enough to talk about central limit theorems and Poisson type ones. This will be done in Chapter 5.

4.4 A combinatorial consequence

The following theorem of Nica and Speicher [56] gives a formula for the moments and free cumulants of product of free random variables.

Theorem 4.4.1. *Let (A, ϕ) be a non-commutative probability space and consider the free random variables $a, b \in A$. The we have*

$$\phi((ab)^n) = \sum_{\pi \in NC(n)} \kappa_\pi(a)\phi_{K(\pi)}(b^n)$$

and

$$\kappa_n(a) = \sum_{\pi \in NC(n)} \kappa_\pi(a)\kappa_{K(\pi)}(b).$$

The observation here is that we can go the other way. Indeed for two multiplicative families f_n and g_n we can find a probability space (A, ϕ), and elements a and b in A such that $\kappa_n^a = f_n$ and $\phi(b^n) = g_n$ and then we can calculate $(f * g)_n$ by the formula $(f * g)_n = \phi((ab)^n)$. Using this idea and the Main Theorem we can generalize broadly Theorem 2.4.3 to any multiplicative family whose first element is not zero.

Theorem 4.4.2. *Let f_n be a multiplicative family in NC, with $f_1 \neq 0$. The following statements are equivalent.*

1) The sequence f_n is given by the k-fold convolution

$$f_n = g_n * \underbrace{h_n * \cdots * h_n}_{k \text{ times}}.$$

2) The dilated sequence $f_n^{(k)}$ is given by the convolution

$$f_n^{(k)} = g_n^{(k)} * h_n.$$

Proof. In the proof of the Main Theorem, from the combinatorial point of view, positivity is not important. So let X, Y be in (A, ϕ) with Y a k-divisible element, and assume that X has cumulants $\kappa_n^x = h_n$ and Y^k has moments $\phi((Y^k)^n) = g_n$ (and therefore $\phi(Y^n) = g_n^{(k)}$). Let $X_1, ..., X_k$ be free elements with the same moments as X. Then $(XY)^k$ and $X_1 \cdots X_k Y^k$ have the same moments, i.e.

$$\phi((XY)^{kn}) = \phi((X_1 \cdots X_k Y^k)^n). \tag{4.4.1}$$

Now, the moments of $X_1 \cdots X_k Y^k$ are given by

$$f_n := \phi((Y^k X_1 \cdots X_k)^n) = g_n * h_n * \cdots * h_n$$

and the moments of XY are given by

$$\tilde{f}_n := \phi((XY)^n) = g_n^{(k)} * h_n.$$

Now Equation (4.4.1) implies that $\tilde{f}_n = f_n^{(k)}$. □

Chapter 5

Limit theorems and free infinite divisibility

In this chapter we will address questions regarding limit theorems. First, we prove central limit theorems for k-symmetric measures. Next, we consider free infinite divisibility. Finally, we study the free multiplicative convolution of measures on the positive real line from the point of view of k-divisible partitions and its connections to the free multiplicative convolution between k-symmetric measures and probability distributions in the positive real line.

5.1 Free central limit theorem for k-divisible measures

We have a new free central limit theorem for k-symmetric measures. Recall that for a measure μ, $D_t(\mu)$ denotes the dilation by t of the measure μ.

Theorem 5.1.1 (Free Central limit theorem for k-symmetric measures)**.** *Let μ be a k-symmetric measure with finite moments and $\kappa_k(\mu) = 1$. Then as N goes to infinity,*

$$D_{N^{-1/k}}(\mu^{\boxplus N}) \to s_k,$$

where s_k is the only k-symmetric measure with free cumulant sequence $\kappa_n(s_k) = 0$ for all $n \neq k$ and $\kappa_k(s_k) = 1$. Moreover,

$$(s_k)^k = \pi^{\boxtimes k-1},$$

where π is a free Poisson measure with parameter 1.

Proof. Convergence in distribution to a measure determined by moments is equivalent to the convergence of the free cumulants. Now, for $i = 1, 2, ..., n-1$ the i-th free cumulant $\kappa_i(\mu^{\boxplus N})$ equals zero and for $i > k$, the i-th free cumulant

$$\kappa_i(D_{N^{-1/k}}(\mu^{\boxplus N})) = (N^{-1/k})^i \kappa_i(\mu^{\boxplus N}) = \frac{N}{N^{i/k}} \kappa_i(\mu) = N^{1-i/k} \kappa_i(\mu) \to 0$$

when N goes to infinity. So, in the limit, the only non vanishing free cumulant is $\kappa_k(D_{N^{-1/k}}(\mu^{\boxplus N})) = \kappa_k(\mu) = 1$. This means that s_k is the only k-symmetric measure with free cumulant sequence $\kappa_n = 0$ for all $n \neq k$ and $\kappa_k = 1$. For the second statement, on one hand, we calculate the moments of s_k using the moment cumulant formula:

$$m_n(s_k^k) = m_{nk}(s_k) = \sum_{\pi \in NC(nk)} \kappa_\pi(s_k) \quad (5.1.1)$$

$$= \sum_{\pi \in NC_k(n)} 1 \quad (5.1.2)$$

$$= \frac{\binom{kn}{n}}{kn-1}. \quad (5.1.3)$$

On the other hand, the moments of $\pi^{\boxtimes k-1}$ are known to be (See [13] or Example 5.2.4 below).

$$m_n(\pi^{\boxtimes k-1}) = \frac{\binom{kn}{n}}{kn-1}.$$

□

Remark 5.1.2. We can derive properties of the limiting distribution s_k from the fact that $(s_k)^k = \pi^{\boxtimes k-1}$. Indeed, let $B(0,r) = \{z \in \mathbb{C} : |z| < r\}$. The measure s_k satisfies the following properties.

(i) There are no atoms.

(ii) The support is $B(0,K) \cap \mathcal{A}_k$, where $K = \sqrt[k]{(k)^k/(k-1)^{k-1}}$.

(iii)The density is analytic on $(0,K)$.

Remark 5.1.3. (1) Note from the proof of Theorem 5.1.1 that in the algebraic sense we only need the first $k-1$ moments to vanish. For $k=1$, this is the law of large numbers and for $k=2$ we obtain the usual free central limit theorem.

(2) Observe that s_k satisfies a stability condition. Indeed,

$$s_k^{\boxtimes 2} = D_{2^{1/k}}(s_k)$$

from where we can interpret s_k as a strictly stable distribution of index k. This raises the question whether there are other k-symmetric stable distributions. Of course, in the presence of moments we can only get a s_k from the free central limit theorem above. Hence, if we expect to find other stable distribution we need to extend the notion of free additive powers to k-symmetric measures without moments. This will be done in Section 7.

(3) The law of small numbers and more generally free compound Poisson type limit theorems are also valid for k-symmetric distributions. Moreover, a notion of free infinite divisibility will be given and studied. This is the content of next parts of this section.

5.2 Compound free Poissons

The analogue of compound Poisson distributions and infinite divisibility are the subjects of this section. Recall the definition of a free compound Poisson on $\mathbb{R}$.

Definition 5.2.1. A probability measure μ is said to be a free compound Poisson of rate λ and jump distribution ν if the free cumulants $(\kappa_n)_{n\geq 1}$ of μ are given by $\kappa_n(\mu) = \lambda m_n(\nu)$. In this case, $\lambda\nu$ coincides with the Lévy measure of μ.

The most important free compound Poisson measure is the Marchenko-Pastur law π whose R-transform is $R_\pi(z) = \frac{z}{1-z}$. π is also characterized by $S_\pi(z) = \frac{1}{z+1}$ in terms of the S-transform.

Following the definition of a free compound Poisson for selfadjoint random variables we can define their analogues for k-symmetric distributions.

Definition 5.2.2. A k-symmetric distribution μ is called a free compound Poisson of rate λ and jump distribution ν if the free cumulants $(\kappa_n)_{n\geq 1}$ of μ are given by $\kappa_n(\mu) = \lambda m_n(\nu)$, for some ν a k-symmetric distribution.

The existence of these measures can be easily proved by finding explicitly $\pi(\lambda,\nu)^k$.As announced we have a limit theorem for the free compound Poisson distributions. We shall mention that, implicitly, Banica et al. [13] treated the case $\nu = \frac{1}{k}\sum_{j=1}^{k}\delta_{q^j}$

Theorem 5.2.3. *We have the Poisson type limit convergence*

$$((1-\frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\nu)^{\boxplus N} \to \pi(\lambda,\nu).$$

Proof. The proof is identical as for the selfadjoint case, see for example [60]. The main observation is that if $\nu_N = ((1-\frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\nu)^{\boxplus N}$ then

$$\kappa_n(\nu_N) = \frac{\lambda}{N} m_n(\nu) + O(1/N^2)$$

and then $\kappa_n(\nu_N^{\boxplus N}) = N\kappa_n(\nu_N)$ converges to $\lambda m_n(\nu)$. □

Example 5.2.4 (Free Bessel laws)**.** Free Bessel laws introduced in [13], are defined by

$$\pi_{kt} = \pi^{\boxtimes k} \boxtimes \pi^{\boxplus k}.$$

We restrict attention to the case $t=1$, for simplicity. They proved using a matrix model that the free Bessel law π_{k1} with $k\in\mathbb{N}$ is given by

$$\pi_{k1} = law\left[\sum_{j=1}^{k}\left[P_j q^j\right]\right]^k, \tag{5.2.1}$$

where $P_1,...,P_k$ s are free random variables, each of them following the free Poisson law of parameter $1/k$. So they were lead to consider the modified free Bessel laws $\hat{\pi}_{s1}$, given by

$$\hat{\pi}_{k1} = law\left[\sum_{j=1}^{k}\left[P_j q^j\right]\right]. \tag{5.2.2}$$

It is important to notice that $\sum_{j=1}^{k}\,[W_j q^j]$ is not a normal operator so the equalities in (5.2.1) and (5.2.2) are just equalities in moments (and not $*$-moments). In our notation means that

$$\pi_{k1} = \hat{\pi}_{k1}^k.$$

A modified free Bessel law is k-symmetric, but moreover it is a compound free Poisson with rate $\lambda = 1$ and jump distribution a k-Haar measure. So we have the representation

$$\hat{\pi}_{k1} = \pi(1, \nu) = \pi \boxtimes \frac{1}{k} \sum_{j=1}^{k} \delta_{q^j}$$

Combining these identities we see that

$$(\pi \boxtimes \frac{1}{k} \sum_{j=1}^{k} \delta_{q^j})^k = \hat{\pi}_{k1}^k = \pi_{k1} = \pi^{\boxtimes k}.$$

which is nothing but Equation (4.2.2) for $\mu = \pi$ and $\nu = \sum_{j=1}^{k} \delta_{q^j}$. Moreover the free cumulants and moments of $\pi^{\boxtimes k}$ are given by

$$m_n(\pi^{\boxtimes k}) = \frac{\binom{(k+1)n}{n}}{kn+1} \quad k_n(\pi^{\boxtimes k}) = \frac{\binom{kn}{n}}{(k-1)n+1}.$$

This is easily seen since the free cumulants of π are given by $k_n(\pi) = 1$ for all $n \in \mathbb{N}$. So calculating the moments and cumulants of $\pi^{\boxtimes k}$ amounts counting the number of k-multichains of $NC(n)$ which was done in Example 2.3.1.

To end this example let us see how the matrix models for free Bessel and modified free Bessel laws given in Theorem 6.2 and 6.3 of [13] are straightforward from our results.

Corollary 5.2.5. *If W is a $W(kN, kN, I_{(kN)^2})$ complex Wishart matrix and*

$$D = \begin{pmatrix} 1_N & & & 0 \\ & w1_N & & \\ & & \ddots & \\ 0 & & & w^{k-1}1_N \end{pmatrix}$$

then the mean empirical distribution of the eigenvalues of $(DW)^k$ converges to $\pi^{\boxtimes k}$, as $N \to \infty$.

Proof. By the Main Theorem if Y is k-divisible and X is positive. then $(XY)^k$ and $X_1 \cdots X_k Y^k$ have the same moments, i.e.

$$\phi((XY)^{kn}) = \phi((X_1 \cdots X_k Y^k)^n). \tag{5.2.3}$$

In particular when Y is a k-Haar unitary and X is a free Poisson, then Y^k are has distribution δ_1 and then the moments of $(XY)^k$ are just the moments of $\pi^{\boxtimes k}$.

Now D_N is a deterministic matrix k-Haar distributed, while it is well known W converges the Marchenko Pastur distribution. Moreover D_N and W are asymptotically free and then DW converges in distribution to XY. □

5.3 Free infinite divisibility

Given the limit theorems above, the concept of free infinite divisibility in $\mathcal{M}_k$ arises naturally.

Definition 5.3.1. A k-symmetric measure is $\boxplus$-infinitely divisible if for any $N > 0$ there exist $\mu_N \in M_k$ such that $\mu_N^{\boxplus N} = \mu$. We will denote the set of freely infinitely divisible distribution in $\mathcal{M}_k$ by $ID^{\boxplus}(\mathcal{M}_k)$

It is easily seen the $ID^{\boxplus}(\mathcal{M}_k)$ is closed under convergence in distribution. Free compound Poissons are $\boxplus$-infinitely divisible, since $\pi(\lambda,\mu)^{\boxplus t} = \pi(\lambda t,\mu)$. Moreover any free infinitely divisible measure can be approximated by free compound Poissons. The proof of this fact follows the same lines as for the selfadjoint case. We will give the main ideas of this proof for the convenience of the reader.

The following is a special case of Lemma 13.2 in Nica Speicher [60].

Lemma 5.3.2. *Let $\{a_N\}_N > 1$ be random variables in some non-commutative probability space $(\mathcal{A}, \phi_N)$ and denote by κ^N the free cumulants w.r.t ϕ_N, then the following statements are equivalent.*

(1)For each $n \geq 1$ the limit

$$\lim_{N\to\infty} N \cdot \phi_N(a_N^n)$$

exists.

(2)For each $n \geq 1$ the limit

$$\lim_{N\to\infty} N \cdot \kappa_n^N(a_N, ..., a_N)$$

exists.

Furthermore the corresponding limits are the same.

Now, we can prove the approximation result.

Proposition 5.3.3. *A k-symmetric measure is freely infinitely divisible if and only if it can be approximated (in distribution) by free compound Poissons.*

Proof. On one hand, since free compound Poissons are freely infinitely divisible any measure approximated by them is also infinitely divisible. On the other hand, let μ be $\boxtimes$-infinitely divisible. Then for any $N > 0$ there exist μ_N such that $\mu_N^{\boxplus N} = \mu$. So by Lemma 5.3.2 we have

$$\kappa_n(\mu) = N \cdot \kappa_n(\mu_N) = \lim_{N\to\infty} N \cdot \kappa_n(\mu_N) = \lim_{N\to\infty} N \cdot m_n(\mu_N) \tag{5.3.1}$$

Now, let ν_N be a free compound Poisson with rate N and jump distribution μ_N then $\kappa_n(\nu_N) = N m_n(\mu_N)$.

$$\lim_{N\to\infty} \kappa_n(\nu_N) = \lim_{N\to\infty} N \cdot m_n(\mu_N). \tag{5.3.2}$$

So $\nu_N \to \mu$ in distribution. □

Next, the results of Chapter 3 can be interpreted in terms of free compound Poissons.

Proposition 5.3.4. *Suppose that x is a k-divisible element and $\alpha_n = \kappa_{kn}(x)$ is the free cumulant sequence of a positive element ($\kappa_n(a) = \alpha_n$) with distribution ν. Then*

$$distr(x^k) = \pi^{\boxtimes k-1} \boxtimes \nu.$$

Proof. By Proposition 3.1.5 we have that the free cumulants of x^k are given by

$$\kappa_n(x^k) = [\alpha * \zeta \cdots * \zeta]_n.$$

On the other hand, by successive application of Equation (1.6.2), we can see that the cumulants of $\pi^{\boxtimes(k-1)} \boxtimes \nu$ are given by

$$\kappa_n(\pi^{\boxtimes(k-1)} \boxtimes \nu) = [\alpha * \zeta \cdots * \zeta]_n,$$

as desired. □

Corollary 5.3.5. *If x is a k-symmetric compound Poisson with rate λ and jump distribution ν, then the distribution of x^k is a compound Poisson with rate 1 and jump distribution $\pi^{\boxtimes k-1} \boxtimes \nu^k$.*

Proof. If x is a k-symmetric compound Poisson with Lévy measure μ, then $\kappa_n(x) = m_n(\nu)$. So, $\alpha_n = \kappa_{kn}(x) = m_{kn}(\nu) = m_n(\nu^k)$, that is, α_n is the free cumulant sequence of $\pi \boxtimes \nu^k$. By Proposition 5.3.4

$$distr(x^k) = \pi^{\boxtimes k} \boxtimes \nu^k.$$

That is, μ_{x^k} is a free compound Poisson with rate 1 and Lévy measure $\pi^{\boxtimes k-1} \boxtimes \nu^k$. □

We prove that free infinite divisibility is maintained under the mapping $\mu \to \mu^k$; this generalizes results of [8] where the case $k = 2$ was considered.

Theorem 5.3.6. *If μ is k-symmetric and $\boxplus$-infinitely divisible, then μ^k is also $\boxplus$-infinitely divisible. Moreover, μ^k has the representation $\pi^{\boxtimes k-1} \boxtimes \nu$ for some measure ν supported on the positive real line.*

Proof. Suppose that μ is infinitely divisible. Then μ can be approximated by free compound Poissons which are k-symmetric. Say $\mu = \lim_{n\to\infty} \mu_n$ where $\mu_n = \pi \boxtimes \nu_n$. By the previous corollary there $\mu_n^k = \pi^{\boxtimes k} \boxtimes \nu_n$. Now $\mu_n^k \to \mu^k$ and since $ID^{\boxplus}(\mathcal{M}_k)$ is closed in the weak convergence topology we have that μ is infinitely divisible. The representation follows from the representation of the approximating measures. □

Remark 5.3.7. Similar arguments as in the proof of Theorem 5.3.6 yield that if μ is k-symmetric and $\boxplus$-infinitely divisible, then μ^n is $\boxplus$-infinitely divisible whenever n divides k.

Corollary 5.3.8. *A k-symmetric $\boxplus$- infinitely divisible measure has at most 1 atom.*

Proof. This follows from the well known result of Bercovici and Voiculescu [25] that a freely infinitely divisible measure on $\mathbb{R}$ has at most 1 atom (which already contains $k = 1$ and $k = 2$).

Indeed, for $k \geq 3$, let μ be k-symmetric $\boxplus$- infinitely divisible measure. From Theorem 5.3.6 we can represent μ^k as $\pi^{\boxtimes k-1} \boxtimes \nu$. Now, we may write μ^k as

$$\mu^k = \pi^{\boxtimes 2} \boxtimes \pi^{\boxtimes k-3} \boxtimes \nu = (\pi \boxtimes \sqrt{\pi^{\boxtimes k-3} \boxtimes \nu})^2,$$

where, for a measure ρ, the measure $\sqrt{\rho}$ denotes the symmetric square root of the measure μ. Noticing that $\pi \boxtimes \sqrt{\pi^{\boxtimes k-3} \boxtimes \nu}$ is freely infinitely divisible and thus has at most one atom (at 0 because of symmetry), we see that μ^k and μ have at most one atom (at 0). $\square$

Finally we come back to the question of defining free convolution. We give a partial answer to the question raised in previous chapter.

Theorem 5.3.9. *Let μ and ν be k-symmetric freely infinitely divisible measures. Then there exists a k-symmetric $\mu \boxplus \nu$ such that*

$$\kappa_n(\mu \boxplus \nu) = \kappa_n(\mu) + \kappa_n(\nu).$$

Moreover $\mu \boxplus \nu$ is also freely infinitely divisible.

Proof. Since the free convolution of k-divisible free compound Poisson is also a k-divisible free compound the by Theorem 5.3.3 this is also true for k-symmetric freely infinitely divisible measures. $\square$

It would be interesting to give a Lévy-Kintchine Formula and study triangular arrays for k-symmetric probability measures.

5.4 Free multiplicative powers of measures on $\mathbb{R}^+$ revisited

In this section, for a probability measure $\mu \in \mathcal{M}_+$ with compact support we will denote by $\mu^{1/k}$ the positive measure with $m_{nk}(\mu^{1/k}) = m_n(\mu)$ and $\mu^{[1/k]}$ the k-symmetric measure such that $m_{nk}(\mu^{[1/k]}) = m_n(\mu)$. Consider Remark 4.2.9 for $\nu = \frac{1}{k}\sum_{j=1}^{k} \delta_{q^j}$, a k-Haar measure. Then

$$(\mu \boxtimes \sum_{j=1}^{k} \delta_{q^j})^k = \mu^{\boxtimes k}. \tag{5.4.1}$$

Using this fact, the moments of $\mu^{\boxtimes k}$ may be calculated using k-divisible non-crossing partitions as we show in the following proposition.

Theorem 5.4.1. *Let μ be a measure with positive support. Then the moments of $\mu^{\boxtimes k}$ are given by*

$$m_n(\mu^{\boxtimes k}) \;\; = \sum_{\pi \in NC^k(n)} \kappa_{Kr(\pi)}(\mu), \tag{5.4.2}$$

where $NC^k(n)$ denotes the k-divisible partitions of $[kn]$.

Proof. Let $\nu = \sum_{j=1}^{k} \delta_{q^j}$, the moments of $\mu \boxtimes \nu$ can be calculated using Theorem 4.4.1:

$$m_n(\mu \boxtimes \nu) = \sum_{\pi \in NC^k(n)} \kappa_{Kr(\pi)}(\mu) m_\pi(\nu) = \sum_{\pi \in NC^k(n)} \kappa_{Kr(\pi)}(\mu)$$

where the last equality follows since $m_\pi(\nu) = 0$ unless π is k-divisible. □

This formula has been generalized for non-identically distributed random variables in [10], where it was used to give new proofs of results in Kargin [40, 42] and Sakuma and Yoshida [68] regarding the asymptotic behaviors of $\mu^{\boxtimes k}$ and $(\mu^{\boxtimes k})^{\boxplus k}$, respectively. Results in [10] will be explained in next chapter.

Moreover, from results of Tucci [76] we know that the k-th root of the measure $\mu^{\boxtimes k}$ converges to a non-trivial measure. More precisely, he proved the following.

Theorem 5.4.2. *Let μ be a probability measure with compact support. If we denote by $\mu_k = (\mu^{\boxtimes k})^{1/k}$, then μ_k converges weakly to $\hat{\mu}$, where $\hat{\mu}$ is the unique measure characterized by $\hat{\mu}([0, \frac{1}{S_\mu(t-1)}]) = t$ for all $t \in (0,1)$. The support of the measure $\hat{\mu}$ is the closure of the interval*

$$(\alpha, \beta) = ((\int_0^\infty x^{-1} d\mu(x))^{-1}, \int_0^\infty x d\mu(x)),$$

where $0 \leq \alpha < \beta \leq \infty$

On the other hand, for R-diagonal operators, Haagerup and Larsen [38] proved the following.

Theorem 5.4.3. *Let T be an R-diagonal operator and $t \in (0,1)$. If $\nu := \mu_{|T|^2}$ is not a Dirac measure then $\mu_T(B(0, \frac{1}{\sqrt{S_\nu(t-1)}})) = t$ where $B(0,r) = \{z \in \mathbb{C} : |z| < r\}$*

If we combine these two results with 5.4.1 we obtain the following interesting interpretation of the limiting distribution.

Theorem 5.4.4. *Let $a, u \in A$ be free elements where a is positive and u a Haar unitary. Moreover, let μ be a probability measure with compact support distributed as a^2. If we denote by*

$$\tilde{\mu}_k = \mu \boxtimes \frac{1}{k} \sum_{j=1}^{k} \delta_{q^j} \tag{5.4.3}$$

then $\tilde{\mu}_k$ converges weakly to μ_∞ where μ_∞ is the rotationally invariant measure such that $\mu_\infty(B(0,t^2)) = \mu_{au}(B(0,t))$, where $\mu_a u$ is the Brown measure of au.

Proof. Let $T = au$, then $|T|^2 = a^2$, so $\mu_{|T|^2} = \mu$. Now, since $(\mu^{\boxtimes k})^{1/k}$ converges to $\hat{\mu}$, then $\mu \boxtimes \sum_{j=1}^{k} \delta_{q^j} = (\mu^{\boxtimes k})^{[1/k]}$ converges to the rotationally invariant measure μ_∞ with $\mu_\infty(B(0,t)) = \hat{\mu}(0,t)$. This implies that

$$\mu_\infty(B(0, \frac{1}{S_\mu(t-1)})) = t = \mu_T(B(0, \frac{1}{\sqrt{S_\mu(t-1)}}))$$

and then $\mu_\infty(B(0,t^2)) = \mu_{au}(B(0,t))$, as desired. □

Remark 5.4.5. (1) Haagerup and Möller [37] have generalized results of [76] to unbounded operators. The previous theorem can be generalized to unbounded operators using the analytic methods of next section.

(2) Recall from Example 3.4.3 that random permutation matrices with cycles of size k are asymptotically free k-Haar unitaries. One can think of a Haar unitary as a limit of k-Haar unitaries. From the previous theorem R-diagonal elements can be thought as the limit of k-divisible ones of the type (5.4.3).

Example 5.4.6 (∞-semicircle)**.** Let s_k be the k-semicircle distribution from Theorem 5.1.1. Then there exist a measure s_∞ such that

$$\lim_{k\to\infty} s_k \to s_\infty.$$

Combining Theorems 5.1.1, and 5.4.4 one can see that $s_\infty(B(0,t)) = t$.

Indeed, since $(s_{k+1})^{k+1} = \pi^{\boxtimes k} = (\pi \boxtimes \frac{1}{k}\sum_{j=1}^{k} \delta_{q^j})^k$ then by Theorem 5.4.4, $s_\infty(B(0,t^2)) = \mu_{au}(B(0,t)) = t^2$, where a is a quarter circular (see Example 5.2 of [38]).

5.5 Squares of random variables with symmetric distributions in $I^{\boxplus}$

We now specialize in the case k=2. This is part of a joint work with Hasebe and Sakuma [8]. Given a probability measure μ, we recall that μ^p for $p \geq 0$ denotes the probability measure in $\mathcal{M}^+$ induced by the map $x \mapsto |x|^p$. For a measure λ on $\mathbb{R}$ we denote by $\mathrm{Sym}(\lambda)$ the symmetric measure $\frac{1}{2}(\lambda(dx) + \lambda(-dx))$.

A particularly important class of freely infinitely distribution are the free regular ones since they correspond to free Lévy processes with positive increments known as free subordinators.

Definition 5.5.1. A probability measure ν is called free regular if $\nu^{\boxplus t} \in \mathcal{M}^+$ for all $t > 0$.

We denote by $I^{\boxplus}$ the freely infinitely divisible measures and by $I^{\boxplus}_{r+}$ the class of free regular measures. Free regular measures are closed under free multiplicative convolution as proved in [8].

Proposition 5.5.2. *Let $\mu \in I^{\boxplus}_{r+}$ and $\nu \in I^{\boxplus}$, then $\mu \boxtimes \nu$ is freely infinitely divisible. Moreover if $\nu \in I^{\boxplus}_{r+}$ then $\mu \boxtimes \nu \in I^{\boxplus}_{r+}$.*

We quote a result from Sakuma and Pérez-Abreu [62, Theorem 12].

Theorem 5.5.3. *A symmetric probability measure μ is $\boxplus$-infinitely divisible if and only if there is a free regular distribution σ such that $\mathcal{C}^{\boxplus}_\mu(z) = \mathcal{C}^{\boxplus}_\sigma(z^2)$. Moreover, the free characteristic triplets $(0, a_\mu, \nu_\mu)$ and $(\eta_\sigma, 0, \nu_\sigma)$ are related as follows: $\nu_\mu = \mathrm{Sym}(\nu_\sigma^{1/2})$ (or equivalently $\nu_\sigma = \nu_\mu^2$), $a_\mu = \eta_\sigma$.*

The following proposition implies that the square of a symmetric measure which is $\boxplus$-infinitely divisible is also $\boxplus$-infinitely divisible. A similar result is proved in [18] for the rectangular free convolution of Benaych-Georges.

Theorem 5.5.4. *Let μ be a $\boxplus$-infinitely divisible symmetric measure then $\mu^2 = m \boxtimes \sigma$, the compound free Poisson with rate 1 and jump distribution σ, where σ is the free regular distribution of Theorem 5.5.3. Conversely, if σ is free regular, then* $\mathrm{Sym}\left((m \boxtimes \sigma)^{1/2}\right)$ *is $\boxplus$-infinitely divisible.*

Proof. We prove that the following are equivalent:
(a) $\mu^2 = m \boxtimes \sigma$,
(b) $\mathcal{C}_\mu^\boxplus(z) = \mathcal{C}_\sigma^\boxplus(z^2)$.
Indeed, if $\mu^2 = m \boxtimes \sigma$, then $S_{\mu^2}(z) = S_m(z)S_\sigma(z) = \frac{1}{1+z}S_\sigma(z)$. Combined with the relation $S_{\mu^2}(z) = \frac{z}{1+z}S_\mu(z)^2$, this implies $zS_\sigma(z) = (zS_\mu(z))^2$. Since the inverse of $zS_\lambda(z)$ is equal to $\mathcal{C}_\lambda^\boxplus$ for a probability measure λ, we conclude that $(\mathcal{C}_\sigma^\boxplus)^{-1}(z) = ((\mathcal{C}_\mu^\boxplus)^{-1}(z))^2$, which is equivalent to (b). Clearly the converse is also true. The desired result immediately follows from the above equivalence and Theorem 5.5.3. □

Now the following result of Sakuma and Pérez-Abreu [62, Theorem 22] follows as a consequence of Theorem 5.5.4.

Theorem 5.5.5. *Let $\sigma \in \mathcal{M}^+$ and w be the standard semicircle law. Then $\sigma \boxtimes \sigma \in I_{r+}^\boxplus$ if and only if $\mu = w \boxtimes \sigma \in I^\boxplus$.*

Remark 5.5.6. It is not true that the square of a symmetric infinitely divisible distribution in the classical sense is also infinitely divisible. For instance, if N_1 and N_2 are independent Poissons then $SN = N_1 - N_2$ is also infinitely divisible and $(SN)^2$ is not infinitely divisible since the support of $(SN)^2$ is $\{0, 1, 4, 9, 25...\}$. (See [75, pp. 51.])

There are two interesting consequences of Theorem 5.5.4. First, Proposition 5.5.4 allows us to identify some non trivial free regular measures which are in $I^* \cap I^\boxplus$: χ^2 and $F(1,1)$. This will be explained in example 5.5.11.

The second consequence concerns the commutator of two free even elements.

Corollary 5.5.7. *Let a_1, a_2 be free, self-adjoint and even elements whose distributions μ_1, μ_2 are $\boxplus$-infinitely divisible. Then the distribution of the free commutator $\mu_1 \square \mu_2 := \mu_{i(a_1a_2 - a_2a_1)}$ is also $\boxplus$-infinitely divisible.*

Remark 5.5.8. If a_1, a_2 are free, even and self-adjoint, the distribution of the anti-commutator $\mu_{a_1a_2+a_2a_1}$ is the same as $\mu_{i(a_1a_2-a_2a_1)}$ [59].

Proof. It was proved by Nica and Speicher [59] that $\mu_1 \square \mu_2$ is also symmetric and satisfies

$$((\mu_1 \square \mu_2)^{\boxplus 1/2})^2 = \mu_1^2 \boxtimes \mu_2^2. \tag{5.5.1}$$

Since, for $i = 1, 2$, the distribution μ_i is symmetric and belongs to $I^\boxplus$, by Theorem 5.5.4, we have the representation $\mu_i^2 = m \boxtimes \sigma_i$, for some σ_i free regular. Then $((\mu_1 \square \mu_1)^{\boxplus 1/2})^2 = m \boxtimes \sigma$ with $\sigma = m \boxtimes \sigma_1 \boxtimes \sigma_2$. Now, by Theorem 5.5.2, σ is free regular and then $(\mu_1 \square \mu_2)^{\boxplus 1/2}$ is $\boxplus$-infinitely divisible. The desired result now follows. □

When we restrict μ_1 to the standard semicircle law, we obtain the analog of Theorem 5.5.5 for the free commutator.

Corollary 5.5.9. *Let σ be a symmetric measure and w be the standard semicircle law. Then $\sigma^2 \in I_{r+}^\boxplus$ if and only if $\mu = w \square \sigma \in I^\boxplus$.*

Proof. It is well known that the $w^2 = m$ and then we get from Equation (5.5.1) that $((w\square\sigma)^{\boxplus 1/2})^2 = m \boxtimes \sigma^2$. The result now follows from Theorem 5.5.4. $\square$

Moreover, Nica and Speicher reduced the problem of calculating the cumulants of the free commutator to symmetric measures. A further analysis of this reduction in combination with Corollary 5.5.7 enables us to omit the assumption of evenness.

Theorem 5.5.10. *Let a_1 and a_2 be free and self-adjoint elements, and let $\mu_1 := \mu_{a_1}$ and $\mu_2 := \mu_{a_2}$ be $\boxplus$-infinitely divisible distributions. Then the distribution of the free commutator $\mu_1\square\mu_2 := \mu_{i(a_1a_2-a_2a_1)}$ is also $\boxplus$-infinitely divisible.*

Proof. By an approximation similar to Proposition 5.3.6, it is enough to consider μ_1 and μ_2 compound free Poissons. Let $\mu_1\square\mu_2$ be the free commutator and $\kappa_n(\mu_i) = \lambda_i m_n(\nu_i)$ the free cumulants of μ_i, for $i = 1, 2$. It is clear that $m_{2n}(\nu_i) = m_{2n}(\mathrm{Sym}(\nu_i))$ and $m_{2n+1}(\mathrm{Sym}(\nu_i)) = 0$. Now, by Theorem 1.2 in [59], the free cumulants of $\mu_1\square\mu_2$ only depend on the even free cumulants of μ_1 and μ_2, and therefore we can change μ_i by the symmetric compound Poisson with Lévy measure $\mathrm{Sym}(\nu_i)$. Thus by Corollary 5.5.7 $\mu_1\square\mu_2$ is $\boxplus$-infinitely divisible as desired. $\square$

Examples

As a first example we use Theorem 5.5.4 to identify measures in $I^* \cap I_{r+}^{\boxplus}$.

Example 5.5.11. The following measures are both classically and freely infinitely divisible.

(1) Let χ^2 be a chi-squared with 1 degree of freedom with density

$$f(x) := \frac{1}{\sqrt{2\pi x}} e^{-x/2}, \qquad x > 0.$$

It is well known that χ^2 is infinitely divisible in the classical sense. It was proved in [23] that a symmetric Gaussian Z is $\boxplus$-infinitely divisible. Hence, by Theorem 5.5.4, Z^2 is free regular. $Z^2 \sim \chi^2$ and then $\chi^2 \in I^* \cap I_{r+}^{\boxplus}$

(2) Let $F(1,n)$ be an F-distribution with density

$$f(x) := \frac{1}{B(1/2, n/2)} \frac{1}{(nx)^{1/2}} \left(1 + \frac{x}{n}\right)^{-(1+n)/2}, \quad x > 0.$$

$F(1,n)$ is classically infinitely divisible,as can be seen in [41]. On the other hand $F(1,n)$ is the square of a t-student with n degrees of freedom $t(n)$. In particular $t(1)$ is the Cauchy distribution, hence by Theorem 5.5.4, $F(1,1)$ belongs to $I^* \cap I_{r+}^{\boxplus}$.

Remark 5.5.12. Numeric computations of free cumulants have shown that the chi-squared with 2 degrees of freedom is not freely infinitely divisible. However, the free infinite divisibility of t-student with n degrees of freedom is still an open question.

Next, we give some examples of free regular measures from known distributions in non-commutative probability.

Example 5.5.13. (1) Free one-sided stable distributions with non-negative drifts. These distributions are found by Biane in Appendix in [20].

(2) The square of a symmetric $\boxplus$-stable law. By Theorem 5.5.4 it is free regular, and moreover, by the results of [9] we can identify the Lévy measure σ of Theorem 5.5.4 with a $\boxplus$-stable law. Indeed, any symmetric stable measure has the representation $w \boxtimes \nu_{\frac{1}{1+t}}$ and then by Equation (1.5.6) the square is $w^2 \boxtimes \nu_{\frac{1}{1+t}} \boxtimes \nu_{\frac{1}{1+t}} = m \boxtimes \nu_{\frac{1}{1+2t}}$.

(3) Free multiplicative, free additive and Boolean powers of the free Poisson m. In particular, for $t \geq 1$the free Bessel laws $m^{\boxtimes t} \boxtimes m^{\boxplus s}$ studied in [13] are free regular.

(4) The free Meixner laws, which are introduced by Saitoh and Yoshida [66] and Anshelevich [1], whose Lévy measures are given by

$$\nu_{a,b,c}(dx) = c\frac{\sqrt{4b-(x-a)^2}}{\pi x^2} 1_{a-2\sqrt{b}<x<a+2\sqrt{b}}(x)dx.$$

If $a-2\sqrt{b} \geq 0$, then the Lévy measure is concentrated on $[0,\infty)$ and $\int_{\mathbb{R}} \min(1,|x|)\nu_{a,b,c}(dx) < \infty$. Thus, if the drift term is non-negative, then it will be free regular. This case includes the free gamma laws, which come from interpretation by orthogonal polynomials not the Bercovici-Pata bijection.

Example 5.5.14. Let w be the standard semicircle law. Then w^2 and w^4 are both free regular. It is well known that $w^2 = m$, which is free regular. From [4], if $\mathbf{b}_s$ is the symmetric beta $(1/2, 3/2)$ distribution, $\mathbf{b}_s$ is freely infinitely divisible and then, by Theorem 5.5.4, $(\mathbf{b}_s)^2$ is free regular.

The symmetric beta distribution $\mathbf{b}_s$ has density

$$\mathbf{b}_s(\mathrm{d}x) = \frac{1}{2\pi}|x|^{-1/2}(2-|x|)^{1/2}\mathrm{d}x, \quad |x|<2.$$

Clearly $m_{2n}(\mathbf{b}_s) = m_{4n}(w)$ and then $(\mathbf{b}_s)^2 = w^4$. Since $w^4 = (\mathbf{b}_s)^2$, we see that w^4 is free regular.

Remark 5.5.15. It is not known if w^{2n} is $\boxplus$-infinitely divisible for all $n > 0$, as in classical probability.

Example 5.5.16 (free commutators)**.** (1) Let σ_s and σ_t be two symmetric free stable distributions of index s and t, respectively. Then by Corollary 5.5.7 the free commutator $\sigma_s \Box \sigma_t$ is $\boxplus$-infinitely divisible. For the case $t = s = 2$ (the Wigner semicircle distribution) the density of $w \Box w$ is given by [59]

$$f(t) = \frac{\sqrt{3}}{2\pi \mid t \mid}\left(\frac{3t^2+1}{9h(t)} - h(t)\right), \qquad \mid t \mid \leq \sqrt{(11+5\sqrt{5})/2}, \tag{5.5.2}$$

where

$$h(t) = \sqrt[3]{\frac{18t^2+1}{27} + \sqrt{\frac{t^2(1+11t^2-t^4)}{27}}}.$$

(2) Let w be the standard semicircle law and let $\nu_{\frac{1}{1+2s}}$ be a positive free stable law, for some $s > 0$. If we denote $\hat{\nu}_{\frac{1}{1+2s}} = \mathrm{Sym}(\nu^{1/2}_{\frac{1}{1+2s}})$ then $\mu := w \Box \hat{\nu}_{\frac{1}{1+2s}}$ is a symmetric free stable distribution with index $\frac{2}{1+2s}$. Indeed, by Equation (5.5.1), μ satisfies

$$(\mu^{\boxplus 1/2})^2 = ((w \Box \hat{\nu}_{\frac{1}{1+2s}})^{\boxplus 1/2})^2 = w^2 \boxtimes \nu_{\frac{1}{1+2s}} = m \boxtimes \nu_{\frac{1}{1+2s}}.$$

From Equation (1.5.6) and results in [9] we see that $m \boxtimes \nu_{\frac{1}{1+2s}} = (w \boxtimes \nu_{\frac{1}{1+s}})^2$. This means that $\mu^{\boxplus 1/2} = w \boxtimes \nu_{\frac{1}{1+s}}$ which is a symmetric free stable distribution with index $\frac{2}{1+2s}$. The case $s = 1/2$ was treated in [59, Example 1.14].

(3) Assume that b is a symmetric Bernoulli distribution $\frac{1}{2}(\delta_{-1}+\delta_1)$. Let μ, ν be symmetric distributions. Then the free commutator $\mu \Box \nu$ is 2-$\boxplus$-divisible, but when $\mu = \nu$ we can identify $(\mu \Box \mu)^{\boxplus 1/2}$. Indeed, by Eq. (5.5.1), $(\mu \Box \mu)^{\boxplus 1/2} = \sqrt{\mu^2 \boxtimes \mu^2}$. On the other hand, by Equation (1.5.6), $(\mu^2 \boxtimes b)^2 = \mu^2 \boxtimes \mu^2$. Hence $(\mu^2 \boxtimes b)^{\boxplus 2} = \mu \Box \mu$.

When $\mu = w$ a strange thing happens: $w^2 = m$, and $m \boxtimes b$ is a compound free Poisson with rate 1 and jump distribution b. This implies that $w \Box w = m \boxplus \widetilde{m}$, where $\widetilde{m}$ is defined by $\widetilde{m}(B) = m(-B)$. It is a free symmetrization of the Poisson distribution (not to be confused with the symmetric beta of Example 5.5.14). As pointed out in [59], this gives another derivation of the density of $w \Box w$ given in Equation (5.5.2).

(4) For the free Poisson with mean 1, the free commutator becomes $m \Box m = (m \boxtimes m \boxtimes b)^{\boxplus 2}$, the compound free Poisson with rate 2 and jump distribution $m \boxtimes b$. Indeed, if we define $\hat{m} := m \boxtimes b$, we have that $m \Box m = \hat{m} \Box \hat{m}$ since the even free cumulants of $\hat{m}$ are all one, the same as those of m, and since the free commutator of measures depends only on the even cumulants of the measures [59, Theorem 1.2]. By Equation (1.5.6) we have $\hat{m}^2 = m \boxtimes m$, and therefore by Equation (5.5.1), we have

$$((m \Box m)^{\boxplus 1/2})^2 = m \boxtimes m \boxtimes m \boxtimes m.$$

Again using Equation (1.5.6) we see that $m \boxtimes m \boxtimes m \boxtimes m = (m \boxtimes m \boxtimes b)^2$. The claim then follows.

Chapter 6

Products of free random variables and k-divisible non-crossing partitions

As we have seen in Theorem 5.4.1 the moments of $\mu^{\boxtimes k}$ may be computed using k-divisible non-crossing partitions. In this chapter we explain results of the joint work with Vargas [10], where we derive formulas for the moments and the free cumulants of the product of k free random variables in terms of k-equal and k-divisible non-crossing partitions. Basically, we exploit the fact that k-divisible and k-equal partitions are linked, by the Kreweras complement, to partitions which are involved in the calculation of moments and free cumulants of the product of k free random variables.

These formulae lead to a very simple proof for the bounds of the right-edge of the support of the free multiplicative convolution $\mu^{\boxtimes k}$, given by Kargin in [40], which show that the growth of this support is at most linear. Moreover, this combinatorial approach generalize the results of Kargin since we do not require the convolved measures to be identical. We also give further applications, such as a new proof of the limit theorem of Sakuma and Yoshida [68].

6.1 Introduction

Recall that given $a, b \in \mathcal{A}$ free random variables, with free cumulants $\kappa_n(a)$ and $\kappa_n(b)$, respectively, one can calculate the free cumulants of ab by

$$\kappa_n(ab) = \sum_{\pi \in NC(n)} \kappa_\pi(a)\kappa_{Kr(\pi)}(b). \tag{6.1.1}$$

where $Kr(\pi)$ is the Kreweras complement of the non-crossing partition π. Therefore, we are able to compute the free cumulants of the free multiplicative convolution of two compactly supported probability measures μ, ν, such that $Supp(\mu) \subseteq [0, \infty)$ by

$$\kappa_n(\mu \boxtimes \nu) = \sum_{\pi \in NC(n)} \kappa_\pi(\mu)\kappa_{Kr(\pi)}(\nu). \tag{6.1.2}$$

In principle, this formula could be inductively used to provide the free cumulants and moments of the convolutions of k (not necessarily equal) positive probability measures.

This approach, however, prevents us from noticing the deeper combinatorial structure behind such products of free random variables.

Our fundamental observation is that, when π and $Kr(\pi)$ are drawn together, the partition $\pi \cup Kr(\pi) \in NC(2n)$ is exactly the Kreweras complement of a 2-equal partition (i.e. a non-crossing pairing). Furthermore, one can show using the previous correspondence that Equation 6.1.1 may be rewritten as

$$\kappa_n(ab) = \sum_{\pi \in NC_2(n)} \kappa_{Kr(\pi)}(a, b, \dots, a, b), \tag{6.1.3}$$

where $NC_2(n)$ denotes the 2-equal partitions of $[2n]$.

Since 2-equal partitions explain the free convolution of two variables, it is natural to try to describe the product of k free variables in terms of k-equal partitions.

The main result of this chapter is the following.

Theorem 6.1.1. *Let $a_1, \dots, a_k \in (\mathcal{A}, \tau)$ be free random variables. Then the free cumulants and the moments of $a := a_1 \dots a_k$ are given by*

$$\kappa_n(a) = \sum_{\pi \in NC_k(n)} \kappa_{Kr(\pi)}(a_1, \dots, a_k) \tag{6.1.4}$$

$$\tau(a^n) = \sum_{\pi \in NC^k(n)} \kappa_{Kr(\pi)}(a_1, \dots, a_k) \tag{6.1.5}$$

where $NC_k(n)$ and $NC^k(n)$ denote, respectively the k-equal and k-divisible partitions of $[kn]$.

The main application of our formulas is a new proof of the fact that for positive measures centered at 1, the support of the free multiplicative convolution $\mu^{\boxtimes k}$ grows at most linearly. More precisely,

Theorem 6.1.2. *Let $\sigma, L > 0$ be given. There exist universal constants $C, c > 0$ such that for all k and any $\mu_1, \dots, \mu_k$ probability measures supported on $[0, L]$, satisfying $E(\mu_i) = 1$ and $Var(\mu_i) > \sigma^2$, for $i = 1, \dots, k$, the supremum L_k of the support of the measure $\mu_1 \boxtimes \cdots \boxtimes \mu_k$ satisfies*

$$ck < L_k < Ck.$$

In other words, for positive free random variables X_i such that $E(X_i) = 1$, and $Var(X_i) > \sigma^2$, (not necessarily identically distributed) we have that

$$\limsup n^{-1} ||X_1 \circ \cdots \circ X_n|| < C$$

and

$$\liminf n^{-1} ||X_1 \circ \cdots \circ X_n|| > c > 0,$$

where, for X, Y positive random variables, we write $X \circ Y := X^{1/2} Y X^{1/2}$.

6.2 Main formulae

The following characterization plays a central role in this work. The proof, elementary but cumbersome, will be detailed in the last section of the present chapter.

Proposition 6.2.1. *i)* $\pi \in NC(kn)$ *is* k*-preserving if and only if* $\pi = Kr(\sigma)$ *for some* k*-divisible partition* $\sigma \in NC^k(n)$.

ii) $\pi \in NC(kn)$ *is* k*-separating if and only if* $\pi = Kr(\sigma)$ *for some* k*-equal partition* $\sigma \in NC_k(n)$.

Remark 6.2.2. In view of the previous characterization, for a k-divisible partition π, the Kreweras complement $Kr(\pi)$ may be divided into k partitions $\pi_1, \pi_2 ..., \pi_k$, with π_j involving only numbers congruent to j mod k. In this case we will write $\pi_1 \cup \cdots \cup \pi_k = Kr(\pi)$ for such decomposition.

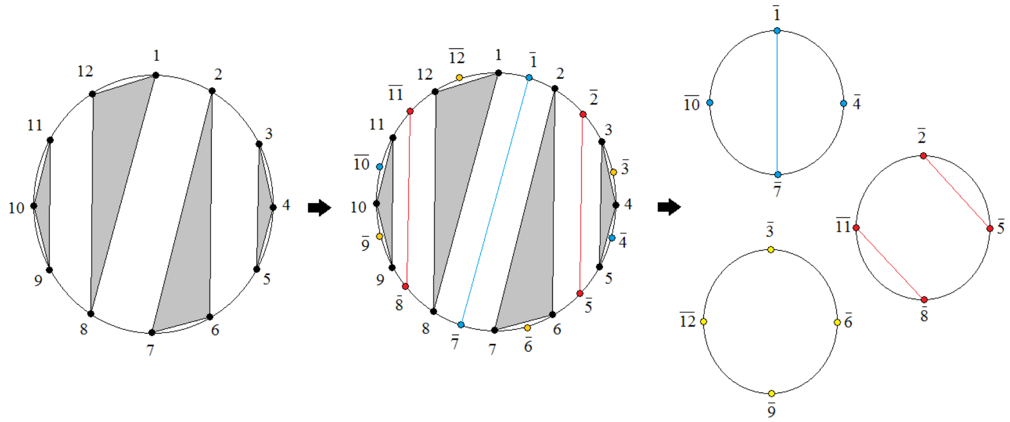


Figure above shows the 3-equal partition $\{(1, 8, 12), (2, 6, 7), (3, 4, 5), 9, 10, 11)\}$ and its Kreweras complement $Kr(\pi) = \pi_1 \cup \pi_2 \cup \pi_3$, with $\pi_1 = \{(1, 7)(4), (10)\}$, $\pi_2 = \{(2, 5), (6, 11)\}$ and $\pi_3 = \{(3), (6), (9), 12)\}$

We are ready to prove the Main Theorem.

Proof of Theorem 6.1.1. By the formula for products as arguments, we have that

$$\kappa_n(a) = \prod_{\substack{\pi \in NC(kn) \\ \pi \wedge \rho_n^k}} \kappa_\pi(a_1, \ldots, a_n).$$

Since the random variables are free, the sum actually runs over k-preserving partitions (otherwise there would be a mixed cumulant). But then by Proposition 6.2.1 ii), the partitions involved in the sum are exactly the Kreweras complements of k-equal partitions, and the formula follows.

For the first formula for the moments, we use the moment cumulant formula

$$\tau(a^n) = \sum_{\pi \in NC(kn)} \kappa_\pi(a_1, \ldots, a_n).$$

Again, the elements involved are free, so only k-preserving partitions matter, and these are the Kreweras complements of k-divisible partitions by Proposition 6.2.1 i). Thus (6.1.5) follows. □

Corollary 6.2.3. *Let $\mu_1, \dots, \mu_k$ be probability measures with positive bounded support and let $\mu = \mu_1 \boxtimes \cdots \boxtimes \mu_k$. Using the notation above, we may rewrite our formulas,*

$$\kappa_n(\mu) = \sum_{\pi \in NC_k(n)} \kappa_{\pi_1}(\mu_1) \dots \kappa_{\pi_k}(\mu_k) \tag{6.2.1}$$

$$m_n(\mu) = \sum_{\pi \in NC^k(n)} \kappa_{\pi_1}(\mu_1) \dots \kappa_{\pi_k}(\mu_k) \tag{6.2.2}$$

where $\pi_1 \cup \cdots \cup \pi_k = Kr(\pi)$ is the decomposition described in Remark 6.2.2.

Remark 6.2.4. From Equation (6.2.2) it is easy to see that for compactly supported measures with mean 1, the variance is additive with respect to free multiplicative convolution, that is

$$Var(\mu_1 \boxtimes \cdots \boxtimes \mu_k) = \kappa_2(\mu_1 \boxtimes \cdots \boxtimes \mu_k) = \sum_{i=1}^{k} \kappa_2(\mu_i) = \sum_{i=1}^{k} Var(\mu_i)$$

6.3 Supports of free multiplicative convolutions

Our main result, Theorem 6.1.1 can be used to compute bounds for the supports of multiplicative free convolutions of positive measures. We recall that the number of non-crossing partitions in $NC(n)$ is given by the Catalan number $C_n \leq 4^n$. We also know that the Möbius function $\mu : NC(n) \times NC(n) \to \mathbb{C}$ is bounded in absolute value by $C_{n-1} \leq 4^{n-1}$. Then we can control the size of the free cumulants.

Lemma 6.3.1. *Let μ be a probability measure supported on $[0, L]$ with variance σ^2, such that $E(\mu) = 1$. Then $\kappa_2 = \sigma^2 \leq L - 1$ and $|\kappa_n^{\mu}| < (26L)^{n-1}$.*

Proof. Its easy to see that $L \geq 1$ and

$$m_n^{\mu} = \int_0^L x^n \mathrm{d}\mu(x) \leq \int_0^L L^{n-1} x \mathrm{d}\mu(x) = L^{n-1}.$$

Then we have that $\kappa_1^{\mu} = 1$, $0 < \kappa_2^{\mu} \leq L - 1 < 26L$, and $|\kappa_3^{\mu}| \leq L^2 + 3L + 1 < 26^2 L^2$, and for $n > 4$ we have

$$|\kappa_n^{\mu}| = \sum_{\pi \in NC(n)} |m_{\pi}^{\mu}||\mu[\pi, 1_n]| \leq \sum_{\pi \in NC(n)} L^{n-1} 4^{n-1} \leq 4^{2n-1} L^{n-1} < (26L)^{n-1}$$

since $4^7 < 26^3$. ☐

Now, we easily see that the growth of the support is no less than linear.

Proposition 6.3.2. *Let $\mu_1, \dots, \mu_k$ be compactly supported probability measures on $\mathbb{R}^+$, satisfying $E(\mu_i) = 1$, $Var(\mu_i) = \sigma^2$ and Let L_k be the supremum of the support of $\mu := \mu_1 \boxtimes \cdots \boxtimes \mu_k$. Then $L_k \geq k\sigma^2 + 1$.*

Proof. It is clear that $E(\mu) = 1$, and hence by Remark 6.2.4 we know that $\kappa_2(\mu) = Var(\mu) = k\sigma^2$. By Lemma 6.3.1 we have that $\kappa_2(\mu) = k\sigma^2 \leq L_k - 1$. ☐

Now we give an upper bound for the support.

Proposition 6.3.3. *There exists a universal constant C such that for all k and all $\mu_1, \ldots, \mu_k$ probability measures supported on $[0, L]$, satisfying $E(\mu_i) = 1$, $i = 1, \ldots, k$,the measure $\mu := \mu_1 \boxtimes \cdots \boxtimes \mu_k$ satisfies*

$$Supp(\mu) \subseteq [0, CL(k+1)].$$

In general, C may be taken to be $26e$. If the measures μ_i, $1 \leq i \leq k$ have non-negative free cumulants, C may be taken to be $e = 2.71...$

Proof. By Equation 6.2.2 we get

$$m_n(\mu) = \sum_{\pi \in NC^k(n)} \kappa_{\pi_1}(\mu_1) \ldots \kappa_{\pi_k}(\mu_k). \tag{6.3.1}$$

Since a k-divisible partition $\pi \in NC^k(n)$ has at most n blocks, we know that

$$|\pi_1| + \cdots + |\pi_k| = |Kr(\pi)| = kn + 1 - |\pi| \geq (k-1)n + 1.$$

Now, let $\tilde{L} = 26L$. By Lemma 6.3.1, we know that $\kappa_{\pi_i}(\mu_i) \leq (\tilde{L})^{n-|\pi_i|}$. Hence

$$\begin{aligned}
\sum_{\pi \in NC_k(n)} \kappa_{\pi_1}(\mu_1) \ldots \kappa_{\pi_k}(\mu_k) &\leq \sum_{\pi \in NC_k(n)} (\tilde{L})^{kn-(|\pi_1|+\cdots+|\pi_k|)} &(6.3.2)\\
&\leq \sum_{\pi \in NC_k(n)} (\tilde{L})^n &(6.3.3)\\
&= \frac{\binom{(k+1)n}{n}}{kn+1} (\tilde{L})^n &(6.3.4)
\end{aligned}$$

By taking the n-th root and the use of Stirling approximation formula, we obtain that

$$\begin{aligned}
\limsup_{n\to\infty} (m_n(\mu))^{1/n} &= \limsup_{n\to\infty} \left(\sqrt{\frac{(k+1)n}{2\pi k n^2}} \frac{((k+1)n)^{(k+1)n} e^{-(k+1)n}}{(kn)^{kn} e^{-kn} n^n e^{-n} (kn+1)} (\tilde{L})^n \right)^{1/n} &(6.3.5)\\
&= \frac{(k+1)^{(k+1)}}{k^k} (\tilde{L}) &(6.3.6)\\
&\leq (k+1)e\tilde{L}. &(6.3.7)
\end{aligned}$$

If μ has non-negative free cumulants we may replace $\tilde{L}$ by L. □

6.4 More applications and examples

In this section we want to show some examples of how Theorem 5.5.4 may be used to calculate free cumulants.

Example 6.4.1. (Product of free Poissons) Theorem 5.5.4 takes a very easy form in the particular case $\mu_i = \mathrm{m}$, where m is the Marchenko-Pastur distribution of parameter 1. Indeed, since $\kappa_n(\mathrm{m}) = 1$, we get

$$\kappa_n(\mathrm{m}^{\boxtimes k}) = \sum_{\pi \in NC^k(n)} 1 = \frac{\binom{(k)n}{n}}{(k-1)n+1},$$

and

$$m_n(\mathrm{m}^{\boxtimes k}) = \sum_{\pi \in NC_k(n)} 1 = \frac{\binom{(k+1)n}{n}}{kn+1}.$$

Moreover, from the last equation one can easily calculate $L_k = (k+1)^{k+1}/k^k$.

Example 6.4.2. (Product of shifted semicirculars) For $\sigma^2 \leq \frac{1}{4}$, let $\omega_+ := \omega_{1,\sigma^2}$ be the shifted Wigner distribution with mean 1 and variance σ^2. The density of ω_+ is given by

$$\omega_{1,\sigma^2}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-1)^2} \cdot 1_{[1-2\sigma,1+2\sigma]}(x)\mathrm{d}x,$$

and its free cumulants are $\kappa_1(\omega_+) = 1$, $\kappa_2(\omega_+) = \sigma^2$ and $\kappa_n(\omega_+) = 0$ for $n > 2$.

We want to calculate the free cumulants of $\omega_+^{\boxtimes k}$, $k \geq 2$. So let $a_1, \ldots, a_k$ be free random variables with distribution ω_+. By Theorem 5.5.4, the free cumulants of $a := a_1 \cdots a_k$ are given by

$$\kappa_n(a) \quad = \sum_{\pi \in NC^k(n)} \kappa_{Kr(\pi)}(a_1, \ldots, a_k). \tag{6.4.1}$$

If $Kr(\pi)$ contains a block of size greater than 2, then $\kappa_{Kr(\pi)} = 0$. Hence the sum runs actually over $NC(k,n)_{2,1}$. Therefore each summand has the common contribution of $(\sigma^2)^{n-1}$ and by Equation (2.1.5) we know the number of summands. Hence the free cumulants are

$$k\frac{((k-1)n)!(\sigma^2)^{n-1}}{(n-1)!((k-2)n+2)!}. \tag{6.4.2}$$

Note that in this example $m_n(a) \geq \kappa_n(a)$ (since all the free cumulants are positive) and $L = 1+2\sigma$. By Proposition 6.3.3 and an application of Stirling's formula to Equation (6.4.2) the supremum L_k of the support of $\omega_+^{\boxtimes k}$ satisfies

$$(k+1)e(1+2\sigma) \geq L_k = \limsup_{n\to\infty}(m_n(a))^{1/n} \geq \limsup_{n\to\infty}(\kappa_n(a))^{1/n} \geq (k-1)e\sigma^2.$$

Hence we obtain a better estimate than the rough bound provided by Proposition 6.3.2.

As another application of our main formula, we show that the free cumulants of $\mu^{\boxtimes k}$ become positive for large k. This is of some relevance if we recall that our estimates of the support of $\mu^{\boxtimes k}$ are better with the presence of non-negative free cumulants.

Theorem 6.4.3. *Let μ be a probability measure supported on $[0,L]$ with mean α and variance σ^2. Then for each $n \geq 1$ there exist a constant N such that for all $k \geq N$, the first n free cumulants of $\mu^{\boxtimes k}$ are non-negative.*

Proof. Clearly it is enough to show that, for each $n \geq 1$, there exist N_0 such that the n-th free cumulant of $\mu^{\boxtimes k}$ is positive for all $k \geq N_0$.

Let $n > 1$ and $\tilde{\alpha} := \max\{\alpha^{n-1}, 1\}$. By the same arguments as in Lemma 6.3.1 one can show that $|\kappa_n(\mu)| \leq 16(L^n)$. Then by Theorem 5.5.4 we have

$$\begin{aligned}
\kappa_n(\mu^{\boxtimes k}) &= \sum_{\pi \in NC(k,n)_{2,1}} \kappa_{Kr(\pi)}(\mu) + \sum_{\substack{\pi \in NC_k(n) \\ \pi \notin NC(k,n)_{2,1}}} \kappa_{Kr(\pi)}(\mu) \\
&\geq \alpha^{(k-2)n+2} \left(\sum_{\pi \in NC(k,n)_{2,1}} \sigma^{2n-2} - \sum_{\substack{\pi \in NC_k(n) \\ \pi \notin NC(k,n)_{2,1}}} (16L)^{n-1}\tilde{\alpha} \right) \\
&= \alpha^{(k-2)n+2}(|NC(k,n)_{2,1}|\sigma^{2n-2} - (|NC_k(n)| - |NC(k,n)_{2,1}|)(16L)^{n-1}\tilde{\alpha}).
\end{aligned}$$

The factor $\alpha^{(k-2)n+2}$ is positive and the rest of the expression becomes positive for all k larger than some N_0, since, by Equation (2.1.6), $NC(k,n)_{2,1}/NC_k(n) \to 1$ as $k \to \infty$. □

It would be interesting to investigate whether or not all free cumulants become positive. Finally, we give a new proof to the recent limit theorem by Sakuma and Yoshida [68, Theorem 9]. We will restrict to the case $E(\mu) = 1$, the general case follows directly from this.

Proposition 6.4.4. *Let μ be a probability measure on supported on $[0, L]$, with $E(\mu) = 1$ and $Var(\mu) = \sigma^2$, then*

$$\lim_{k\to\infty} D_{1/k}\left((\mu^{\boxtimes k})^{\boxplus k}\right) = \mathfrak{h}_{\sigma^2}$$

where $\mathfrak{h}_{\sigma^2}$ is the unique measures such that $\kappa_n(\mathfrak{h}_{\sigma^2}) = \frac{(\sigma^2 n)^{n-1}}{n!}$.

Proof. First, we see by Stirling's Formula that

$$\lim_{k\to\infty} \frac{1}{k^{n-1}} \sum_{\pi \in NC(k,n)_{2,1}} \kappa_{Kr(\pi)}(\mu) = \lim_{k\to\infty} \frac{1}{k^{n-1}} \sum_{\pi \in NC(k,n)_{2,1}} \sigma^{2n-2} = \frac{n^{n-1}}{n!}\sigma^{2n-2}. \tag{6.4.3}$$

Therefore, by Equation (2.1.6), $|NC(k,n)_{2,1}|$ and $|NC_k(n)|$ are of order k^{n-1} as $k \to \infty$ and

$$\frac{|NC_k(n)| - |NC(k,n)_{2,1}|}{|NC_k(n)|} \to 0.$$

By using the bound $|\kappa_n(\mu)| \leq 16(L^n)$, we obtain that

$$\lim_{k\to\infty} \frac{1}{k^{n-1}} \sum_{\substack{\pi \in NC_k(n) \\ \pi \notin NC(k,n)_{2,1}}} \kappa_{Kr(\pi)}(\mu) \leq \lim_{k\to\infty} \frac{1}{k^{n-1}} \sum_{\substack{\pi \in NC_k(n) \\ \pi \notin NC(k,n)_{2,1}}} (16L)^n = 0. \tag{6.4.4}$$

Hence

$$\begin{aligned}
\lim_{k\to\infty} \kappa_n\left(D_{1/k}\left((\mu^{\boxtimes k})^{\boxplus k}\right)\right) &= \lim_{k\to\infty} \frac{1}{k^{n-1}} \kappa_n(\mu^{\boxtimes k}) \\
&= \lim_{k\to\infty} \frac{1}{k^{n-1}} \sum_{\pi \in NC(k,n)_{2,1}} \kappa_{Kr(\pi)}(\mu) \\
&+ \lim_{k\to\infty} \frac{1}{k^{n-1}} \sum_{\substack{\pi \in NC^k(n) \\ \pi \notin NC(k,n)_{2,1}}} \kappa_{Kr(\pi)}(\mu) \\
&= \frac{n^{n-1}}{n!}\sigma^{2n-2}.
\end{aligned}$$

□

6.5 Proof of Proposition 6.2.1

For a partition $\pi \in NC(n)$ will often write $r \sim_\pi s$, meaning that r, s belong to the same block of π.

Let us introduce two operations on non-crossing partitions. For $n, k \geq 1$ and $r \leq n$, we define $I_r^k : NC(n) \to NC(n+k)$, where $I_r^k(\pi)$ is obtained from π by duplicating the element in the position r, identifying the copies and inserting $k-1$ singletons between the two copies. More precisely, for $\pi \in NC(n)$, $I_k^r(\pi) \in NC(n+k)$ is the partition given by the relations

- for $1 \leq m_1, m_2 \leq r$,
$$m_1 \sim_{I_k^r(\pi)} m_2 \ \Leftrightarrow m_1 \sim_\pi m_2$$
- for $r+k \leq m_1, m_2 \leq n+k$,
$$m_1 \sim_{I_k^r(\pi)} m_2 \ \Leftrightarrow m_1 - k \sim_\pi m_2 - k$$
- for $1 \leq m_1 \leq r$ and $r+k+1 \leq m_2 \leq n+k$,
$$m_1 \sim_{I_k^r(\pi)} m_2 \ \Leftrightarrow m_1 \sim_\pi m_2 - k$$
- $r \sim_{I_k^r(\pi)} r+k$,

The operation $\tilde{I}_r^k : NC(n) \to NC(n+k)$ consists of inserting an interval block of size k between the positions $r-1$ and r in π. We will skip the explicit definition.

The importance of these operations is that they are linked by the relation

$$Kr(I_r^k(\pi)) = \tilde{I}_r^k(Kr(\pi)). \tag{6.5.1}$$

Our operations preserve properties of partitions, as shown in the following lemma.

Lemma 6.5.1. *Let $\pi \in NC(nk)$, $r \leq nk$, $s \geq 1$. Then*
i) π is k-preserving if and only if $I_r^{sk}(\pi)$ is k-preserving.
ii) π is k-separating if and only if $I_r^k(\pi)$ is k-separating.
iii) π is k-divisible if and only if $\tilde{I}_r^{sk}(\pi)$ is k-divisible.
iv) π is k-equal if and only if $\tilde{I}_r^k(\pi)$ is k-equal.

Proof. i) By definition of $I_k^r(\pi)$, the relations indicated by $I_k^r(\pi)$ are obtained by relations indicated by π, with possible shifts by ks (which do not modify congruences modulo k). Hence the equivalence follows.

ii) One should think of the block intervals of ρ_n^k as vertices of a graph. For $\pi \in NC(nk)$, an edge will join two vertices V, W, if there are elements $r \in V$, $s \in W$ such that $r \sim_\pi s$. Then $\pi \wedge \rho_n^k = 1_{nk}$ if and only if the graph is connected.

It is easy to see that the effect of I_r^k on the graph of π is just splitting the vertex corresponding to the block V containing r into 2 vertices V_1, V_2. The edges between all other vertices are preserved, while the edges which were originally joined to V will now be joined either to V_1 or V_2. Finally, the last additional relation $r \sim_{I_r^k(\pi)} r+k$ means an edge joining V_1 to V_2. Therefore, it is clear that the connectedness of the two graphs are equivalent.

iii) and iv) are trivial. □

Now we want to show that we can produce all partitions of our interest by applying our operations to elementary partitions.

Lemma 6.5.2. *i) Let $\pi \in NC(kn)$ be k-preserving. Then there exist $m \geq 0$ and numbers $q_0, q_1, \ldots, q_m$, $r_1, \ldots, r_m$ such that*

$$\pi = I_{r_m}^{kq_m} \circ \cdots \circ I_{r_1}^{kq_1}(0_{q_0}). \tag{6.5.2}$$

ii) Let $\pi \in NC(kn)$ be k-separating. Then there exist $m \geq 0$ and numbers $r_1, \ldots, r_m$ such that

$$\pi = I_{r_m}^{k} \circ \cdots \circ I_{r_1}^{k}(0_k).$$

iii) Let $\pi \in NC(kn)$ be k-divisible. Then there exist $m \geq 0$ and numbers $q_0, q_1, \ldots, q_m$, $r_1, \ldots, r_m$ such that

$$\pi = \tilde{I}_{r_m}^{kq_m} \circ \cdots \circ \tilde{I}_{r_1}^{kq_1}(1_{q_0}).$$

iv) Let $\pi \in NC(kn)$ be k-equal. Then there exist $m \geq 0$ and numbers $r_1, \ldots, r_m$ such that

$$\pi = \tilde{I}_{r_m}^{k} \circ \cdots \circ \tilde{I}_{r_1}^{k}(1_k).$$

Proof. i) We use induction on n. For $n = 1$ the only k-preserving partition is 0_k, so the statement holds. So assume that i) holds for $n \leq m$. For $\pi \in NC_k(m)$ suppose that there exist $1 \leq r < r + sk \leq km$ such that $r \sim_\pi r + sk$ and ,$r + 1, \ldots r + sk - 1$ are singletons of π (if no such pair (r, s) exist, necessarily $\pi = 0_{mk}$ and we are done). Then its easy to see that $\pi = I_r^{sk}(\pi')$ for some $\pi' \in NC((n - s)k)$. By Lemma 6.5.1 i) π' is k-preserving. By induction hypothesis π' has a representation as in Equation (6.5.2) and hence, so does $\pi = I_r^{sk}(\pi')$.

The proof of ii) is similar. The proofs of iii) and iv) are trivial using Remark 2.1.2. □

Now we can prove Proposition 6.2.1.

Proof. We only show the first implication of i). The converse and ii) are similar.

Let $\pi \in NC(n)$ be k-preserving. Then by Lemma 6.5.2 i) we can express it as

$$\pi = I_{r_m}^{kq_m} \circ \cdots \circ I_{r_1}^{kq_1}(1_{q_0})$$

but then we can apply Equation 6.5.1 at every step, obtaining

$$\begin{aligned}
Kr(\pi) &= Kr(I_{r_m}^{kq_m} \circ \cdots \circ I_{r_1}^{kq_1}(0_{q_0})) && (6.5.3)\\
&= \tilde{I}_{r_m}^{kq_m} \circ Kr(I_{r_{m-1}}^{kq_{m-1}} \cdots \circ I_{r_2}^{kq_2} \circ I_{r_1}^{kq_1}(1_{q_0})) && (6.5.4)\\
&= \tilde{I}_{r_m}^{kq_m} \circ \tilde{I}_{r_{m-1}}^{kq_{m-1}} \circ Kr(I_{r_{m-2}}^{kq_{m-2}} \circ \cdots \circ I_{r_2}^{kq_2} \circ I_{r_1}^{kq_1}(1_{q_0})) && (6.5.5)\\
&\;\;\vdots && (6.5.6)\\
&= \tilde{I}_{r_m}^{kq_m} \circ \cdots \circ \tilde{I}_{r_2}^{kq_2} \circ \tilde{I}_{r_1}^{kq_1}(Kr(1_{q_0})) && (6.5.7)\\
&= \tilde{I}_{r_m}^{kq_m} \circ \cdots \circ \tilde{I}_{r_2}^{kq_2} \circ \tilde{I}_{r_1}^{kq_1}(0_{q_0}), && (6.5.8)
\end{aligned}$$

which, by Lemma 6.5.1 iii) is k-divisible. □

Chapter 7

The unbounded case

We end by generalizing some of our results to k-symmetric measures without moments. The *free multiplicative convolution* $\boxtimes$ for general measures in $\mathcal{M}^+$ was defined in [22] using operators affiliated with a W^*-algebra. This convolution is characterized by S-transforms defined as follows. Recall that for a general probability measure μ on $\mathbb{R}$ we define

$$\Psi_\mu(z) = \int_{\mathbb{R}} \frac{zt}{1-zt}\mu(\mathrm{d}t) = \frac{1}{z}G_\mu\left(\frac{1}{z}\right) - 1, \quad z \in \mathbb{C}\backslash\mathbb{R}_+. \tag{7.0.1}$$

Recall also that, for $\chi_\mu : \Psi_\mu(i\mathbb{C}_+) \to i\mathbb{C}_+$ the inverse function of Ψ_μ, the S-transform of μ is the function

$$S_\mu(z) = \chi(z)\frac{1+z}{z}.$$

The importance of the S-transform is the following.

Proposition 7.0.3 ([22]). *Let μ_1 and μ_2 be probability measures in $\mathcal{M}^+$ with $\mu_i \neq \delta_0$, $i = 1, 2$. Then $\mu_1 \boxtimes \mu_2 \neq \delta_0$ and*

$$S_{\mu_1 \boxtimes \mu_2}(z) = S_{\mu_1}(z)S_{\mu_2}(z)$$

in that component of the common domain which contains $(-\varepsilon, 0)$ for small $\varepsilon > 0$. Moreover, $(\mu_1 \boxtimes \mu_2)(\{0\}) = \max\{\mu_1(\{0\}), \mu_2(\{0\})\}$.

Free multiplicative convolution $\mu_1 \boxtimes \mu_2$ can be defined for any two probability measures μ_1 and μ_2 on $\mathbb{R}$, provided that one of them is supported on $[0, \infty)$. However, it is not known whether an S-transform can be defined for every probability measure.

In this chapter we will define the free multiplicative convolution between measures $\mu \in \mathcal{M}_k$ and $\nu \in \mathcal{M}^+$. We generalize the S-transform to k-symmetric measures; we follow similar strategies as in [9] and show the multiplicative property still holds for this S-transform.

7.1 Analytic aspects of S-transforms

Recall that for a k-symmetric probability measure μ on $\mathbb{R}$, we denote by μ^k be the probability measure in $\mathcal{M}^+$ induced by the map $t \to t^k$ and q a primitive k-th root of unity, we consider the k-semiaxes $A_k := \{x \in \mathbb{C} \mid x = tq^s$ for some $t > 0$ and $s \in \mathbb{N}\}$.

We define the Cauchy transform a k-symmetric distribution μ by the formula

$$G_\mu(z) = \int_{\mathbb{C}} \frac{1}{z-t}\mu(\mathrm{d}t) \quad z \in \mathbb{C}\backslash A_k.$$

and the function Ψ in a similar way as (7.0.1),

$$\Psi_\mu(z) = \int_{\mathbb{C}} \frac{zt}{1-zt}\mu(\mathrm{d}t) = \frac{1}{z}G_\mu\left(\frac{1}{z}\right) - 1. \tag{7.1.1}$$

The following two important relations between the Cauchy transforms and the Ψ functions of μ and μ^k were proved in [9] for $k=2$. The proof presented here is the same with obvious changes; we present it for the convenience of the reader.

Proposition 7.1.1. *Let μ be a k-symmetric probability measure μ on $\mathbb{R}$. Then*
a) $G_\mu(z) = z^{k-1}G_{\mu^k}(z^k)$, $z \in \mathbb{C}\backslash A_k$.
b) $\Psi_\mu(k) = \Psi_{\mu^k}(z^k)$, $1/z \in \mathbb{C}\backslash A_k$.

Proof. a) Use the k-symmetry of μ twice to obtain

$$\begin{aligned}
G_\mu(z) &= \int_{\mathbb{R}} \frac{1}{z-t}\mu(\mathrm{d}t) = \sum_{i=1}^{\kappa}\int_{\mathbb{R}_+} \frac{1}{z-t\omega_i}\mu(\mathrm{d}t)\\
&= kz^{k-1}\int_{\mathbb{R}_+} \frac{1}{z^k-t^k}\mu(\mathrm{d}t) = z\int_{supp(\mu)} \frac{1}{z^k-t^k}\mu(\mathrm{d}t)\\
&= z^{k-1}\int_{\mathbb{R}_+} \frac{1}{z^k-t}\mu^k(\mathrm{d}t) = zG_{\mu^k}(z^k).
\end{aligned}$$

b) Use (7.1.1) twice and (a) to obtain

$$\begin{aligned}
\Psi_\mu(z) &= \frac{1}{z}G_\mu\left(\frac{1}{z}\right) - 1\\
&= \frac{1}{z^k}G_{\mu^k}\left(\frac{1}{z^k}\right) - 1 = \Psi_{\mu^k}(z^k)
\end{aligned}$$

which shows (b). □

An important consequence is that the function G_μ determines the measure μ uniquely since the Cauchy transform G_{μ^k} determines μ^k and thus μ. Also, the function Ψ_μ determines the measure μ uniquely since the Cauchy transform G_μ does.

Theorem 7.1.2. *Let μ be a k-symmetric probability measure μ on $\mathbb{R}$.*
a) If $\mu \neq \delta_0$, the function Ψ_μ is univalent on $\mathbb{H}_k := \{z \in \mathbb{C}_+ : \arg z \in \pi/2k < \arg z < 3\pi/2k\}$. Therefore Ψ_μ has a unique inverse on $\mathbb{H}_k$, $\chi_\mu : \Psi_\mu(\mathbb{H}_k) \to \mathbb{H}_k$.
b) If $\mu \neq \delta_0$, the S-transform

$$S_\mu(z) = \frac{1+z}{z}\chi_\mu(z) \tag{7.1.2}$$

satisfies

$$S^k_\mu(z) = (\frac{1+z}{z})^{k-1}S_{\mu^k}(z) \tag{7.1.3}$$

for z in $\Psi_\mu(\mathbb{H}_k)$.

Proof. a) On one hand, let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^k$. Then $f(\mathbb{H}_k) = i\mathbb{C}_+$ and therefore f is univalent in $\mathbb{H}_k$. On the other hand, since $\mu^k \in \mathcal{M}^+$, by Proposition 1.5.3, $\Psi_{\mu^k}(z)$ is univalent in $i\mathbb{C}_+$ and therefore $\Psi_{\mu^k}(z^k)$ is univalent in $\mathbb{H}_k$.

b) Since $\mu^k \in \mathcal{M}^+$, from Proposition 1.5.3, the unique inverse χ_{μ^k} of Ψ_{μ^k} is such that $\chi_{\mu^k} : \Psi_{\mu^k}(i\mathbb{C}_+) \to i\mathbb{C}_+$. Thus, use (a) to obtain $\Psi_{\mu^k}(\chi_\mu^k(z)) = \Psi_\mu(\chi_\mu(z)) = z$ for $z \in \Psi_\mu(\mathbb{H}_k)$ and the uniqueness of χ_{μ^k} gives $\chi_{\mu^k}(z) = \chi_\mu^k(z)$, $z \in \Psi_\mu(\mathbb{H}_k)$. Hence

$$\begin{aligned} S_\mu^k(z) = \chi_\mu^k(z)(\frac{1+z}{z})^k &= \chi_{\mu^k}(z)(\frac{1+z}{z})^k \\ &= S_{\mu^k}(z)\frac{1+z}{z}, \quad z \in \Psi_\mu(H). \end{aligned}$$

as desired. □

7.2 Free multiplicative convolution

Now, we are in position to define free multiplicative convolution for measures with unbounded support. We will use Equation (4.2.2) as our definition.

Definition 7.2.1. Let μ be k-symmetric and ν be a measure in $\mathcal{M}^+$. The free multiplicative convolution of μ and ν is defined to be the unique k-symmetric measure $\mu \boxtimes \nu$ such that

$$(\mu \boxtimes \nu)^k = \mu^k \boxtimes \nu^{\boxtimes k}$$

Remark 7.2.2. The fact that the last definition makes sense is justified as follows: μ^k and $\nu^{\boxtimes k}$ are in $\mathcal{M}^+$ and then $\mu^k \boxtimes \nu^{\boxtimes k}$ also belongs to $\mathcal{M}^+$. So the symmetric pull back under x^k of the measure is unique and well defined. Moreover, clearly this definition is consistent to the one for compactly supported measures given above.

Now we show how to compute the free multiplicative convolution of a k-symmetric probability measure and a probability measure supported on $[0, \infty)$. No existence of moments or bounded supports for the measures are assumed.

Theorem 7.2.3. *Let μ be k-symmetric and ν be a measure in $\mathcal{M}^+$ with respective S transforms $S_\mu(z)$ and $S_\nu(z)$ then*

$$S_{\mu\boxtimes\nu}(z) = S_\mu(z)S_\nu(z)$$

Proof. From Equation (7.1.3) and the multiplicative property of the S-transform for measures in $\mathcal{M}^+$ we see that

$$\begin{aligned} S_{\mu\boxtimes\nu}^k(z) &= (\frac{1+z}{z})^{k-1}S_{(\mu\boxtimes\nu)^k}(z) = (\frac{1+z}{z})^{k-1}S_{\mu^k\boxtimes\nu^{\boxtimes k}}(z) \\ &= (\frac{1+z}{z})^{k-1}S_{\mu^k}(z)S_\nu^k = S_\mu^k(z)S_\nu^k. \end{aligned}$$

□

7.3 Free additive powers

In this short section we define free additive powers for k-symmetric distributions. The tool that permits to define free additive powers is the S-transform. This is not the standard approach but it is the most suitable for our purposes.

Definition 7.3.1. Let μ be a k-symmetric distribution and $t > 1$. The free additive power $\mu^{\boxplus t}$ is defined to be the unique k-symmetric measure such that

$$S_{\mu^{\boxplus t}}(z) = \frac{1}{t} S_\mu(z/t)$$

Remark 7.3.2. (1) As mentioned above, this definition is consistent to the one for compactly supported measures.

(2) The fact that there is a k-symmetric measure $\mu^{\boxplus t}$ is proved as follows: The S-transform is continuous with respect to weak convergence. Thus taking a sequence of k-symmetric measures μ_n with compact support such that $\mu_n \to \mu$ we easily see that $\mu_n^{\boxplus t} \to \nu$ for some ν and that $\mu^{\boxplus t} := \nu$ satisfies the desired equation.

As we mentioned, free infinite divisibility is closed under convergence in distribution. By standard approximation arguments all the theorems regarding freely infinite divisibility are valid for the unbounded case. Let us state again the most important results of Section 5.3. The first explains how free infinite divisibility is preserved under the push-forward $t \to t^k$.

Theorem 7.3.3. *If μ is k-symmetric and $\boxplus$-infinitely divisible, then μ^k is also $\boxplus$-infinitely divisible.*

The second considers free additive convolution.

Theorem 7.3.4. *Let μ and ν be k-symmetric freely infinitely divisible measures. Then there exists a k-symmetric $\mu \boxplus \nu$ such that*

$$\phi_{\mu\boxplus\nu}(z) = \phi_\mu(z) + \phi_\nu(z).$$

Moreover $\mu \boxplus \nu$ is also freely infinitely divisible.

Note here that we have not defined the Voiculescu's transform ϕ, but this can be done without any problem following arguments of [22].

7.4 Stable distributions

Now we come back to the question of stability. A real probability measure σ_α is said to be $\boxplus$- stable of *index* α if $\sigma_\alpha^{\boxplus 2} \boxplus \delta_t = D_{2^{1/\alpha}}(\sigma_\alpha)$ for some t. If $t = 0$, we say that σ_α is $\boxplus$-strictly stable. Note that among k-symmetric stable measures we can only have strictly stable laws since adding non-trivial Dirac measure to a k-symmetric distribution would not yield another k-symmetric measure.

Closely related to the notion of stability is that of domains of attraction. Recall that for a probability measure μ we say that ν is in the *free domain of attraction* of ν if there exists α such that $D_{N^\alpha}(\mu^{\boxplus N}) \to \nu_i$. The following theorem explains the relation between domains of attraction and stable laws.

Theorem 7.4.1. *Assume that $\mu \in \mathcal{M}$ is not a point mass. Then ν is $\boxplus$-stable if and only if the free domain of attraction of ν is not empty.*

As we have mentioned before, s_k is strictly stable of index k. We begin by showing that for each k and each $\alpha \in (0, k]$ there is a k-symmetric strictly stable law of index α (that we will denote $\sigma_{k,\alpha}$). In fact, we have an explicit representation of $\sigma_{k,\alpha}$ as the free multiplicative convolution between a k-semicircular distribution and strictly stable distribution on $\mathbb{R}$. This result was proved in [9] for symmetric distributions on the real line and in [24] for positive measures.

Theorem 7.4.2. *For $k > 0$ and $0 < \alpha \leq 1$, let $\beta = \frac{k\alpha}{\alpha+k-k\alpha}$. The measure $\sigma_\beta^k := w_k \boxtimes \nu_\alpha$ is stable of index β. The S-transform of σ_β^k is given by*

$$S_\beta = \theta_\beta e^{i(1-\beta)\frac{\pi}{\beta}}. \tag{7.4.1}$$

Proof. The S-transform for positive strictly stable laws is found in [9] and can be easily derived from the appendix in [20]:

$$S_\alpha = \theta_\alpha e^{i(1-\alpha)\frac{\pi}{\alpha}} z^{\frac{1-\alpha}{\alpha}}.$$

A direct calculation shows that the S-transform of w_k is

$$S_{w_k} = z^{\frac{1-k}{k}}.$$

Thus, the S transform of $w_k \boxtimes \nu_\alpha$ is given by

$$S_{w_k \boxtimes \nu_\alpha}(z) = \theta_\alpha e^{i(1-\alpha)\frac{\pi}{\alpha}} z^{\frac{1-\alpha}{\alpha}+\frac{1-k}{k}}.$$

Hence, on one hand, from (1.5.8) we get

$$\begin{aligned} S_{(w_k \boxtimes \nu_\alpha)^{\boxplus 2}}(z) &= \frac{1}{2} S_{(w_k \boxtimes \nu_\alpha)}(z/2) &(7.4.2)\\ &= 1/2 \cdot \theta_\alpha e^{i(1-\alpha)\frac{\pi}{\alpha}} (\frac{z}{2})^{\frac{1-\alpha}{\alpha}+\frac{1-k}{k}} &(7.4.3)\\ &= 1/2^{1/\beta} \cdot \theta_\alpha e^{i(1-\alpha)\frac{\pi}{\alpha}} z^{\frac{1-\alpha}{\alpha}+\frac{1-k}{k}}. &(7.4.4)\end{aligned}$$

On the other hand, from (1.5.9) we have

$$S_{D_{2^{1/\beta}}(w_k \boxtimes \nu_\alpha)}(z) = \frac{1}{2^{1/\beta}} \cdot \theta_\alpha e^{i(1-\alpha)\frac{\pi}{\alpha}} z^{\frac{1-\alpha}{\alpha}+\frac{1-k}{k}}.$$

□

Conjecture 7.4.3. For $k > 2$, the k-symmetric measures σ_β^k defined in Theorem 7.4.2 are the only k-symmetric $\boxtimes$-stable distributions.

The following reproducing property was proved in [24] for one sided free stable distributions:

$$\nu_{1/(1+t)} \boxtimes \nu_{1/(1+s)} = \nu_{1/(1+t+s)}, \tag{7.4.5}$$

while for the real symmetric free stable distribution the analog relation was proved in [9].

$$\sigma_{1/(1+t)} \boxtimes \nu_{1/(1+s)} = \sigma_{1/(1+t+s)}. \tag{7.4.6}$$

A generalization for k-symmetric distributions is also true. The proofs in [9] and [24] rely on an explicit calculation of the S-transform and can be easily modified to this framework.

Theorem 7.4.4. *For any $s, r > 0$, let $\sigma^k_{1/(1+r)}$ be a k-symmetric strictly stable distribution of index $1/(1+r)$ and $\nu_{1/(1+s)}$ be a positive strictly stable distribution of index $1/(1+s)$. Then*

$$\sigma^k_{1/(1+t)} \boxtimes \nu_{1/(1+s)} = \sigma^k_{1/(1+t+s)}. \tag{7.4.7}$$

Proof. This follows from Theorem 7.4.2, indeed letting $\beta = (k - t + kt)/(tk)$

$$\begin{aligned} \sigma^k_{1/(1+t)} \boxtimes \nu_{1/(1+s)} &= w_k \boxtimes \nu_{1/(1+\beta)} \boxtimes \nu_{1/(1+s)} \\ &= w_k \boxtimes \nu_{1/(1+\beta+s)} \\ &= \sigma^k_{1/(1+t+s)}. \end{aligned}$$

We used 7.4.5 in the second inequality. □

We have the following conjecture regarding domains of attraction.

Conjecture 7.4.5. Assume that $\mu \in \mathcal{M}_k$ is not a point mass. Then ν is $\boxplus$-stable if and only if the free domain of attraction of ν is not empty.

Now, Theorem 7.4.4 may be explained by the following observation.

Lemma 7.4.6. *Let μ_1 and μ_2 be in the $\boxplus$-domain of attraction of ν_1 and μ_2, respectively. Then $\mu_1 \boxtimes \mu_2$ is in the $\boxtimes$-domain of attraction of $\nu_1 \boxtimes \nu_2$.*

Proof. For $i = 1, 2$, since $\mu_i \in \mathcal{D}^{\boxplus}(\nu_i)$, there are some α_i´s such that $D_{N^{\alpha_i}}(\mu^{\boxplus N}) \to \nu_i$. Now using Equation (1.5.10) we have

$$(\mu_1 \boxtimes \mu_2)^{\boxplus N} = D_N(\mu_1^{\boxplus N} \boxtimes \mu_2^{\boxplus N})$$

and dilating by $N^{\alpha_1+\alpha_2-1}$ we get

$$D_{N^{\alpha_1+\alpha_2-1}}((\mu_1 \boxtimes \mu_2)^{\boxplus N}) = D_{N^{\alpha_1+\alpha_2}}(\mu_1^{\boxplus N} \boxtimes \nu_2^{\boxplus N}) = D_{N^{\alpha_1}}(\mu_1^{\boxplus N}) \boxtimes D_{N^{\alpha_2}}(\mu_2^{\boxplus N}). \tag{7.4.8}$$

The RHS of the Equation (7.4.8) tends to $\nu_1 \boxtimes \nu_2$, and then also the LHS. This of course means that $\mu_1 \boxtimes \mu_2 \in \mathcal{D}^{\boxplus}(\nu_1 \boxtimes \nu_2)$. □

Remark 7.4.7. A closer look at the proof of Lemma 7.4.6 gives another proof of the reproducing property for $k = 1, 2$ and for general k if Conjectures 7.4.3 and 7.4.3 are true.

Indeed, for any $s, t > 0$, let $\sigma^k_{1/(1+t)}$ a k-symmetric strictly stable distribution of index $1/(1+s)$ and $\nu_{1/(1+s)}$ be a positive strictly stable distribution of index $1/(1+t)$. The measure $\sigma^k_{1/(1+t)} \boxtimes \nu_{1/(1+s)}$ is clearly k-symmetric and strictly stable since $\mathcal{D}(\sigma^k_{1/(1+t)} \boxtimes \nu_{1/(1+s)})$ is non-empty by the last lemma. The index of stability can be easily calculated from Equation (7.4.8), since in this

$$D_{N^{1+s+1+t-1}}(\sigma^k_{1/(1+t)} \boxtimes \nu_{1/(1+s)}) \to \sigma^k_{1/(1+t)} \boxtimes \nu_{1/(1+s)}$$

which means that $\sigma^k_{1/(1+t)} \boxtimes \nu_{1/(1+s)}$ is a k-symmetric strictly stable distribution of index $1/(1+s+t)$.

Finally, recall from Theorem 5.3.6 that the k-power of a freely infinitely divisible measure in $\mathcal{M}_k$ is also freely infinitely divisible. In the case of stable laws we can identify explicitly the Lévy measure, for $k \geq 2$. Indeed, since

$$(w_k \boxtimes \nu_{1/(1+s)})^k = w_k^k \boxtimes (\nu_{1/(1+s)})^{\boxtimes k} = \pi^{\boxtimes k-1} \boxtimes \nu_{1/(1+ks)},$$

the Lévy measure is given by $\pi^{\boxtimes k-2} \boxtimes \nu_{1/(1+ks)}$.

Bibliography

[1] M. Anshelevich. Free martingale polynomials. *J. Funct. Anal.*, **201**, (2003), 228–261.

[2] O. Arizmendi, Statistics of blocks in k-divisible non-crossing partitions. *Electronic. J. Combin.* **19** 2, Research Paper 47 ,(2012), (electronic)

[3] O. Arizmendi. k-divisible elements in free probability. Preprint (ArXiv:1203.4780)

[4] O. Arizmendi, O. E. Barndorff-Nielsen and V. Pérez-Abreu. On free and classical type G distributions. Special number, *Braz. J. Probab. Stat.*, **24**, (2010), 106–127.

[5] O. Arizmendi and T. Hasebe. Semigroups related to additive and multiplicative, free and Boolean convolutions. Preprint (ArXiv:1105.3344v2).

[6] O. Arizmendi and T. Hasebe. On a class of explicit Cauchy-Stieltjes transforms related to monotone stable and free Poisson laws. Preprint (ArXiv:1108.3438v1).

[7] O. Arizmendi and T. Hasebe. On Boolean Stable laws. Preprint (ArXiv:1205.1575).

[8] O. Arizmendi, T. Hasebe and N. Sakuma, Free Regular Infinite Divisibility and Squares of Random Variables with $\boxplus$-infinitely Divisible Distributions. Preprint (ArXiv:1201.0311)

[9] O. Arizmendi and V. Pérez-Abreu, The S-transform of symmetric probability measures with unbounded supports, Proc. Amer. Math. Soc. **137** (2009), 3057–3066.

[10] O. Arizmendi and C. Vargas, Products of free random variables and k-divisible non-crossing partitions *Elect. Comm. in Probab.* 17 (2012)

[11] D. Armstrong, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Mem. Amer. Math. Soc. 2002, no. 949. (2009)

[12] O. E. Barndorff-Nielsen and S. Thorbjørnsen, Classical and Free Infinite Divisibility and Lévy Processes. In U. Franz and M. Schürmann (Eds.): *Quantum Independent Increment Processes II. Quantum Lévy processes, Classical Probability and Applications to Physics*, Springer, pp 33-160, 2006.

[13] T. Banica, S. T. Belinschi, M. Capitane and B. Collins. *Free Bessel Laws.*,Canad. J. Math. **63** (2011), 337.

[14] T. Banica T and A. Skalski, Quantum isometry groups of dual of free powers of cyclic groups *Int. Math. Res. Not.*, to appear.

[15] S. T. Belinschi and A. Nica, On a remarkable semigroup of homomorphisms with respect to free multiplicative convolution. *Indiana Univ. Math. J.* **57,** 1679-1713.

[16] S.T. Belinschi and D. Shlyakhtenko. Free probability of type B: analytic aspects and applications, preprint (arXiv:0903.2721.37)

[17] A. Ben Ghorbal and M. & Schürman Non-commutative notions of stochastic independence, *Mathematical Proceedings of the Cambridge Philosophical Society* **133** (2002), 531-561.

[18] F. Benaych-Georges. On a surprising relation between the Marchenko-Pastur law, rectangular and square free convolutions. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics.* **46**, (2010), 644–652.

[19] H. Bercovici and V. Pata, A free analogue of Kinčin characterization of infinite divisibility, *Proc. Amer. Math. Soc.* **128** (2000), 1011-1015.

[20] H. Bercovici and V. Pata, with an appendix by P. Biane, Stable laws and domains of attraction in free probability theory, *Ann.Math.* **149** (1999), 1023-1060.

[21] H. Bercovici and D. Voiculescu, Lévy-Kinčin type theorems for multiplicative and additive free convolutions, *Pacific J. Math.* **153** (1992), 217-248.

[22] H. Bercovici and D. Voiculescu, Free convolution of measures with unbounded supports, *Indiana Univ. Math. J.* **42** (1993), 733-773.

[23] S.T. Belinschi, M. Bożejko, F. Lehner and R. Speicher, The normal distribution is $\boxplus$-infinitely divisible, Adv. Math. **226**, No. 4 (2011), 3677–3698.

[24] H. Bercovici and V. Pata, Stable laws and domains of attraction in free probability theory (with an appendix by Philippe Biane), Ann. of Math. (2) **149**, No. 3 (1999), 1023–1060.

[25] H. Bercovici and D. Voiculescu, Free convolution of measures with unbounded support, Indiana Univ. Math. J. **42**, No. 3 (1993), 733–773.

[26] D. Bessis, The dual braid monoid, *Ann. Sci. Ec. Norm. Super.* **36** (2003) 647–683

[27] P. Biane, F. Goodman and A. Nica. Non-crossing cumulants of type B, *Trans. Amer. Math. Soc.* **355** (2003), 2263–2303.

[28] P. Biane, Processes with free increments, Math. Z. **227** (1998), 143–174.

[29] T. Brady, C. Watt, Non-crossing partition lattices in finite reflection groups, *Trans. Amer. Math. Soc.* **360** (2008) 1983–2005.

[30] L. G. Brown, Lidskiis theorem in the type II case, Geometric methods in operator algebras (Kyoto, 1983), Longman Sci. Tech., Harlow, 1986, pp. 1-35.

[31] Z. Burda, R. A. Janik and M. A. Nowak, Multiplication law and S transform for non-hermitian random matrices

[32] Doubilet, P., Rota, G.-C. and Stanley, R., *On the foundations of combinatorial theory. VI. The idea of generating function*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory (Berkeley, Calif.), Univ. California Press, 1972, pp. 267318.

[33] P. H. Edelman, Chain enumeration and non-crossing partitions,Discrete Math. 31 (1980), 171–18

[34] M. Février and A. Nica. Infinitesimal non-crossing cumulants and free probability of type B.*J. Funct. Anal.* **258** (2010), no. 9, 29833023.

[35] F. M. Goodman, $\mathbb{Z}_n$–graded Independence, Indiana University Mathematics Journal, **53** (2004), 515–532;

[36] F. Hiai and D. Petz, *The Semicircle Law, Free Random Variables and Entropy*, Mathematical Surveys and Monographs 77, Amer. Math. Soc., Providence, 2000.

[37] U. Haagerup and S. Möller, *A law of large numbers for the free multiplicative convolution*, in preparation.

[38] U. Haagerup and F. Larsen, Brown's spectral distribution measure for *R*- diagonal elements in finite von Neumann algebras. *J. Funct. Anal.*, 176(2):331–367, (2000.)

[39] U. Haagerup and H. Schultz. Brown measures of unbounded operators affiliated with a finite von Neumann algebra. *Math. Scand.*, **100**(2) 209–263, (2007).

[40] V. Kargin. The norm of products of free random variables, *Probab. Theory Relat. Fields* **139**, (2007) 397–413.

[41] M. E. H. Ismail and D. H. Kelker. Special Functions, Stieltjes Transforms and Infinite Divisibility *SIAM J. Math. Anal.* **10**(5), (1979), 884–901.

[42] V. Kargin. On asymtotic growth of the support of free multiplicative convolutions *Elect. Comm. in Probab.* **13** (2008), 415–421

[43] Jang Soo Kim, Chain enumeration of k-divisible noncrossing partitions of classical types. *J. Combin. Theory Ser. A* **118** (2011), No.3, 879–898

[44] B. Krawczyk and R. Speicher. Combinatorics of free cumulants, *Journal of Combinatorial Theory* Series A **90** (2000), 267-292.

[45] C. Krattenthaler, Non-crossing partitions on an annulus, in preparation.

[46] C. Krattenthaler and T. W. Müller, Decomposition numbers for finite Coxeter groups and generalised non-crossing partitions. *Trans. Amer. Math. Soc.* **362** (2010), no. 5, 2723-2787

[47] G. Kreweras, Sur les partitions non croisés d´un cycle, *Discrete Math.* **1**, (1972) 333-350

[48] F. Lehner, Free cumulants and enumeration of connected partitions. *Europ. J. Combinatorics* **23** (2002), 1025-1031,

[49] F. Lehner, On the computation of spectra in free probability, *J. Funct. Anal.* **183** (2001) 451-471

[50] V. A. Marchenko, and Pastur, L A . "Distribution of eigenvalues for some sets of random matrices". Mathematics of the USSR-Sbornik 1 (4) (1967) 457483.

[51] W. Młotkowski, Fuss-Catalan numbers in noncommutative probability, *Doc. Math.* **15** (2010), 939–955.

[52] N. Muraki, Monotonic independence, monotonic central limit theorem and monotonic law of small numbers, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **4** (2001), 39–58.

[53] Muraki, N. (2003). The five independences as natural products. *Infinite dimensional Analysis, Quantum Probability and Related Topics* **6**, 337–371.

[54] M. G. Neagu, Asymptotic Freeness of Random Permutation Matrices with Restricted Cycle Lengths. arXiv:math/0512067v2

[55] A. Nica, D. Shlyakhtenko and R. Speicher, *R-cyclic families of matrices in free probability* J. Funct. Anal. **188** (2002), 227–271.

[56] A. Nica and R. Speicher, On the multiplication of free n-tuples of non-commutative random variables (with an Appendix by D. Voiculescu). *Amer. J. Math.* 118 (1996), 799-837

[57] A. Nica and R. Speicher, *R-diagonal pairs-a common approach to Haar unitaries and circular elements*, Fields Institute Communications **12** (1997), 149-188

[58] A. Nica and R. Speicher, A "Fourier transform" for multiplicative functions of noncrossing partitions, *J. Algebraic Combin.* **6** (1997), 141-160

[59] A. Nica and R. Speicher. Commutators of free random variables. *Duke Math. J.*, **92**, (1998), 553–592.

[60] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability,* London Mathematical Society Lecture Notes Series **335**, Cambridge University Press, Cambridge, 2006.

[61] J. Ortmann. Large deviations for non-crossing paritition. preprint (arXiv:1107.0208)

[62] V. Pérez-Abreu and N. Sakuma, Free infinite divisibility of free multiplicative mixtures of the Wigner distributions, *J. Theoret. Probab.*, to appear.

[63] M. Popa. Freeness with amalgamation, limit theorems and S-transform in noncommutative probability spaces of type B, preprint.(ArXiv:0709.0011).

[64] N.R. Rao, R. Speicher, Multiplication of free random variables and the S-transform: the case of vanishing mean Elect. Comm. in Probab. 12 (2007), 248-258

[65] G.-C. Rota, The number of partitions of a set, *Amer. Math. Monthly* 71 (1964), 498504.

[66] N. Saito and H. Yoshida, The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory, Probab. Math. Statist. **21**, Fasc. 1 (2001), 159–170.

[67] N. Sakuma. On free regular infinitely divisible distributions. *RIMS Kokyuroku Bessatsu*, **B27**, (2011), 115–122.

[68] N. Sakuma and H. Yoshida. *New limit theorems related to free multiplicative convolution.* 2011. arXiv:1103.6156v1.

[69] K. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, (1999).

[70] D. Shlyakhtenko, Notes on Free Probability Theory. (ArXiv: 0504063)

[71] P. Śniady P and R. Speicher, Continuous family of invariant subspaces for *R*-diagonal operator. *Invent. math.* **146**, 329–363 (2001)

[72] R. Speicher. Multiplicative functions on the lattice of noncrossing partitions and free convolution, *Mathematische Annalen* **298**, 611–628 (1994).

[73] Speicher, R. On universal products. *Fields Institute Communications*, D. Voiculescu, (ed). **12**, (1997) 257-266.

[74] R. Speicher and R. Woroudi, Boolean convolution, in Free Probability Theory, Ed. D. Voiculescu, Fields Inst. Commun., vol. 12 (Amer. Math. Soc., 1997), 267–280.

[75] F. W. Steutel and K. Van Harn. *Infinite Divisibility of Probability Distributions on the Real Line.* Marcel-Dekker, New York, 2003.

[76] G. H. Tucci, Limit Laws for Geometric Means of Free Random Variables *Indiana Journal of Mathematics*, **59** no. 1, 1-13, 2010.

[77] E. Wishart, The Generalized product moment distribution in samples from a normal multivariate populations. *Biometrika* **20** (1928) 3552.

[78] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Annals of Mathematics* **62** (1955), 548-564.

[79] D. Voiculescu, Addition of certain non-commutative random variables, *J. Funct. Anal.* **66** (1986), 323–346.

[80] D. Voiculescu, Dual algebraic structures on operator algebras related to free products, *J. Operator Theory* **17** (1987), 85-98.

[81] D. Voiculescu, Symmetries of some reduced free product C^*-álgebras. In: *Operator Algebras and their Connections with Topology in Ergodic Theory. Lecture Notes in Mathematics* **1132**, Springer Verlag (1985), 556-588.

[82] D. Voiculescu, Multiplication of certain non-commuting random variables, *J. Operator Theory* **18** (1987), 223-235.

[83] D. Voiculescu, K. Dykema and A. Nica, *Free Random Variables,* CRM Monograph Series **1,** Amer. Math. Soc., Providence, 1992.