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Abstract

Besides other things we prove that if $u \in L^\infty_{loc}(\Omega;\mathbb{R}^M)$, $\Omega \subset \mathbb{R}^n$, locally minimizes the energy

$$\int_\Omega \big[a(|\tilde{\nabla} u|) + b(|\partial_n u|)\big]\,\mathrm{d}x,$$

$\tilde{\nabla} := (\partial_1, \dots, \partial_{n-1})$, with N-functions $a \le b$ having the Δ_2-property, then $|\partial_n u|^2 b(|\partial_n u|) \in L^1_{loc}(\Omega)$. Moreover, the condition

$$b(t) \le const\, t^2 a(t^2) \tag{$*$}$$

for all large values of t implies $|\tilde{\nabla} u|^2 a(|\tilde{\nabla} u|) \in L^1_{loc}(\Omega)$. If $n = 2$, then these results can be improved up to $|\nabla u| \in L^s_{loc}(\Omega)$ for all $s < \infty$ without the hypothesis $(*)$. If $n \ge 3$ together with $M = 1$, then higher integrability for any exponent holds under more restrictive assumptions than $(*)$.

1 Introduction

As a first step towards the question of (partial) regularity of weak local minimizers u: $\mathbb{R}^n \supset \Omega \to \mathbb{R}^M$ of the variational integral

$$I[u,\Omega] = \int_\Omega F(\nabla u)\,\mathrm{d}x$$

we want to analyze the local higher integrability properties of ∇u concentrating on the so-called anisotropic case. The most prominent example leading to anisotropic energies is given by integrands F of anisotropic (p,q)-growth with exponents $1 < p \le q < \infty$, which by definition satisfy an estimate of the form

$$m_1\big[|Z|^p - 1\big] \le F(Z) \le m_2\big[|Z|^q + 1\big], \qquad Z \in \mathbb{R}^{nM}, \tag{1.1}$$

m_1, m_2 denoting positive constants. As it was discovered by Giaquinta [Gi] (and later re-investigated by Hong [Ho]) one can not expect any regularity of local minimizers, if p and q are too far apart, and this even concerns the scalar situation, i.e. the case $M = 1$. Observing that (1.1) follows from the anisotropic convexity condition

$$\lambda(1+|Z|^2)^{\frac{p-2}{2}}|Y|^2 \le D^2F(Z)(Y,Y) \le \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2, \tag{1.2}$$

Y, $Z \in \mathbb{R}^{nM}$, Marcellini [Ma1] and Fusco and Sbordone [FS] showed: if $M = 1$ and if (1.2) or some weaker variant hold, then the gradient of a local minimizer is locally bounded provided

$$q \le c(n)p \tag{1.3}$$

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for a constant $c(n) \to \infty$ as $n \to \infty$, whereas, e.g., for $n = 2$ (1.3) can be dropped. If we pass to the vector case, then there are strong regularity results due to Marcellini [Ma3] and Marcellini and Papi [MP] for integrands of the special form $F = F(|Z|)$, whereas Esposito, Leonetti and Mingione [ELM1] studied more general densities F and proved

$$\nabla u \in L^q_{loc}(\Omega;\mathbb{R}^{nM}) \tag{1.4}$$

working with a relaxed version of (1.2) and assuming

$$q < p + 2\min\{1, p/n\}, \tag{1.5}$$

so that as in (1.3) the range of anisotropy becomes smaller as $n \to \infty$, if (1.5) is imposed.

An intermediate situation occurs if in addition to (1.2) F is of the form $F(|\partial_1 u|, \dots, |\partial_n u|)$. Then – by the maximum principle proved in [DLM] – it makes sense to consider local minima of class $L^\infty_{loc}(\Omega;\mathbb{R}^M)$, and in [ELM2] it is shown that now the dimensionless condition

$$q < p + 2 \tag{1.6}$$

implies

$$\nabla u \in L^r_{loc}(\Omega;\mathbb{R}^{nM}) \quad \text{for all } r < \frac{np}{n-p+q-2}\,. \tag{1.7}$$

However note that for large n (1.7) is a weaker result than (1.4), i.e. (1.7) does not give (1.4). The local integrability property (1.4) under the hypothesis (1.6) together with $u \in L^\infty_{loc}(\Omega;\mathbb{R}^M)$ has been proved in [Bi], Theorem 5.12. for integrands of the form $F(\nabla u) = F(|\partial_1 u|, \dots, |\partial_n u|)$, and it is further shown that this requirement concerning F even can be dropped if $M = 1$. For completeness we like to mention an earlier contribution of Choe [Ch] concerning bounded local minima in the scalar case but replacing (1.6) by the stronger condition $q < p+1$ and imposing the structure $F = F(|\nabla u|)$.

If we continue our discussion of local minima u from the space $L^\infty_{loc}(\Omega;\mathbb{R}^M)$, then the results described above can be improved by adjusting the class of integrands F to anisotropic power growth which means that for example we have an additive decomposition of the integrand F in the sense that ($\tilde{\nabla} u := (\partial_1 u, \dots, \partial_{n-1} u)$)

$$F(\nabla u) = f(\tilde{\nabla} u) + g(\partial_n u) \tag{1.8}$$

where f is of p-growth and g is of q-growth with $p \le q$, and where in case $M > 1$ we require in addition that

$$f(\tilde{\nabla} u) = f_1(|\partial_1 u|, \dots, |\partial_{n-1} u|)\,, \qquad g(\partial_n u) = g_1(|\partial_n u|)\,.$$

Then we proved in [BF2] and [BFZ]:

- $|\partial_n u| \in L^{q+2}_{loc}(\Omega)$;
- $q \le 2p + 2 \Rightarrow |\tilde{\nabla} u| \in L^{p+2}_{loc}(\Omega)$;

- $M = 1$ or $n = 2 \Rightarrow |\nabla u| \in L^t_{loc}(\Omega)$ for all $t < \infty$.

Moreover, we used these higher integrability results to obtain (partial) interior $C^{1,\alpha}$-regularity (see also [BF3]) in the general vector case $n \geq 3$ together with $M \geq 2$.

Inspired by Marcellini's paper [Ma2] we are now going to analyze the integrability properties of ∇u for local minimizers $u \in L^\infty_{loc}(\Omega; \mathbb{R}^M)$ if F is of splitting-type (1.8) with f and g generated by N-functions a, b: $[0, \infty) \to [0, \infty)$. Let us suppose for simplicity of the exposition that

$$F(\nabla u) = a(|\tilde{\nabla} u|) + b(|\partial_n u|)$$

with N-functions $a \leq b$ having the Δ_2-property (see Section 2 for details). Then we have (compare Theorem 2.1 – 2.3):

- $b(|\partial_n u|)|\partial_n u|^2 \in L^1_{loc}(\Omega)$;
- $b(t) \leq ct^2 a(t^2)$ for large $t \Rightarrow a(|\tilde{\nabla} u|)|\tilde{\nabla} u|^2 \in L^1_{loc}(\Omega)$;
- $n = 2$ and we have at least quadratic growth $\Rightarrow |\nabla u| \in L^s_{loc}$ for all $s < \infty$,

where now "$b(t) \leq ct^2 a(t^2)$" replaces "$q \leq 2p + 2$".

If the case $M = 1$ is considered, then – apart from the particular choice $a(t) = t^2$ – we did not succeed to obtain the local integrability of ∇u for any exponent without a condition relating a and b. In fact, this is not surprising since N-functions are allowed to differ essentially from power-growth behaviour. A more detailed explanation will be given in Section 6.

We think that our results are even new in the isotropic case $a = b$: if we assume

$$F(\nabla u) = a(|\tilde{\nabla} u|) + a(|\partial_n u|)$$

together with $M = 1$, then we get that $|\nabla u| \in L^t_{loc}(\Omega)$ for any $t < \infty$, and this cannot be deduced from Marcellini's work [Ma2] since his contributions just cover the case $F(\nabla u) = a(|\nabla u|)$ but allowing N-functions a being more general than the ones considered here.

Our paper is organized as follows: in Section 2 we fix our notation and state our results precisely, Section 3 contains the general vector case, in Section 4 we study the case $\Omega \subset \mathbb{R}^2$, and in Section 5 we investigate the scalar situation. A list of examples together with a discussion of our hypotheses can be found in Section 6. Finally, some technical details concerning N-functions are summarized in an appendix.

2 Notation and results

Suppose that we are given N-functions a, b: $[0,\infty) \to [0,\infty)$ of class C^2 which according to [Ad] means that for $h := a$, $h := b$ it holds

$$h \text{ is strictly increasing and convex satisfying} \quad \lim_{t\downarrow 0} \frac{h(t)}{t} = 0\,, \quad \lim_{t\to\infty} \frac{h(t)}{t} = \infty\,. \qquad (H1)$$

Our second hypothesis reads as: there exist $\bar{\varepsilon} > 0$ and $\bar{h} > 0$ such that for all $t \geq 0$

$$\bar{\varepsilon}\frac{h'(t)}{t} \leq h''(t) \leq \bar{h}\frac{h'(t)}{t}\,. \qquad (H2)$$

A discussion of $(H2)$ and several examples of functions h satisfying $(H1)$ and $(H2)$ are given in Section 6, here we just collect some elementary consequences of our hypotheses.

Remark 2.1. *a) Hypothesis $(H1)$ implies*

$$h(0) = 0 = h'(0)\,, \qquad h'(t) > 0 \quad \textit{for all } t > 0\,,$$

where the strict positive sign of h' follows from the convexity and the strict monotonicity of h. Note that $h''(0) = \lim_{t\to 0} h'(t)/t$, and therefore $(H2)$ means for $t = 0$ that

$$\bar{\varepsilon}h''(0) \leq h''(0) \leq \bar{h}h''(0)\,,$$

hence $\bar{\varepsilon} \leq 1 \leq \bar{h}$ in case $h''(0) \neq 0$.

b) The l.h.s. inequality of $(H2)$ gives with $p := 1 + \bar{\varepsilon}$

$$h(t) \geq ct^p\,.$$

In fact we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln(h'(t)) \geq \bar{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\ln(t)$$

which implies that the function $\ln(h'(t)) - \bar{\varepsilon}\ln(t)$ is increasing, thus $(t \geq 1)$

$$h'(t) \geq h'(1)t^{\bar{\varepsilon}}$$

and the claim follows by integrating this inequality.

c) According to Lemma A.1, a), it follows from $(H1)$ and the r.h.s. of inequality $(H2)$ that h fullfils a global Δ_2-condition, i.e.

$$h(2t) \leq \mu h(t) \quad \textit{for all } t \geq 0 \qquad (\Delta_2)$$

for a suitable constant $\mu > 0$. In particular, by Lemma A.2 there exists an exponent q such that for large t

$$h(t) \leq ct^q\,.$$

This is also a direct consequence of the r.h.s. of $(H2)$ with the choice $q = 1 + \bar{h}$.

d) Conversely, if h satisfies (H1) and has the Δ_2-property, then the r.h.s. of inequality (H2) holds under the additional assumption that h'' is increasing (see Lemma A.1, b)) which is equivalent to the convexity of h'. At the same time convexity of h' implies

$$0 = h'(0) \geq h'(t) + h''(t)(-t)\,,$$

and this inequality shows that the l.h.s. inequality of (H2) is always satisfied under the extra assumption that h' is convex. Thus, if $h \in C^3([0,\infty))$ is any N-function with the Δ_2-property and $h^{(3)} \geq 0$, then we have (H2).

e) Letting $H(Z) := h(|Z|)$, $Z \in \mathbb{R}^k$, we have by elementary calculations

$$\min\Big\{h''(|Z|), \frac{h'(|Z|)}{|Z|}\Big\}|Y|^2 \leq D^2H(Z)(Y,Y) \leq \max\Big\{h''(|Z|), \frac{h'(|Z|)}{|Z|}\Big\}|Y|^2\,,$$

and (H2) gives for all Y, $Z \in \mathbb{R}^k$

i) $\lambda\frac{h'(|Z|)}{|Z|}|Y|^2 \leq D^2H(Z)(Y,Y) \leq \Lambda\frac{h'(Z)}{|Z|}|Y|^2$.
In particular we observe that the function H is strictly convex.

ii) $|D^2H(Z)| \leq c(1+|Z|^2)^{\frac{q-2}{2}}$.

Here ii) is a consequence of i) and the growth of h, see Remark 3.1 for details.

Now given $n \geq 2$, $M \geq 1$ we write

$$Z = (Z_1,\dots,Z_n) = (\tilde{Z}, Z_n)\,,\quad \tilde{Z} := (Z_1,\dots,Z_{n-1})\,,\quad Z_i \in \mathbb{R}^M\,,\quad i = 1,\,\dots,\,n\,,$$

for an arbitrary matrix $Z \in \mathbb{R}^{nM}$. If Ω is an open set and if $u\colon \Omega \to \mathbb{R}^M$ is a (weakly) differentiable function, then the Jacobian matrix $\nabla u = (\partial_1 u,\dots,\partial_n u)$ is decomposed as $\nabla u = (\tilde{\nabla} u, \partial_n u)$ with $\tilde{\nabla} u := (\partial_1 u,\dots,\partial_{n-1} u)$. To our N-functions a and b we associate the functions $\mathcal{A}\colon \mathbb{R}^{(n-1)M} \to [0,\infty)$, $\mathcal{B}\colon \mathbb{R}^M \to [0,\infty)$,

$$\mathcal{A}(\tilde{Z}) := a(|\tilde{Z}|)\,,\quad \mathcal{B}(Z_n) := b(|Z_n|)\,,\quad Z \in \mathbb{R}^{nM}\,,$$

and define the strictly convex energy density

$$F(Z) := \mathcal{A}(\tilde{Z}) + \mathcal{B}(Z_n)\,,\quad Z \in \mathbb{R}^{nM}\,. \tag{2.1}$$

Recalling Remark 2.1, c), we have the upper bound

$$F(Z) \leq C\big[|Z|^q + 1\big] \quad \text{for all } Z \in \mathbb{R}^{nM}\,. \tag{2.2}$$

Let us finally assume

$$a(t) \leq b(t) \tag{2.3}$$

for large values of t.

Introducing the variational integral

$$I[u,\Omega] := \int_\Omega F(\nabla u)\,\mathrm{d}x \tag{2.4}$$

it is reasonable to call a function u from the space $W^1_{1,loc}(\Omega;\mathbb{R}^M)$ (compare [Ad] for a definition of Sobolev and related spaces) a local minimizer of the functional from (2.4) if and only if $I[u,\Omega'] < \infty$ and $I[u,\Omega'] \le I[v,\Omega']$ for all subdomains Ω' with compact closure in Ω and all $v \in W^1_{1,loc}(\Omega;\mathbb{R}^M)$ s.t. $\operatorname{spt}(u-v) \subset \Omega'$.

Let us now state our results:

Theorem 2.1. *(general vector case) Suppose that a, b satisfy (H1) and (H2). Consider a local minimizer $u \in W^1_{1,loc}(\Omega;\mathbb{R}^M)$ of the energy (2.4) with F defined in (2.1). Suppose further that u is locally bounded. Then we have:*

a) $b(|\partial_n u|)|\partial_n u|^2$ is in the space $L^1_{loc}(\Omega)$.

b) Let us further assume that we have

$$b(t) \le ct^2 a(t^2) \quad \text{for large } t \ge 0 \text{ and a constant } c > 0\,. \tag{2.5}$$

Then we obtain $a(|\tilde{\nabla} u|)|\tilde{\nabla} u|^2 \in L^1_{loc}(\Omega)$.

c) If $a = b$, then $a(|\nabla u|)|\nabla u|^2 \in L^1_{loc}(\Omega)$.

Remark 2.2. *a) The restriction to the particular variational integral*

$$\int_\Omega \big[a(|\tilde{\nabla} u|) + b(|\partial_n u|)\big]\,\mathrm{d}x$$

is just for the simplicity of the exposition. Of course we can consider more general integrals of splitting type, e.g.

$$\int_\Omega \big[f(\tilde{\nabla} u) + g(\partial_n u)\big]\,\mathrm{d}x\,,$$

provided the growth and convexity properties of f and g can be described in terms of N-functions a, b in an obvious way. Moreover, in this more general case we must have $f(\tilde{\nabla} u) = f(|\partial_1 u|,\dots,|\partial_{n-1} u|)$, $g(\partial_n u) = g(|\partial_n u|)$ in order to apply the maximum-principle of [DLM]during the proof. Other extensions of Theorem 2.1 concern alternative decompositions of ∇u: if for example ∇u is formed by the two submatrices $(\nabla u)_1$, $(\nabla u)_2$ or if we replace $\tilde{\nabla} u$ by ∇u and $\partial_n u$ by some part $\hat{\nabla} u$ of ∇u, then we have corresponding results for locally bounded local minimizers of

$$\int_\Omega \big[a(|(\nabla u)_1|) + b(|(\nabla u)_2|)\big]\,\mathrm{d}x$$

and of

$$\int_\Omega \big[a(|\nabla u|) + b(|\hat{\nabla} u|)\big]\,\mathrm{d}x\,.$$

b) Theorem 2.1 corresponds to Theorem 1, a), b), in [BF2], where the anisotropic (p, q)-case is considered and where (2.5) reads as $q \le 2p + 2$.

Theorem 2.2. *(2D vector case) Consider a domain $\Omega \subset \mathbb{R}^2$. Suppose that a, b satisfy (H1), (H2) and in addition: there exists $h_0 > 0$ such that*

$$\frac{h'(t)}{t} \ge h_0 \quad on\ [0,\infty)\,. \tag{2.6}$$

Moreover, let (2.3) hold. Then, if $u \in W^1_{1,loc}(\Omega;\mathbb{R}^M)$ denotes an arbitrary local minimizer of the energy from (2.4), we have $|\nabla u| \in L^t_{loc}(\Omega)$ for any finite t.

Remark 2.3. *a) We have the same comments as in Remark 2.2, a).*

b) If should be emphasized that (2.5) is not required if $n = 2$.

c) (2.6) implies that F is of superquadratic growth, i.e.

$$c\big[|Z|^2 - 1\big] \le F(Z) \quad for\ all\ Z \in \mathbb{R}^{nM}\,,$$

in particular we have $u \in W^1_{2,loc}(\Omega;\mathbb{R}^M)$ for the local minimizer in Theorem 2.2.

Theorem 2.3. *(scalar case) Let $M = 1$ and suppose that the functions a, b satisfy (H1), (H2) and (2.3). Consider a local minimizer u from the class $W^1_{1,loc} \cap L^\infty_{loc}(\Omega)$.*

a) If (2.5) holds, then we have

$$\begin{aligned} b(|\partial_n u|)|\partial_n u|^r &\in L^1_{loc}(\Omega) \quad for\ all\ r < 6\,,\\ a(|\tilde\nabla u|)|\tilde\nabla u|^r &\in L^1_{loc}(\Omega) \quad for\ all\ r < 4\,. \end{aligned}$$

b) For the particular case $a(t) = t^2$ it follows $|\nabla u| \in L^r_{loc}(\Omega)$ for all $r < \infty$ and this is true without (2.5).

c) If (2.5) is replaced by the stronger assumption

$$b(t) \le const\, t^2 a(t) \quad for\ large\ t\,, \tag{2.7}$$

then we have $|\nabla u| \in L^r_{loc}(\Omega)$ for all $r < \infty$, so that local higher integrability for any finite exponent holds in the "isotropic" case $a = b$.

Remark 2.4. *a) The results of Theorem 2.3 extend to the cases described in Remark 2.1, a).*

b) If we compare Theorem 2.3 with the anisotropic power-growth case studied in [BFZ], then in the present setting of N-functions we have as expected much weaker results: we need condition (2.5) to gain some higher integrability of $\partial_n u$ and $\tilde\nabla u$, whereas the local higher integrability of ∇u for any finite exponent can only be achieved under stronger assumptions or by specifying a or b. For instance, if $a(t) = t^2$, then we do not need additional hypotheses for b.

c) The reader should note that (2.7) is a (weaker) variant of (1.6) formulated in terms of N-functions which means that with Theorem 2.3, c) we have an extension of Theorem 5.12 from [Bi] to the class of splitting functionals being in addition not necessarily of power growth.

3 Proof of Theorem 2.1

We proceed as in [BF2] by fixing a ball $B := B_R(x_0) \Subset \Omega$. For small $\varepsilon > 0$ let $(u)_\varepsilon$ denote the mollification of u. By Remark 2.1, c), we have with $q = 1 + \bar{h}$, $\bar{h}$ being defined in $(H2)$,

$$b(t) \leq c(t^q + 1) \quad \text{for all } \ t \geq 0\,. \tag{3.1}$$

Fixing $\tilde{q} > \max\{2, q\}$, we let

$$\delta := \delta(\varepsilon) := \Big[1 + \varepsilon^{-1} + \|\nabla (u)_\varepsilon\|^{2\tilde{q}}_{L^{\tilde{q}}(B)}\Big]^{-1}\,,$$

and define

$$F_\delta(Z) = \delta(1 + |Z|^2)^{\frac{\tilde{q}}{2}} + F(Z)\,, \qquad Z \in \mathbb{R}^{nM}\,.$$

We further consider the unique solution u_δ of

$$I_\delta[w, B] := \int_B F_\delta(\nabla w)\,\mathrm{d}x \to \min \qquad \text{in} \ \ \overset{\circ}{W}{}^1_{\tilde{q}}(B; \mathbb{R}^M) + (u)\varepsilon\,.$$

Lemma 3.1. *a) We have as $\varepsilon \to 0$: $u_\delta \rightharpoonup u$ in $W^1_p(B; \mathbb{R}^M)$, where $p = 1 + \bar{\varepsilon}$ with $\bar{\varepsilon}$ from (H2);*

$$\delta \int_B (1 + |\nabla u_\delta|^2)^{\frac{\tilde{q}}{2}}\,\mathrm{d}x \to 0\,; \qquad \int_B F(\nabla u_\delta)\,\mathrm{d}x \to \int_B F(\nabla u)\,\mathrm{d}x\,.$$

b) $\|u_\delta\|_{L^\infty(B)}$ is bounded independent of ε.

c) ∇u_δ is in the space $L^\infty_{loc} \cap W^1_{2,loc}(B; \mathbb{R}^{nM})$.

Proof of Lemma 3.1. a) is standard, compare, e.g., [BF1]. b) follows from the maximum principle of [DLM], for c) we can quote [GM] and [Ca]. □

Remark 3.1. *(3.1) combined with [Da], Lemma 2.2, p. 156, gives*

$$|b(t + \varepsilon) - b(t)| \leq c\big(1 + |t + \varepsilon|^{q-1} + |t|^{q-1}\big)|\varepsilon|\,,$$

hence

$$0 \leq b'(t) \leq c(1 + t^{q-1}) \quad \textit{for } \ t \geq 0\,.$$

Applying Remark 2.1, e), i), to $\mathcal{B}$ and the vectors $\tau \in \mathbb{R}^M$, $|\tau| \geq 1$, $\sigma \in \mathbb{R}^M$ we therefore get

$$\begin{aligned} D^2\mathcal{B}(\tau)(\sigma, \sigma) &\leq c\frac{b'(|\tau|)}{|\tau|}|\sigma|^2 \\ &\leq c|\tau|^{-1}(1 + |\tau|^{q-1})|\sigma|^2 \\ &\leq c(1 + |\tau|^2)^{\frac{q-2}{2}}|\sigma|^2\,, \end{aligned}$$

and for $|\tau| \le 1$ the bound

$$D^2\mathcal{B}(\tau)(\sigma,\sigma) \le c(1+|\tau|^2)^{\frac{q-2}{2}}|\sigma|^2$$

follows from Remark 2.1, e), i) and the l.h.s. of (H2). Analogous calculations using (2.3) imply

$$D^2\mathcal{A}(\tau)(\sigma,\sigma) \le c(1+|\tau|^2)^{\frac{q-2}{2}}|\sigma|^2$$

now for all τ, $\sigma \in \mathbb{R}^{(n-1)M}$, so that by (2.1)

$$D^2F(Z)(Y,Y) \le c(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2 \quad \text{for all } Z,\, Y \in \mathbb{R}^{nM}\,.$$

Since we have chosen $\tilde{q} > q$, we see from this inequality that the arguments of [GM] actually can be applied.

Lemma 3.2. *(Caccioppoli-type inequality) For any $\eta \in C_0^\infty(B)$ and any $\gamma \in \{1,\dots,n\}$ we have*

$$\int_B \eta^2 D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta, \partial_\gamma\nabla u_\delta)\,\mathrm{d}x \le c\int_B D^2F_\delta(\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta, \nabla\eta\otimes\partial_\gamma u_\delta)\,\mathrm{d}x\,. \tag{3.2}$$

(No summation w.r.t. γ, $\otimes$ denotes the tensor product and c is independent of ε and η.)

Proof of Lemma 3.2. Compare, e.g. [BF1], proof of Lemma 3.1. Inequality (3.2) follows from this reference by applying the Cauchy-Schwarz inequality to the bilinear form $D^2F_\delta(\nabla u_\delta)$. □

We let

$$\Gamma_\delta := 1+|\nabla u_\delta|^2\,, \quad \tilde{\Gamma}_\delta := 1+|\tilde{\nabla} u_\delta|^2\,, \quad \Gamma_{n,\delta} := 1+|\partial_n u_\delta|^2$$

and consider $\eta \in C_0^\infty(B)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$, $|\nabla\eta| \le c/(R-r)$, where $r < R$. For any $k \in \mathbb{N}$ we have using integration by parts as well as the bound for u_δ

$$\begin{aligned}
\int_B \eta^{2k} b(|\partial_n u_\delta|)|\partial_n u_\delta|^2\,\mathrm{d}x &= -\int_B u_\delta\cdot\partial_n\big[\eta^{2k}b(|\partial_n u_\delta|)\partial_n u_\delta\big]\,\mathrm{d}x\\
&\le c\Bigg[\int_B \eta^{2k}|\partial_n\partial_n u_\delta| b(|\partial_n u_\delta|)\,\mathrm{d}x\\
&\quad + \int_B \eta^{2k-1}|\nabla\eta| b(|\partial_n u_\delta|)|\partial_n u_\delta|\,\mathrm{d}x\\
&\quad + \int_B \eta^{2k} b'(|\partial_n u_\delta|)|\partial_n\partial_n u_\delta||\partial_n u_\delta|\,\mathrm{d}x\Bigg]\\
&=: c[T_1+T_2+T_3]\,, \quad c = c(n,N,k,\|u\|_{L^\infty(B)})\,.
\end{aligned} \tag{3.3}$$

We discuss the terms T_i: from Young's inequality we get

$$T_2 \le \tau\int_B \eta^{2k}|\partial_n u_\delta|^2 b(|\partial_n u_\delta|)\,\mathrm{d}x + c(\tau)\int_B \eta^{2k-2}|\nabla\eta|^2 b(|\partial_n u_\delta|)\,\mathrm{d}x$$

for any $\tau > 0$, and the first term on the r.h.s. can be absorbed into the l.h.s. of (3.3) for small τ, whereas the second integral is bounded by a local constant on account of Lemma 3.1. This together with (3.3) shows

$$\int_B \eta^{2k} b(|\partial_n u_\delta|)\Gamma_{n,\delta}\,\mathrm{d}x \le c\Bigg[c_{loc} + \underbrace{\int_B \eta^{2k} b(|\partial_n u_\delta|)\,\mathrm{d}x}_{\le c_{loc}} + T_1 + T_3\Bigg]\,. \tag{3.4}$$

Here c_{loc} denotes a local constant depending in particular on R and r but being independent of ε. Again with Young's inequality we get

$$T_1 \le \tau \int_B \eta^{2k} b(|\partial_n u_\delta|)\Gamma_{n,\delta}\,\mathrm{d}x + c(\tau)\int_B \eta^{2k} b(|\partial_n u_\delta|)|\partial_n\partial_n u_\delta|^2\Gamma_{n,\delta}^{-1}\,\mathrm{d}x\,.$$

Observing

$$b(|\partial_n u_\delta|) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} b(t|\partial_n u_\delta|)\,\mathrm{d}t = |\partial_n u_\delta|\int_0^1 b'(t|\partial_n u_\delta|)\,\mathrm{d}t \le |\partial_n u_\delta| b'(|\partial_n u_\delta|)$$

(note: b' is increasing) we find

$$T_1 \le \tau \int_B \eta^{2k} b(|\partial_n u_\delta|)\Gamma_{n,\delta}\,\mathrm{d}x + c(\tau)\int_B \eta^{2k}\frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}|\partial_n\partial_n u_\delta|^2\,\mathrm{d}x.$$

Now we use Remark 2.1, e), i), for $\mathcal{B}$ to estimate

$$\int_B \eta^{2k}\frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}|\partial_n\partial_n u_\delta|^2\,\mathrm{d}x \le \int_B \eta^{2k} D^2\mathcal{B}(\partial_n u_\delta)(\partial_n\partial_n u_\delta,\partial_n\partial_n u_\delta)\,\mathrm{d}x$$

and get for $\tau \ll 1$ from (3.4)

$$\int_B \eta^{2k} b(|\partial_n u_\delta|)\Gamma_{n,\delta}\,\mathrm{d}x \le c\Bigg[c_{loc} + \int_B \eta^{2k} D^2\mathcal{B}(\partial_n u_\delta)(\partial_n\partial_n u_\delta,\partial_n\partial_n u_\delta)\,\mathrm{d}x + T_3\Bigg]\,. \tag{3.5}$$

Finally we observe (using Young's inequality)

$$T_3 \le \tau \int_B \eta^{2k} b'(|\partial_n u_\delta|)|\partial_n u_\delta|^3\,\mathrm{d}x + c(\tau)\int_B \eta^{2k}\frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}|\partial_n\partial_n u_\delta|^2\,\mathrm{d}x\,,$$

where the second term on the r.h.s. has already been estimated before (3.5). For discussing the first term we claim

$$b'(t)t \le c b(t) \quad \text{for all } t \ge 0\,. \tag{3.6}$$

In fact we have

$$b(2t) = \int_0^2 \frac{\mathrm{d}}{\mathrm{d}s} b(st)\,\mathrm{d}s = t\int_0^2 b'(st)\,\mathrm{d}s \ge t\int_1^2 b'(st)\,\mathrm{d}s \ge t b'(t)$$

by the monotonicity of b'. If we use the Δ_2-property for b, then we get (3.6), and this inequality implies

$$\tau \int_B \eta^{2k} b'(|\partial_n u_\delta|)|\partial_n u_\delta|^3 \,\mathrm{d}x \le c\tau \int_B \eta^{2k} b(|\partial_n u_\delta|)\Gamma_{n,\delta} \,\mathrm{d}x \,,$$

so that we can absorb this term. Summing up it is shown that

$$\int_B \eta^{2k} b(|\partial_n u_\delta|)\Gamma_{n,\delta} \,\mathrm{d}x \le c\Bigg[c_{loc} + \int_B \eta^{2k} D^2\mathcal{B}(\partial_n u_\delta)(\partial_n\partial_n u_\delta, \partial_n\partial_n u_\delta) \,\mathrm{d}x\Bigg] . \qquad (3.7)$$

By the Caccioppoli inequality (3.2) we have

$$\begin{aligned}
&\int_B \eta^{2k} D^2\mathcal{B}(\partial_n u_\delta)(\partial_n\partial_n u_\delta, \partial_n\partial_n u_\delta) \,\mathrm{d}x \\
\le\ & \int_B \eta^{2k} D^2F_\delta(\nabla u_\delta)(\partial_n\nabla u_\delta, \partial_n\nabla u_\delta) \,\mathrm{d}x \\
\le\ & c\int_B D^2F_\delta(\nabla u_\delta)(\nabla\eta\otimes\partial_n u_\delta, \nabla\eta\otimes\partial_n u_\delta)\eta^{2k-2} \,\mathrm{d}x \\
\le\ & c\Bigg[\int_B \delta\Gamma_\delta^{\frac{\tilde q}{2}}|\nabla\eta|^2\eta^{2k-2} \,\mathrm{d}x \\
& + \int_B D^2\mathcal{A}(\tilde\nabla u_\delta)(\nabla\eta\otimes\partial_n u_\delta, \nabla\eta\otimes\partial_n u_\delta)\eta^{2k-2} \,\mathrm{d}x \\
& + \int_B D^2\mathcal{B}(\partial_n u_\delta)(\nabla\eta\otimes\partial_n u_\delta, \nabla\eta\otimes\partial_n u_\delta)\eta^{2k-2} \,\mathrm{d}x\Bigg] \\
=:\ & c[S_1 + S_2 + S_3] \,,
\end{aligned}$$

and Lemma 3.1 implies

$$S_1 \to 0 \quad \text{as } \varepsilon \to 0 \,.$$

¿From Remark 2.1, e), i), and from (3.6) we get

$$\begin{aligned}
S_3 \ &\le\ c\int_B |\nabla\eta|^2\eta^{2k-2}\frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}|\partial_n u_\delta|^2 \,\mathrm{d}x \\
&\le\ c\int_B \eta^{2k-2}|\nabla\eta|^2 b(|\partial_n u_\delta|) \,\mathrm{d}x \le c_{loc} \,.
\end{aligned}$$

Again by Remark 2.1, e), i), we see

$$S_2 \le c\int_B |\nabla\eta|^2\eta^{2k-2}\frac{a'(|\tilde\nabla u_\delta|)}{|\tilde\nabla u_\delta|}|\partial_n u_\delta|^2 \,\mathrm{d}x \,,$$

and in order to proceed further let

$$\mathcal{N}(t) := b(\sqrt{t})t \,, \qquad t \ge 0 \,.$$

Since

$$\begin{aligned}\mathcal{N}'(t) &= b(\sqrt{t}) + \frac{1}{2}b'(\sqrt{t})\sqrt{t}\,,\\ \mathcal{N}''(t) &= \frac{1}{2\sqrt{t}}b'(\sqrt{t}) + \frac{1}{4\sqrt{t}}b'(\sqrt{t}) + \frac{1}{4}b''(t)\,,\end{aligned}$$

we see that $\mathcal{N}$ is a N-function (with the Δ_2-property). For $\tau > 0$ let $\mathcal{N}_\tau(t) := \tau\mathcal{N}(t)$ and define

$$\rho := \eta^{2k-2}|\nabla\eta|^2\frac{a'(|\tilde{\nabla}u_\delta|)}{|\tilde{\nabla}u_\delta|}|\partial_n u_\delta|^2\,.$$

On the set $B \cap [|\tilde{\nabla}u_\delta| \le 1]$ we estimate (using $(H2)$)

$$\rho \le c\eta^{2k-2}|\nabla\eta|^2 a''(|\tilde{\nabla}u_\delta|)|\partial_n u_\delta|^2 \le c\eta^{2k-2}|\nabla\eta|^2|\partial_n u_\delta|^2 \le c_{loc}\eta^{2k-2}\Gamma_{n,\delta}\,,$$

i.e.

$$\int_{B\cap[|\tilde{\nabla}u_\delta|\le 1]}\rho\,\mathrm{d}x \le c_{loc}\int_B \eta^{2k-2}\Gamma_{n,\delta}\,\mathrm{d}x\,,$$

whereas (by Young's inequality for N-functions)

$$\begin{aligned}\int_{B\cap[|\tilde{\nabla}u_\delta|\ge 1]}\rho\,\mathrm{d}x &\le \int_{B\cap[|\tilde{\nabla}u_\delta|\ge 1]}\mathcal{N}_\tau(\eta^{2k-2}|\partial_n u_\delta|^2)\,\mathrm{d}x\\ &\quad + \int_{B\cap[|\tilde{\nabla}u_\delta|\ge 1]}\mathcal{N}^*_\tau\Big(|\nabla\eta|^2\frac{a'(|\tilde{\nabla}u_\delta|)}{|\tilde{\nabla}u_\delta|}\Big)\,\mathrm{d}x\\ &= \tau\int_{B\cap[|\tilde{\nabla}u_\delta|\ge 1]}\eta^{2k-2}|\partial_n u_\delta|^2 b(\eta^{k-1}|\partial_n u_\delta|)\,\mathrm{d}x\\ &\quad + \int_{B\cap[|\tilde{\nabla}u_\delta|\ge 1]}\mathcal{N}^*_\tau\Big(|\nabla\eta|^2\frac{a'(|\tilde{\nabla}u_\delta|)}{|\tilde{\nabla}u_\delta|}\Big)\,\mathrm{d}x\\ &=: \tau U_1 + U_2\,.\end{aligned}$$

Since b is convex with $b(0) = 0$, we have

$$b(\eta^{k-1}|\partial_n u_\delta|) \le \eta^{k-1}b(|\partial_n u_\delta|)\,,$$

which means that for k large and τ small the term τU_1 can be absorbed in the l.h.s. of (3.7). By definition the conjugate function $\mathcal{N}^*_\tau$ satisfies

$$\begin{aligned}\mathcal{N}^*_\tau(t) &= \sup_{s\ge 0}[st - \tau b(\sqrt{s})s] = \sup_{s\ge 0}[t - \tau b(\sqrt{s})]s = \sup_{s\le[b^{-1}(t/\tau)]^2}[t - \tau b(\sqrt{s})]s\\ &\le \big[b^{-1}(t/\tau)\big]^2\sup[t - \tau b(\sqrt{s})]\\ &\le t\big[b^{-1}(t/\tau)\big]^2\,.\end{aligned}$$

Applying (3.6) to the function a we see

$$\int_{B\cap[|\tilde{\nabla}u_\delta|\ge 1]}\mathcal{N}^*_\tau\Big(|\nabla\eta|^2\frac{a'(|\tilde{\nabla}u_\delta|)}{|\tilde{\nabla}u_\delta|}\Big)\,\mathrm{d}x \le \int_{B\cap[|\tilde{\nabla}u_\delta|\ge 1]}\mathcal{N}^*_\tau(|\nabla\eta|^2|\tilde{\nabla}u_\delta|^{-2}a(|\tilde{\nabla}u_\delta|))\,\mathrm{d}x\,,$$

and by the convexity of $\mathcal{N}_\tau^*$ we have on the set of integration

$$\mathcal{N}_\tau^*(|\nabla\eta|^2|\tilde{\nabla}u_\delta|^{-2}a(|\tilde{\nabla}u_\delta|)) \leq |\tilde{\nabla}u_\delta|^{-2}\mathcal{N}_\tau^*(|\nabla\eta|^2a(|\tilde{\nabla}u_\delta|))\,,$$

whereas the Δ_2-property of $\mathcal{N}_\tau^*$ can be used to control the last term through the quantity

$$c(\tau,\eta)|\tilde{\nabla}u_\delta|^{-2}\mathcal{N}_\tau^*(\tau a(|\tilde{\nabla}u_\delta|))\,.$$

Now we can apply the upper bound for $\mathcal{N}_\tau^*$ to get

$$\begin{aligned}
&\int_{B\cap[|\tilde{\nabla}u_\delta|\geq 1]} \mathcal{N}_\tau^*\Big(|\nabla\eta|^2\frac{a'(|\tilde{\nabla}u_\delta|)}{|\tilde{\nabla}u_\delta|}\Big)\,\mathrm{d}x \\
&\leq\ c(\tau,\eta)\int_{B\cap[|\tilde{\nabla}u_\delta|\geq 1]} |\tilde{\nabla}u_\delta|^{-2}a(|\tilde{\nabla}u_\delta|)\big[b^{-1}(a(|\tilde{\nabla}u_\delta|))\big]^2\,\mathrm{d}x \\
&\leq\ c(\tau,\eta)\int_{B\cap[|\tilde{\nabla}u_\delta|\geq 1]} a(|\tilde{\nabla}u_\delta|)\,\mathrm{d}x \leq c_{loc}\,,
\end{aligned}$$

where we have used the inequality (2.3). Thus it is shown that

$$\int_B \eta^{2k}b(|\partial_n u_\delta|)\Gamma_{n,\delta}\,\mathrm{d}x \leq c_{loc}\left[1+\int_B \eta^{2k-2}\Gamma_{n,\delta}\,\mathrm{d}x\right]\,,$$

and for $k>3$ and τ sufficiently small Young's inequality gives

$$\begin{aligned}
\int_B \eta^{2k}b(|\partial_n u_\delta|)\Gamma_{n,\delta}\,\mathrm{d}x\ &\leq\ c_{loc}\left[1+\tau\int_B \eta^{2k}\Gamma_{n,\delta}^{\frac{3}{2}}\,\mathrm{d}x + c(\tau)\right] \\
&\leq\ c_{loc}\left[c(\tau)+\int_{B\cap[|\partial u_n|\leq K]}\eta^{2k}\Gamma_{n,\delta}^{\frac{3}{2}}\,\mathrm{d}x+\tau\int_{B\cap[|\partial u_n|>K]}\eta^{2k}\Gamma_{n,\delta}^{\frac{3}{2}}\,\mathrm{d}x\right]\,,
\end{aligned}$$

where K is chosen such that $b(t)\geq(1+t^2)^{1/2}$ for $t\geq K$, i.e. the last integral can be absorbed into the l.h.s. and the other integral trivially is bounded. Altogether we end up with

$$\int_B \eta^{2k}b(|\partial_n u_\delta|)\Gamma_{n,\delta}\,\mathrm{d}x \leq c_{loc}\,, \tag{3.8}$$

and this proves Theorem 2.1, a), by passing to the limit $\varepsilon\to 0$ and recalling Lemma 3.1.

For proving part b) we keep our notation and get analogous to (3.7)

$$\int_B \eta^{2k}a(|\tilde{\nabla}u_\delta|)\tilde{\Gamma}_\delta\,\mathrm{d}x \leq c\left[c_{loc}+\int_B \eta^{2k}D^2\mathcal{A}(\tilde{\nabla}u_\delta)(\partial_\gamma\tilde{\nabla}u_\delta,\partial_\gamma\tilde{\nabla}u_\delta)\,\mathrm{d}x\right]\,, \tag{3.9}$$

where here and in what follows we always take the sum w.r.t. $\gamma=1,\,\dots,\,n-1$. In fact, (3.9) is established along the same lines as (3.7) by performing an integration by parts on the r.h.s. of the following equation

$$\int_B \eta^{2k}a(|\tilde{\nabla}u_\delta|)|\tilde{\nabla}u_\delta|^2\,\mathrm{d}x = \int_B \partial_\gamma u_\delta\cdot[\eta^{2k}a(|\tilde{\nabla}u_\delta|)\partial_\gamma u_\delta]\,\mathrm{d}x$$

using the uniform boundedness of u_δ.

Inequality (3.2) gives

$$\begin{aligned}
&\int_B \eta^{2k} D^2\mathcal{A}(\tilde\nabla u_\delta)(\partial_\gamma \tilde\nabla u_\delta, \partial_\gamma \tilde\nabla u_\delta)\,\mathrm{d}x\\
&\le \int_B \eta^{2k} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\,\mathrm{d}x\\
&\le c\Bigg[\delta\int_B \eta^{2k-2}|\nabla\eta|^2\Gamma_\delta^{\frac{\tilde q}{2}}\,\mathrm{d}x + \int_B \eta^{2k-2} D^2\mathcal{A}(\tilde\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta, \nabla\eta\otimes\partial_\gamma u_\delta)\,\mathrm{d}x\\
&\qquad + \int_B \eta^{2k-2} D^2\mathcal{B}(\partial_n u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta, \nabla\eta\otimes\partial_\gamma u_\delta\,\mathrm{d}x\Bigg],
\end{aligned}$$

and if we use Remark 2.1, e), i), for $\mathcal{A}$ and $\mathcal{B}$ together with

$$\delta\int_B \eta^{2k-2}|\nabla\eta|^2\Gamma_\delta^{\frac{\tilde q}{2}}\,\mathrm{d}x \to 0 \quad \text{as } \delta\to 0,$$

we see

$$\begin{aligned}
&\int_B \eta^{2k} D^2\mathcal{A}(\tilde\nabla u_\delta)(\partial_\gamma \tilde\nabla u_\delta, \partial_\gamma \tilde\nabla u_\delta)\,\mathrm{d}x\\
&\le c\Bigg[c_{\mathrm{loc}} + \int_B \eta^{2k-2}|\nabla\eta|^2\frac{a'(|\tilde\nabla u_\delta|)}{|\tilde\nabla u_\delta|}|\tilde\nabla u_\delta|^2 + \int_B \eta^{2k-2}|\nabla\eta|^2\frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}|\tilde\nabla u_\delta|^2\,\mathrm{d}x\Bigg]\\
&=: c[c_{loc} + W_1 + W_2]\,. \qquad (3.10)
\end{aligned}$$

Using (3.6) for a we deduce

$$W_1 \le c\int_B \eta^{2k-2}|\nabla\eta|^2 a(|\tilde\nabla u_\delta|)\,\mathrm{d}x \le c_{loc}\,. \qquad (3.11)$$

For discussing W_2 we consider the N-functions

$$\mathcal{M}(t) := ta(\sqrt{t})\,, \qquad \mathcal{M}_\tau(t) := \tau\mathcal{M}(t)$$

with small $\tau > 0$ and observe first (recalling $(H2)$)

$$\begin{aligned}
&\int_{B\cap[|\partial_n u_\delta|\le 1]} \eta^{2k-2}|\nabla\eta|^2\frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}|\tilde\nabla u_\delta|^2\,\mathrm{d}x\\
&\le c\int_{B\cap[|\partial_n u_\delta|\le 1]} \eta^{2k-2}|\nabla\eta|^2 b''(|\partial_n u_\delta|^2)|\tilde\nabla u_\delta|^2\,\mathrm{d}x\\
&\le c_{loc}\max_{0\le t\le 1} b''(t)\int_{B\cap[|\partial_n u_\delta|\le 1]} \eta^{2k-2}\tilde\Gamma_\delta\,\mathrm{d}x
\end{aligned}$$

whereas

$$\begin{aligned}
&\int_{B\cap[|\partial_n u_\delta|\geq 1]} \eta^{2k-2}|\nabla\eta|^2 \frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}|\tilde{\nabla} u_\delta|^2 \,\mathrm{d}x \\
&\leq \int_{B\cap[|\partial_n u_\delta|\geq 1]} \mathcal{M}_\tau(\eta^{2k-2}|\tilde{\nabla} u_\delta|^2)\,\mathrm{d}x + \int_{B\cap[|\partial_n u_\delta|\geq 1]} \mathcal{M}_\tau^*\Big(|\nabla\eta|^2 \frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}\Big)\,\mathrm{d}x \\
&\leq \tau\int_B \eta^{2k-2}|\tilde{\nabla} u_\delta|^2 \underbrace{a(\eta^{k-1}|\tilde{\nabla} u_\delta|)}_{\leq \eta^{k-1}a(|\tilde{\nabla} u_\delta|)}\,\mathrm{d}x + \int_{B\cap[|\partial_n u_\delta|\geq 1]} \mathcal{M}_\tau^*\Big(|\nabla\eta|^2 \frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}\Big)\,\mathrm{d}x\,,
\end{aligned}$$

and for $\tau \ll 1$ and $k \in \mathbb{N}$ large enough we can put the τ-term to the l.h.s. of (3.9). In the same way as before for $\mathcal{N}_\tau^*$ we find

$$\mathcal{M}_\tau^*(t) \leq t\big[a^{-1}(t/\tau)\big]^2\,,$$

and using the Δ_2-property of $\mathcal{M}_\tau^*$ we have for $t \geq 1$ by (3.6)

$$\begin{aligned}
\mathcal{M}_\tau^*\Big(|\nabla\eta|^2\frac{b'(t)}{t}\Big) &\leq c(\eta)\mathcal{M}_\tau^*\Big(\frac{b'(t)}{t}\Big) \leq c(\eta)\mathcal{M}_\tau^*(t^{-2}b(t)) \leq c(\tau,\eta)\mathcal{M}_\tau^*(\tau b(t)t^{-2}) \\
&\leq c(\tau,\eta)t^{-2}b(t)\big[a^{-1}(t^{-2}b(t))\big]^2\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{B\cap[|\partial_n u_\delta|\geq 1]} \mathcal{M}_\tau^*\Big(|\nabla\eta|^2 \frac{b'(|\partial_n u_\delta|)}{|\partial_n u_\delta|}\Big)\,\mathrm{d}x \\
&\leq c(\tau,\eta)\int_{\operatorname{spt}\eta\cap[|\partial_n u_\delta|\geq 1]} |\partial_n u_\delta|^{-2} b(|\partial_n u_\delta|)\big[a^{-1}(|\partial_n u_\delta|^{-2}b(|\partial_n u_\delta|))\big]^2\,\mathrm{d}x\,,
\end{aligned}$$

and we can apply (3.8) provided

$$\big[a^{-1}(|\partial_n u_\delta|^{-2}b(|\partial_n u_\delta|))\big]^2 \leq c|\partial_n u_\delta|^4\,,$$

but this follows from assumption (2.5) (w.l.o.g. assuming the validity of (2.5) for $t \geq 1$), i.e. we can handle W_2 in an appropriate way. By combining the above estimates with (3.8), (3.10) and (3.11) and returning to (3.9) it is proved by repeating the calculations before (3.8) that

$$\int_B \eta^{2k} a(|\tilde{\nabla} u_\delta|)\tilde{\Gamma}_\delta \,\mathrm{d}x \leq c_{loc}\,, \tag{3.12}$$

and b)of Theorem 2.1 follows. The last part is immediate. □

4 Proof of Theorem 2.2

We first give a slight modification of the approximation from Section 3: we now start from a local minimizer $u \in W^1_{2,loc}(\Omega;\mathbb{R}^M)$ (recall Remark 2.3, c)) being a priori unbounded.

Then we select a disc B' such that $B \Subset B' \Subset \Omega$ and such that $u_{|\partial B'} \in W^1_2(\partial B';\mathbb{R}^M) \subset C^0(\partial B';\mathbb{R}^M)$ which is possible by [Mo], Theorem 3.6.1, c). The maximum principle of [DLM] gives $u \in L^\infty(B';\mathbb{R}^M)$, thus $(u)_\varepsilon \in L^\infty(B;\mathbb{R}^M)$ uniformly and again by quoting [DLM] we deduce

$$\|u_\delta\|_{L^\infty(B)} \leq const < \infty\,.$$

We proceed as in [BF2] by first showing

$$\partial_2 u_\delta \in W^1_{2,loc}(B;\mathbb{R}^M) \tag{4.1}$$

uniformly w.r.t. ε. We have by Remark 2.1, e), i), and by (3.2) with $\gamma = 2$ and for $\eta \in C^\infty_0(B)$

$$\begin{aligned}
&\int_B \eta^2 D^2F_\delta(\nabla u_\delta)(\partial_2\nabla u_\delta, \partial_2\nabla u_\delta)\,\mathrm{d}x\\
&\quad\leq\; c\int_B D^2F_\delta(\nabla u_\delta)(\nabla\eta\otimes\partial_2 u_\delta, \nabla\eta\otimes\partial_2 u_\delta)\,\mathrm{d}x\\
&\quad\leq\; c\Bigg[\int_B |\nabla\eta|^2\Gamma_\delta^{\frac{\tilde q}{2}}\,\mathrm{d}x + \int_B |\nabla\eta|^2\frac{b'(|\partial_2 u_\delta|)}{|\partial_2 u_\delta|}|\partial_2 u_\delta|^2\,\mathrm{d}x + \int_B |\nabla\eta|^2\frac{a'(|\partial_1 u_\delta|)}{|\partial_1 u_\delta|}|\partial_2 u_\delta|^2\,\mathrm{d}x\Bigg]\,.
\end{aligned}$$

The first term on the r.h.s. goes to zero as $\varepsilon \to 0$, the third one corresponds to the quantity S_2 introduced in the previous section, and as demonstrated in Section 3 (compare the discussion of $\int_B \rho\,\mathrm{d}x$) we can control

$$\int_B |\nabla\eta|^2\frac{a'(|\partial_1 u_\delta|)}{|\partial_1 u_\delta|}|\partial_2 u_\delta|^2\,\mathrm{d}x$$

in terms of local constants and the quantity

$$\int_{\operatorname{spt}\eta} b(|\partial_2 u_\delta|)|\partial_2 u_\delta|^2\,\mathrm{d}x\,.$$

But this term is bounded by c_{loc} on account of (3.8). The second term on the r.h.s. corresponds to S_3 in Section 3, and in Section 3 we showed $S_3 \leq c_{loc}$. Therefore we get

$$\int_B \eta^2 D^2F_\delta(\nabla u_\delta)(\partial_2\nabla u_\delta, \partial_2\nabla u_\delta)\,\mathrm{d}x \leq c_{loc}$$

without using (2.5). Combining (2.6) and Remark 2.1, e), i), we deduce from this inequality that

$$\int_B \eta^2|\partial_2\nabla u_\delta|^2\,\mathrm{d}x \leq c_{loc}\,,$$

and (4.1) follows. Sobolev's embedding theorem then implies

$$\partial_2 u_\delta \in L^s_{loc}(B;\mathbb{R}^M) \tag{4.2}$$

for all $s < \infty$ uniformly w.r.t. ε.

In a second step we want to prove (3.12), i.e.

$$a(|\partial_1 u_\delta|)|\partial_1 u_\delta|^2 \in L^1_{loc}(B) \tag{4.3}$$

uniformly in ε *without* (2.5). This can be achieved starting from (3.9) by bounding the integral W_2 defined in (3.10) in a different way: to this purpose we recall Remark 2.1, c), hence we can estimate for $t \geq 1$ (once more by (3.6))

$$\begin{aligned}\mathcal{M}^*_\tau\Big(|\nabla\eta|^2\frac{b'(t)}{t}\Big) &\leq c(\eta)\mathcal{M}^*_\tau\Big(\frac{b'(t)}{t}\Big) \leq c(\eta)\mathcal{M}^*_\tau(t^{-2}b(t)) \leq c(\eta)\mathcal{M}^*_\tau(t^{q-2})\\ &\leq c(\eta,\tau)\mathcal{M}^*_\tau(t^{q-2}\tau) \leq c(\eta,\tau)t^{q-2}\big[a^{-1}(t^{q-2})\big]^2 .\end{aligned}$$

Recalling $a'(0) = 0$ and using $a''(t) \geq a_0 > 0$ we get that $a(t) \geq ct^2$, i.e. $a^{-1}(t) \leq c\sqrt{t}$, and in conclusion

$$\mathcal{M}^*_\tau\Big(|\nabla\eta|^2\frac{b'(t)}{t}\Big) \leq c(\eta,\tau)t^{2q-4} .$$

This shows

$$\int_{B\cap[|\partial_2 u_\delta|\geq 1]} \mathcal{M}^*_\tau\Big(|\nabla\eta|^2\frac{b'(|\partial_2 u_\delta|)}{|\partial_2 u_\delta|}\Big)\,\mathrm{d}x \leq c(\eta,\tau)\int_{\operatorname{spt}\eta}\Gamma^{q-2}_{2,\delta}\,\mathrm{d}x ,$$

and to the latter integral we can apply (4.2), hence we get (4.3).

Let

$$H_\delta := D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma\nabla u_\delta)^{\frac{1}{2}} ,$$

where here in what follows the sum is taken w.r.t. $\gamma = 1$, 2. Remark 2.1, e), i), together with (2.6) applied to a and b gives

$$c\big[\partial_\gamma\partial_1 u_\delta\cdot\partial_\gamma\partial_1 u_\delta + \partial_\gamma\partial_2 u_\delta\cdot\partial_\gamma\partial_2 u_\delta\big] \leq H^2_\delta ,$$

i.e.

$$|\nabla^2 u_\delta|^2 \leq cH^2_\delta .$$

¿From (3.2) it follows

$$\begin{aligned}\int_B \eta H^2_\delta\,\mathrm{d}x &\leq c\int_B D^2F_\delta(\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta,\nabla\eta\otimes\partial_\gamma u_\delta)\,\mathrm{d}x\\ &\leq c\Bigg[\int_B|\nabla\eta|^2\Gamma_\delta^{\frac{\tilde q}{2}}\,\mathrm{d}x + \int_B a'(|\partial_1 u_\delta|)|\partial_1 u_\delta||\nabla\eta|^2\,\mathrm{d}x\\ &\quad+\int_B b'(|\partial_2 u_\delta|)|\partial_2 u_\delta||\nabla\eta|^2\,\mathrm{d}x + \int_B\frac{a'(|\partial_1 u_\delta|)}{|\partial_1 u_\delta|}|\partial_2 u_\delta|^2|\nabla\eta|^2\,\mathrm{d}x\\ &\quad+\int_B\frac{b'(|\partial_2 u_\delta|)}{|\partial_2 u_\delta|}|\partial_1 u_\delta|^2|\nabla\eta|^2\,\mathrm{d}x\Bigg] ,\end{aligned}$$

and the first three integrals on the r.h.s. are bounded by a local constant: for the first one we use Lemma 3.1, the second and the third one are bounded by (3.6) applied to a and b combined with Lemma 3.1. The fourth one occurs as an upper bound for S_2 and the calculations from Section 3 show

$$\int_B \frac{a'(|\partial_1 u_\delta|)}{|\partial_1 u_\delta|} |\partial_2 u_\delta|^2 |\nabla \eta|^2 \, dx \le c_{loc}$$

on account of (3.8). The fifth integral corresponds to W_2 from Section 3 and has already been discussed after (4.3), where it was outlined how the calculations of Section 3 can be modified to give (recall (4.2))

$$\int_B \frac{b'(|\partial_2 u_\delta|)}{|\partial_2 u_\delta|} |\partial_1 u_\delta|^2 |\nabla \eta|^2 \, dx \le c \left[c_{loc} + \int_{\operatorname{spt} \eta} \Gamma_{2,\delta}^{q-2} \, dx \right] \le c_{loc} \, .$$

Altogether it follows

$$H_\delta \in L^2_{loc}(B)$$

uniformly in $\varepsilon > 0$, hence $\nabla u_\delta \in W^1_{2,loc}(B; \mathbb{R}^{2M})$ uniformly, and Sobolev's embedding theorem implies the uniform local higher integrability of ∇u_δ for any finite exponent. The proof of Theorem 2.2 is complete. □

5 Proof of Theorem 2.3

In the scalar case we choose a different way of regularization avoiding the introduction of an extra power-growth energy. Proceeding as in [BFZ] we first fix a ball $B := B_R(x_0) \Subset \Omega$ and consider the mollification $(u)_\varepsilon$ of our local minimizer $u \in L^\infty_{loc}(\Omega)$. Let u_ε denote the unique Lipschitz function minimizing $I[\cdot, B]$ among all Lipschitz maps w: $\overline{B} \to \mathbb{R}$ for boundary values $(u)_\varepsilon$, i.e. u_ε is the Hilbert-Haar solution (see, e.g., [MM], Theorem 4, p. 162). For the next auxiliary results we refer to [BFZ].

Lemma 5.1. *a) Passing to the limit $\varepsilon \to 0$ we have ($p := 1 + \bar{\varepsilon}$)*

$$u_\varepsilon \rightharpoondown u \quad \text{in } W^1_p(B) \, , \qquad \int_B F(\nabla u_\varepsilon) \, dx \to \int_B F(\nabla u) \, dx \, .$$

b) $\|u_\varepsilon\|_{L^\infty(B)}$ is bounded independent of ε.

Lemma 5.2. *The functions u_ε are of class $C^{1,\alpha}(B) \cap W^2_{2,loc}(B)$ for any $\alpha < 1$.*

Lemma 5.3. *(Variants of Caccioppoli's inequality) For any numbers α, $\beta \ge 0$ and for all $\eta \in C^\infty_0(B)$ s.t. $0 \le \eta \le 1$ we have*

$$\begin{aligned} &\int_B D^2F(\nabla u_\varepsilon)(\partial_n \nabla u_\varepsilon, \partial_n \nabla u_\varepsilon) \Gamma_{n,\varepsilon}^{\frac{\alpha}{2}} \eta^2 \, dx \\ &\le \; c(\alpha) \int_\Omega D^2F(\nabla u_\varepsilon)(\nabla \eta, \nabla \eta) \Gamma_{n,\varepsilon}^{\frac{\alpha}{2}} |\partial_n u_\varepsilon|^2 \, dx \, , \end{aligned} \tag{5.1}$$

and

$$\int_B D^2F(\nabla u_\varepsilon)(\partial_\gamma \nabla u_\varepsilon, \partial_\gamma \nabla u_\varepsilon)\tilde{\Gamma}_\varepsilon^{\frac{\beta}{2}}\eta^2 \,\mathrm{d}x$$
$$\leq \; c(\beta)\int_B D^2F(\nabla u_\varepsilon)(\nabla\eta,\nabla\eta)\tilde{\Gamma}_\varepsilon^{\frac{\beta}{2}}|\tilde{\nabla} u_\varepsilon|^2\,\mathrm{d}x\,. \tag{5.2}$$

In (5.2) (and in what follows) we always take the sum w.r.t. γ from 1 to $n-1$. $c(\alpha)$, $c(\beta)$ denote positive constants independent of ε, and we have set: $\Gamma_{n,\varepsilon} = 1 + (\partial_n u_\varepsilon)^2$, $\tilde{\Gamma}_\varepsilon = 1 + |\tilde{\nabla} u_\varepsilon|^2$, $\tilde{\nabla} := (\partial_1, \dots, \partial_{n-1})$.

We fix some $\alpha \geq 0$ and a function $\eta \in C_0^\infty(B)$ such that $0 \leq \eta \leq 1$. Writing

$$\int_B \eta^2 b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha+2}{2}}\,\mathrm{d}x$$
$$= \int_B \eta^2 b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}\,\mathrm{d}x + \int_B \eta^2 b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}\partial_n u_\varepsilon \partial_n u_\varepsilon\,\mathrm{d}x$$

and performing an integration by parts in the second integral on the r.h.s., i.e.

$$\int_B \eta^2 b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}\partial_n u_\varepsilon \partial_n u_\varepsilon\,\mathrm{d}x$$
$$= -\int_B u_\varepsilon \partial_n\big[\partial_n u_\varepsilon \eta^2 b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}\big]\,\mathrm{d}x\,,$$

analogous calculations as carried out in Section 3 together with Lemma 5.1, b), lead to the result (compare (3.7))

$$\int_B \eta^2 b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha+2}{2}}\,\mathrm{d}x \tag{5.3}$$
$$\leq \; c\left[\int_B (\eta^2 + |\nabla\eta|^2) b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}\,\mathrm{d}x + \int_B \eta^2 D^2\mathcal{B}(\partial_n u_\varepsilon)(\partial_n\partial_n u_\varepsilon, \partial_n\partial_n u_\varepsilon)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}\,\mathrm{d}x\right],$$

whereas for any $\beta \geq 0$ we obtain (see (3.9))

$$\int_B \eta^2 a(|\tilde{\nabla} u_\varepsilon|)\tilde{\Gamma}_\varepsilon^{\frac{\beta+2}{2}}\,\mathrm{d}x \tag{5.4}$$
$$\leq \; c\left[\int_B (\eta^2 + |\nabla\eta|^2) a(|\tilde{\nabla} u_\varepsilon|)\tilde{\Gamma}_\varepsilon^{\frac{\beta}{2}}\,\mathrm{d}x + \int_B \eta^2 D^2\mathcal{A}(\tilde{\nabla} u_\varepsilon)(\partial_\gamma \tilde{\nabla} u_\varepsilon, \partial_\gamma \tilde{\nabla} u_\varepsilon)\tilde{\Gamma}_\varepsilon^{\frac{\beta}{2}}\,\mathrm{d}x\right].$$

On the r.h.s. of (5.3) and (5.4), respectively, we apply (5.1) and (5.2) in order to get

$$\int_B \eta^2 D^2\mathcal{B}(\partial_n u_\varepsilon)(\partial_n\partial_n u_\varepsilon, \partial_n\partial_n u_\varepsilon)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}\,\mathrm{d}x \leq c(\alpha)\int_B D^2F(\nabla u_\varepsilon)(\nabla\eta,\nabla\eta)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}|\partial_n u_\varepsilon|^2\,\mathrm{d}x\,,$$

as well as

$$\int_B \eta^2 D^2\mathcal{A}(\tilde{\nabla} u_\varepsilon)(\partial_\gamma \tilde{\nabla} u_\varepsilon, \partial_\gamma \tilde{\nabla} u_\varepsilon)\tilde{\Gamma}_\varepsilon^{\frac{\beta}{2}}\,\mathrm{d}x \leq c(\beta)\int_B D^2F(\nabla u_\varepsilon)(\nabla\eta,\nabla\eta)\tilde{\Gamma}_\varepsilon^{\frac{\beta}{2}}|\tilde{\nabla} u_\varepsilon|^2\,\mathrm{d}x\,.$$

Inserting these inequalities in (5.3), (5.4) and using Remark 2.1, e), i), to obtain an upper bound for $D^2F(\nabla u_\varepsilon)(\nabla\eta,\nabla\eta)$ we find

$$\begin{aligned}
&\int_B \eta^2 b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha+2}{2}}\\
&\le\; c(\alpha)\Bigg[\int_B (\eta^2+|\nabla\eta|^2) b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}} + \int_B |\nabla\eta|^2 \frac{b'(|\partial_n u_\varepsilon|)}{|\partial_n u_\varepsilon|}\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}|\partial_n u_\varepsilon|^2\,\mathrm{d}x\\
&\qquad + \int_B |\nabla\eta|^2\frac{a'(|\tilde\nabla u_\varepsilon|)}{|\tilde\nabla u_\varepsilon|}\Gamma_{n,\varepsilon}^{\frac{\alpha+2}{2}}\,\mathrm{d}x\Bigg]\,.
\end{aligned}$$

Recalling (3.6) we have

$$\frac{b'(|\partial_n u_\varepsilon|)}{|\partial_n u_\varepsilon|}\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}|\partial_n u_\varepsilon|^2 \le c b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}\,,$$

hence

$$\begin{aligned}
&\int_B \eta^2 b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha+2}{2}}\,\mathrm{d}x\\
&\le\; c(\alpha)\Bigg[\int_B (\eta^2+|\nabla\eta|^2) b(|\partial_n u_\varepsilon|)\Gamma_{n,\varepsilon}^{\frac{\alpha}{2}}\,\mathrm{d}x + \int_B |\nabla\eta|^2\frac{a'(|\tilde\nabla u_\varepsilon|)}{|\tilde\nabla u_\varepsilon|}\Gamma_{n,\varepsilon}^{\frac{\alpha+2}{2}}\,\mathrm{d}x\Bigg]\,, \qquad (5.5)
\end{aligned}$$

and in the same way

$$\begin{aligned}
&\int_B \eta^2 a(|\tilde\nabla u_\varepsilon|)\tilde\Gamma_{\varepsilon}^{\frac{\beta+2}{2}}\,\mathrm{d}x\\
&\le\; c(\beta)\Bigg[\int_B (\eta^2+|\nabla\eta|^2) a(|\tilde\nabla u_\varepsilon|)\tilde\Gamma_{\varepsilon}^{\frac{\beta}{2}}\,\mathrm{d}x + \int_B |\nabla\eta|^2\frac{b'(|\partial_n u_\varepsilon|)}{|\partial_n u_\varepsilon|}\tilde\Gamma_{\varepsilon}^{\frac{\beta+2}{2}}\,\mathrm{d}x\Bigg]\,. \qquad (5.6)
\end{aligned}$$

The next calculations can be made precise easily along the lines of Section 3 by replacing η^2 in (5.5) and (5.6) by η^{2k} for $k\in\mathbb{N}$ large enough and by using Young's inequality with an additional factor τ in order to absorb terms in the l.h.s.'s. In what follows the domain of integration always is the support of a "hidden testfunction". If we reduce (5.5) and (5.6) to the core, then we have

$$\begin{aligned}
&\int b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^{\alpha+2}\,\mathrm{d}x\\
&\le\; c(\alpha)\Bigg[\int b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^{\alpha}\,\mathrm{d}x + \int \frac{a'(|\tilde\nabla u_\varepsilon|)}{|\tilde\nabla u_\varepsilon|}|\partial_n u_\varepsilon|^{\alpha+2}\,\mathrm{d}x\Bigg]\,, \qquad (5.7)
\end{aligned}$$

and

$$\begin{aligned}
&\int a(|\tilde\nabla u_\varepsilon|)|\tilde\nabla u_\varepsilon|^{\beta+2}\,\mathrm{d}x\\
&\le\; c(\beta)\Bigg[\int a(|\tilde\nabla u_\varepsilon|)|\tilde\nabla u_\varepsilon|^{\beta}\,\mathrm{d}x + \int \frac{b'(|\partial_n u_\varepsilon|)}{|\partial_n u_\varepsilon|}|\tilde\nabla u_\varepsilon|^{\beta+2}\,\mathrm{d}x\Bigg]\,. \qquad (5.8)
\end{aligned}$$

We discuss the r.h.s. of (5.7): since

$$\begin{aligned} b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^\alpha &= b(|\partial_n u_\varepsilon|)^{\frac{\alpha}{\alpha+2}}|\partial_n u_\varepsilon|^\alpha b(|\partial_n u_\varepsilon|)^{\frac{2}{2+\alpha}} \\ &\le \big[b(|\partial_n u_\varepsilon|)^{\frac{\alpha}{\alpha+2}}|\partial_n u_\varepsilon|^\alpha\big]^{\frac{\alpha+2}{\alpha}} + b(|\partial_n u_\varepsilon|)\,, \end{aligned}$$

the first integral on the r.h.s. of (5.7) can be absorbed in the l.h.s. ("use τ") producing on the r.h.s. a term being bounded by a local constant. Let

$$K(t) := tb\big(t^{\frac{1}{\alpha+2}}\big)\,, \qquad t \ge 0\,.$$

It is easy to check that K is an N-function, and we have an estimate for the conjugate function:

$$\begin{aligned} K^*(s) &= \sup_{t\ge 0}[ts - K(t)] = \sup_{t\ge 0}\big[s - b\big(t^{\frac{1}{\alpha+2}}\big)\big]t = \sup_{t\le \left[b^{-1}(s)\right]^{\alpha+2}}\big[s - b\big(t^{\frac{1}{\alpha+2}}\big)\big]t \\ &\le s\big[b^{-1}(s)\big]^{\alpha+2}\,. \end{aligned}$$

This gives for the second term on the r.h.s. of (5.7)

$$\int \frac{a'(|\tilde\nabla u_\varepsilon|)}{|\tilde\nabla u_\varepsilon|}|\partial_n u_\varepsilon|^{\alpha+2}\,\mathrm{d}x \le \int K(|\partial_n u_\varepsilon|^{\alpha+2})\,\mathrm{d}x + \int K^*\Big(\frac{a'(|\tilde\nabla u_\varepsilon|)}{|\tilde\nabla u_\varepsilon|}\Big)\,\mathrm{d}x\,,$$

and using (3.6) and (Δ_2) we find

$$\begin{aligned} \int K^*\Big(\frac{a'(|\tilde\nabla u_\varepsilon|)}{|\tilde\nabla u_\varepsilon|}\Big)\,\mathrm{d}x &\le c\int K^*\big(a(|\tilde\nabla u_\varepsilon|)|\tilde\nabla u_\varepsilon|^{-2}\big)\,\mathrm{d}x \\ &\le c\int a(|\tilde\nabla u_\varepsilon|)|\tilde\nabla u_\varepsilon|^{-2}\big[b^{-1}\big(a(|\tilde\nabla u_\varepsilon|)|\tilde\nabla u_\varepsilon|^{-2}\big)\big]^{\alpha+2}\,\mathrm{d}x\,. \end{aligned}$$

We therefore deduce from (5.7)

$$\begin{aligned} &\int b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^{\alpha+2}\,\mathrm{d}x \\ &\le c(\alpha)\bigg[\int a(|\tilde\nabla u_\varepsilon|)|\tilde\nabla u_\varepsilon|^{-2}\big[b^{-1}\big(a(|\tilde\nabla u_\varepsilon|)|\tilde\nabla u_\varepsilon|^{-2}\big)\big]^{\alpha+2}\,\mathrm{d}x + \ldots\bigg]\,, \end{aligned} \tag{5.9}$$

and in an analogous way (5.8) implies

$$\begin{aligned} &\int a(|\tilde\nabla u_\varepsilon|)|\tilde\nabla u_\varepsilon|^{\beta+2}\,\mathrm{d}x \\ &\le c(\beta)\bigg[\int b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^{-2}\big[a^{-1}\big(b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^{-2}\big)\big]^{\beta+2}\,\mathrm{d}x + \ldots\bigg]\,, \end{aligned} \tag{5.10}$$

where "..." represent terms being bounded by local constants. Let

$$\begin{aligned} m(\alpha) &:= \int b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^{\alpha+2}\,\mathrm{d}x\,, \\ M(\alpha) &:= \int a(|\tilde{\nabla} u_\varepsilon|)|\tilde{\nabla} u_\varepsilon|^{-2}\big[b^{-1}\big(a(|\tilde{\nabla} u_\varepsilon|)|\tilde{\nabla} u_\varepsilon|^{-2}\big)\big]^{\alpha+2}\,\mathrm{d}x\,, \\ n(\beta) &:= \int a(|\tilde{\nabla} u_\varepsilon|)|\tilde{\nabla} u_\varepsilon|^{\beta+2}\,\mathrm{d}x\,, \\ N(\beta) &:= \int b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^{-2}\big[a^{-1}\big(b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^{-2}\big)\big]^{\beta+2}\,\mathrm{d}x\,. \end{aligned}$$

(5.9) and (5.10) then turn into the inequalities

$$m(\alpha) \le c(\alpha)[M(\alpha) + \dots] \tag{5.9$_\alpha$}$$

and

$$n(\beta) \le c(\beta)[N(\beta) + \dots]\,. \tag{5.10$_\beta$}$$

Suppose for the moment that $a(t) = t^2$. Then $M(\alpha) \le c(\alpha)$ for any $\alpha \ge 0$, so that by (5.9$_\alpha$) the same is true for $m(\alpha)$, and this implies

$$|\partial_n u_\varepsilon| \in L^r_{loc}(B)$$

for any finite r uniformly in ε.

This together with Remark 2.1, c), gives $N(\beta) \le c(\beta)$ for any $\beta \ge 0$, and (5.10$_\beta$) shows $n(\beta) \le c(\beta)$ for all β, i.e.

$$|\tilde{\nabla} u_\varepsilon| \in L^r_{loc}(B)\,,$$

again for any finite r uniformly in ε.

We return to the general case and claim the existence of $\alpha_0 > 0$ s.t.

$$M(\alpha_0) \le c_0\,. \tag{5.11}$$

Clearly (5.11) will follow if we have for large enough t the estimate

$$t^{-2}\big[b^{-1}\big(a(t)t^{-2}\big)\big]^{\alpha_0+2} \le c\,.$$

By the Δ_2-property this inequality will hold if we can prove

$$a(t) \le ct^2 b\big(t^{\frac{2}{2+\alpha_0}}\big)\,, \qquad t \gg 1\,. \tag{5.12}$$

Let us discuss the validity of (5.12): from

$$b(2s) \le \mu b(s) \quad \text{for all } \ s \ge 0$$

we get according to Lemma A.3

$$b(\lambda s) \leq \Big[1 + \mu^{1+\frac{\ln(\lambda)}{\ln(2)}}\Big] b(s) \,.$$

Letting $\lambda = t^{\alpha_0/(2+\alpha_0)}$, $s = t^{2/(2+\alpha_0)}$ for some α_0 being specified later this inequality gives for $t \gg 1$

$$\begin{aligned} b(t) &\leq c\big[1 + t^{\gamma_0}\big] b\Big(t^{\frac{2}{2+\alpha_0}}\Big) \\ &\leq c t^{\gamma_0} b\Big(t^{\frac{2}{2+\alpha_0}}\Big) \,, \\ \gamma_0 &:= \frac{\alpha_0}{2+\alpha_0} \frac{\ln(\mu)}{\ln(2)} \,. \end{aligned}$$

In particular we see

$$b(t) \leq c t^2 b\Big(t^{\frac{2}{2+\alpha_0}}\Big) \tag{5.13}$$

as long as $\gamma_0 \leq 2$. So if we define α_0 through the equation

$$\frac{\alpha_0}{2+\alpha_0} \frac{\ln(\mu)}{\ln(2)} = 2 \,, \tag{5.14}$$

then (5.13) together with $a(t) \leq b(t)$ guarantees (5.12) and hence (5.11).

(5.11) and (5.9_{α_0}) show that $m(\alpha_0) \leq c_0$, and by the definition of $N(\beta)$ we will get

$$N(\beta_0) \leq c_0 \tag{5.15}$$

provided that β_0 is chosen in such a way that for large t

$$t^{-2}\big[a^{-1}\big(b(t)t^{-2}\big)\big]^{\beta_0+2} \leq c t^{\alpha_0+2} \,.$$

This inequality in turn follows from

$$b(t) \leq c t^2 a\Big(t^{\frac{4+\alpha_0}{2+\beta_0}}\Big)$$

and by (2.5) we may take $\beta_0 = \alpha_0/2$ to get the above estimate leading to (5.15). Next we claim

$$M(\alpha_l) + N(\beta_l) \leq c_l \tag{5.16_l}$$

for suitable sequences α_l, β_l, c_l. For $l = 0$ this is true by (5.11) and (5.15) and the choices of α_0, β_0. Suppose now that $l \geq 1$ and that (5.16_{l-1}) is valid. From $N(\beta_{l-1}) \leq c_{l-1}$ we deduce quoting ($5.10_{\beta_{l-1}}$) that

$$n(\beta_{l-1}) \leq c_{l-1}$$

and this together with the definition of M shows

$$M(\alpha_l) \leq c_l \,, \tag{5.17}$$

provided we have for large t

$$t^{-2}\big[b^{-1}\big(a(t)t^{-2}\big)\big]^{\alpha_l+2} \le ct^{\beta_{l-1}+2}$$

or (which is the same)

$$a(t) \le ct^2 b\Big(t^{\frac{4+\beta_{l-1}}{2+\alpha_l}}\Big) . \tag{5.18}$$

Clearly (5.18) is satisfied for the choice

$$\alpha_l = 2 + \beta_{l-1} , \tag{5.19}$$

and (5.19) implies (5.17). Now, (5.17) and (5.9_{α_l}) give $m(\alpha_l) \le c_l$, and

$$N(\beta_l) \le c_l \tag{5.20}$$

will follow if we require (see the definition of N)

$$t^{-2}\big[a^{-1}\big(t^{-2}b(t)\big)\big]^{\beta_l+2} \le ct^{\alpha_l+2}$$

for $t \gg 1$, i.e.

$$b(t) \le ct^2 a\Big(t^{\frac{4+\alpha_l}{2+\beta_l}}\Big) , \tag{5.21}$$

and we may take

$$\beta_l = \frac{1}{2}\alpha_l$$

on account of (2.5). In conclusion, by (5.17) and (5.20) we have established (5.16_l), and (5.16_l) holds for all l if we define α_0 according to (5.14) and (recall (5.19)) take

$$\alpha_l = 2 + \beta_{l-1} , \quad \beta_l = \frac{1}{2}\alpha_l .$$

This gives the recursion

$$\alpha_l = 2 + \frac{1}{2}\alpha_{l-1} ,$$

hence $\alpha_l \to 4$ and $\beta_l \to 2$ as $l \to \infty$, and we have shown (recall that (5.9_{α_l}) and (5.10_{β_l}) together with (5.16_l) give $m(\alpha_l) + n(\beta_l) \le c_l$)

$$b(|\partial_n u_\varepsilon|)|\partial_n u_\varepsilon|^\rho \in L^1_{loc}(B) , \quad \rho < 6 ,$$
$$a(|\tilde\nabla u_\varepsilon|)|\tilde\nabla u_\varepsilon|^\rho \in L^1_{loc}(B) , \quad \rho < 4 ,$$

uniformly w.r.t. ε. In the particular case $a = b$ or if $b(t) \le ct^2 a(t)$ is assumed we may choose $\beta_l = 2 + \alpha_l$ in (5.21) replacing the requirement $\beta_l = \alpha_l/2$, and at the same time we may keep the choice of α_0 and the relation $\alpha_l = 2 + \beta_{l-1}$. This implies

$$\alpha_l = 4 + \alpha_{l-1} , \quad \alpha_0 > 0 ,$$

hence $\alpha_l \to \infty$ and $\beta_l \to \infty$ so that for $a = b$ or $b(t) \le ct^2 a(t)$ we arrive at

$$|\nabla u_\varepsilon| \in L^s_{loc}(B) \quad \text{for all } s < \infty$$

uniformly in ε. □

6 Examples

We start with a rather standard example of a N-function h being very close to the power growth case. Here h is of nearly s-growth provided that

$$ct^{s-\varepsilon} \leq h(t) \leq Ct^{s+\varepsilon}$$

for all $t \gg 1$, for positive constants c, C and for any $\varepsilon > 0$.

Example 6.1. *a) For $s \geq 2$ the function*

$$h(t) = \left[(1+t^2)^{\frac{s}{2}} - 1\right] \ln(1+t)\,, \qquad t \geq 0\,,$$

satisfies (H1), (H2) and (2.6).

b) If $s > 1$, then

$$h(t) = t^s \ln(1+t) \quad , \quad t \geq 0\,,$$

fulfills (H1) and (H2).

Remark 6.1. *Of course it is possible to replace* $\ln(1+t)$ *by iterated variants.*

Example 6.2. *(compare Remark 2.1, d)) Suppose that the continuous function θ: $[0,\infty) \to [0,\infty)$ is increasing and satisfies (Δ_2). Suppose further that $\theta(0) > 0$ and let*

$$h(t) = \int_0^t \left[\int_0^u \theta(s)\,\mathrm{d}s\right] \mathrm{d}u\,, \qquad t \geq 0\,.$$

Then (H1), (H2) and (2.6) hold for the function h.

In fact, since

$$h'(t) = \int_0^t \theta(s)\,\mathrm{d}s\,, \qquad h''(t) = \theta(t) \geq \theta(0) > 0\,,$$

$(H1)$ clearly holds. We observe

$$\frac{h'(t)}{t} = \frac{1}{t}\int_0^t \theta(s)\,\mathrm{d}s \geq \frac{1}{t}\int_0^t \theta(0)\,\mathrm{d}s = \theta(0)\,,$$

which gives (2.6), and at the same time

$$\frac{h'(t)}{t} = \frac{1}{t}\int_0^t \theta(s)\,\mathrm{d}s = \theta(\xi) \leq \theta(t) = h''(t)\,,$$

where ξ denotes a suitable number in $(0,t)$. This proves the first part of $(H2)$. For the second part we argue as follows: we have

$$\frac{h'(t)}{t} = \frac{1}{t}\int_0^t \theta(s)\,\mathrm{d}s \geq \frac{1}{2}\theta(t/2)\,,$$

i.e.

$$\theta(t) \le \mu\theta(t/2) \le \frac{2}{t}\mu h'(t)$$

and in conclusion

$$h''(t) \le 2\mu \frac{h'(t)}{t}\,.$$

In order to construct "explicit" examples which really "oscillate" between $\bar{\varepsilon}+1$ and $\bar{h}+1$-growth and still satisfy $(H1)$ and $(H2)$ we need an equivalent formulation of $(H2)$ which clarifies the geometric structure of $(H2)$ in terms of h'.

Suppose there exist $0 < \bar{\varepsilon} \le \bar{h}$ such that on $(0,\infty)$

$$\frac{h'(t)}{t^{\bar{\varepsilon}}} \text{ increases} \quad \text{and} \quad \frac{h'(t)}{t^{\bar{h}}} \text{ decreases}\,. \tag{$H2^*$}$$

Then we have $(H2) \Leftrightarrow (H2^*)$, where the equivalence

$$\bar{\varepsilon}\frac{h'(t)}{t} \le h''(t) \quad \Leftrightarrow \quad \frac{h'(t)}{t^{\bar{\varepsilon}}} \text{ is increasing}$$

is stated in Remark 2.1, b), and where the second equivalence is just a similar observation.

Example 6.3. *Suppose that $\bar{\varepsilon} < \varepsilon_1 < h_1 < \bar{h}$ and that $\mathbb{R}^+$ is the disjoint union of Intervalls, $\mathbb{R}^+ = \bigcup_i I_i$. Then we let*

$$h' = c_1 t^{h_1} \text{ on } I_1\,, \quad h' = c_2 t^{\varepsilon_1} \text{ on } I_2\,, \quad h' = c_3 t^{h_1} \text{ on } I_3 \quad \dots\,,$$

where the positive constants c_i are chosen s.t. h' is of class C^0. Then $(H2^*)$ is satisfied, i.e. we have $(H2)$. Integrating h' we obtain a function h which satisfies depending on the choice of the intervalls

$$ct^{\varepsilon_2} \le h(t) \le Ct^{h_2}$$

with positive constants c, C and with optimal exponents $\varepsilon_1 \le \varepsilon_2 < h_2 \le h_1$. In this sense the function h is far away from being of power growth.

Remark 6.2. *Of course the energy density considered in Example 6.3 is not of class C^2. To overcome this difficulty let us consider the endpoint of one fixed intervall I_i of the construction. If $(\cdot)_\gamma$ denotes a local mollification around this point with radii less than $\gamma > 0$, then we observe that the a.e. identity*

$$\varepsilon_1 \frac{h'}{t} \le h'' \le h_1 \frac{h'}{t}$$

implies

$$\varepsilon_1 \Big(\frac{h'}{t}\Big)_\gamma \le (h'')_\gamma \le h_1 \Big(\frac{h'}{t}\Big)_\gamma\,.$$

Since the function h'/t is of class C^0 we have for γ sufficiently small

$$\Big(\frac{h'}{t}\Big)_\gamma \approx \frac{h'}{t} \approx \frac{(h')_\gamma}{t}$$

and since h' weakly differentiable we have in addition

$$(h'')_\gamma = ((h')_\gamma)' ,$$

thus $(h')_\gamma$ is a smooth function satisfying

$$\varepsilon_0 \frac{(h')_\gamma}{t} \le ((h')_\gamma)' \le h_0 \frac{(h')_\gamma}{t}$$

with exponents $\bar\varepsilon \le \varepsilon_0 < \varepsilon_1 < h_1 < h_0 \le \bar h$.

Example 6.4. *Let us finally mention an example of a N-function which does not satisfy $(H2)$. Here we choose*

$$\theta(t) = \cos^2(t) + t\sin^2(t)$$

and integrate twice to obtain a N-function h which is not covered by our assumptions. We leave the details to the reader.

Appendix. Elementary properties of N-functions

Consider a N-function h: $[0,\infty) \to [0,\infty)$ of class C^2, i.e. we have assumption $(H1)$.

Lemma A.1. *a) If we know for all $t \ge 0$*

$$t h''(t) \le \bar h h'(t) \tag{A.1}$$

for a non-negative constant $\bar h$, then h satisfies a Δ_2-condition, i.e. we have (Δ_2) of Section 2.

b) Conversely, if we have (Δ_2) and if in addition h'' is increasing, i.e. h' is convex, then (A.1) holds.

Proof of Lemma A.1.
ad a). According to the non-vanishing of h' on $(0,\infty)$ we can rewrite (A.1) in the form

$$\frac{h''(t)}{h'(t)} \le \frac{\bar h}{t} \quad \text{for all } t > 0$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\ln(h'(t)) - \bar h \ln(t)\big] \le 0 \quad \text{on } (0,\infty) .$$

Thus the function $t \mapsto \ln(h'(t)) - \bar h \ln(t)$ is decreasing, in particular

$$\ln(h'(2t)) - \bar h \ln(2t) \le \ln(h'(t)) - \bar h \ln(t) ,$$

i.e.

$$\ln\Big(h'(2t)/h'(t)\Big) \le \bar h \ln(2)$$

and in conclusion

$$h'(2t) \le 2^{\bar h} h'(t) \quad \text{for all } \; t > 0\,. \tag{A.2}$$

¿From $h(0) = 0$ we get using (A.2)

$$h(2t) = \int_0^{2t} h'(s)\,\mathrm{d}s = 2\int_0^t h'(2s)\,\mathrm{d}s \le 2\int_0^t 2^{\bar h} h'(s)\,\mathrm{d}s = 2^{\bar h+1} h(t)\,.$$

Therefore we have (Δ_2) with $\mu = 2^{\bar h+1}$.

ad b). We show that (Δ_2) for h implies a similar condition for h': we have

$$h(t) = \int_0^t h'(s)\,\mathrm{d}s \ge \int_{t/2}^t h'(s)\,\mathrm{d}s \ge \frac{t}{2} h'\big(t/2\big)\,,$$

since h' is nonnegative and increasing. This gives

$$th'(t) \le h(2t)$$

and in conclusion by the Δ_2-property of h $(s > 0)$

$$h'(2s) \le \frac{1}{2s} h(4s) \le \frac{1}{2}\frac{1}{s}\mu^2 h(s) = \frac{\mu^2}{2}\frac{1}{s}\int_0^s h'(t)\,\mathrm{d}t \le \frac{\mu^2}{2} h'(s)\,. \tag{A.3}$$

Next we use our additional assumption that h'' is increasing: as usual it holds

$$h'(s) = \int_0^s h''(t)\,\mathrm{d}t \ge \int_{s/2}^s h''(t)\,\mathrm{d}t$$

(recall $h'(0) = \lim_{t\to 0} h(t)/t = 0$) and now we can estimate

$$\int_{s/2}^s h''(t)\,\mathrm{d}t \ge \frac{s}{2} h''(s/2)$$

with the result

$$th''(t) \le h'(2t)\,.$$

But with (A.3) this inequality implies (A.1). □

Lemma A.2. *If the Δ_2-condition (Δ_2) holds for the function h, then we have*

$$h(t) \le h(1)t^{\mu} \quad \textit{for all } \; t \ge 1\,. \tag{A.4}$$

Proof of Lemma A.2. Similar to the last step in the proof of b) of Lemma A.1 we have

$$h(t) = \int_0^t h'(s)\,\mathrm{d}s \ge \int_{t/2}^t h'(s)\,\mathrm{d}s \ge \frac{t}{2} h'(t/2)\,,$$

i.e.

$$sh'(s) \le h(2s)\,.$$

Using (Δ_2) we see

$$sh'(s) \le \mu h(s)$$

so that for $t > 0$

$$\frac{h'(t)}{h(t)} \le \frac{\mu}{t}\,,$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\ln(h(t)) - \mu\ln(t)\big] \le 0\,.$$

Thus the function $t \mapsto \ln(h(t)) - \mu\ln(t)$ is decreasing, for $t \ge 1$ it follows

$$\ln(h(t)) - \mu\ln(t) \le \ln(h(1))\,,$$

and (A.4) is established. □

Lemma A.3. *If the Δ_2-condition holds for the function h, then we get*

$$h(\lambda s) \le \Big(1 + \mu^{1+\frac{\ln(\lambda)}{\ln(2)}}\Big)h(s) \tag{A.5}$$

for all λ, $s > 0$.

Proof of Lemma A.3. If $\lambda \le 1$, then we just observe $h(\lambda s) \le h(s)$. Let $\lambda > 1$. Then we select $l \in \mathbb{N}$ s.t. $\lambda \in [2^{l-1}, 2^l]$ and get

$$h(\lambda s) \le h(2^l s) \le \mu h(2^{l-1}s) \le \mu^l h(s)\,.$$

By the choice of l we have $\lambda \ge 2^{l-1}$, i.e. $l \le 1 + \frac{\ln(\lambda)}{\ln(2)}$, and (A.5) follows by combining both cases. □

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