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Abstract

We discuss the weak form of the Ramberg-Osgood equations (also known as the Norton-Hoff model) for nonlinear elastic materials and prove functional type a posteriori error estimates for the difference of the exact stress tensor and any tensor from the admissible function space. These equations are of great importance since they can be used as an approximation for elastic-perfectly plastic Hencky materials.

1 Introduction

In our note we study a geometrically linear and physically nonlinear elastic model whose constitutive equations are of power-law type. This model also known as Norton-Hoff model was suggested by Ramberg and Osgood for aluminium alloys (compare [OR]), and nowadays it is frequently used as an approximation for elastic-plastic material behaviour. We refer to the works of Temam [Te] and of Bensoussan and Frehse [BeF] where it is shown that the stress fields for elastic-perfectly plastic Hencky materials can be approximated by the stress fields which are solutions of the Ramberg-Osgood equations. In this model equilibrium configurations are characterized through the following set of equations (see Section 2 for details concerning the notation): to find a stress tensor σ_0 and a displacement field u_0 such that

$$A\sigma_0 + \alpha|\sigma_0^D|^{q-2}\sigma_0^D = \varepsilon(u_0) \qquad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}\sigma_0 + f = 0 \qquad \text{in } \Omega, \tag{1.2}$$

$$u_0 = u_b \qquad \text{on } \partial\Omega, \tag{1.3}$$

where for simplicity in (1.3) we just consider given Dirichlet boundary data u_b. The existence and some initial regularity properties of weak solutions (σ_0, u_0) to (1.1) – (1.3) have been investigated in the recent thesis [Kn] of Knees, further regularity properties are discussed in [BeF], [BiF1] and [BiF2]. In the present paper we use the fact that the stress tensor σ_0 and the displacement field u_0 are solutions of variational problems being complementary to each other in order to obtain error estimates which are important to verify the accuracy of a numerical approximation. More precisely we will establish the estimate

$$\operatorname{dev}_{W_c}(\sigma_0, \sigma) \le \mathcal{M}(\sigma, u, f, \Omega, \alpha, A, q) \tag{1.4}$$

valid for all tensors σ and all vector fields u satisfying $u = u_b$ on $\partial\Omega$. Here the deviation dev_{W_c} w.r.t. the energy class is induced by the natural norms (acting on the space of tensors) involved in the problem and it is a suitable measure for the distance between σ_0 and σ (see Section 2 for the precise definition). The deviation is controlled by a functional $\mathcal{M}$ acting on σ and u and merely depending on the given data of the problem. The essential properties of $\mathcal{M}$ are: there is no overestimation in (1.4), the functional is explicitly computable and fulfills

$$\mathcal{M}(\sigma_k, u_k, f, \Omega, \alpha, A, q) \to 0 \quad \text{as } k \to \infty$$

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if $\sigma_k \to \sigma_0$, $u_k \to u_0$ in the corresponding function spaces. Moreover, $\mathcal{M}(\sigma, u, f, \Omega, \alpha, A, q)$ vanishes if and only if $\sigma = \sigma_0$ and $u = u_0$.

A posteriori error estimates for approximations of various nonlinear models in continuum mechanics were investigated by many authors. In an abstract form (for nonlinear mappings) the estimates derived in the framework of the residual approach were considered by Pousin and Rappaz [PR] and Verführt (see, e.g., [Ve1, Ve2]). A posteriori estimates for variational inequalities related to problems with obstacles were analyzed in, e.g., the work of Ainsworth, Oden and Lee [AO], [AOL], Braess [Br], Chen and Nochetto [CN], and Hoppe and Kornhuber [HK]. A different group of a posteriori error estimation methods, which nowadays is widely used in finite element computations is based upon using adjoint problems. At this point we first refer to the so-called dual-weighted residual method (see, e.g., Rannacher [Ra], Becker and Rannacher [BeR]). One of the advantages of this method is that the consideration of the adjoint problem allows to avoid difficulties with the evaluation of the interpolation constants. Readers will find a detailed exposition of the method and applications to various problems in the book of Bangerth and Rannacher [BaR].

Another (functional) approach to a posteriori error estimation was suggested in [Re1], [Re2] and [Re3] and some other papers cited therein. In it, the estimates are derived on purely functional grounds without attracting special properties of an approximate solution (such as, e.g., Galerkin orthogonality or extra regularity). Therefore, such estimates do not involve mesh dependent constants and are valid for any approximation from the admissible (energy) class. This approach for various nonlinear problems related to problems in continuum mechanics was derived in [Br], [BFR], [FR], [Re4], [Re5], [RX] (see also the book [NR]).
Now, with respect to these comments, the estimate (1.4) which we are going to derive in the present paper, is an a posteriori error estimate of functional type for the stress tensor σ_0 based on the observation that this tensor solves a maximization problem being the dual problem to the minimization problem for the displacement field u_0.

Our paper is organized as follows: in Section 2 we introduce our notation and give a precise formulation of the estimate (1.4) as well as a discussion of its consequences. In Section 3 we prove a first variant of (1.4) for tensors $\widetilde{\sigma}$ satisfying the equation $\operatorname{div}\widetilde{\sigma} + f = 0$. In Section 4 this restriction is removed which also requires to estimate the distance of an arbitrary tensor to the set of tensors for which the above equation is valid. During this procedure we need an auxiliary function whose properties are discussed in the Appendix. In Section 5 we shortly discuss some aspects of the minimization problem for the displacement field u_0.

2 Notation and results

Consider a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, fix an exponent $q \geq 2$, and let $p := q/(q-1)$. We define the spaces

$$\begin{aligned} L^{q,2}(\Omega) &:= \left\{\tau : \Omega \to \mathbb{S}^d : \tau^D \in L^q(\Omega),\, \operatorname{tr}\tau \in L^2(\Omega)\right\}, \\ U^{p,2}(\Omega) &:= \left\{v : \Omega \to \mathbb{R}^d : \varepsilon^D(v) \in L^p(\Omega),\, \operatorname{div} v \in L^2(\Omega)\right\}, \end{aligned}$$

where

$$\begin{aligned}
\mathbb{S}^d &:= \text{the space of all symmetric } (d\times d)\text{-matrices}\,,\\
\tau^D &:= \tau - \frac{1}{d}\mathrm{tr}\,\tau\mathbf{1}\,, \quad \mathrm{tr}\,\tau := \tau_{ii} \ \text{ for } \tau\in\mathbb{S}^d\,,\\
\varepsilon(v) &:= \frac{1}{2}(\nabla v + \nabla v^T) = \frac{1}{2}(\partial_i v^j + \partial_j v^i) \ \text{ for } v\colon\, \Omega\to\mathbb{R}^d\,.
\end{aligned}$$

Here and in what follows we will use summation convention. The spaces $L^{q,2}(\Omega)$ and $U^{p,2}(\Omega)$ are discussed for example in [GS] (see also [FS]), and following common practice we will endow them with the norms

$$\begin{aligned}
\|\tau\|_{L^{q,2}(\Omega)} &:= \|\tau^D\|_{L^q(\Omega)} + \|\mathrm{tr}\,\tau\|_{L^2(\Omega)}\,,\\
\|v\|_{U^{p,2}(\Omega)} &:= \|v\|_{L^p(\Omega)} + \|\varepsilon^D(v)\|_{L^p(\Omega)} + \|\mathrm{div}\,v\|_{L^2(\Omega)}\,.
\end{aligned}$$

Let further $U_0^{p,2}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $U^{p,2}(\Omega)$ w.r.t. $\|\cdot\|_{U^{p,2}(\Omega)}$ and fix a function $u_b\in U^{p,2}(\Omega)$ acting as a boundary datum.

REMARK 2.1. *In order to use intuitive symbols we will denote by*

$$(\sigma_0,u_0)\in L^{q,2}(\Omega)\times\big(u_b+U_0^{p,2}(\Omega)\big)$$

the exact solution, $(\sigma,u)\in L^{q,2}(\Omega)\times\big(u_b+U_0^{p,2}(\Omega)\big)$ stands for an approximation of the exact solution, τ is an arbitrary tensor of class $L^{q,2}(\Omega)$. If σ is assumed to satisfy in addition condition (2.5) below, then we use the symbol $\widetilde{\sigma}$. In what follows, $v\in U^{p,2}(\Omega)$ and $w\in U_0^{p,2}(\Omega)$ denote arbitrary functions of these classes.

Suppose now that we are given a system of volume forces $f\colon\,\Omega\to\mathbb{R}^d$ of class $L^q(\Omega)$. Following [Kn] we introduce the complementary energy density

$$W_c(\sigma) := \frac{1}{2}A\sigma:\sigma + \frac{\alpha}{q}|\sigma^D|^q\,,\qquad \sigma\in\mathbb{S}^d\,, \tag{2.1}$$

where α is a positive constant and where A is of the form

$$A\sigma := \lambda_1\mathrm{tr}\,\sigma\mathbf{1} + \lambda_2\sigma^D\,,\qquad \sigma\in\mathbb{S}^d\,, \tag{2.2}$$

with constants λ_1, $\lambda_2>0$. Note that this particular choice of A: $\mathbb{S}^d\to\mathbb{S}^d$ corresponds to isotropic linear elasticity, and we will make use of this special form in an essential way. Now we give a precise formulation of the equations (1.1) – (1.3): find $\sigma_0\in L^{q,2}(\Omega)$ and $u_0\in u_b+U_0^{p,2}(\Omega)$ such that

$$\int_\Omega DW_c(\sigma_0):\tau\,\mathrm{d}x = \int_\Omega \varepsilon(u_0):\tau\,\mathrm{d}x\,, \tag{2.3}$$

$$\int_\Omega [\sigma_0:\varepsilon(w) - f\cdot w]\,\mathrm{d}x = 0 \tag{2.4}$$

hold for all $\tau\in L^{q,2}(\Omega)$ and for any $w\in U_0^{p,2}(\Omega)$. According to [Kn] (Theorem 1.19.3) we know that there exists a unique pair $(\sigma_0,u_0)\in L^{q,2}(\Omega)\times\big(u_b+U_0^{p,2}(\Omega)\big)$ satisfying (2.3) and (2.4).

Theorem 2.1 below presents the first a posteriori error estimate of the type (1.4). In it, we estimate the deviation of a tensor $\tau \in L^{q,2}(\Omega)$ from the exact solution σ_0 with the help of the quantity

$$\mathrm{dev}_{W_c}(\sigma_0,\tau) := \frac{\alpha}{q}2^{1-q}\int_\Omega |\sigma_0^D - \tau^D|^q \,\mathrm{d}x + \frac{\lambda_1}{4}\int_\Omega |\mathrm{tr}\,\sigma_0 - \mathrm{tr}\,\tau|^2 \,\mathrm{d}x + \frac{\lambda_2}{4}\int_\Omega |\sigma_0^D - \tau^D|^2 \,\mathrm{d}x,$$

which is a weighted "norm" in the space of stresses with the weights given by the elasticity coefficients and the power growth parameter.

THEOREM 2.1. *Suppose that the hypotheses stated above are valid. Then, for any* $u \in u_b + U_0^{p,2}(\Omega)$, *for all* $\tau \in L^{q,2}(\Omega)$ *and for any* $\widetilde{\sigma} \in L^{q,2}(\Omega)$ *such that*

$$\widetilde{\sigma} \in Q_f^{p,2} := \left\{ \widetilde{\tau} \in L^{q,2}(\Omega) : \int_\Omega [\widetilde{\tau} : \varepsilon(w) - f \cdot w] \,\mathrm{d}x = 0 \quad \text{for all } w \in U_0^{p,2}(\Omega) \right\} \tag{2.5}$$

the following estimate holds

$$\mathrm{dev}_{W_c}(\sigma_0, \widetilde{\sigma}) \;\le\; \mathcal{M}_1(\widetilde{\sigma};\tau) + \mathcal{M}_2(\widetilde{\sigma}, u;\tau), \tag{2.6}$$

where

$$\mathcal{M}_1(\widetilde{\sigma};\tau) := \int_\Omega [W_c(\widetilde{\sigma}) - W_c(\tau) + (\tau - \widetilde{\sigma}) : DW_c(\tau)] \,\mathrm{d}x,$$

$$\begin{aligned}
&\mathcal{M}_2(\widetilde{\sigma}, u;\tau) := \mathcal{N}_1[\widetilde{\sigma}, u]\mathcal{G}_1[\tau, u] + \mathcal{N}_2[\widetilde{\sigma}, u]\mathcal{G}_2[\tau, u],\\
&\mathcal{N}_1[\widetilde{\sigma}, u] := \frac{1}{d}\|\mathrm{tr}\,\widetilde{\sigma}\|_{L^2(\Omega)} + \frac{1}{d^2}\frac{1}{\lambda_1}\|\mathrm{div}\,u\|_{L^2(\Omega)}\,,\ \mathcal{G}_1[\tau, u] := \|\mathrm{div}\,u - \mathrm{tr}\,DW_c(\tau)\|_{L^2(\Omega)}\,,\\
&\mathcal{N}_2[\widetilde{\sigma}, u] := \|\widetilde{\sigma}^D\|_{L^q(\Omega)} + \alpha^{-\frac{1}{q-1}}\|\varepsilon^D(u)\|^{p-1}_{L^p(\Omega)},\ \ \mathcal{G}_2[\tau, u] := \|\varepsilon^D(u) - DW_c(\tau)^D\|_{L^p(\Omega)}\,.
\end{aligned}$$

It is clear that $\mathcal{M}_1(\widetilde{\sigma};\tau) \ge 0$ for any τ. Assume that $\mathcal{M}_1(\widetilde{\sigma};\tau) = 0$. Since

$$\int_\Omega (W_c(\widetilde{\sigma}) - W_c(\tau))dx \ge \int_\Omega (\widetilde{\sigma} - \tau) : DW_c(\widetilde{\sigma}))dx \tag{2.7}$$

we obtain

$$\int_\Omega (DW_c(\widetilde{\sigma}) - DW_c(\tau)) : (\widetilde{\sigma} - \tau)dx \le 0. \tag{2.8}$$

Recall that DW_c is strictly monotone. Therefore, the last relation means that $\widetilde{\sigma} = \tau$ a.e. in Ω. If in addition $\mathcal{M}_2(\widetilde{\sigma}, u;\tau) = 0$, then

$$\mathcal{G}_1[\widetilde{\sigma}, u] = \mathcal{G}_1[\tau, u] = 0 \;\Rightarrow\quad \mathrm{div}\,u = \mathrm{tr}\,DW_c(\widetilde{\sigma}), \tag{2.9}$$

$$\mathcal{G}_2[\widetilde{\sigma}, u] = \mathcal{G}_2[\tau, u] \;\Rightarrow\quad \varepsilon^D(u) = DW_c(\widetilde{\sigma})^D \tag{2.10}$$

Jointly (2.9) and (2.10) show the *constitutive relation*

$$\varepsilon(u) = DW_c(\tau). \tag{2.11}$$

Thus, the r.h.s. of (2.6) can be thought of as a measure of the error in the constitutive relation. Since $\widetilde{\sigma}$ satisfies the equilibrium equations, the r.h.s. of (2.6) vanishes only if $\widetilde{\sigma}$ coincides with σ_0 and $u = u_0$.

The applicability of Theorem 2.1 suffers from the fact that the admissible tensors $\widetilde{\sigma}$ have to satisfy condition (2.5). In order to remove this restriction we let

$$\begin{aligned}\operatorname{dist}_{W_c}(\widetilde{\sigma},\sigma) &:= \int_\Omega W_c(\widetilde{\sigma}-\sigma)\,\mathrm{d}x \\ &= \frac{\alpha}{q}\|\widetilde{\sigma}^D-\sigma^D\|^q_{L^q(\Omega)} + \frac{\lambda_1}{2}\|\operatorname{tr}\widetilde{\sigma}-\operatorname{tr}\sigma\|^2_{L^2(\Omega)} + \frac{\lambda_2}{2}\|\widetilde{\sigma}^D-\sigma^D\|^2_{L^2(\Omega)}\end{aligned}$$

for any $\widetilde{\sigma}\in Q_f^{p,2}$ and for any $\sigma\in L^{q,2}(\Omega)$, i.e.

$$\operatorname{dist}_{W_c}(\sigma, Q_f^{p,2}) := \inf_{\widetilde{\sigma}\in Q_f^{p,2}} \int_\Omega W_c(\sigma-\widetilde{\sigma})\,\mathrm{d}x$$

measures the distance of any arbitrary approximation of class $L^{q,2}(\Omega)$ to the affine manifold $Q_f^{p,2}$. Now we can formulate our main

THEOREM 2.2. *Let the hypotheses of Theorem 2.1 hold.*

a) For any $\sigma\in L^{q,2}(\Omega)$, for all $\tau\in L^{q,2}(\Omega)$ and arbitrary $u\in u_b+U_0^{p,2}(\Omega)$ we have the estimate

$$\operatorname{dev}_{W_c}(\sigma_0,\sigma)\le 2^{q-1}(\mathcal{M}_1(\sigma;\tau)+\mathcal{M}_2(\sigma,u;\tau)+\mathcal{M}_3(\sigma,u;\tau)), \tag{2.12}$$

where

$$\begin{aligned}\mathcal{M}_3(\sigma,u;\tau) = &\Big[3+c_1\frac{q}{\alpha}\Big]\operatorname{dist}_{W_c}(\sigma,Q_f^{p,2}) + \\ &+\operatorname{dist}_{W_c}(\sigma,Q_f^{p,2})^{\frac12}\left(\frac{1}{d}\sqrt{\frac{2}{\lambda_1}}\mathcal{G}_1[\tau,u]+\sqrt{2\lambda_1}\big[\|\operatorname{tr}\tau\|_{L^2(\Omega)}+\|\operatorname{tr}\sigma\|_{L^2(\Omega)}\big]\right)+ \\ &+\Big[\frac{q}{\alpha}\operatorname{dist}_{W_c}(\sigma,Q_f^{p,2})\Big]^{\frac1q}\Big[\mathcal{G}_2[\tau,u]+\|DW_c(\tau)^D\|_{L^p(\Omega)}+\|DW_c(\sigma)^D\|_{L^p(\Omega)}\Big]+ \\ &\qquad + c_1\|\sigma^D\|^{1-\frac2q}_{L^q(\Omega)}\Big[\frac{q}{\alpha}\operatorname{dist}_{W_c}(\sigma,Q_f^{p,2})\Big]^{\frac2q}\Bigg],\end{aligned}$$

$c_1:=\alpha(q-1)2^{q-2}$, and $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{G}_1$, $\mathcal{G}_2$ are defined as in Theorem 2.1.

b) There is an explicitly computable constant C_0 depending on d, α, q, λ_1, λ_2 and Ω such that for all $\sigma\in L^{q,2}(\Omega)$ with the property $\operatorname{div}\sigma\in L^q(\Omega)$ it holds

$$\operatorname{dist}_{W_c}(\sigma,Q_f^{p,2})\le C_0\max\left\{\|f+\operatorname{div}\sigma\|_{L^q(\Omega)}, \|f+\operatorname{div}\sigma\|^q_{L^q(\Omega)}\right\}.$$

REMARK 2.2. *Estimate (2.12) differs from (2.6) by the term $\mathcal{M}_3$ which evidently vanishes if σ satisfies the equilibrium equation. Therefore, the r.h.s. of (2.12) is zero if and only if $\sigma=\sigma_0$ and $u=u_0$. Note however that (2.12) does not reproduce (2.6) if $\widetilde{\sigma}$ is equilibrated since dev_{W_c} is not a norm. A more transparent explanation is given by the first estimate in the proof of Theorem 2.2.*

REMARK 2.3. *The functionals dev_{W_c} and $\operatorname{dist}_{W_c}$ in principle measure the same quantities, their definitions just differ by constants. But in order to keep our estimates as sharp as possible, we work with both definitions.*

3 Proof of Theorem 2.1

A first estimate for the deviation is given in

LEMMA 3.1. *For any $\widetilde{\sigma} \in Q_f^{p,2}$ we have*

$$\mathrm{dev}_{W_c}(\sigma_0, \widetilde{\sigma}) \le K[\sigma_0] - K[\widetilde{\sigma}]\,, \tag{3.1}$$

where on the set $Q_f^{p,2}$ of admissible stress fields

$$K[\widetilde{\sigma}] := \int_\Omega \widetilde{\sigma} : \varepsilon(u_b)\,\mathrm{d}x - \int_\Omega W_c(\widetilde{\sigma})\,\mathrm{d}x - \int_\Omega f \cdot u_b\,\mathrm{d}x\,.$$

Proof of Lemma 3.1. From [Kn], Theorem 1.19.2, we deduce that σ_0 is the unique solution of the problem

$$K[\cdot] \to \max \quad \text{in } Q_f^{p,2}\,,$$

and we want to use this fact in order to estimate $\mathrm{dev}_{W_c}(\sigma_0, \widetilde{\sigma})$ for tensors $\widetilde{\sigma} \in Q_f^{p,2}$ as stated in (3.1). We begin with a version of Clarkson's inequality (see, e.g. [Cl]) which can be found in [MM1]: for any exponent $s \ge 2$ and arbitrary vector-valued functions y_1, $y_2 \in L^s(\Omega)$ it holds

$$\int_\Omega \left[\left|\frac{y_1 + y_2}{2}\right|^s + \left|\frac{y_1 - y_2}{2}\right|^s\right] \mathrm{d}x \le \frac{1}{2}\|y_1\|^s_{L^s(\Omega)} + \frac{1}{2}\|y_2\|^s_{L^s(\Omega)}\,. \tag{3.2}$$

For τ_1, $\tau_2 \in L^{q,2}(\Omega)$ we deduce from (3.2)

$$\begin{aligned}
K\Big[\frac{\tau_1+\tau_2}{2}\Big] &= \int_\Omega \frac{\tau_1+\tau_2}{2} : \varepsilon(u_b)\,\mathrm{d}x - \frac{\alpha}{q}\int_\Omega \Big|\frac{\tau_1^D + \tau_2^D}{2}\Big|^q \mathrm{d}x \\
&\quad -\frac{1}{2}\int_\Omega A\frac{\tau_1+\tau_2}{2} : \frac{\tau_1+\tau_2}{2}\,\mathrm{d}x - \int_\Omega f\cdot u_b\,\mathrm{d}x \\
&\ge \frac{\alpha}{q}\int_\Omega \Big|\frac{\tau_1^D - \tau_2^D}{2}\Big|^q \mathrm{d}x - \frac{1}{2}\frac{\alpha}{q}\int_\Omega |\tau_1^D|^q\,\mathrm{d}x - \frac{1}{2}\frac{\alpha}{q}\int_\Omega |\tau_2^D|^q\,\mathrm{d}x \\
&\quad + \int_\Omega \frac{\tau_1+\tau_2}{2} : \varepsilon(u_b)\,\mathrm{d}x - \int_\Omega f\cdot u_b\,\mathrm{d}x - \mathcal{A}\Big(\frac{\tau_1+\tau_2}{2}, \frac{\tau_1+\tau_2}{2}\Big)\,,
\end{aligned}$$

where we have abbreviated for any τ, η of class $L^2(\Omega)$

$$\mathcal{A}(\tau,\eta) := \frac{1}{2}\int_\Omega A\tau : \eta\,\mathrm{d}x\,.$$

It follows that

$$\begin{aligned}
\frac{\alpha}{q}\int_\Omega \Big|\frac{\tau_1^D - \tau_2^D}{2}\Big|^q \mathrm{d}x &\le \frac{1}{2}\Big[\frac{\alpha}{q}\int_\Omega |\tau_1^D|^q\,\mathrm{d}x - \int_\Omega \tau_1 : \varepsilon(u_b)\,\mathrm{d}x + \int_\Omega f\cdot u_b\,\mathrm{d}x \\
&\quad + \frac{\alpha}{q}\int_\Omega |\tau_2^D|^q\,\mathrm{d}x - \int_\Omega \tau_2 : \varepsilon(u_b)\,\mathrm{d}x + \int_\Omega f\cdot u_b\,\mathrm{d}x\Big] \\
&\quad + \mathcal{A}\Big(\frac{\tau_1+\tau_2}{2}, \frac{\tau_1+\tau_2}{2}\Big) + K\Big[\frac{\tau_1+\tau_2}{2}\Big]\,.
\end{aligned}$$

Observing the identity

$$\mathcal{A}\Big(\frac{\tau_1+\tau_2}{2},\frac{\tau_1+\tau_2}{2}\Big)=\frac{1}{2}\mathcal{A}(\tau_1,\tau_1)+\frac{1}{2}\mathcal{A}(\tau_2,\tau_2)-\mathcal{A}\Big(\frac{\tau_1-\tau_2}{2},\frac{\tau_1-\tau_2}{2}\Big)$$

we find that

$$\begin{aligned}&\frac{\alpha}{q}\int_\Omega\Big|\frac{\tau_1^D-\tau_2^D}{2}\Big|^q\,\mathrm{d}x+\frac{1}{2}\int_\Omega A\frac{\tau_1-\tau_2}{2}:\frac{\tau_1-\tau_2}{2}\,\mathrm{d}x\\ &\quad\leq\ \frac{1}{2}\big[-K[\tau_1]-K[\tau_2]\big]+K\Big[\frac{\tau_1+\tau_2}{2}\Big]\,. \end{aligned}\tag{3.3}$$

If we choose $\tau_1=\sigma_0$ and $\tau_2=\widetilde{\sigma}\in Q_f^{p,2}$, then the K-maximality of σ_0 implies that

$$\frac{1}{2}\big[-K[\sigma_0]-K[\widetilde{\sigma}]\big]+K\Big[\frac{\sigma_0+\widetilde{\sigma}}{2}\Big]\leq\frac{1}{2}\big[K[\sigma_0]-K[\widetilde{\sigma}]\big]\,,$$

and according to equation (2.2) we have

$$\frac{1}{2}\int_\Omega A\frac{\tau_1-\tau_2}{2}:\frac{\tau_1-\tau_2}{2}\,\mathrm{d}x=\frac{1}{2}\int_\Omega\bigg[\lambda_1\Big(\mathrm{tr}\,\frac{\tau_1-\tau_2}{2}\Big)^2+\lambda_2\Big|\frac{\tau_1^D-\tau_2^D}{2}\Big|^2\bigg]\,\mathrm{d}x\,.$$

Returning to (3.3) we have shown the lemma. □

In order to continue we remove σ_0 from the r.h.s. of (3.1) with the help of a duality argument. To this purpose we consider the conjugate function W_c^* of the density W_c from (2.1), i.e. we consider

$$W_c^*(\varepsilon):=\sup_{\varkappa\in\mathbb{S}^d}\big[\varepsilon:\varkappa-W_c(\varkappa)\big]\,,\qquad\varepsilon\in\mathbb{S}^d\,.$$

Following [Kn] (note that Knees uses the symbol W_{el}) we define the elastic strain energy as

$$J[v]:=\int_\Omega W_c^*(\varepsilon(v))\,\mathrm{d}x-\int_\Omega f\cdot v\,\mathrm{d}x\,,$$

which according to the estimate (1.54) of [Kn] makes sense on the space $U^{p,2}(\Omega)$. Moreover, in Lemma 1.24 of [Kn] it is shown that $u_0\in u_b+U_0^{p,2}(\Omega)$ is the unique J-minimizer in $u_b+U_0^{p,2}(\Omega)$ and that $J[u_0]=K[\sigma_0]$ is true. If $u\in u_b+U_0^{p,2}(\Omega)$ is arbitrary, we get for $\widetilde{\sigma}\in Q_f^{p,2}$

$$\begin{aligned}K[\sigma_0]-K[\widetilde{\sigma}]&\leq J[u]-K[\widetilde{\sigma}]\\ &=\int_\Omega W_c^*(\varepsilon(u))\,\mathrm{d}x-\int_\Omega f\cdot u\,\mathrm{d}x+\int_\Omega f\cdot u_b\,\mathrm{d}x\\ &\quad+\int_\Omega W_c(\widetilde{\sigma})\,\mathrm{d}x-\int_\Omega\widetilde{\sigma}:\varepsilon(u_b)\,\mathrm{d}x\\ &=\int_\Omega\big[W_c^*(\varepsilon(u))+W_c(\widetilde{\sigma})-\varepsilon(u):\widetilde{\sigma}\big]\,\mathrm{d}x\end{aligned}$$

and (3.1) together with the latter estimate shows

$$\mathrm{dev}_{W_c}(\sigma_0,\widetilde{\sigma})\leq\int_\Omega\big[W_c^*(\varepsilon(u))+W_c(\widetilde{\sigma})-\varepsilon(u):\widetilde{\sigma}\big]\,\mathrm{d}x\,,\tag{3.4}$$

and (3.4) holds for all $\widetilde{\sigma} \in Q_f^{p,2}$, $u \in u_b + U_0^{p,2}(\Omega)$. Unfortunately we cannot give an explicit formula for W_c^* (compare also the discussion after (4.2)) and therefore we argue as follows: with u and $\widetilde{\sigma}$ being fixed for the moment we consider an arbitrary tensor $\gamma \in L^{p,2}(\Omega)$ and let $\delta := \varepsilon(u) - \gamma$. Moreover, we define $\bar{\eta} \in L^{q,2}(\Omega)$ through the relation

$$DW_c(\bar{\eta}) = \varepsilon(u)\,, \tag{3.5}$$

i.e. we have

$$W_c^*(\varepsilon(u)) = \varepsilon(u) : \bar{\eta} - W_c(\bar{\eta})\,. \tag{3.6}$$

From (3.5) and (3.6) we see

$$\begin{aligned} W_c^*(\varepsilon(u)) &= W_c^*(DW_c(\bar{\eta})) = \bar{\eta} : DW_c(\bar{\eta}) - W_c(\bar{\eta}) \\ &= \bar{\eta} : (\gamma + \delta) - W_c(\bar{\eta}) = \bar{\eta} : \gamma - W_c(\bar{\eta}) + \delta : \bar{\eta} \\ &\leq W_c^*(\gamma) + \delta^D : \bar{\eta}^D + \frac{1}{d}\operatorname{tr}\delta \operatorname{tr}\bar{\eta}\,, \end{aligned}$$

and if we choose $\gamma = DW_c(\tau)$ with $\tau \in L^{q,2}(\Omega)$ we have shown

$$W_c^*(\varepsilon(u)) \leq \tau : DW_c(\tau) - W_c(\tau) + |\delta^D||\bar{\eta}^D| + \frac{1}{d}|\operatorname{tr}\delta||\operatorname{tr}\bar{\eta}|\,. \tag{3.7}$$

We apply (3.7) on the r.h.s. of (3.4) and get

$$\begin{aligned} \operatorname{dev}_{W_c}(\sigma_0, \widetilde{\sigma}) \leq & \int_\Omega \Big[W_c(\widetilde{\sigma}) - W_c(\tau) + \tau : DW_c(\tau) - \varepsilon(u) : \widetilde{\sigma}\Big]\,\mathrm{d}x \\ & + \int_\Omega \frac{1}{d}|\operatorname{tr}\bar{\eta}||\operatorname{div} u - \operatorname{tr} DW_c(\tau)|\,\mathrm{d}x \\ & + \int_\Omega |\bar{\eta}^D||\varepsilon^D(u) - DW_c(\tau)^D|\,\mathrm{d}x\,. \end{aligned} \tag{3.8}$$

Finally we use the special form of the fourth order tensor A (see (2.2)): we have

$$\begin{aligned} \operatorname{tr}\varepsilon(u) &= \operatorname{tr}(A\bar{\eta}) = d\lambda_1 \operatorname{tr}\bar{\eta}\,, \\ \varepsilon^D(u) &= \lambda_2\bar{\eta}^D + \alpha|\bar{\eta}^D|^{q-2}\bar{\eta}^D\,, \end{aligned}$$

hence

$$\begin{aligned} |\operatorname{tr}\varepsilon(u)| &= d\lambda_1|\operatorname{tr}\bar{\eta}|\,, \\ |\varepsilon^D(u)| &= |\bar{\eta}^D|(\lambda_2 + \alpha|\bar{\eta}^D|^{q-2}) \geq \alpha|\bar{\eta}^D|^{q-1} \end{aligned}$$

and the r.h.s. of (3.8) is bounded from above by the expression

$$\begin{aligned} & \int_\Omega \Big[W_c(\widetilde{\sigma}) - W_c(\tau) + (\tau - \widetilde{\sigma}) : DW_c(\tau)\Big]\,\mathrm{d}x \\ & \quad + \int_\Omega \Big[DW_c(\tau) - \varepsilon(u)\Big] : \widetilde{\sigma}\,\mathrm{d}x + \int_\Omega \frac{1}{d^2}\frac{1}{\lambda_1}|\operatorname{div} u||\operatorname{div} u - \operatorname{tr} DW_c(\tau)|\,\mathrm{d}x \\ & \quad + \int_\Omega \Big(\frac{1}{\alpha}\Big)^{\frac{1}{q-1}}|\varepsilon^D(u)|^{\frac{1}{q-1}}|\varepsilon^D(u) - DW_c(\tau)^D|\,\mathrm{d}x \\ & =: T_1 + T_2 + T_3 + T_4\,. \end{aligned}$$

Obviously $T_1 \geq 0$ and $T_1 = 0$ if and only if $\tau = \widetilde{\sigma}$. We have by Hölder's inequality

$$
\begin{aligned}
T_2 &= \frac{1}{d}\int_\Omega \operatorname{tr}\widetilde{\sigma}\operatorname{tr}\left[DW_c(\tau) - \varepsilon(u)\right]\mathrm{d}x + \int_\Omega \widetilde{\sigma}^D : \left[DW_c(\tau)^D - \varepsilon^D(u)\right]\mathrm{d}x \\
&\leq \frac{1}{d}\|\operatorname{tr}\widetilde{\sigma}\|_{L^2(\Omega)}\mathcal{G}_1[\tau,u] + \|\widetilde{\sigma}^D\|_{L^q(\Omega)}\mathcal{G}_2[\tau,u]\,, \\
T_3 &\leq \frac{1}{d^2}\frac{1}{\lambda_1}\|\operatorname{div} u\|_{L^2(\Omega)}\mathcal{G}_1[\tau,u]\,, \\
T_4 &\leq \alpha^{-\frac{1}{q-1}}\|\varepsilon^D(u)\|^{p-1}_{L^p(\Omega)}\mathcal{G}_2[\tau,u]\,.
\end{aligned}
$$

Collecting our estimates and recalling that $T_1 + T_2 + T_3 + T_4$ is an upper bound for $\operatorname{dev}_{W_c}(\sigma_0, \widetilde{\sigma})$, the proof of Theorem 2.1 is complete. □

4 Proof of Theorem 2.2

Suppose that we are given $u \in u_b + U_0^{p,2}(\Omega)$, $\tau \in L^{q,2}(\Omega)$ and $\sigma \in L^{q,2}(\Omega)$ Moreover, consider $\widetilde{\sigma} \in Q_f^{p,2}$. Then we obviously have ($\operatorname{dev}_{W_c}(\sigma,\cdot)$ is defined analogous to $\operatorname{dev}_{W_c}(\sigma_0,\cdot)$)

$$
\begin{aligned}
\operatorname{dev}_{W_c}(\sigma_0,\sigma) \leq\; & \frac{\alpha}{q}2^{1-q}\left[2^{q-1}\int_\Omega |\sigma_0^D - \widetilde{\sigma}^D|^q\,\mathrm{d}x + 2^{q-1}\int_\Omega |\sigma^D - \widetilde{\sigma}^D|^q\,\mathrm{d}x\right] \\
& + \frac{\lambda_1}{4}\left[2\int_\Omega |\operatorname{tr}\sigma_0 - \operatorname{tr}\widetilde{\sigma}|^2\,\mathrm{d}x + 2\int_\Omega |\operatorname{tr}\sigma - \operatorname{tr}\widetilde{\sigma}|^2\,\mathrm{d}x\right] \\
& + \frac{\lambda_2}{4}\left[2\int_\Omega |\sigma_0^D - \widetilde{\sigma}^D|^2 + 2\int_\Omega |\sigma^D - \widetilde{\sigma}^D|^2\,\mathrm{d}x\right] \\
\leq\; & 2^{q-1}\left[\operatorname{dev}_{W_c}(\sigma_0,\widetilde{\sigma}) + \operatorname{dev}_{W_c}(\sigma,\widetilde{\sigma})\right]
\end{aligned}
$$

and with (2.6) from Theorem 2.1 we conclude

$$
\begin{aligned}
\operatorname{dev}_{W_c}(\sigma_0,\sigma) \leq\; & 2^{q-1}\int_\Omega \left[W_c(\widetilde{\sigma}) - W_c(\tau) + (\tau - \widetilde{\sigma}) : DW_c(\tau)\right]\mathrm{d}x \\
& + 2^{q-1}\left[\frac{1}{d}\|\operatorname{tr}\widetilde{\sigma}\|_{L^2(\Omega)} + \frac{1}{d^2\lambda_1}\|\operatorname{div} u\|_{L^2(\Omega)}\right]\mathcal{G}_1[\tau,u] \\
& + 2^{q-1}\left[\|\widetilde{\sigma}^D\|_{L^q(\Omega)} + \alpha^{-\frac{1}{q-1}}\|\varepsilon^D(u)\|^{p-1}_{L^p(\Omega)}\right]\mathcal{G}_2[\tau,u] \\
& + 2^{q-1}\operatorname{dev}_{W_c}(\sigma,\widetilde{\sigma})\,.
\end{aligned}
$$

Applying the triangle inequality we arrive at

$$\begin{aligned}
\mathrm{dev}_{W_c}(\sigma_0,\sigma) &\le 2^{q-1}\int_\Omega \big[W_c(\sigma)-W_c(\tau)+(\tau-\sigma):DW_c(\tau)\big]\,\mathrm{d}x\\
&\quad +2^{q-1}\int_\Omega \big[W_c(\widetilde{\sigma})-W_c(\sigma)+(\sigma-\widetilde{\sigma}):DW_c(\tau)\big]\,\mathrm{d}x\\
&\quad +2^{q-1}\Big[\frac{1}{d}\|\mathrm{tr}\,\sigma\|_{L^2(\Omega)}+\frac{1}{d}\|\mathrm{tr}\,\sigma-\mathrm{tr}\,\widetilde{\sigma}\|_{L^2(\Omega)}+\frac{1}{d^2\lambda_1}\|\mathrm{div}\,u\|_{L^2(\Omega)}\Big]\\
&\quad \cdot\mathcal{G}_1[\tau,u]\\
&\quad +2^{q-1}\Big[\|\sigma^D\|_{L^q(\Omega)}+\|\sigma^D-\widetilde{\sigma}^D\|_{L^q(\Omega)}+\alpha^{-\frac{1}{q-1}}\|\varepsilon^D(u)\|^{p-1}_{L^p(\Omega)}\Big]\\
&\quad \cdot\mathcal{G}_2[\tau,u]+2^{q-1}\mathrm{dev}_{W_c}(\sigma,\widetilde{\sigma})\\
&=: 2^{q-1}[\bar{T}_1+\bar{T}_2+\bar{T}_3+\bar{T}_4+\bar{T}_5]\,.
\end{aligned}$$

Clearly $\bar{T}_1\ge 0$ and $\bar{T}_1=0$ if and only if $\sigma=\tau$. For $\bar{T}_2$ we observe that

$$W_c(\sigma)\ge W_c(\widetilde{\sigma})+DW_c(\widetilde{\sigma}):(\sigma-\widetilde{\sigma})\,,$$

so that

$$\begin{aligned}
\bar{T}_2 &\le \int_\Omega(\sigma-\widetilde{\sigma}):(DW_c(\tau)-DW_c(\widetilde{\sigma}))\,\mathrm{d}x\\
&= \int_\Omega(\sigma-\widetilde{\sigma}):(DW_c(\sigma)-DW_c(\widetilde{\sigma}))\,\mathrm{d}x+\int_\Omega(\sigma-\widetilde{\sigma}):(DW_c(\tau)-DW_c(\sigma))\,\mathrm{d}x\,.
\end{aligned}$$

For the first integral on the r.h.s. of the foregoing inequality we observe

$$\begin{aligned}
(\sigma-\widetilde{\sigma}):(DW_c(\sigma)-DW_c(\widetilde{\sigma})) &= \int_0^1 D^2W_c(\widetilde{\sigma}+t(\sigma-\widetilde{\sigma}))(\sigma-\widetilde{\sigma},\sigma-\widetilde{\sigma})\,\mathrm{d}t\\
&\le \lambda_1(\mathrm{tr}\,\sigma-\mathrm{tr}\,\widetilde{\sigma})^2+\lambda_2|\sigma^D-\widetilde{\sigma}^D|^2\\
&\quad +c_1\big[|\sigma^D|^{q-2}|\sigma^D-\widetilde{\sigma}^D|^2+|\sigma^D-\widetilde{\sigma}^D|^q\big]
\end{aligned}$$

with

$$c_1:=\alpha(q-1)2^{q-2}\,,$$

which is a consequence of inequality (A.18) in [Kn]. Recalling the definition of dist_{W_c} (stated before Theorem 2.2) we thus have the upper bound

$$\begin{aligned}
&\lambda_1\int_\Omega|\mathrm{tr}\,\sigma-\mathrm{tr}\,\widetilde{\sigma}|^2\,\mathrm{d}x+\lambda_2\int_\Omega|\sigma^D-\widetilde{\sigma}^D|^2\,\mathrm{d}x\\
&+\ c_1\left[\int_\Omega|\sigma^D-\widetilde{\sigma}^D|^q\,\mathrm{d}x+\int_\Omega|\sigma^D|^{q-2}|\sigma^D-\widetilde{\sigma}^D|^2\,\mathrm{d}x\right]\\
&\le\ 2\mathrm{dist}_{W_c}(\sigma,\widetilde{\sigma})+c_1\Big[\frac{q}{\alpha}\mathrm{dist}_{W_c}(\sigma,\widetilde{\sigma})+\|\sigma^D\|^{1-\frac{2}{q}}_{L^q(\Omega)}\Big[\frac{q}{\alpha}\mathrm{dist}_{W_c}(\sigma,\widetilde{\sigma})\Big]^{\frac{2}{q}}\Big]
\end{aligned}$$

for the first integral. For handling the second one we estimate

$$\begin{aligned}
&\int_\Omega (\sigma-\widetilde{\sigma}) : (DW_c(\tau) - DW_c(\sigma))\,\mathrm{d}x \\
&\le \left[\lambda_1 \int_\Omega |\operatorname{tr}\sigma - \operatorname{tr}\widetilde{\sigma}|\big[|\operatorname{tr}\tau| + |\operatorname{tr}\sigma|\big]\,\mathrm{d}x + \int_\Omega |\sigma^D - \widetilde{\sigma}^D|\big[|DW_c(\tau)^D| + |DW_c(\sigma)^D|\big]\,\mathrm{d}x\right] \\
&\le \sqrt{2\lambda_1}\operatorname{dist}_{W_c}(\sigma,\widetilde{\sigma})^{\frac12}\big[\|\operatorname{tr}\tau\|_{L^2(\Omega)} + \|\operatorname{tr}\sigma\|_{L^2(\Omega)}\big] \\
&\quad + \left[\frac{q}{\alpha}\operatorname{dist}_{W_c}(\sigma,\widetilde{\sigma})\right]^{\frac1q}\big[\|DW_c(\tau)^D\|_{L^p(\Omega)} + \|DW_c(\sigma)^D\|_{L^p(\Omega)}\big]\,.
\end{aligned}$$

Collecting terms we get

$$\begin{aligned}
\operatorname{dev}_{W_c}(\sigma_0,\sigma) \le\ & 2^{q-1}\Bigg[\int_\Omega \big[W_c(\sigma) - W_c(\tau) + (\tau-\sigma) : DW_c(\tau)\big]\,\mathrm{d}x \\
& + \left[\frac{1}{d}\|\operatorname{tr}\sigma\|_{L^2(\Omega)} + \frac{1}{d^2\lambda_1}\|\operatorname{div} u\|_{L^2(\Omega)}\right]\mathcal{G}_1[\tau,u] \\
& + \left[\|\sigma^D\|_{L^q(\Omega)} + \alpha^{-\frac{1}{q-1}}\|\varepsilon^D(u)\|^{p-1}_{L^p(\Omega)}\right]\mathcal{G}_2[\tau,u]\Bigg] \\
& + 2^{q-1}\left[3 + c_1\frac{q}{\alpha}\right]\operatorname{dist}_{W_c}(\sigma,\widetilde{\sigma}) \\
& + 2^{q-1}\frac{1}{d}\sqrt{\frac{2}{\lambda_1}}\operatorname{dist}_{W_c}(\sigma,\widetilde{\sigma})^{\frac12}\mathcal{G}_1[\tau,u] \\
& + 2^{q-1}\sqrt{2\lambda_1}\operatorname{dist}_{W_c}(\sigma,\widetilde{\sigma})^{\frac12}\big[\|\operatorname{tr}\tau\|_{L^2(\Omega)} + \|\operatorname{tr}\sigma\|_{L^2(\Omega)}\big] \\
& + 2^{q-1}\left[\frac{q}{\alpha}\operatorname{dist}_{W_c}(\sigma,\widetilde{\sigma})\right]^{\frac1q}\Big[\mathcal{G}_2[\tau,u] + \|DW_c(\tau)^D\|_{L^p(\Omega)} \\
& \quad + \|DW_c(\sigma)^D\|_{L^p(\Omega)}\Big] \\
& + 2^{q-1}c_1\|\sigma^D\|^{1-\frac2q}_{L^q(\Omega)}\left[\frac{q}{\alpha}\operatorname{dist}_{W_c}(\sigma,\widetilde{\sigma})\right]^{\frac2q}\,, \qquad (4.1)
\end{aligned}$$

and if we take the infimum w.r.t. $\widetilde{\sigma} \in Q^{p,2}_f$, then (4.1) gives the first part of Theorem 2.2.

Now we are now going to prove the second part of Theorem 2.2, i.e. we want to find an explicitly computable reasonable quantity which controls the distance $\operatorname{dist}_{W_c}(\sigma, Q^{p,2}_f)$ of tensors σ to the affine manifold $Q^{p,2}_f$: suppose that $\sigma \in L^{q,2}(\Omega)$ with the property $\operatorname{div}\sigma \in L^q(\Omega)$ is fixed. We have

$$\inf_{\widetilde{\sigma}\in Q^{p,2}_f} \operatorname{dist}_{W_c}(\sigma,\widetilde{\sigma}) = -\sup_{\widetilde{\sigma}\in Q^{p,2}_f}\big[-\operatorname{dist}_{W_c}(\sigma,\widetilde{\sigma})\big] = -\sup_{\widetilde{\eta}\in Q^{p,2}_{\widetilde{f}}}\big[-\operatorname{dist}_{W_c}(\widetilde{\eta},0)\big]\,,$$

where

$$Q^{p,2}_{\widetilde{f}} := \left\{\widetilde{\eta} \in L^{q,2}(\Omega) : \int_\Omega \widetilde{\eta} : \varepsilon(w)\,\mathrm{d}x = \int_\Omega \widetilde{f}\cdot w\,\mathrm{d}x \ \text{ for all } \ w \in U^{p,2}_0(\Omega)\right\}$$

and where

$$\widetilde{f} := f + \operatorname{div}\sigma\,.$$

Let

$$\begin{aligned}\widetilde{K}[\widetilde{\eta}] &:= -\mathrm{dist}_{W_c}(\widetilde{\eta},0) = -\int_\Omega W_c(\widetilde{\eta})\,\mathrm{d}x\,,\\ \widetilde{J}[v] &:= \int_\Omega W_c^*(\varepsilon(v))\,\mathrm{d}x - \int_\Omega \widetilde{f}\cdot v\,\mathrm{d}x\,.\end{aligned}$$

Then, with the same arguments as in Section 3, we deduce that

$$\sup_{\widetilde{\eta}\in Q^{p,2}_{\widetilde{f}}} \widetilde{K}[\widetilde{\eta}] = \inf_{w\in U_0^{p,2}} \widetilde{J}[w]\,,$$

hence

$$\inf_{\widetilde{\sigma}\in Q^{p,2}_f} \mathrm{dist}_{W_c}(\sigma,\widetilde{\sigma}) = -\inf_{w\in U_0^{p,2}(\Omega)} \widetilde{J}[w]\,. \tag{4.2}$$

Equation (4.2) shows that we must find a lower bound for $\widetilde{J}[w]$, which in turn requires information about the behaviour of the conjugate function

$$W_c^*(\varepsilon) = \sup_{\varkappa\in\mathbb{S}^d}[\varkappa:\varepsilon - W_c(\varkappa)]\,.$$

Given $\varepsilon\in\mathbb{S}^d$, the supremum is attained at $\varkappa\in\mathbb{S}^d$ satisfying

$$\varepsilon = DW_c(\varkappa) = \alpha|\varkappa^D|^{q-2}\varkappa^D + \lambda_1\mathrm{tr}\,\varkappa\mathbf{1} + \lambda_2\varkappa^D\,.$$

As in Section 3 we clearly have $\mathrm{tr}\,\varepsilon = \lambda_1 d\,\mathrm{tr}\,\varkappa$ and

$$\varepsilon^D = \left[\alpha|\varkappa^D|^{q-2} + \lambda_2\right]\varkappa^D\,,\qquad |\varepsilon^D| = \alpha|\varkappa^D|^{q-1} + \lambda_2|\varkappa^D|\,.$$

Now let

$$\varphi(t) := \alpha t^{q-1} + \lambda_2 t$$

for $t\geq 0$ and let

$$\psi(s) := \varphi^{-1}(s)$$

denote the inverse function. Then we write

$$\begin{aligned}W_c^*(\varepsilon) &= \varkappa:\varepsilon - \left[\frac{\lambda_1}{2}(\mathrm{tr}\,\varkappa)^2 + \frac{\lambda_2}{2}|\varkappa^D|^2 + \frac{\alpha}{q}|\varkappa^D|^q\right]\\ &= \frac{1}{d}\mathrm{tr}\,\varkappa\mathrm{tr}\,\varepsilon + \varkappa^D:\varepsilon^D - \left[\frac{\lambda_1}{2}(\mathrm{tr}\,\varkappa)^2 + \frac{\lambda_2}{2}|\varkappa^D|^2 + \frac{\alpha}{q}|\varkappa^D|^q\right]\\ &= \frac{1}{\lambda_1 d^2}(\mathrm{tr}\,\varepsilon)^2 + \lambda_2|\varkappa^D|^2 + \alpha|\varkappa^D|^q - \frac{1}{2}\frac{1}{\lambda_1 d^2}(\mathrm{tr}\,\varepsilon)^2 - \frac{\lambda_2}{2}|\varkappa^D|^2 - \frac{\alpha}{q}|\varkappa^D|^q\\ &= \frac{1}{2}\frac{1}{\lambda_1 d^2}(\mathrm{tr}\,\varepsilon)^2 + \frac{\lambda_2}{2}|\varkappa^D|^2 + \frac{\alpha}{p}|\varkappa^D|^q\\ &= \frac{1}{2}\frac{1}{\lambda_1 d^2}(\mathrm{tr}\,\varepsilon)^2 + \frac{\lambda_2}{2}\psi(|\varepsilon^D|)^2 + \frac{\alpha}{p}\psi(|\varepsilon^D|)^q\,,\end{aligned}$$

and by introducing the auxiliary function

$$\Phi(t) := \frac{\lambda_2}{2}\psi(t)^2 + \frac{\alpha}{p}\psi(t)^q\,,\qquad t\geq 0\,, \tag{4.3}$$

we obtain the representation

$$W_c^*(\varepsilon) = \frac{1}{2}\frac{1}{\lambda_1 d^2}(\operatorname{tr}\varepsilon)^2 + \Phi(|\varepsilon^D|)\,, \qquad \varepsilon \in \mathbb{S}^d\,. \tag{4.4}$$

Due to the stucture of W_c^* being expressed in (4.4) it seems more appropriate in the following to work in the Orlicz-space generated by Φ rather than in Lebesgue-classes, we refer to Lemma A.1. Note that the only embedding constant that we will need to argue with Luxemburg-norms instead of Lebesgue-norms is explicitely computable (see [Ad], proof of Theorem 8.12, (b)). Now, from $\varphi(t) \geq \lambda_2 t$ it follows $s \geq \lambda_2 \Psi(s)$ so that according to formula (4.4)

$$W_c^*(\varepsilon) \geq \frac{1}{2}\frac{1}{\lambda_1 d^2}|\operatorname{tr}\varepsilon|\lambda_2\psi(|\operatorname{tr}\varepsilon|) + \Phi(|\varepsilon^D|)\,.$$

On the other hand, we have by the definition of φ

$$s = \varphi(\psi(s)) = \alpha\psi(s)^{q-1} + \lambda_2\psi(s)\,,$$

thus

$$s\psi(s) = \alpha\psi(s)^q + \lambda_2\psi(s)^2 \geq \Phi(s)\,,$$

and we get the lower bound

$$\begin{aligned} W_c^*(\varepsilon) &\geq \Phi(|\varepsilon^D|) + \frac{1}{2d^2}\frac{\lambda_2}{\lambda_1}\Phi(|\operatorname{tr}\varepsilon|) \\ &\geq \min\Big\{2, \frac{\lambda_2}{\lambda_1 d^2}\Big\}\Big[\frac{1}{2}\Phi(|\varepsilon^D|) + \frac{1}{2}\Phi(|\operatorname{tr}\varepsilon|)\Big] \\ &\geq \min\Big\{2, \frac{\lambda_2}{\lambda_1 d^2}\Big\}\Phi\Big(\frac{1}{2}|\varepsilon^D| + \frac{1}{2}|\operatorname{tr}\varepsilon|\Big)\,, \qquad \varepsilon \in \mathbb{S}^d\,, \end{aligned}$$

by the convexity of Φ. Next we apply inequality (A.13) from the appendix which gives

$$\Phi\Big(\frac{1}{2}|\varepsilon^D| + \frac{1}{2}|\operatorname{tr}\varepsilon|\Big) \geq a_7^{-1}\Phi(|\varepsilon^D| + |\operatorname{tr}\varepsilon|) \geq a_7^{-1}\Phi(|\varepsilon|)\,,$$

thus

$$W_c^*(\varepsilon) \geq \min\Big\{2, \frac{\lambda_2}{\lambda_1 d^2}\Big\}a_7^{-1}\Phi(|\varepsilon|)\,, \qquad \varepsilon \in \mathbb{S}^d\,. \tag{4.5}$$

Now let us choose $w \in U_0^{p,2}(\Omega)$. Then (4.5) gives

$$\widetilde{J}[w] \geq \min\Big\{2, \frac{\lambda_2}{\lambda_1 d^2}\Big\}a_7^{-1}\int_\Omega \Phi(|\varepsilon(w)|)\,\mathrm{d}x - \|\widetilde{f}\|_{L^q(\Omega)}\|w\|_{L^p(\Omega)}\,,$$

and we may use Poincarè's inequality

$$\|w\|_{L^p(\Omega)} \leq P_p(\Omega)\|\nabla w\|_{L^p(\Omega)}$$

as well as Korn's inequality (see [MM2])

$$\|\nabla w\|_{L^p(\Omega)} \leq K_p(\Omega)\|\varepsilon(w)\|_{L^p(\Omega)}$$

to deduce

$$\widetilde{J}[w] \geq \min\Big\{2, \frac{\lambda_2}{\lambda_1 d^2}\Big\} a_7^{-1} \int_\Omega \Phi(|\varepsilon(w)|)\,\mathrm{d}x - P_p(\Omega)K_p(\Omega)\|\widetilde{f}\|_{L^q(\Omega)}\|\varepsilon(w)\|_{L^p(\Omega)}\,. \tag{4.6}$$

Let $L_\Phi(\Omega)$ denote the Orlicz-space generated by Φ equipped with the norm (see [Ad] for details)

$$\|\rho\|_{L_\Phi(\Omega)} := \inf\Big\{k>0 : \int_\Omega \Phi\Big(\frac{|\rho|}{k}\Big)\,\mathrm{d}x \leq 1\Big\}\,.$$

According to (A.12) the N-function Φ dominates the N-function $t \mapsto \frac{1}{p}t^p$ near infinity, and since Ω has finite volume, we can use [Ad], Theorem 8.12, (b), and have

$$\|\zeta\|_{L^p(\Omega)} \leq C_p(\Omega)\|\zeta\|_{L_\Phi(\Omega)}\,, \quad \zeta \in L_\Phi(\Omega)\,, \tag{4.7}$$

for a positive constant $C_p(\Omega)$. Combining (4.6) and (4.7) with the choice $\zeta = \varepsilon(w)$ we end up with

$$\begin{aligned} \widetilde{J}[w] \;\geq\; & \min\Big\{2, \frac{\lambda_2}{\lambda_1 d^2}\Big\} a_7^{-1} \int_\Omega \Phi(|\varepsilon(w)|)\,\mathrm{d}x \\ & - C_p(\Omega)P_p(\Omega)K_p(\Omega)\|\widetilde{f}\|_{L^q(\Omega)}\|\varepsilon(w)\|_{L_\Phi(\Omega)}\,. \end{aligned} \tag{4.8}$$

For simplicity let

$$\nu := \min\Big\{1, \frac{\lambda_2}{\lambda_1 d^2}\Big\} a_7^{-1}\,, \quad \mu := C_p(\Omega)P_p(\Omega)K_p(\Omega)\,.$$

Let further $l := \|\varepsilon(w)\|_{L_\Phi(\Omega)}$ and assume that $l > 0$. By the definition of the $L_\Phi(\Omega)$-norm we have

$$\int_\Omega \Phi\Big(\frac{|\varepsilon(w)|}{l/2}\Big)\,\mathrm{d}x \geq 1\,,$$

and (A.12) implies

$$a_6 \int_\Omega \min\Big\{\frac{4}{l^2}|\varepsilon(w)|^2, \frac{2^p}{l^p}|\varepsilon(w)|^p\Big\}\,\mathrm{d}x \geq 1\,. \tag{4.9}$$

By considering the cases $l \geq 1$ and $l < 1$, and with (A.12) it is easy to check that (4.9) gives the estimate

$$\int_\Omega \Phi(|\varepsilon(w)|)\,\mathrm{d}x \geq 4^{-1}a_6^{-1}a_5 \min\big\{l^2, l^p\big\}\,, \tag{4.10}$$

and (4.10) clearly is true for $l = 0$. Returning to (4.8) and using (4.10), (A.12) we get

$$\begin{aligned} \inf_{w \in U_0^{p,2}(\Omega)} \widetilde{J}[w] \;\geq\; & \inf_{t \geq 0}\Big\{\nu 4^{-1}a_6^{-1}a_5 \min\big\{t^2, t^p\big\} - \mu\|\widetilde{f}\|_{L^q(\Omega)}t\Big\} \\ =:\; & \inf_{t \geq 0}\Big\{\widetilde{\nu}\min\big\{t^2, t^p\big\} - \mu\|\widetilde{f}\|_{L^q(\Omega)}t\Big\} \end{aligned}$$

The function

$$g(t) := \widetilde{\nu}\min\{t^2, t^p\} - \mu\|\widetilde{f}\|_{L^q(\Omega)}t\,, \quad t \geq 0\,,$$

attains its minimum at $t_0 > 0$.

Case 1. $t_0 = 1$. Then

$$g(t_0) \geq -\mu \|\widetilde{f}\|_{L^q(\Omega)}$$

is immediate.

Case 2. $t_0 < 1$. Then we must have

$$\widetilde{\nu} 2 t_0 - \mu \|\widetilde{f}\|_{L^q(\Omega)} = 0 \,,$$

i.e.

$$t_0 = \widetilde{\nu}^{-1} \frac{1}{2} \mu \|\widetilde{f}\|_{L^q(\Omega)} \,,$$

so that

$$g(t_0) \geq -\mu^2 \widetilde{\nu}^{-1} \frac{1}{2} \|\widetilde{f}\|^2_{L^q(\Omega)} \,.$$

Case 3. $t_0 > 1$. In this case it holds

$$p \widetilde{\nu} t_0^{p-1} = \mu \|\widetilde{f}\|_{L^q(\Omega)} \,,$$

and therefore

$$t_0 = (p^{-1} \widetilde{\nu}^{-1} \mu \|\widetilde{f}\|_{L^q(\Omega)})^{\frac{1}{p-1}} \,.$$

This gives

$$g(t_0) \geq -\mu (p^{-1} \widetilde{\nu}^{-1} \mu)^{\frac{1}{p-1}} \|\widetilde{f}\|^q_{L^q(\Omega)} \,.$$

Summarizing all cases, letting

$$C_0 := \max \left\{ \mu, \mu^2 \widetilde{\nu}^{-1} \frac{1}{2}, \mu [p^{-1} \widetilde{\nu}^{-1} \mu]^{\frac{1}{p-1}} \right\}$$

and recalling (4.2) we have proved the second part of the theorem. □

5 Remarks on the variational problem for the displacement fields

During our foregoing analysis we used the variational problem

$$K[\sigma_0] = \sup_{\widetilde{\sigma} \in Q^{p,2}_f} K[\widetilde{\sigma}] \tag{5.1}$$

which is rather convenient from the analytical point of view but rather unpleasant from the viewpoint of numerical analysis because the approximations must exactly satisfy the differential relation required in the definition of $Q^{p,2}_f$. If this hypothesis is dropped, then (compare Theorem 2.2) we have to estimate the distance of the approximations to $Q^{p,2}_f$. This situation is typical for many linear and nonlinear models related to problems in continuum mechanics, where for this reason usually the variational problem for the displacement field is studied leading to a posteriori error estimates for the solution u_0 of the problem

$$J[u_0] = \inf_{u \in u_b + U^{p,2}_0(\Omega)} J[u] \,, \tag{5.2}$$

which is the dual problem of (5.1) having the advantage that it is defined on an affine subspace of a function space not being restricted through differential relations. From the theoretical point of view (5.2) is investigated in great detail in [Kn] where the existence of solutions as well as various properties of the density W_c^* are established. Unfortunately the structure of W_c^* is not very explicit, moreover due to a lack of uniform convexity we have to work with the problem (5.1) in order to obtain a posteriori error estimates for the stress tensor σ_0 (see, e.g., [BiR] for a more detailled discussion of this problem).

On the other hand, in order to approximate the displacement field by a sequence

$$\min_{u_h \in V_h} I[v_h]\,, \qquad I[v_h] := \int_\Omega \left[W_c^*(\varepsilon(v_h)) - f \cdot v_h\right] \mathrm{d}x \tag{5.3}$$

of finite dimensional problems one usually applies finite element methods which means that the functions v_h are piecewise affine and continuous. Then on each element $\varepsilon(v_h)$ is constant and by (4.4) we can exactly calculate the part of the functional associated with the element. In fact, if we let

$$\mu(t) := \frac{\psi(t)}{t}\,,$$

then by the definitions of φ and ψ stated after (4.2) it holds

$$t = \varphi(\psi(t)) = \alpha t^{q-1} \mu(t)^{q-1} + \lambda_2 t \mu(t)\,,$$

which means that μ satisfies the algebraic equation

$$1 = \alpha t^{q-2} \mu(t)^{q-1} + \lambda_2 \mu(t)\,, \tag{5.4}$$

and by (4.4) we have

$$\begin{aligned} W_c^*(\varepsilon(v_h)) &= \frac{1}{2}\frac{1}{\lambda_1 d^2}(\operatorname{div} v_h)^2 + \frac{\lambda_2}{2}|\varepsilon^D(v_h)|^2 \mu(|\varepsilon^D(v_h)|)^2 \\ &\quad + \frac{\alpha}{p}|\varepsilon^D(v_h)|^q \mu(|\varepsilon^D(v_h)|)^q\,. \end{aligned} \tag{5.5}$$

Therefore, if we want to calculate $W_c^*(\varepsilon(v_h))$ we first solve (5.4) for the value $t = |\varepsilon^D(v_h)|$ giving the number $\mu(|\varepsilon^D(v_h)|)$ which we then insert into (5.5). In conclusion, the semi-explicit formulas (5.4) and (5.5) can be used to solve the approximate minimization problems (5.3) leading to a sequence $\{v_h\}$ of approximations of the displacement field u_0.

Appendix. An auxiliary lemma

LEMMA A.1. *The function Φ from (4.3) is strictly increasing and convex. Moreover, it satisfies the Δ_2-property, i.e. there is a constant c such that for all $t \geq 0$ we have $\Phi(2t) \leq c\Phi(t)$.*

Proof. Obviously φ is a stictly increasing function so that the same is true for ψ^2 and ψ^q, which means that Φ has the same property. We have

$$\begin{aligned} \Phi'(t) &= \lambda_2 \psi(t)\psi'(t) + \frac{1}{p}\alpha q \psi(t)^{q-1}\psi'(t)\,, \\ \Phi''(t) &= \lambda_2 \psi'(t)^2 + \lambda_2 \psi(t)\psi''(t) + \frac{1}{p}\alpha q(q-1)\psi(t)^{q-2}\psi'(t)^2 + \frac{1}{p}\alpha q \psi(t)^{q-1}\psi''(t)\,, \end{aligned}$$

and

$$\psi'(t) = \frac{1}{\varphi'(\psi(t))}\,, \qquad \psi''(t) = -\varphi'(\psi(t))^{-3}\varphi''(\psi(t))\,.$$

Letting $s := \psi(t)$ it follows that

$$\begin{aligned}
\Phi''(t) &= \lambda_2\varphi'(s)^{-2} + \lambda_2 s(-\varphi'(s))^{-3}\varphi''(s) + \frac{1}{p}\alpha q(q-1)s^{q-2}\varphi'(s)^{-2} \\
&\quad -\frac{1}{p}\alpha q s^{q-1}\varphi'(s)^{-3}\varphi''(s) \\
&= \varphi'(s)^{-3}\big[\lambda_2\varphi'(s) - \lambda_2 s\varphi''(s) + \frac{1}{p}\alpha q(q-1)s^{q-2}\varphi'(s) \\
&\quad -\frac{1}{p}\alpha q s^{q-1}\varphi''(s)\big] \\
&= \varphi'(s)^{-3}\big[\lambda_2^2 + \lambda_2\alpha s^{q-2}(q-1) - \lambda_2\alpha(q-1)(q-2)s^{q-2} + \frac{1}{p}\alpha q(q-1)s^{q-2}\lambda_2 \\
&\quad +\frac{1}{p}\alpha q(q-1)s^{q-2}\alpha(q-1)s^{q-2} - \frac{1}{p}\alpha q s^{q-1}\alpha(q-1)(q-2)s^{q-3}\big] \\
&= \varphi'(s)^{-3}\big[\lambda_2^2 + s^{q-2}\big[\lambda_2\alpha(q-1) - \lambda_2\alpha(q-1)(q-2) + \lambda_2\alpha(q-1)^2\big] \\
&\quad +s^{2q-4}\big[(q-1)^3\alpha^2 - (q-1)^2(q-2)\alpha^2\big]\big]\,,
\end{aligned}$$

and from this representation of Φ'' the strict convexity of Φ follows. Next we discuss the growth properties of Φ: if $t \geq 1$, then

$$\begin{aligned}
t &= \psi(\varphi(t)) = \psi(\lambda_2 t + \alpha t^{q-1}) \\
&\leq \psi\big([\lambda_2+\alpha]t^{q-1}\big)\,,
\end{aligned}$$

which means

$$\psi(y) \geq [\lambda_2+\alpha]^{-\frac{1}{q-1}} y^{\frac{1}{q-1}} \tag{A.1}$$

in case that $y \geq \lambda_2 + \alpha$. At the same time we have for any $x \geq 0$

$$x = \psi(\lambda_2 x + \alpha x^{q-1}) \geq \psi(\alpha x^{q-1})\,,$$

hence

$$\psi(y) \leq y^{\frac{1}{q-1}}\alpha^{-\frac{1}{q-1}}\,, \qquad y \geq 0\,. \tag{A.2}$$

Let $t \leq 1$. Then

$$t \leq \psi\big([\lambda_2+\alpha]t\big)\,,$$

and we deduce

$$\psi(y) \geq [\lambda_2+\alpha]^{-1}y \tag{A.3}$$

for all $y \leq \lambda_2 + \alpha$. Finally, again for $t \leq 1$, it holds

$$t = \psi(\varphi(t)) \geq \psi(\lambda_2 t)\,,$$

thus

$$\frac{1}{\lambda_2}y \geq \psi(y)\,, \qquad y \leq \lambda_2\,. \tag{A.4}$$

If $y \in [\lambda_2, \lambda_2 + \alpha]$, then according to (A.2)

$$\psi(y) \le \alpha^{-\frac{1}{q-1}} [\lambda_2 + \alpha]^{\frac{1}{q-1}} \le \alpha^{-\frac{1}{q-1}} [\lambda_2 + \alpha]^{\frac{1}{q-1}} y \frac{1}{\lambda_2} .$$

Putting together (A.1) – (A.4) and observing the latter inequality we have shown that

$$[\lambda_2 + \alpha]^{-\frac{1}{q-1}} y^{\frac{1}{q-1}} \le \psi(y) \le \alpha^{-\frac{1}{q-1}} y^{\frac{1}{q-1}} , \quad y \ge \lambda_2 + \alpha , \tag{A.5}$$

$$[\lambda_2 + \alpha]^{-1} y \le \psi(y) \le \frac{1}{\lambda_2} \alpha^{-\frac{1}{q-1}} [\lambda_2 + \alpha]^{\frac{1}{q-1}} y , \quad y \le \lambda_2 + \alpha . \tag{A.6}$$

Let

$$\begin{aligned} a_1 &:= \min \left\{ [\lambda_2 + \alpha]^{-1}, [\lambda_2 + \alpha]^{-\frac{1}{q-1}} \right\} , \\ a_2 &:= \max \left\{ \alpha^{-\frac{1}{q-1}}, \frac{1}{\lambda_2} \alpha^{-\frac{1}{q-1}} [\lambda_2 + \alpha]^{\frac{1}{q-1}} \right\} . \end{aligned}$$

Then (A.5), (A.6) give the estimates

$$\left. \begin{aligned} a_1 y^{\frac{1}{q-1}} &\le \psi(y) \le a_2 y^{\frac{1}{q-1}} , \quad y \ge \lambda_2 + \alpha , \\ a_1 y &\le \psi(y) \le a_2 y , \quad y \le \lambda_2 + \alpha , \end{aligned} \right\} \tag{A.7}$$

and from (A.7) it is immediate that

$$a_1 \min \left\{ y, y^{\frac{1}{2q-1}} \right\} \le \psi(y) , \quad y \ge 0 . \tag{A.8}$$

Let $y \le 1$. If also $y \le \lambda_2 + \alpha$, then

$$\psi(y) \le a_2 y = a_2 \min \left\{ y, y^{\frac{1}{q-1}} \right\} .$$

If $y \ge \lambda_2 + \alpha$, then

$$\psi(y) \le a_2 y^{\frac{1}{q-1}} = a_2 y y^{\frac{1}{q-1} - 1} \le a_2 [\lambda_2 + \alpha]^{\frac{2-q}{q-1}} y ,$$

so that now

$$\psi(y) \le a_3 y , \quad a_3 := \max \left\{ a_2, a_2 [\lambda_2 + \alpha]^{\frac{2-q}{q-1}} \right\} . \tag{A.9}$$

Combining (A.7) – (A.9) we have shown that

$$a_1 \min \left\{ y, y^{\frac{1}{q-1}} \right\} \le \psi(y) \le a_4 \min \left\{ y, y^{\frac{1}{q-1}} \right\} , \quad y \ge 0 , \tag{A.10}$$

where the second inequality in (A.10) for $y \ge 1$ follows in the same way as for $y \le 1$, provided we choose

$$a_4 := \max \left\{ a_3, a_2 [\lambda_2 + \alpha]^{\frac{q-2}{q-1}} \right\} . \tag{A.11}$$

From (A.10) the Δ_2-property of Φ now immediately follows: if $y \le 1$, then

$$\begin{aligned} \Phi(y) &\le \lambda_2 a_4^2 y^2 + \frac{1}{p} \alpha y^q a_4^q \le \left[\lambda_2 a_4^2 + \frac{1}{p} \alpha a_4^q \right] y^2 , \\ \Phi(y) &\ge \frac{\lambda_2}{2} a_1^2 y^2 + \frac{1}{p} \alpha a_1^q y^q \ge \frac{\lambda_2}{2} a_1^2 y^2 , \end{aligned}$$

and if $y \geq 1$ we get

$$\begin{aligned}\Phi(y) &\leq \frac{\lambda_2}{2} a_4^2 y^{\frac{2}{q-1}} + \frac{1}{p}\alpha a_4^q y^p \leq \Big[\frac{\lambda_2}{2} a_4^2 + \frac{1}{p}\alpha a_4^q\Big] y^p\,,\\ \Phi(y) &\geq \frac{\lambda_2}{2} a_1^2 y^{\frac{2}{q-1}} + \frac{1}{p}\alpha a_1^q y^p \geq \frac{1}{p}\alpha a_1^q y^p\,.\end{aligned}$$

This implies with new constants

$$a_5 = a_5(\lambda_1, \lambda_2, , q, \alpha)\,, \qquad a_6 = a_6(\lambda_1, \lambda_2, q, \alpha)$$

the estimate

$$a_5 \min\left\{t^2, t^p\right\} \leq \Phi(t) \leq a_6 \min\left\{t^2, t^p\right\}, \qquad t \geq 0\,. \tag{A.12}$$

Let $t \geq 1$. Then (A.12) gives

$$\Phi(2t) \leq a_6 (2t)^p = a_6 2^p t^p \leq a_5^{-1} a_6 2^p \Phi(t)\,.$$

In case $t \leq 1$ we have

$$\Phi(2t) \leq a_6 \min\left\{2^2 t^2, 2^p t^p\right\} \leq 4 a_6 \min\left\{t^2, t^p\right\} \leq a_5^{-1} a_6 4 \Phi(t)\,,$$

thus Φ satisfies the global Δ_2-condition

$$\Phi(2t) \leq 4 a_5^{-1} a_6 \Phi(t)\,, \qquad t \geq 0\,. \tag{A.13}$$

Note that the constant

$$a_7 := 4 a_5^{-1} a_6$$

just depends on the parameters λ_1, λ_2, q and α and can be calculated explicitely.

References

[Ad] Adams, R. A., Sobolev spaces. Academic Press, New York-San Francisco-London 1975.

[AO] Ainsworth, M., Oden, J.T., A posteriori error estimation in finite element analysis. Wiley and Sons, New York, 2000.

[AOL] Ainsworth, M., Oden, J.T., Lee, C.Y., Local a posteriori error estimators for variational inequalities. Num. Meth. for PDE 9 (1993), 23-33.

[BaR] Bangerth, W., Rannacher, R., Adaptive finite element methods for differential equations. Birkhäuser, Berlin, 2003.

[BeR] Becker, R., Rannacher, R., A feed-back approach to error control in finite element methods: basic approach and examples. East-West J. Num. Math. 4 (1996), 237-264.

[BeF] Bensoussan, A., Frehse, J., Asymptotic behaviour of Norton-Hoff's law in plasticity theory and H^1 regularity. In Boundary Value Problems for Partial Differential Equations and Applications. J.L. Lions and C. Baiocchi, Eds, no. 29 in Research Notes in Applied Mathematics. Masson, Paris, 1993, 3–26.

[BiF1] Bildhauer, M., Fuchs, M., Smoothness of weak solutions of the Ramberg/Osgood equations on plane domains. Z. Angew. Math. Mech. 87 (2007), 70–76.

[BiF2] Bildhauer, M., Fuchs, M., Continuity properties of the stress tensor in the 3-dimensional Ramberg/Osgood model. J. Appl. Anal. 13 (2007).

[BiR] Bildhauer, M., Repin, S., Estimates for the deviation from exact solutions of variational problems with power growth functionals. Zap. Nauch. Semi. (POMI), in press.

[BFR] Bildhauer, M., Fuchs, M., Repin, S., A posteriori error estimates for stationary slow flows of power-law fluids. Int. J. Non-Newtonian Fluids, in press.

[Br] Braess, D., A posteriori error estimators for obstacle problems – another look. Num. Math. 101 (2005), 415-421.

[Cl] Clarkson, J.A., Uniformly convex spaces. Trans. Am. Math. Soc. 40, 396–414 (1936).

[CN] Chen, Z., Nochetto, R.H., Residual type a posteriori error estimates for elliptic obstacle problems. Num. Math. 84 (2000), 527-548.

[FR] Fuchs, M., Repin, S., Estimates for the deviation from the exact solutions of variational problems modeling certain classes of generalized Newtonian fluids. Math. Meth. Appl. Sci. (M2AS) 29 (2006), 2225–2244.

[FS] Fuchs, M., Seregin, G., Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Lecture Notes in Mathematics 1749, Springer, Berlin-Heidelberg 2000.

[GS] Geymonat, G., Suquet, P., Functional spaces for Norton-Hoff Materials. Math. Meth. Appl. Sci 8 (1986), 206–222.

[HK] Hoppe, R.H.W., Kornhuber, R., Adaptive multilevel methods for obstacle problems. SIAM J. Num. Anal. 31 (1994), 301-323.

[Kn] Knees, D.,Regularity results for quasilinear elliptic systems of power-law growth in nonsmooth domains – boundary, transmission and crack problems. Dissertation, Fakultät Mathematik und Physik, Universität Stuttgart, 2004.

[MM1] P.P. Mosolov and V.P. Mjasnikov, Mechanics of rigid plastic media. Nauka, Moscow (1981) (in Russian).

[MM2] P.P. Mosolov, V.P. Mjasnikov, On well-posedness of boundary value problems in the mechanics of continuous media. Mat. Sbornik 88 (130), 256–284 (1972). English translation in Math. USSR Sbornik 17, 257–268 (1972).

[NR] Neittaanmäki, P., Repin, S., Reliable methods for computer simulation, error control and a posteriori estimates. Elsevier, New York, 2004.

[OR] Osgood, W.R., Ramberg, W., Description of stress-strain curves by three parameters. NACA Technical Note 902, National Bureau of Standards, Washington, 1943.

[PR] J. Pousin and J. Rappaz, *Consistance, stabilité, erreurs a priori et a posteriori pour des probl emes non linéaries,* C. R. Acad. Sci. Paris 312(1991), pp. 699-703.

[Ra] Rannacher, R., The dual-weighted-residual method for error control and mesh adaption in finite element methods. The mathematics of finite elements and applications X, MAFELAP 1999. Proceedings of the 10th conference, Brunel Univ., Uxbridge, Middlesex, GB, June 22-25, 1999 (J. Whiteman ed.), Elsevier Amsterdam 2000, 97-116.

[Re1] Repin, S., A posteriori error estimation for nonlinear variational problems by duality theory. Zap. Nauch. Sem. (POMI) 243 (1997), 201-214.

[Re2] Repin, S., A posteriori error estimation for variational problems with uniformly convex functionals. Math. Comp. 69 (2000), 481-500.

[Re3] Repin, S., Two-sided estimates of deviation from exact solutions of uniformly elliptic equations. Proc. St. Petersburg Math. Soc. IX (2001), 143-171. Translation: Amer. Math. Soc. Tansl. Ser. 2, 209, Amer. Math. Soc., Providence, RI, 2003.

[Re4] Repin, S., Estimates of deviations from exact solutions for some boundary-value problems with incompressibility condition. Algebra i Analiz 16 (2004), 124-161.

[Re5] Repin, S., Aposteriori estimates of the accuracy of variational methods for problems with nonconvex functionals. Algebra i Analiz 11 (1999), 151-182.

[RX] Repin, S., Xanthis, L.S., A posteriori error estimation for elasto-plastic problems based on duality theory. Comp. Meth. Appl. Math. Eng. 138 (1996), 317-339.

[SW1] Schulz, M., Wendland, W.L., A general approach to a posteriori error estimates for strictly monotone and Lipschitz continuous nonlinear operators illustrated in elasto-plasticity. Proc. 2nd European Conf. Num. Math. Advanced Appl 1997 (Heidelberg), World Sci. Publ., River Edge, New York, 1998, 572-579.

[SW2] Schulz, M., Wendland, W.L., A posteriori error estimates for nonlinear problems. Finite element discretisations of elliptic equations. Math. Comp. 62 (1994), 445-475.

[SW3] Schulz, M., Wendland, W.L., A review of a posteriori error estimation and adaptive mesh-refinement techniques. Wiley ans Sons, Teubner, New York, 1996.

[Te] Temam, R., Mathematical problems in plasticity. Gauthier-Villars, Paris, 1985.

[Ve1] Verfürth, R., A review of a posteriori error estimation and adaptive mesh-refinement techniques. Wiley and Sons, Teubner, New York 1996.

[Ve2] Verfürth, R., *A posteriori error estimates for nonlinear problems. Finite element discretisations of elliptic equations,* Math. Comput. 62(1994) pp. 445-475.