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Heinz König

Dedicated to the Memory of GÜNTER LUMER

Abstract. The present article aims to describe the main ideas and developments in the theory of measure and integral in the course and at the end of the first century of its existence.

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The theory of measure and integral created by BOREL and LEBESGUE around 1900 was the *concrete* theory of the Lebesgue measure on $\mathbb{R}$. Its decisive feature was a small collection of entirely new and powerful theorems: the theorems of type Beppo Levi-Fatou-Lebesgue, Fubini-Tonelli, ... The theory soon became a kind of foundation of mathematical analysis.

Also this theory soon turned into an *abstract* one. This is the usual fate of mathematical theories, but in the case of measure and integral a powerful impetus came from the fact that the whole of mathematical analysis, like the whole of mathematics, went through a continuous chain of dramatic abstractions throughout the 20th century: Each new step of abstraction required its specific class of measures, in order that those powerful theorems could be put into action. Examples of first rank were the *locally compact topological groups* (HAAR 1933, VON NEUMANN 1934/36, ANDRÉ WEIL 1940) and the *mathematical theory of probability* (WIENER 1923, KOLMOGOROV 1933, DOOB 1953).

It so happened that for measure and integral the process of abstraction involved a *particular twofold task*: It is, in order to develop the theory for some abstract field, the task to discover on the one hand the appropriate *concepts* and *classes* of measures, and on the other hand the appropriate *procedures* which produce these measures from basic data of preconceived nature. As a rule these are hard problems, but decisive for the success of the respective enterprise.

In the course of the 20th century thus two comprehensive abstract theories of measure and integral came into existence: the *traditional abstract theory*, as presented for example in the famous 1950 textbook of Halmos [5], and the theory of *Radon measures* on Hausdorff topological spaces, developed in particular in the 1952-69 treatise of Bourbaki [2]. For all their power and splendour, both theories came to show some *essential weaknesses* with respect to the above particular tasks. We shall attempt to describe these weaknesses in sections 1 and 2.

The time of relief then came at the end of the 20th century. The second part of this article will describe the systematization due to the present author, based on ideas which date back to 1968-70. Another development of a different nature is the monumental treatise 2000-2003 of Fremlin [4]. Its basic aim is the comprehensive presentation of measure and integral in both the abstract and topological theories, rather than their unification under new concepts like the present *premeasures*. In the fundamentals there are substantial overlaps in facts and spirit, in particular in the emphasis on *inner regular* and *nonsequential* procedures. But there are also certain differences, as it will be sketched at the end of section 4.

1. The Two Abstract Theories of the 20th Century

The Traditional Abstract Theory. The basic notion is that of a *measure* $\alpha : \mathfrak{A} \to [0, \infty]$, understood to be defined on a σ algebra $\mathfrak{A}$ of subsets of a nonvoid set X. The fundamental weakness of the theory is its total limitation to *sequential procedures* and its neglect of *regularity*: In the main parts of the textbooks, devoted to the abstract situation, there is the ubiquitous *sequential* (=: σ) upward/downward continuity (as a rule even defined in the disguise of *countable additivity* - unfortunate because it ties continuity to additivity), but there is no *nonsequential* (=: τ) upward/downward continuity (defined via directed set systems) and no outer/inner *regularity*. Examples of immediate consequences of this impoverishment are the lack of uniqueness results, for instance for finite products of measures and for Daniell-Stone representation, and the smallness of the domains of certain fundamental constructions, for instance for finite and infinite products.

Then in the back of the textbooks there are specific chapters where X is assumed to be a Hausdorff topological space, with its usual set systems $\mathrm{Op}(X)$ and $\mathrm{Cl}(X) \supset \mathrm{Comp}(X)$ and its Borel σ algebra $\mathrm{Bor}(X)$. Here one finds, for the Borel measures $\alpha : \mathrm{Bor}(X) \to [0, \infty]$ and related ones, the concepts which were absent so far, but as before attached to specific set systems, in most cases to $\mathrm{Op}(X)$ and $\mathrm{Comp}(X)$, as it had been in the concrete case of the Borel-Lebesgue measure $\lambda : \mathrm{Bor}(\mathbb{R}) \to [0, \infty]$:

λ outer regular $\mathrm{Op}(\mathbb{R})$ $\quad$ λ inner regular $\mathrm{Comp}(\mathbb{R})$,
$\lambda|\mathrm{Op}(\mathbb{R})$ upward τ continuous $\quad$ $\lambda|\mathrm{Comp}(\mathbb{R})$ downward τ continuous.

In the course of time it became clear that *inner* regularity is much more important than outer, due to the predominant rôle of compactness in topology. This was the point of departure for the opponent *Radon measure* theory.

THE RADON MEASURE THEORY. Here one assumes X to be a Hausdorff topological space. A measure $\alpha : \mathfrak{A} \rightarrow [0, \infty]$ on X is called *Radon* iff $\mathfrak{A} \supset \mathrm{Comp}(X)$ such that $\alpha|\mathrm{Comp}(X) < \infty$ (in part of the literature strengthened to local finiteness) and such that α is *inner regular* $\mathrm{Comp}(X)$. One deduces from these properties that $\alpha|\mathrm{Comp}(X)$ is downward τ continuous. A simple extension procedure permits to assume that $\mathfrak{A} \supset \mathrm{Bor}(X)$. One then proves that $\alpha|\mathrm{Op}(X)$ is upward τ continuous. But as a rule α is not outer regular $\mathrm{Op}(X)$.

The most common particular cases are: 1) λ is a Radon measure. 2) When X is compact, not all finite Borel measures must be Radon. 3) When X is Polish, all locally finite Borel measures are Radon.

The present definition of Radon measures is not the one in BOURBAKI [2], but the two definitions are equivalent (up to local finiteness). The explanation is the belief of Bourbaki that the theory of measure and integral must be based on *integrals* and not on *set functions*. But of course *set functions are the bones* in the body of measure and integral, and hence an essential part of the basic labour is predestined to produce the fundamental set functions from whatever had been declared to be the basic entities. In his treatise Bourbaki was able to develop his conception in the frame of *locally compact* spaces X: we call this the *initial version* of the Radon measure theory. But in his last chapter, where X is an arbitrary Hausdorff space, Bourbaki seemed to have made peace with the basic rôle of set functions. The final end of the initial conception then came with the 1973 treatise of LAURENT SCHWARTZ [20] - which does not mean that all authors of textbooks have realized this fact.

We list a few achievements of the Radon measure theory: 1) Existence and uniqueness of finite products. 2) Existence and uniqueness in the Riesz representation theorem. 3) The notion of *support* for Radon measures. 4) The existence of (countable or uncountable) decompositions of measure spaces based on compact subsets, the so-called *concassages*.

It appears that success in these points results from inner regularity *under strict attachment to topological compactness* - like the resultant downward τ continuity emphasized above. However, for the Radon measure theory the strict attachment to topological compactness can be a severe obstacle. An important instance is the area of infinite products. We shall see that here neither theory is satisfactory.

INFINITE PRODUCTS AND PROJECTIVE LIMITS. Let T be an infinite index set and $(Y_t)_{t \in T}$ be a family of nonvoid sets with product set $X = \prod_{t \in T} Y_t$.

The *traditional abstract* theory assumes a family of probability (=:prob) measure spaces $(Y_t, \mathfrak{B}_t, \beta_t)_{t \in T}$ (defined to mean that $\beta_t(Y_t) = 1$). In X one forms the product σ algebra $\mathfrak{A}$, defined to be generated by the product sets

$$A = \prod_{t\in T} B_t \text{ with } B_t \in \mathfrak{B}_t \text{ and } B_t = Y_t \text{ for almost all } t \in T,$$

where as usual *almost all* means *all except finitely many.* Then there exists a unique *product measure* $\alpha : \mathfrak{A} \to [0,\infty]$, in the sense that

$$\alpha(A) = \prod_{t\in T} \beta_t(B_t) \text{ for the above } A = \prod_{t\in T} B_t.$$

However, in the case of an uncountable T this result has the basic defect that its domain $\mathfrak{A}$ tends to be much *too small*: its members $A \in \mathfrak{A}$ are all *countably determined* in the intuitive sense. Thus let for example $T = [0,\infty[$ and $Y_t = \mathbb{R}$ for $t \in T$, so that the members of $X = \mathbb{R}^T$ are the *paths* $x : T = [0,\infty[\to \mathbb{R}$. Then the subset $\mathrm{C}(T,\mathbb{R}) \subset X$ of the *continuous* paths is not countably determined, and even worse, any countably determined $A \subset \mathrm{C}(T,\mathbb{R})$ must be $A = \varnothing$.

On the other side, the *Radon measure* theory starts from a family of Hausdorff topological spaces $(Y_t)_{t\in T}$, with the product topology on X. One assumes $\mathfrak{B}_t = \mathrm{Bor}(Y_t)$ and Borel-Radon prob measures β_t for $t \in T$. Then the previous $\mathfrak{A}$ satisfies $\mathfrak{A} \subset \mathrm{Bor}(X)$, and hence the desired result would be that the previous product measure $\alpha : \mathfrak{A} \to [0,\infty[$ has an extension to a Radon measure $\beta : \mathrm{Bor}(X) \to [0,\infty[$. However, this is far from true: It is obvious that such an extension does not exist when T is uncountable and $\beta_t|\mathrm{Comp}(Y_t) < 1$ for all $t \in T$, because then $\alpha^\star(K) := \inf\{\alpha(A) : A \in \mathfrak{A} \text{ with } A \supset K\} = 0$ for all compact $K \subset X$: the compact subsets of X are *too small.*

After this we turn to the context of *projective limits.* Let $I = I(T)$ consist of the nonvoid finite subsets $p, q, \cdots$ of T. For $p \in I$ we form the product set $Y_p = \prod_{t\in p} Y_t$ and the canonical projection $H_p : X \to Y_p$, and for $p \subset q$ in I the canonical projection $H_{pq} : Y_q \to Y_p$. For the sequel we assume the *traditional abstract* situation: From $(\mathfrak{B}_t)_{t\in T}$ as above we form the family $(\mathfrak{B}_p)_{p\in I}$ of the product σ algebras $\mathfrak{B}_p$ in Y_p, and then from $(\beta_t)_{t\in T}$ as above the (unique) family $(\beta_p)_{p\in I}$ of the product measures β_p on $\mathfrak{B}_p$. Then on the one hand the family $(\beta_p)_{p\in I}$ is *consistent* in the sense that

$$(\leftarrow) \quad \beta_p(B) = \beta_q(H_{pq}^{-1}(B)) \;\forall\, B \in \mathfrak{B}_p \quad \text{for all } p \subset q \text{ in } I,$$

and on the other hand the above characterization of the product measure $\alpha : \mathfrak{A} \to [0,\infty[$ of $(\beta_t)_{t\in T}$ can be written

$$(\Leftarrow) \quad \beta_p(B) = \alpha(H_p^{-1}(B)) \;\forall B \in \mathfrak{B}_p \quad \text{for all } p \in I,$$

that is in terms of the family $(\beta_p)_{p\in I}$. All this evokes a natural *variant* of the previous product formation: From a prescribed family of prob measures $(\beta_p)_{p\in I}$, assumed to be consistent $(\leftarrow)$, one is asked to produce a prob measure $\alpha : \mathfrak{A} \to [0,\infty[$ which satisfies $(\Leftarrow)$. Then α is unique, and is called the *projective limit* of the family $(\beta_p)_{p\in I}$.

It is clear first of all from $(\leftarrow)(\Leftarrow)$ that each prob measure $\alpha : \mathfrak{A} \to [0,\infty[$ is the projective limit of a unique consistent family $(\beta_p)_{p\in I}$, much in contrast to the previous product formation which furnishes but a small portion of these α. But of course the essential point is to determine those consistent families $(\beta_p)_{p\in I}$ which produce prob measures $\alpha : \mathfrak{A} \to [0,\infty[$, that is, those which via $(\Leftarrow)$ come from such prob measures α. Let us call them *solvable*. It is known that not all consistent families $(\beta_p)_{p\in I}$ are solvable; it seems that some kind of *compactness* is involved. For the moment we quote the famous positive result due to KOLMOGOROV [7]: *If* Y_t *is a Polish topological space and* $\mathfrak{B}_t = \mathrm{Bor}(Y_t)$ $\forall$ $t \in T$ *then all consistent families* $(\beta_p)_{p\in I}$ *are solvable.* The fundamental fact behind this result is that in a Polish space all finite (and all locally finite) Borel measures are Radon measures. A much more comprehensive result will be presented at the end of this article; it will at the same time be able to overcome the barrier of those domains $\mathfrak{A}$ in X which consist of countably determined subsets alone.

The present context of projective limits is the basis of the traditional theory of *stochastic processes*. Here one assumes that $Y_t = Y$ and $\mathfrak{B}_t = \mathfrak{B}$ independent of $t \in T$. A *stochastic process for* T *and* $(Y,\mathfrak{B})$ can be defined to be a prob measure $\alpha : \mathfrak{A} \to [0,\infty[$ as above, so that it is equivalent to be a solvable consistent family $(\beta_p)_{p\in I}$ (the usual definition looks quite different, it is in the guise of so-called *versions* of α). After this definition the members $A \in \mathfrak{A}$ are those sets of paths $x : T \to Y$ in the path space $X = Y^T$ which the stochastic process α is *able to measure*. Thus the fact that all $A \in \mathfrak{A}$ are countably determined can lead to misfortune in the case that T is uncountable. A specific problem is caused by those subsets of the path space which support the *essential features* of a stochastic process α and could be named the *essential* sets for α; they can a priori be far from obvious. The most prominent example is the stochastic process of *Brownian motion* = the *Wiener measure* α, with $T = [0,\infty[$ and (in one dimension) $Y = \mathbb{R}$. Here the prime candidate for an essential set is $\mathrm{C}(T,\mathbb{R}) \subset X$. But it must be recalled that the idea for this candidate came from *experimental observations without participation of mathematics*:*"all Brownian paths are continuous"*. In contrast, on the mathematical side the set $\mathrm{C}(T,\mathbb{R}) \subset X$ is not in $\mathfrak{A}$ and hence beyond immediate consideration. In its more than 50 years the traditional theory of stochastic processes has not been able to produce an adequate concept of essential sets.

INSERTION: SET-THEORETICAL COMPACTNESS. We recall the set-theoretical notions of compactness initiated in MARCZEWSKI [17]. These notions are weaker and more flexible than topological compactness, and in our new development all aspects of compactness will be based on them. Let X be a nonvoid set. A nonvoid set system $\mathfrak{S}$ in X is called σ/τ *compact* iff each nonvoid *countable/arbitrary* subsystem of $\mathfrak{S}$ with intersection $\varnothing$ contains a nonvoid *finite* subsystem with intersection $\varnothing$.

We list some immediate properties: 1) If X is a Hausdorff topological space then $\mathrm{Comp}(X)$ is τ compact. 2) If $\mathfrak{S}$ is σ/τ compact then $\mathfrak{S} \cup \{X\}$ is σ/τ compact as well. 3) If X is a *non-compact* Hausdorff space then the τ compact set system $\mathrm{Comp}(X) \cup \{X\}$ does not come via 1) from any Hausdorff topology on X. The difference thus expressed will turn out to be a decisive one.

We use the occasion to introduce some further notations. For a nonvoid set system $\mathfrak{M}$ in X we define $\mathfrak{M}^\star \subset \mathfrak{M}^\sigma \subset \mathfrak{M}^\tau$ to consist of the *unions* of the nonvoid *finite/countable/arbitrary* subsystems of $\mathfrak{M}$, and define $\mathfrak{M}_\star \subset \mathfrak{M}_\sigma \subset \mathfrak{M}_\tau$ to consist of the respective *intersections*. Likewise for a nonvoid function system $M \subset \overline{\mathbb{R}}^X$ we define $M^\star \subset M^\sigma \subset M^\tau$ to consist of the *pointwise suprema* of the nonvoid *finite/countable/arbitrary* subsystems of M, and define $M_\star \subset M_\sigma \subset M_\tau$ to consist of the respective *infima*.

In conclusion we want to introduce the *shorthand notation* $\bullet = \star\sigma\tau$, to mean that $\bullet$ can in a fixed context be read as one and the same of the symbols $\star/\sigma/\tau$, or of the words *finite/countable/arbitrary*, like *variables* are in common use all over mathematics.

2. The Generation of Measures in the two Previous Theories

The Traditional Abstract Theory: Carathéodory 1914. In the traditional abstract theory the method of Carathéodory [3] is the most fundamental source of nontrivial measures. Let X be a nonvoid set. The basic idea is to form for a set function $\Theta : \mathfrak{P}(X) \to [0,\infty]$ with $\Theta(\varnothing) = 0$ the so-called Carathéodory class

$$\mathfrak{C}(\Theta) := \{A \subset X : \Theta(M) = \Theta(M \cap A) + \Theta(M \cap A') \; \forall\, M \subset X\} \subset \mathfrak{P}(X),$$

with $A' := X \setminus A$, the members of which are called *measurable* Θ. One proves that $\Theta|\mathfrak{C}(\Theta)$ is a content on an algebra. Beyond $\Theta(\varnothing) = 0$ the class $\mathfrak{C}(\Theta) \subset \mathfrak{P}(X)$ can be defined after [9] section 4, but we shall not need the explicit definition.

Moreover one defines for a set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a set system $\mathfrak{S}$ in X with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ the so-called *outer envelope* $\varphi^\circ : \mathfrak{P}(X) \to [0,\infty]$ to be

$$\varphi^\circ(A) = \inf\{\sum_{l=1}^{\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } A \subset \bigcup_{l=1}^{\infty} S_l\},$$

which is a familiar construction since Borel and Lebesgue. These two concepts then furnish the Carathéodory theorem: *Assume that $\mathfrak{S}$ is a ring. If φ is a content and upward σ continuous, then $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ is a measure and an extension of φ.* Thus a set function on a ring can be extended to a measure iff it is a content and upward σ continuous.

For all its power the above theorem has experienced quite some criticism. In the traditional frame the attacks are directed at the construction $\mathfrak{C}(\cdot)$, as an unmotivated and artificial one, while as a rule no doubt falls upon the definition of the outer envelope. However, we shall see that the opposite is true: There are in fact quite some weaknesses around the theorem, but it is the particular form of $\varphi \mapsto \varphi^\circ$ which must be blamed for them, whereas the construction $\mathfrak{C}(\cdot)$ remains the decisive methodical idea and even improves when put into the appropriate context.

We formulate the main deficiencies of the Carathéodory theorem as follows.

1) The construct φ° and hence the measure $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ are *outer regular* $\mathfrak{S}^\sigma$ by the very definition. It is mysterious how an *inner regular* counterpart could look.

2) The *sequential character* of the construct φ° is the reason that *sequential continuity* carries over from φ to $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$. It is mysterious how a *nonsequential* counterpart could look. Both times the *sum* in the definition of φ° appears to be a crucial obstacle.

3) The domains $\mathfrak{S}$ of basic data $\varphi : \mathfrak{S} \to [0,\infty]$ are as a rule not *rings* (and not even so-called *semirings*), but at most *lattices*. This becomes even more obvious when regularity and nonsequential continuity come on the scene. But the proof of the Carathéodory theorem suffers a total breakdown when one attempts to pass from rings to lattices $\mathfrak{S}$.

With respect to 3) the present author produced a certain relief in an analysis course 1969/70: Instead of φ° he defined for an *isotone* set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a set system $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ the set function $\varphi^\sigma : \mathfrak{P}(X) \to [0,\infty]$ to be

$$\varphi^\sigma(A) = \inf\{ \lim_{l\to\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ isotone with } A \subset \bigcup_{l=1}^{\infty} S_l\}.$$

It is obvious that $\varphi^\sigma = \varphi^\circ$ when φ is a content on a ring $\mathfrak{S}$, so that the Carathéodory theorem persists when formulated with φ^σ instead of φ°. But for φ^σ the same proof furnishes a much more comprehensive theorem: Let us define a set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ to be a *content* iff it is isotone with $\varphi(\varnothing) = 0$ and *modular* in the sense that

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \quad \text{for all } A, B \in \mathfrak{S},$$

which is the previous notion when $\mathfrak{S}$ is a ring. Then for φ^σ the Carathéodory theorem carries over from rings to the class of lattices $\mathfrak{S}$ with the condition

$$B \setminus A \in \mathfrak{S}^\sigma \quad \text{for all pairs } A \subset B \text{ in } \mathfrak{S}.$$

Note for example that this condition is fulfilled for the lattices $\mathrm{Cl}(X)$ and $\mathrm{Comp}(X)$ in a metric space X! Yet the author could not perceive any trace of the extended theorem in the traditional abstract theory.

Much later then he realized that the construct φ^σ is superior to φ° with respect to the other deficiencies 1) and 2) as well. This will be one of the two decisive points in the present enterprise. The author dares say that the world of measure and integral in the 20th century would have been another one if Carathéodory in 1914 had implemented his ideas with φ^σ instead of φ°!

Both Theories: Positive Linear Functionals. In the *traditional abstract* theory another fundamental source of nontrivial measures is the area of positive linear functionals via the Daniell-Stone theorem. Later the *initial version of the Radon measure* theory adopted and adapted the idea. The common set-up is as follows: On the nonvoid set X one assumes a vector space of real-valued functions $F \subset \mathbb{R}^X$ which is a lattice under the pointwise max and min operations $\vee\wedge$ and *Stonean*, defined to mean that $f \in F \Rightarrow f \wedge t \in F$ for $0 < t < \infty$. One considers the linear functionals $J : F \to \mathbb{R}$ which are *isotone* (=: *positive*). The *traditional* *abstract* theory assumes J to be σ *continuous*, defined to mean that pointwise convergence $f_n \downarrow 0$ implies that $J(f_n) \downarrow 0$, or that each *countable* nonvoid $M \subset F$ which is downward directed with pointwise infimum 0, in symbols $M \downarrow 0$, satisfies $\inf\limits_{f \in M} J(f) = 0$. The *initial Radon measure* theory assumes X to be a *locally compact* Hausdorff topological space and $F = \mathrm{CK}(X, \mathbb{R})$ to consist of the continuous real-valued functions with compact support. The *Dini* theorem then implies that the $J : F \to \mathbb{R}$ are τ *continuous*, defined to mean that an *arbitrary* nonvoid $M \subset F$ such that $M \downarrow 0$ in the above sense satisfies $\inf\limits_{f \in M} J(f) = 0$. Bourbaki in fact *defines* these J to be the *Radon measures* on X. We want to combine the two cases, and thus assume the functional J to be $\bullet$ *continuous* for some $\bullet = \sigma\tau$. The procedure then runs as described below: the fundamental point is that it is of *outer regular* character like the previous Carathéodory 1914 procedure.

One defines the *outer* $\bullet$ *envelope* $J^\bullet : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ to be

$$J^\bullet(f) = \inf\{\sup_{u \in M} J(u) : M \subset F \text{ nonvoid } \bullet \text{ with } M \uparrow\geqq f\},$$

where $\bullet$ restricts the cardinality of M, and $M \uparrow\geqq f$ means that M is upward directed with $\sup\limits_{u \in M} u \geqq f$; in case $\bullet = \star$ the formula produces the *crude outer envelope* $J^\star : J^\star(f) = \inf\{J(u) : u \in F \text{ with } u \geqq f\}$. The natural counterpart of $J^\bullet$ is the *inner* $\bullet$ *envelope* $J_\bullet : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$, defined to be $J_\bullet(f) = -J^\bullet(-f)$ or

$$J_\bullet(f) = \sup\{\inf_{u \in M} J(u) : M \subset F \text{ nonvoid } \bullet \text{ with } M \downarrow\leqq f\},$$

with the obvious $M \downarrow\leqq f$. One notes that $J^\bullet \geqq J_\bullet$ and $J^\bullet|F = J_\bullet|F = J$.

After this one defines the function $f \in \overline{\mathbb{R}}^X$ to be $\bullet$ *integrable* J iff it fulfils

$$(\circ)\ \inf_{u \in F} J^\bullet(|f - u|) = 0, \text{ which is equivalent to } J^\bullet(f) = J_\bullet(f) \in \mathbb{R}.$$

Then one passes from J to a *set function*: One forms the set system $\mathfrak{a}^\bullet := \{A \subset X : \chi_A \text{ is } \bullet \text{ integrable } J\}$, which turns out to be a lattice, and its *transporter* $\mathfrak{A}^\bullet := \{A \subset X : A \cap M \in \mathfrak{a}^\bullet\ \forall M \in \mathfrak{a}^\bullet\} \supset \mathfrak{a}^\bullet$, and on $\mathfrak{A}^\bullet$ one defines $\beta^\bullet(A) = J^\bullet(\chi_A)$. In both cases $\bullet = \sigma\tau$ one obtains the result: *The set function* $\beta^\bullet : \mathfrak{A}^\bullet \to [0, \infty]$ *is a measure. A function* $f \in \overline{\mathbb{R}}^X$ *is* $\bullet$ *integrable* J *iff it is integrable* $\beta^\bullet$*, and then* $J^\bullet(f) = \int f d\beta^\bullet$*. In particular the functions* $f \in F$ *are integrable* $\beta^\bullet$*, and fulfil* $J(f) = \int f d\beta^\bullet$. Moreover one proves that the set system

$$\mathfrak{N} := \{[f > t] : f \in F \text{ and } 0 < t < \infty\}$$

satisfies $\mathfrak{N}^\bullet \subset \mathfrak{A}^\bullet$, and that $J^\bullet(\chi_\cdot)$ and hence $\beta^\bullet$ are *outer regular* $\mathfrak{N}^\bullet$.

In the *traditional abstract* theory one has $\bullet = \sigma$. Thus the measure $\beta^\sigma : \mathfrak{A}^\sigma \to [0, \infty]$ furnishes the usual *Daniell-Stone* theorem and is *outer regular* $\mathfrak{N}^\sigma$. As a rule the traditional abstract theory is content with this result, as it is content with the result from the Carathéodory method, even though both results produce *outer regular* and not *inner regular* measures. Both times the ideas of the traditional theory do not suffice to provide the construction of an *inner regular* measure with respect to an appropriate set system. In particular it is not clear whether and how in place of J^σ the *inner* envelope J_σ could be used: note that in ($\circ$) one cannot simply replace the *subadditive* J^σ with the *superadditive* J_σ!

In the *initial Radon measure* theory $J : F = \mathrm{CK}(X, \mathbb{R}) \to \mathbb{R}$ one has $\bullet = \tau$ and $\mathfrak{N}^\tau = \mathrm{Op}(X)$, so that the measure β^τ is *outer regular* $\mathrm{Op}(X) \subset \mathrm{Bor}(X) \subset \mathfrak{A}^\tau$. Thus as a rule β^τ is *not Radon*! There are several textbooks which are content with this result as it is, and thus formulate the *Riesz representation* theorem with β^τ in place of a true Radon measure. Not so Bourbaki: *Faute de mieux* one continued to utilize the *outer* envelope J^τ as the basic construction, but then went on to put a *second* construction on top of it, named the *essential* one: One passed from J^τ to its so-called *essential upper integral* $J^\tau_\circ : [0, \infty]^X \to [0, \infty]$, defined to be

$$J^\tau_\circ(f) = \sup\{J^\tau(f\chi_K) : K \in \mathrm{Comp}(X)\}.$$

The new expression looks somewhat more sensible when one notes that

$$J^\tau_\circ(f) = \sup\{J^\tau(u) : 0 \leqq u \leqq f \text{ with } J^\tau(u) < \infty\}.$$

Thus $J^\tau_\circ(f) = J^\tau(f)$ when $J^\tau(f) < \infty$, and in particular $J^\tau_\circ(f) = J(f)$ for $f \geqq 0$ in F. The construct $J^\tau_\circ$ happens to work in place of J^τ: Since it remains subadditive one can in fact follow the former procedure based on ($\circ$). One comes back to the former $\mathfrak{A}^\tau$, and on $\mathfrak{A}^\tau$ one defines this time $\alpha^\tau(A) = J^\tau_\circ(\chi_A)$, to obtain the result which follows: *The set function $\alpha^\tau : \mathfrak{A}^\tau \to [0, \infty]$ is a Radon measure. A function $f \in \overline{\mathbb{R}}^X$ is τ integrable with respect to $J^\tau_\circ$* (=: essentially τ integrable J) *iff it is integrable α^τ, and in case $f \geqq 0$ then $J^\tau_\circ(f) = \int f d\alpha^\tau$. In particular the functions $f \in F$ are integrable α^τ, and fulfil $J(f) = \int f d\alpha^\tau$.* It follows that the map $J \mapsto \alpha^\tau$ is one-to-one to all Borel-Radon measures $\mathrm{Bor}(X) \to [0, \infty]$. This is the true *Riesz representation* theorem: it identifies the Radon measures in the sense of Bourbaki with the *true Borel-Radon measures*.

All that is restricted to *locally compact* spaces X. In the final chapter of Bourbaki [2] the development continues with the definition and construction of the resultant measures on *arbitrary* Hausdorff spaces X and their identification with the *true Borel-Radon measures*.

Summary. The overall picture at the end of the 20th century shows that the foundations of measure and integral are in inconsistent conditions: One knows from both old concrete facts and the Radon measure theory that *regularity*, above all *inner regularity*, and *nonsequential continuity* are fundamental and indispensable concepts and tools. Yet as we have said the textbooks in the unspoilt traditional abstract field pass over these concepts in complete silence. However, in an unbelievable contrast, the two central methods which serve to produce measures from basic data are such that the resultant measures are all equipped with a natural *outer regular* structure. Thus *regularity exists in silent omnipresence* - in the form of *outer* regularity.

But that is about all in these foundational matters: In the traditional abstract theory the method of Carathéodory 1914 shows no hint at all how to produce inner regularity nor nonsequential continuity. And in the representation theory for positive linear functionals it is, in order to produce inner regular outcomes, far from appropriate to stick to the weapons of the outer arsenal - and this must in fact at once be paid for with the appearance of that unfortunate *essential construction*, which degrades inner regularity to a subordinate one. We shall come back to this point in our final section. The subsequent Radon measure theories on arbitrary Hausdorff topological spaces X in Bourbaki 1969 and Schwartz 1973 improved the access to inner regular set functions; but after all, the exposition of Bourbaki is based on the initial version of the theory, and Schwartz insisted that a Radon measure be tied to an outer regular companion. Above all, the development remained restricted to the topological context. In the *abstract* context there were a few lines of research with an emphasis on inner regularity, in particular the somewhat isolated area around the *compact* and *perfect* measures, for example in [4] sections 342 and 451. But on the whole the fundamental relevance of inner regularity and nonsequential continuity had been left without adequate structure.

3. The Origin of the New Systematization

The resolution started with two *natural ideas*: The first idea is to consider, in a Hausdorff topological space X, those set functions $\varphi : \mathrm{Comp}(X) \to [0,\infty[$ that can be extended to Radon measures, and to *characterize* these set functions. Of course one can assume that φ is isotone with $\varphi(\varnothing) = 0$. Then the second idea is to *extend* this characterization to the *abstract situation* of a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in a nonvoid set X, that is to characterize those isotone set functions $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(\varnothing) = 0$ that can be extended to measures which are *inner regular* $\mathfrak{S}$. It is immediate that the respective extensions $\alpha : \mathfrak{A} \to [0,\infty]$ are *unique*: If one defines the *crude inner envelope* $\varphi_\star : \mathfrak{P}(X) \to [0,\infty]$ of φ to be

$$\varphi_\star(A) = \sup\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \subset A\},$$

then each such $\alpha : \mathfrak{A} \to [0,\infty]$ must be $\alpha = \varphi_\star|\mathfrak{A}$.

As for the first idea, we note three theorems in the literature which characterize those isotone set functions $\varphi : \mathrm{Comp}(X) \to [0,\infty[$ with $\varphi(\varnothing) = 0$ that can be extended to Radon measures, henceforth called the *Radon premeasures.*

Choquet 1953: *φ is a* locally bounded Radon premeasure *iff it is modular and* continuous from above : *for any $A \in \mathrm{Comp}(X)$ and $\varepsilon > 0$ there exists an open $U \supset A$ such that all compact $K \subset U$ fulfil $\varphi(K) < \varphi(A) + \varepsilon$.* Note that the last condition implies that φ is *downward τ continuous.*

Bourbaki 1969: *Assume that φ is* locally bounded (which in fact can be dispensed with)*. Then φ is a* Radon premeasure *iff it is modular and downward τ continuous.*

Kisyński 1968: *φ is a* Radon premeasure *iff*

$$\varphi(B) = \varphi(A) + \varphi_\star(B \setminus A) \quad \textit{for all } A \subset B \textit{ in } \mathrm{Comp}(X).$$

As for the second idea, the three theorems are quite different. The Choquet condition, where besides $\mathrm{Comp}(X)$ also $\mathrm{Op}(X)$ comes in, is so close to the topological context that the natural attempts at extension lead back to that context. The Bourbaki condition breaks down even for certain bounded φ on certain lattices $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ which fulfil $\mathfrak{S} = \mathfrak{S}_\tau$ and are τ compact, like the former $\mathfrak{S} = \mathrm{Comp}(X)$ (for an example see [10] remark 3.3). The miraculous event is the Kisyński [6] theorem: It was not recorded in Bourbaki 1969 and Schwartz 1973. But in no time Topsøe [21][22] realized that this theorem is capable of an abstract extension. His basic achievement is for both $\bullet = \sigma\tau$.

Topsøe 1970: *Let* $\varphi : \mathfrak{S} \to [0, \infty[$ *on the lattice* $\mathfrak{S}$ *with* $\varnothing \in \mathfrak{S}$ *be isotone with* $\varphi(\varnothing) = 0$. *Consider the properties*

1) φ *can be extended to a measure* α *on a* σ *algebra* $\mathfrak{A} \supset \mathfrak{S}$ *which is* inner regular $\mathfrak{S}$*, and has* $\alpha|\mathfrak{S} = \varphi$ *downward* $\bullet$ *continuous.*
2) $\varphi(B) = \varphi(A) + \varphi_\star(B \setminus A)$ *for all* $A \subset B$ *in* $\mathfrak{S}$*, and* φ *is downward* $\bullet$ *continuous at* $\varnothing$.

Then 1)$\Rightarrow$2) *is obvious, and* 2)$\Rightarrow$1) *holds true when* $\mathfrak{S} = \mathfrak{S}_\bullet$.

However, at this point a serious problem comes up: Without $\mathfrak{S} = \mathfrak{S}_\bullet$ the implication 2)$\Rightarrow$1) becomes false. But true is the implication 2)$\Rightarrow$1$\bullet$), where

1$\bullet$) φ *can be extended to a measure* α *on a* σ *algebra* $\mathfrak{A} \supset \mathfrak{S}_\bullet$ *which is inner regular* $\mathfrak{S}_\bullet$*, and has* $\alpha|\mathfrak{S}_\bullet$ *downward* $\bullet$ *continuous.*

However, this time without $\mathfrak{S} = \mathfrak{S}_\bullet$ the converse 1$\bullet$)$\Rightarrow$2) becomes false. Thus it appears that beyond $\mathfrak{S} = \mathfrak{S}_\bullet$ the formulation of 2) in terms of the *crude* inner envelope $\varphi_\star$ of φ ceases to be adequate and prevents an equivalence assertion. As a further evidence we invoke the *outer* situation of Carathéodory 1914: this situation *did not use the obvious crude outer counterpart* $\varphi^\star$ *of* $\varphi_\star$, but the more subtle φ° or the later φ^σ.

All this underlines that *new envelopes* are required as the fundamental tools for systematization - while the basic ideas of Kisyński and Topsøe must remain in force. These new envelopes and the subsequent systematization are the contribution of the present author around the end of the 20th century.

4. The New Systematization

The new systematization is structured in order to cope with the two particular tasks formulated in the introduction. Its first and central aim is to produce certain *distinguished classes of measures* from certain particular classes of basic data, in the spirit that can be expected from what has been said so far. The foundational part consists of an *inner* and an *outer* theory, which are parallel in almost all essentials and have been developed in parallel at the outset [9]. But it soon became clear that the *inner* version is the superior one in the most decisive points. Therefore in the present article the explicit description will be restricted to the inner theory. The development will be almost uniform in the three columns $\bullet = \star\sigma\tau$, thanks to an appropriate formulation of the basic notions.

In the sequel we assume that $\mathfrak{S}$ is a lattice in a nonvoid set X with $\varnothing \in \mathfrak{S}$ and that $\varphi : \mathfrak{S} \to [0, \infty[$ is an isotone set function with $\varphi(\varnothing) = 0$. The basic definition is as follows: We define φ to be an *inner* $\bullet$ *premeasure* iff it can be extended to a content $\alpha : \mathfrak{A} \to [0, \infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}_\bullet$ which is *inner regular* $\mathfrak{S}_\bullet$, and has $\alpha|\mathfrak{S}_\bullet$ *downward* $\bullet$ *continuous* (which is void for $\bullet = \star$). We call these set functions α the *inner* $\bullet$ *extensions* of φ.

The subsequent *inner extension theorem* characterizes those φ which are inner $\bullet$ premeasures, and then *describes* all inner $\bullet$ extensions of φ. The decisive weapons are the new *inner* $\bullet$ *envelopes* $\varphi_\bullet : \mathfrak{P}(X) \to [0, \infty]$ announced above and defined to be

$$\varphi_\bullet(A) = \sup\{ \inf_{M \in \mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid} \bullet \text{with } \mathfrak{M} \downarrow\subset A\},$$

where $\mathfrak{M} \downarrow\subset A$ means that $\mathfrak{M}$ is downward directed with $\bigcap\limits_{M \in \mathfrak{M}} M \subset A$. It follows that $\varphi_\bullet$ is *inner regular* $\mathfrak{S}_\bullet$. For $A \in \mathfrak{S}$ we have $\varphi(A) \leqq \varphi_\bullet(A)$, and $\varphi(A) = \varphi_\bullet(A)$ iff φ is downward $\bullet$ continuous at A. Furthermore $\varphi_\star \leqq \varphi_\sigma \leqq \varphi_\tau$, and $\varphi_\star$ is the previous *crude* inner envelope, while φ_σ can be defined via *sequences* like the previous *outer* counterpart. We also need the *satellites* $\varphi_\bullet^B : \mathfrak{P}(X) \to [0, \infty]$ with $B \subset X$, defined by restricting the above $\mathfrak{M} \subset \mathfrak{S}$ to those which consist of subsets $M \subset B$.

Inner Extension Theorem ($\bullet = \star\sigma\tau$): *Let $\varphi : \mathfrak{S} \to [0, \infty[$ be isotone with $\varphi(\varnothing) = 0$. Then the following are equivalent.*

0) *φ is an inner $\bullet$ premeasure.*
1) *φ is supermodular and downward $\bullet$ continuous, and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*
1') *$\varphi(B) = \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*
2) *φ is supermodular and downward $\bullet$ continuous at $\varnothing$, and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*
2') *$\varphi(B) = \varphi(A) + \varphi_\bullet^B(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*
3) *The set function $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extension of φ.*

In this case $\Phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an inner $\bullet$ extension of φ; it is a complete *content, and a* measure *when $\bullet = \sigma\tau$. All inner $\bullet$ extensions of φ are restrictions of Φ.*

The present inner extension theorem is a perfect confirmation for the desired rôle of the new $\bullet$ envelopes. Its conditions 1')2') are of course more handsome than 1)2), but as a rule conditions 1)2) will be easier to establish. The prominent rôle of $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ as the *unique maximal* inner $\bullet$ extension of φ emphasizes the *fundamental nature* of the construction $\mathfrak{C}(\cdot)$ due to Carathéodory [3]. It appears that not until the present new theory has this construction achieved its appropriate position. There is no such position in the traditional abstract theory!

The inner extension theorem has several important addenda. First of all, the Localization Principle: *If $A \subset X$ fulfils $A \cap S \in \mathfrak{C}(\varphi_\bullet)$ for all $S \in \mathfrak{S}$ then*

$A \in \mathfrak{C}(\varphi_\bullet)$; that is $\mathfrak{S}\top\mathfrak{C}(\varphi_\bullet) \subset \mathfrak{C}(\varphi_\bullet)$ with the *transporter* $\top$ as defined in [9] section 1. Also note that $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$, and in particular in case $\bullet = \tau$ that $\mathfrak{S}_\tau \subset \mathfrak{C}(\varphi_\tau)$ can be of *immense* size.

An important special case for $\bullet = \sigma\tau$ is that $\mathfrak{S}$ is $\bullet$ *compact* in the set-theoretical sense defined at the end of section 1. In this case the above set functions φ are all downward $\bullet$ continuous at $\varnothing$. Thus the equivalent condition 2) in the inner extension theorem becomes much simpler.

The most familiar example is the *topological* situation: Let X be a Hausdorff topological space and $\mathfrak{S} = \mathrm{Comp}(X)$. For each $\bullet = \star\sigma\tau$ then the inner $\bullet$ premeasures φ are identical with the *Radon premeasures*, and for such φ the envelopes $\varphi_\bullet$ and hence the measures $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ are the same for $\bullet = \star\sigma\tau$. The localization principle quoted above shows that $\mathrm{Cl}(X) \subset \mathfrak{C}(\varphi_\bullet)$ and hence $\mathrm{Bor}(X) \subset \mathfrak{C}(\varphi_\bullet)$. Thus the common Φ is the *unique maximal* Radon measure extension of φ.

A brief word on the parallel new *outer* theory $\bullet = \star\sigma\tau$ and on the connections between the two theories: The outer theory starts with $\varphi : \mathfrak{S} \to [0,\infty]$, but the deviation caused by the value ∞ finds its natural explanation in the extended versions of the two theories developed in [9] - in those extended versions the theories are even *identical*! The outer theory is based on the *outer* $\bullet$ *envelopes* $\varphi^\bullet : \mathfrak{S} \to [0,\infty]$ of φ, of which $\varphi^\star$ is the obvious *crude* outer one and φ^σ the previous 1969/70 variant of the Carathéodory construct φ°. The resultant *outer extension theorem* corresponds to the present inner one in the essentials, except of course that it has no extra condition with the senseless *upward* $\bullet$ *continuity at* $\varnothing$ and hence no satellites, but in return is provided with a certain *safety barrier* at ∞ in case $\bullet = \tau$. The case $\bullet = \sigma$ contains the result of Carathéodory 1914 and its 1969/70 extension, but goes far beyond.

We return to the fundamentals and recall the two *decisive ideas* which combine to form the basis of the present new development: The first one is the idea of Kisyński and Topsøe how to express the existence of *inner regular* extensions for set functions defined on *lattices*. The second one is the 1969/70 idea to pass from the Carathéodory construct φ° to its variant φ^σ. It is remarkable that these two ideas came up in the same small period of time before 1970. Much later then the present author returned to the context and noticed that, in contrast to φ°, the construct φ^σ has an obvious *inner* counterpart φ_σ, and that the two of them have obvious *nonsequential* counterparts φ^τ and φ_τ, defined via directed set systems. There are pleasant supports in favour of these constructs - one of the nicest is that all of the constructs are σ continuous in the proper sense, even when no such assumption had been imposed on φ (see [8] and [11] 2.8.2)). What remained was the *systematization*, to start off with the proper formulation of the basic concepts, in order to arrive at our inner and outer extension theorems with their *equivalent conditions* and their connection with the traditional substance, manifested in $\mathfrak{C}(\cdot)$.

A final note on the comparison with the traditional theories: In the treatise of Fremlin [4] the construction of measures is based on the old concept of *outer measures* due to Carathéodory [3] and on a new concept of *inner measures* [4]

413A, which is of quite different type. As a rule one produces the outer measures from appropriate set functions $\varphi : \mathfrak{S} \to [0,\infty]$ via the Carathéodory construct φ° as before. The inner measures do not seem to be of substantial importance or to contribute to more appropriate proofs (compare [4] 413 I+J with Pollard [19] Appendix A 3+4). The decisive point is that there are no counterparts of the present *inner and outer* $\bullet$ *envelopes* for $\bullet = \sigma\tau$. As a consequence there are no universal inner and outer $\bullet$ extension theorems which are *best possible* in the sense that there are *equivalent conditions* like in the present ones. A similar observation applies to the treatise of Bogachev [1].

5. The Further Development in a Few Examples

In the last few years the present author was pleased to demonstrate that the inner and outer extension theorems - and in particular the nature of their basic concepts - opened the road for an extensive development in measure and integration and beyond, the results of which are not more complicated and at times even simpler to formulate, but can be much more powerful and comprehensive than the earlier ones. In particular the author thinks it is the first time that an *abstract* theory of measure and integral contains the respective *topological* theory as an *explicit special case*. He developed a number of topics in [9] and in subsequent papers. We mention in particular the basic points treated in [15][16] and the survey articles [11][13]. The present section wants to offer a few examples, related to the points of criticism in sections 1 and 2.

The Choquet Integral. We shall need the concept of the *integral* due to Choquet 1953/54. Our version will be adapted to our situation of *two parallel* theories. Let $\mathfrak{S}$ be a lattice in the nonvoid set X with $\varnothing \in \mathfrak{S}$. We define the function classes $\mathrm{Inn}(\mathfrak{S})$ and $\mathrm{Out}(\mathfrak{S})$ to consist of the functions $f \in [0,\infty]^X$ with $[f \geqq t] \in \mathfrak{S}$ and $[f > t] \in \mathfrak{S}$, respectively, for $0 < t < \infty$. Then for $\varphi : \mathfrak{S} \to [0,\infty]$ isotone with $\varphi(\varnothing) = 0$ the *Choquet integral* $⨍ f d\varphi \in [0,\infty]$ is defined to be

$$= \int_{0\leftarrow}^{\to\infty} \varphi([f \geqq t])dt \text{ for } f \in \mathrm{Inn}(\mathfrak{S}) \quad \text{and} \quad = \int_{0\leftarrow}^{\to\infty} \varphi([f > t])dt \text{ for } f \in \mathrm{Out}(\mathfrak{S}),$$

both times as an improper Riemann integral of a decreasing function $\geqq 0$. One verifies that for $f \in \mathrm{Inn}(\mathfrak{S}) \cap \mathrm{Out}(\mathfrak{S})$ the two second members are equal. In particular $⨍ \chi_A d\varphi = \varphi(A)$ for $A \in \mathfrak{S}$. When $\mathfrak{S}$ is a σ algebra, $\mathrm{Inn}(\mathfrak{S}) = \mathrm{Out}(\mathfrak{S})$ consists of the usual $f \in [0,\infty]^X$ *measurable* $\mathfrak{S}$, and when moreover φ is a measure then $⨍ f d\varphi$ is the usual integral $\int f d\varphi$. This notion of an integral is so natural and simple that one could wonder why it did not become the foundation for all of integration theory. But the basic trouble with the Choquet integral is that it is a priori obscure whether and when it is *additive*. For this issue we refer to [12].

Positive Linear Functionals. Our first point is the representation of positive linear functionals as discussed in section 2. Let as before $F \subset \mathbb{R}^X$ be a Stonean vector lattice and $J : F \to \mathbb{R}$ be a positive linear functional, assumed to be $\bullet$ continuous for some $\bullet = \sigma\tau$. In section 2 the basic problem was to produce *inner* *regular* representations of J. We considered the particular *initial Radon measure* situation with $\bullet = \tau$ and described the route via $J_\circ^\tau$ due to Bourbaki. Now in the present new systematization the *inner extension theorem* produces the result which follows, in the *full situation* and in striking contrast to the former one. It is based on the counterpart

$$\mathfrak{M} := \{[f \geqq t] : f \in F \text{ and } 0 < t < \infty\}$$

of the former $\mathfrak{N} := \{[f > t] : f \in F \text{ and } 0 < t < \infty\}$. Both of them are lattices with $\varnothing$. Note that $F^+ := \{f \in F : f \geqq 0\} \subset \mathrm{Inn}(\mathfrak{M}) \cap \mathrm{Out}(\mathfrak{N})$.

Inner Representation Theorem ($\bullet = \sigma\tau$): *There is a unique inner $\bullet$ premeasure $\varphi : \mathfrak{M} \to [0, \infty[$ which represents J in the sense that $J(f) = ⨍ f d\varphi$ for all $f \in F^+$. This is $\varphi = J_\bullet(\chi_.)|\mathfrak{M}$, which also is $\varphi = J^\star(\chi_.)|\mathfrak{M}$. It even fulfils $J_\bullet(f) = ⨍ f d\varphi_\bullet$ for all $f \in [0, \infty]^X$, and hence $J_\bullet(\chi_.) = \varphi_\bullet = \Phi_\star$.* It follows that $\Phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ represents J in the sense that all $f \in F$ are integrable Φ with $J(f) = \int f d\Phi$.

In the particular *initial Radon measure* case one has $\mathfrak{M}_\tau = \mathrm{Comp}(X)$, so that Φ is the unique maximal Radon measure which represents J. One proves that in fact $\Phi = \alpha^\tau$. In this context a final word on the old construct $J_\circ^\tau$: One proves that $J_\circ^\tau(f) = ⨍ f d\Phi^\star$ for all $f \in [0, \infty]^X$, and hence $J_\circ^\tau(\chi_.) = \Phi^\star$, in contrast to the above $J_\bullet(\chi_.) = \varphi_\bullet = \Phi_\star$. This shows that the construct is of *hybrid type*: From its intention and definition it is of *inner* type, but its properties are more like those of *outer* type. For example, as a rule $J_\circ^\tau(\chi_.)$ is far from inner regular $\mathrm{Comp}(X)$. Therefore $J_\circ^\tau$ has no place in the new systematization.

The new *outer procedure* is parallel to the new inner one. But it is of course closer to the old procedure, which after all has been of *outer* character: it is in terms of $\mathfrak{N}$ and ends up at the former $\beta^\bullet$.

On the whole it seems clear that we have arrived at the appropriate method of representation. The two new representation theorems are in [11] in much more comprehensive versions than described above: thus their domains are subsets of $[0, \infty[^X$ and $[0, \infty]^X$, assumed to be positive-homogeneous with 0 and to be *Stonean* lattices in the appropriate sense, but not even to be *stable under addition*. The two theorems are the precise counterparts of the earlier inner and outer extension theorems for set functions. The *inner* development culminates in the definitive *Daniell-Stone-Riesz representation theorem* in [14]. It specializes to the class of *all* Hausdorff topological spaces X. We note that all these results have substantial predecessors in Pollard-Topsøe [18] and Topsøe [23].

Finite Products. It is well-known and has been noted in section 1 that the two abstract theories of the 20th century are quite different in their treatment of *finite products* of measures: thus the *Radon product measure* of two Radon measures is out of reach of the traditional abstract theory. Our next point is to show that with the new systematization the situation becomes totally different. We note at once that this point - like the subsequent one - is a matter of the *inner* theory: there is no full outer counterpart.

We fix nonvoid sets X and Y. For nonvoid set systems $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y we have the usual *product* set system $\mathfrak{S} \times \mathfrak{T} := \{S \times T : S \in \mathfrak{S}$ and $T \in \mathfrak{T}\}$ in $X \times Y$. For *lattices* $\mathfrak{S}$ and $\mathfrak{T}$ with $\varnothing$ then $\mathfrak{R} := (\mathfrak{S} \times \mathfrak{T})^\star$ is a lattice containing $\varnothing$ as well (and the same for rings and algebras). Now let $\varphi : \mathfrak{S} \to [0, \infty]$ and $\psi : \mathfrak{T} \to [0, \infty]$ be isotone set functions with $\varphi(\varnothing) = \psi(\varnothing) = 0$. One proves for $E \in \mathfrak{R}$ that the function $x \mapsto \psi(E(x))$, where $E(x) := \{y \in Y : (x, y) \in E\} \in \mathfrak{T}$ is the vertical section of E at $x \in X$, is in $\mathrm{Inn}(\mathfrak{S}) \cap \mathrm{Out}(\mathfrak{S})$. We define the *product set function*

$$\vartheta = \varphi \times \psi : \mathfrak{R} \to [0, \infty] \quad \text{to be} \quad \vartheta(E) = ⨍\psi(E(\cdot))d\varphi.$$

It follows that ϑ is isotone with $\vartheta(\varnothing) = 0$ and satisfies $\vartheta(S \times T) = \varphi(S)\psi(T)$ for $S \in \mathfrak{S}$ and $T \in \mathfrak{T}$ (with $0\infty = 0$ as usual). Also ϑ inherits from φ and ψ the properties of being *modular*, of being *finite*, and of being *finite and downward* $\bullet$ *continuous*. The fundamental fact is the

Product Theorem ($\bullet = \star\sigma\tau$): *Assume that* $\varphi : \mathfrak{S} \to [0, \infty[$ *and* $\psi : \mathfrak{T} \to [0, \infty[$ *are inner* $\bullet$ *premeasures. Then* $\vartheta = \varphi \times \psi : \mathfrak{R} \to [0, \infty[$ *is an inner* $\bullet$ *premeasure as well, and* $\Theta := \vartheta_\bullet | \mathfrak{C}(\vartheta_\bullet)$ *is an extension of the product* $\Phi \times \Psi$ *of* $\Phi := \varphi_\bullet | \mathfrak{C}(\varphi_\bullet)$ *and* $\Psi := \psi_\bullet | \mathfrak{C}(\psi_\bullet)$.

If in particular X and Y are Hausdorff topological spaces with $\mathfrak{S} = \mathrm{Comp}(X)$ and $\mathfrak{T} = \mathrm{Comp}(Y)$, then one notes that $\mathfrak{R}_\tau = \mathrm{Comp}(X \times Y)$. Thus if φ and ψ are Radon premeasures on X and Y with $\vartheta = \varphi \times \psi$, then $\pi := \vartheta_\tau | \mathfrak{R}_\tau$ is a Radon premeasure on $X \times Y$ and satisfies $\pi_\tau = \pi_\star = (\vartheta_\tau | \mathfrak{R}_\tau)_\star = \vartheta_\tau$, so that $\Theta := \vartheta_\tau | \mathfrak{C}(\vartheta_\tau) = \pi_\tau | \mathfrak{C}(\pi_\tau)$ is an extension of $\Phi \times \Psi$ which is *maximal Radon* on $X \times Y$.

Projective Limits. Our final point is the concept of projective limits as discussed in section 1. The aim is a comprehensive *projective limit theorem* in terms of the new *inner theory*. As before we fix an infinite index set T and a family $(Y_t)_{t \in T}$ of nonvoid sets with product set X, and we recall the index set $I = I(T)$ and the family $(Y_p)_{p \in I}$ of partial product sets, with the projections $H_p : X \to Y_p$ and $H_{pq} : Y_q \to Y_p$ for $p \subset q$ in I.

This time we assume, in the spirit of the new inner systematization, a family $(\mathfrak{K}_t)_{t \in T}$ of *lattices* $\mathfrak{K}_t$ in Y_t, such that $\mathfrak{K}_t$ contains the finite subsets of Y_t and is $\bullet$ *compact*. We form the family $(\mathfrak{K}_p)_{p \in I}$ of partial product lattices $\mathfrak{K}_p = \{\prod_{t \in p} K_t : K_t \in \mathfrak{K}_t\}^\star$ in Y_p, which retain these properties. The decisive construct is

$$\mathfrak{S} := \{\prod_{t \in T} S_t : S_t \in \mathfrak{K}_t \cup \{Y_t\} \text{ with } S_t = Y_t \text{ for almost all } t \in T\}^\star,$$

which is a lattice in X with $\varnothing, X \in \mathfrak{S}$ and likewise $\bullet$ *compact*. Our theorem then reads as follows.

Projective Limit Theorem: *There is a one-to-one correspondence between*

the families $(\varphi_p)_{p \in I}$ *of inner* $\bullet$ prob *premeasures* $\varphi_p : \mathfrak{K}_p \to [0, \infty[$ *which are* consistent *in the sense that* $\varphi_p = (\varphi_q)_\bullet (H_{pq}^{-1}(\cdot)) | \mathfrak{K}_p$ *for* $p \subset q$ *in* I,

and the inner $\bullet$ prob *premeasures* $\varphi : \mathfrak{S} \to [0, \infty[$, *via* $\varphi_p = \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p$ *for* $p \in I$. *The correspondence fulfils* $(\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))$ *and* $\mathfrak{C}((\varphi_p)_\bullet) = \{B \subset Y_p : H_p^{-1}(B) \in \mathfrak{C}(\varphi_\bullet)\}$ *for* $p \in I$. *Moreover for* $A \in \mathfrak{S}_\bullet$ *one has*

$$H_p(A) \in \mathfrak{K}_p \top (\mathfrak{K}_p)_\bullet \subset \mathfrak{C}((\varphi_p)_\bullet)\ \forall\, p \in I \quad \textit{and} \quad \Phi(A) = \inf_{p \in I} \Phi_p(H_p(A)).$$

The present projective limit theorem has been, thanks to the concepts involved, unachieved in its scope so far. Thus its version $\bullet = \tau$, specialized to an appropriate (almost) topological context, extends the Kolmogorov theorem cited in section 1 to arbitrary Hausdorff topological spaces. However, most important is the fact that its version $\bullet = \tau$ produces on the product set X in the form of $\Phi := \varphi_\tau|\mathfrak{C}(\varphi_\tau)$ a measure which is of an *immense size*: The subclass $\mathfrak{S}_\tau \subset \mathfrak{C}(\varphi_\tau)$ alone contains all product sets $S = \prod_{t \in T} S_t$ with $S_t \in \mathfrak{K}_t \cup \{Y_t\}\ \forall\, t \in T$, and hence in case of an uncountable T reaches far beyond the frame of countably determined subsets of X.

In the context of *stochastic processes* one assumes that $Y_t = Y$ and $\mathfrak{K}_t = \mathfrak{K}$ independent of $t \in T$. In the spirit of the new inner systematization the adequate concept is to *define a stochastic process for* T *and* $(Y, \mathfrak{K})$ to be an inner τ prob premeasure $\varphi : \mathfrak{S} \to [0, \infty[$ - of course all the time connected with its maximal extension $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$. It is fundamental to take $\bullet = \tau$, in order that the measure Φ on the path space $X = Y^T$ attains that immense domain $\mathfrak{C}(\varphi_\tau)$. This situation offers the chance for an adequate definition: we define the *essential sets* for the stochastic process $\varphi : \mathfrak{S} \to [0, \infty[$ to be those subsets $E \in \mathfrak{C}(\varphi_\tau)$ which have full measure $\Phi(E) = 1$. At the same time we have all the benefits of the $\bullet = \tau$ version of our inner systematization.

In case Y is a *Polish topological space* with $\mathfrak{B} = \mathrm{Bor}(Y)$ and $\mathfrak{K} = \mathrm{Comp}(Y)$, one proves that there is a one-to-one correspondence between

the traditional stochastic processes $\alpha : \mathfrak{A} \to [0, \infty[$ for T and $(Y, \mathfrak{B})$ and
the new stochastic processes $\varphi : \mathfrak{S} \to [0, \infty[$ for T and $(Y, \mathfrak{K})$.

The correspondence rests upon $\mathfrak{S} \subset \mathfrak{A} \subset \mathfrak{C}(\varphi_\tau)$ and reads $\varphi = \alpha|\mathfrak{S}$ and $\alpha = \Phi|\mathfrak{A}$. Moreover $\varphi_\tau = (\alpha^\star|\mathfrak{S}_\tau)_\star$.

In the example of the *Brownian motion* with $T = [0, \infty[$ and $Y = \mathbb{R}$ one proves that $\mathrm{C}(T, \mathbb{R}) \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(\mathrm{C}(T, \mathbb{R})) = 1$, so that $\mathrm{C}(T, \mathbb{R})$ is in fact an *essential set* for this stochastic process. Thus the *Wiener measure* appears in its full size $\Phi = \varphi_\tau|\mathfrak{C}(\varphi_\tau)$, and $\mathrm{C}(T, \mathbb{R})$ is a member of its domain $\mathfrak{C}(\varphi_\tau)$ from the start and must no longer be pushed in via $\alpha^\star(\mathrm{C}(T, \mathbb{R})) = 1$. The present development has been summarized in [13]. It should be compared with the previous ones, for example in Fremlin [4] chapter 45.

In conclusion we want to specialize the new projective limit theorem to the case of *infinite products*. Assume that $(\vartheta_t)_{t \in T}$ is a family of inner $\bullet$ prob premeasures $\vartheta_t : \mathfrak{K}_t \to [0, \infty[$ for some $\bullet = \sigma\tau$. For $p \in I$ we define the inner $\bullet$ prob premeasure $\varphi_p : \mathfrak{K}_p \to [0, \infty[$ to be the *product* of the finite family $(\vartheta_t)_{t \in p}$ under the obvious extension of the product formation in the last example to any finite number of factors. One verifies that $(\varphi_p)_{p\in I}$ is a *projective family* in the sense of the present theorem. Thus it produces an inner $\bullet$ prob premeasure $\varphi : \mathfrak{S} \to [0,\infty[$. Then $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ has the obvious position of the natural *infinite product* of the family $(\Theta_t)_{t\in T}$ of the prob measures $\Theta_t = (\vartheta_t)_\bullet|\mathfrak{C}((\vartheta_t)_\bullet)$. This construction is far more comprehensive than the former one for Radon prob measures. It makes clear that in the present context the appropriate notion of *compactness* is not the *topological* but the *set-theoretical* $\bullet$ one.

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In Remembrance of Günter Lumer

Günter Lumer was a close friend of mine for several decades. We had the same age: our dates of birth were but 13 days apart. We met for the first time in the fall of 1962 at a functional analysis conference in Oberwolfach. The year before Günter had published two of his most important papers: the joint paper with Ralph PHILLIPS on dissipative operators and the paper on semi-inner products.

The subsequent years were the grand period in the development of the functional analytic theory of abstract analytic functions, known under the key words of uniform algebras and Hardy spaces. We were both deeply involved, with quite often different methods but close results. Günter obtained fundamental breakthroughs in two situations: The first time in Bulletin Amer.Math.Soc. 70(1964), where he was able to develop the abstract counterpart of the classical unit disk situation on an arbitrary uniform algebra and for an individual multiplicative linear functional, under the basic assumption that the functional in question has a unique representing measure. Before that one needed global assumptions on the algebra, such as to be Dirichlet or logmodular. After his work then in 1965 Kenneth HOFFMAN and Hugo ROSSI, and independently myself, obtained the final abstract version of the classical unit disk situation in terms of a fixed so-called Szegö measure for an individual multiplicative linear functional.

The second breakthrough was in his 1968 Lecture Notes, this time for an arbitrary multiplicative linear functional on any uniform algebra. Günter defined its universal Hardy class and was able to transfer the classical concepts and results to an amazing extent, in particular to establish an abstract conjugation operation via extension of the classical Kolmogorov estimations. He then left the field in the early seventies. I myself returned to it in connection with the extended concept of Daniell-Stone integration due to Michael Leinert 1982, which produced a definitive theory in the late eighties. But it is clear that to an essential extent the basic contributions are due to Günter Lumer in the sixties.

In all these years we had close contacts. During the academic year 1967/68 Günter stayed at Strasbourg University, thus close to my home University Saarbrücken. In the summer term 1967 he gave a series of lectures in Saarbrücken, and in the winter term 1967/68, which I spent at Caltech in Pasadena, a little bus supplied by our University brought my students to his lectures in Strasbourg every week. In the academic year 1969/70 Günter Lumer together with Irving Glicksberg organized a Research Seminar on function algebras at their home University, the University of Washington in Seattle. I had the good fortune to participate for three months on his invitation.

After his move to Belgium in 1973/74 Günter was a regular visitor to Saarbrücken, both private and for a further series of lectures and several colloquium talks. He wrote a comprehensive survey article on evolution equations for our Annales Universitatis Saraviensis and published several papers in the Archiv der Mathematik of which I had been the editor for abstract analysis. Our relations became even closer because of the sequence of the North-West European Analysis Seminars 1992-1997, of which Günter was the unique creator and driving force. We were common chairmen of the second seminar 1993 at Schloss Dagstuhl in the Saarland, which is the Informatics counterpart of the Oberwolfach Institute. Thus we two are in the tiny group of "outside" mathematicians who have ever been chairpersons of conferences at Schloss Dagstuhl. Unfortunately, in 1997 a serious hip joint operation forced Günter to discontinue the beautiful enterprise. There was no successor.

For me the first of the seminars 1992 in Saint-Amand-les-Eaux near Lille was a moving event: Near its end I experienced heart trouble, and my doctor said on the telephone that I should come to his hospital right away but must not drive a car. What then happened was that Günter asked Luc Paquet to take his car back to Brussels, and took the steering-wheel of my car (which was new at the time) to drive us for at least 400 kilometers to Saarbrücken. We arrived late at night, and my wife said later that I looked radiant with health but Günter grey with exhaustion. This was the deepest evidence of friendship which I ever experienced in my life.

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