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Abstract

We prove a partial regularity result for local minimizers $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^M$ of the variational integral $J(u,\Omega) = \int_\Omega f(\nabla^k u)\, dx$, where k is any integer and f is a strictly convex integrand of anisotropic (p,q)–growth with exponents satisfying the condition $q < p(1+\frac{2}{n})$. This is some extension of the regularity theorem obtained in [BF2] for the case $n = 2$.

1 Introduction

In this note we study the regularity properties of local minimizers $u : \Omega \to \mathbb{R}^M$ of higher order variational integrals of the form

$$J(w,\Omega) = \int_\Omega f(\nabla^k w)\, dx,$$

where Ω is a domain in $\mathbb{R}^n, n \geq 2$, and $k \geq 2$ denotes a given integer. The symbol $\nabla^k w$ stands for the tensor of all k^{th} order (weak) partial derivatives of the function w, i.e. $\nabla^k w = (D^\alpha w^i)_{|\alpha|=k, 1\leq i\leq M, \alpha\in\mathbb{N}_0^n}$. Our main assumption concerns the energy density f : we consider $f \geq 0$ of class C^2 satisfying with given exponents $1 < p \leq q < \infty$ and with positive constants λ, Λ the anisotropic ellipticity condition

$$\lambda(1+|\sigma|^2)^{\frac{p-2}{2}}|\tau|^2 \leq D^2 f(\sigma)(\tau,\tau) \leq \Lambda(1+|\sigma|^2)^{\frac{q-2}{2}}|\tau|^2 \tag{1.1}$$

being valid for all tensors σ and τ. Note that the left–hand side of (1.1) implies the strict convexity of f, moreover, it is easy to see that

$$a|\tau|^p - b \leq f(\tau) \leq A|\tau|^q + B \tag{1.2}$$

is true with constants $a,\ A > 0,\ b,\ B \geq 0$.
According to (1.2) the appropriate space for local minimizers is the energy class consisting of all Sobolev functions $u \in W^k_{p,\mathrm{loc}}(\Omega;\mathbb{R}^M)$ such that $J(u,\Omega') < \infty$ for any subdomain

$\Omega' \subset\subset \Omega$, and we say that a function u with these properties is a local J–minimizer if and only if

$$J(u,\Omega') \leq J(v,\Omega')$$

for any $v \in W^k_{p,\mathrm{loc}}(\Omega;\mathbb{R}^M)$ such that $\mathrm{spt}(u-v) \subset\subset \Omega'$, where as above Ω' is an arbitrary subdomain of Ω with compact closure in Ω. For a definition of the Sobolev classes $W^k_p, W^k_{p,\mathrm{loc}}$, etc., we refer the reader to the book of Adams [Ad]. Now we can state our main result:

THEOREM 1.1. *Let u denote a local J–minimizer where f satisfies (1.1). Suppose further that*

$$q < p(1+\frac{2}{n}) \tag{1.3}$$

is true. Then there is an open subset $\Omega_\circ$ of Ω such that $\Omega - \Omega_\circ$ is of Lebesgue measure zero and $u \in C^{k,\nu}_{\mathrm{loc}}(\Omega_\circ;\mathbb{R}^M)$ for any exponent $0 < \nu < 1$.

REMARK 1.1. *i) In the twodimensional case, i.e. $n = 2$, the partial regularity result of Theorem 1.1 can be improved to everywhere regularity which means that actually we have $\Omega_\circ = \Omega$. This is outlined in the recent paper [BF2].*
ii) The anisotropic first order case, i.e. we have $k = 1$ and f satisfies conditions similar to (1.1), is well investigated: without being complete we mention the papers of Acerbi and Fusco [AF], of Esposito, Leonetti and Mingione [ELM1,2,3] and the results obtained by the second author in collaboration with Bildhauer, see e.g. [BF1]. Further references are contained in the monograph [Bi]. Clearly the list above addresses the case of vectorvalued functions. The anisotropic scalar situation for first order problems has been discussed before mainly by Marcellini, compare e.g. [Ma1,2,3], with the major result that conditions of the form (1.3) are in fact sufficient for excluding the occurrence of singular points.
iii) If $n \geq 3$ together with $k \geq 2$, then partial $C^{k,\nu}$–regularity of minimizers of the variational integral $\int_\Omega f(\nabla^k u)\,dx$ has been studied in the paper [Kr1] of Kronz. Here the main feature however is the quasiconvexity assumption imposed on f, i.e. the right–hand side of (1.1) is required to hold with $q = p$ and the first inequality in (1.1) is replaced by the hypothesis of uniform strict quasiconvexity with exponent $p \geq 2$. A related result concerning quasimonotone nonlinear systems of higher order with p–growth ($p \geq 2$) is established in [Kr2]. Of course the theorems of Kronz imply our regularity result if we consider (1.1) in the isotropic case $p = q$ together with $p \geq 2$.
For completeness we also like to mention the work of Duzaar, Gastel and Grotowski [DGG] dealing with partial regularity of certain higher order nonlinear elliptic systems and improving earlier results of Giaquinta and Modica established in [GM2].
iv) If the non–autonomous case $I(w,\Omega) := \int_\Omega F(x,\nabla^k w)\,dx$ is considered with integrand $F(x,\sigma)$ satisfying (1.1) uniformly w.r.t. σ, and if in addition we require

$$|D_x D_\sigma F(x,\sigma)| \leq c_1(1+|\sigma|^2)^{\frac{q-1}{2}}$$

then Theorem 1.1 remains valid, provided (1.3) is replaced by the stronger condition $q < p\,(1+1/n)$ and if for example we assume that $F(x,\sigma)$ is given by $F(x,\sigma) = g(x,|\sigma|)$ for a suitable function g. The details are left to the reader, we refer to [ELM3] and [BF3].

The proof of Theorem 1.1 is organized in two steps. First we introduce a suitable regularization of our variational problem following the lines of [BF2] which leads us to uniform higher integrability and higher weak differentiability results for the solutions of the approximate problems which then extend to our local minimizer. In a second step we combine this initial regularity with a blow–up procedure which will give partial regularity as stated in Theorem 1.1. From now on and just for notational simplicity we will assume that $k = 2$ together with $M = 1$. Moreover, we let $n \geq 3$ for obvious reasons. If necessary, we pass to subsequences without explicit indications, and we use the same symbol to denote various constants with different numerical values.

2 Approximation and initial regularity

Let the assumptions of Theorem 1.1 hold and consider a local J–minimizer u. We proceed as in [BF2] by fixing two open domains $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$. Then we consider the mollification $\overline{u}_m$ of u with radius $1/m, m \in \mathbb{N}$, and let $u_m \in \overline{u}_m + \overset{\circ}{W}{}^2_q(\Omega_2)$ denote the unique solution of the problem
$J_m(w,\Omega_2) := J(w,\Omega_2) + \rho_m \int_{\Omega_2}(1+|\nabla^2 w|^2)^{q/2}\,dx \longrightarrow \min$ in $\overline{u}_m + \overset{\circ}{W}{}^2_q(\Omega_2)$,
where we have set $\rho_m := \|\overline{u}_m - u\|_{W^2_p(\Omega_2)}\Big[\int_{\Omega_2}(1+|\nabla^2\overline{u}_m|^2)^{q/2}\,dx\Big]^{-1}$.
It is easy to see that (compare [BF2])

$$\begin{aligned} &u_m \rightharpoondown u \text{ in } W^2_p(\Omega_2), J(u_m,\Omega_2) \to J(u,\Omega_2),\\ &J_m(u_m,\Omega_2) \to J(u,\Omega_2)\end{aligned}$$

as $m \to \infty$. Next we use the Euler equation

$$\int_\Omega Df_m(\nabla^2 u_m) : \nabla^2\varphi\,dx = 0,\ \varphi \in \overset{\circ}{W}{}^2_q(\Omega_2), \tag{2.1}$$

$f_m := \rho_m(1+|\cdot|^2)^{q/2} + f$, with the choice $\partial_i(\eta^6\partial_i u_m)$, $i = 1,\dots,n$, $\eta \in C^\infty_\circ(\Omega_2)$, $0 \leq \eta \leq 1$, $\eta = 1$ on Ω_1, and get (from now on summation w.r.t. i) with the help of the Cauchy–Schwarz inequality for the bilinear form $D^2f_m(\nabla^2 u_m)$

$$\begin{aligned} &\int_{\Omega_2}\eta^6 D^2f_m(\nabla^2u_m)(\partial_i\nabla^2u_m,\partial_i\nabla^2u_m)\,dx\\ &\leq\ c\Big\{(\|\nabla^2\eta\|^2_\infty + \|\nabla\eta\|^4_\infty)\int_{\operatorname{spt}\nabla\eta}|D^2f_m(\nabla^2u_m)|\,|\nabla u_m|^2\,dx \\ &\quad +\|\nabla\eta\|^2_\infty\int_{\operatorname{spt}\nabla\eta}|D^2f_m(\nabla^2u_m)|\,|\nabla^2u_m|^2\,dx\Big\}\end{aligned} \tag{2.2}$$

where c denotes a finite constant independent of m. Of course this calculation has to be justified with the help of the difference quotient technique using $\varphi = \Delta_{-h}(\eta^6\Delta_h u_m)$ in (2.1), $\Delta_h u_m(x) := \frac{1}{h}[u_m(x+he_i) - u_m(x)]$. In case that $q \geq 2$, the reader can follow the steps in [BF2] leading from (2.6) to (2.13) where (2.12) has to be adjusted for dimensions $n \geq 3$. If $q < 2$, then we refer to [BF1] or [Bi], p. 55–57.

Inequality (2.2) implies local uniform higher integrability of the sequence $\{\nabla^2 u_m\}$: let $\chi := \frac{n}{n-2}$ and $s := \frac{p}{2}\chi$. For concentric balls $B_r \subset\subset B_R \subset\subset \Omega_2$ and $\eta \in C_\circ^\infty(B_R)$, $0 \leq \eta \leq 1$, $\eta = 1$ on B_r, $|\nabla^\ell \eta| \leq c/(R-r)^\ell$, $\ell = 1, 2$, we have by Sobolev's inequality

$$\begin{aligned} \int_{B_r} (1 + |\nabla^2 u_m|^2)^s \, dx \; &\leq \; \int_{B_R} \Big(\eta^3 [1 + |\nabla^2 u_m|^2]^{s\frac{n-2}{2n}}\Big)^{2\chi} \, dx \\ = \; \int_{B_R} (\eta^3 h_m)^{2\chi} \, dx \; &\leq \; c\Big(\int_{B_R} |\nabla(\eta^3 h_m)|^2 \, dx\Big)^{\frac{n}{n-2}}. \end{aligned}$$

Here $h_m := (1 + |\nabla^2 u_m|^2)^{p/4}$ is known to be of class $W^1_{2,\mathrm{loc}}(\Omega_2)$ on account of (2.2), and with Young's inequality we deduce

$$\begin{aligned} \int_{B_r} (1 + |\nabla^2 u_m|^2)^s \, dx \leq c\Big[\int_{B_R} \eta^6 |\nabla h_m|^2 \, dx \\ + \int_{B_R} |\nabla \eta^3|^2 h_m^2 \, dx\Big]^\chi =: c[T_1 + T_2]^\chi. \end{aligned} \tag{2.3}$$

From (1.1) and (2.2) we get ($T_{R,r} := B_R - B_r$)

$$\begin{aligned} T_1 \;&\leq\; c(r,R) \int_{T_{R,r}} (1 + |\nabla^2 u_m|^2)^{\frac{q-2}{2}} \Big[|\nabla^2 u_m|^2 + |\nabla u_m|^2\Big] \, dx \\ &\leq\; c(r,R)\Big[\int_{T_{R,r}} (1 + |\nabla^2 u_m|^2)^{\frac{q}{2}} \, dx + \int_{T_{R,r}} |\nabla u_m|^q \, dx\Big], \end{aligned}$$

moreover

$$T_2 \leq c(r,R) \int_{T_{R,r}} (1 + |\nabla^2 u_m|^2)^{p/2} \, dx.$$

Inserting these estimates into (2.3) we find that

$$\begin{aligned} &\int_{B_r} (1 + |\nabla^2 u_m|^2)^s \, dx \\ &\leq c(r,R)\Big[\int_{T_{R,r}} (1 + |\nabla^2 u_m|^2)^{q/2} \, dx + \int_{T_{R,r}} |\nabla u_m|^q \, dx\Big]^\chi \end{aligned} \tag{2.4}$$

for a constant $c(r,R) = c(R-r)^{-\beta}$ with suitable exponent $\beta > 0$. Fix $\Theta \in (0,1)$ such that

$$\frac{1}{q} = \frac{\Theta}{p} + \frac{1-\Theta}{2s}$$

(note: $2s = p\chi > q$ on account of $q < p(1+\frac{2}{n})$). Then the interpolation inequality implies

$$\|\nabla^2 u_m\|_q \le \|\nabla^2 u_m\|_p^{\Theta} \ \ \|\nabla^2 u_m\|_{2s}^{1-\Theta}$$

where the norms are taken over $T_{R,r}$, and we get:

$$\int_{T_{R,r}} |\nabla^2 u_m|^q \, dx \le \Big(\int_{B_R} |\nabla^2 u_m|^p \, dx\Big)^{\Theta q/p} \Big(\int_{T_{R,r}} |\nabla^2 u_m|^{2s} \, dx\Big)^{(1-\Theta)\frac{q}{2s}}. \tag{2.5}$$

Before applying (2.5) to the first integral on the r.h.s. of (2.4) we discuss the second one: we have (for any $0 < \varepsilon < 1$)

$$\int_{T_{R,r}} |\nabla u_m|^q \, dx \le \varepsilon \int_{T_{R,r}} |\nabla^2 u_m|^q \, dx + c(\varepsilon, R, r) \int_{T_{R,r}} |u_m|^q \, dx, \tag{2.6}$$

which follows for example from [Mo], Theorem 3.6.9. For the ε–term on the r.h.s. of (2.6) we may use (2.5). By construction we know that $\sup_m \|u_m\|_{W_p^2(\Omega_2)} < \infty$. If $p \ge n$, then the sequence $\{u_m\}$ is uniformly bounded in any space $W_t^1(\Omega_2), t < \infty$, thus we clearly have the boundedness of $\int_{\Omega_2} |u_m|^q \, dx$. So let us assume that $p < n$. Then

$$\sup_m \|u_m\|_{W_t^1(\Omega_2)} < \infty$$

for $t \le \frac{np}{n-p} =: \overline{p}$. In case $\overline{p} \ge n$ we are done. If $\overline{p} < n$, then we obtain

$$\sup_m \|u_m\|_{L^t(\Omega_2)} < \infty$$

for $t \le \frac{n\overline{p}}{n-\overline{p}} = \frac{np}{n-2p}$. Obviously $q \le \frac{np}{n-2p}$ which is a consequence of (1.3) since $p(1+\frac{2}{n}) \le \frac{np}{n-2p}$. Altogether we have shown that

$$\int_{T_{R,r}} |u_m|^q \, dx \le \overline{c} \tag{2.7}$$

for a constant $\overline{c}$ depending also on Ω_2 and $\sup_m \|u_m\|_{W_p^2(\Omega_2)}$. Returning to (2.4), inserting (2.6) combined with (2.7) and applying (2.5) we have shown that

$$\int_{B_r} (1+|\nabla^2 u_m|^2)^s \, dx \le \tag{2.8}$$

$$c(R-r)^{-\beta}\Big[\Big(\int_{\Omega_2} (1+|\nabla^2 u_m|^2)^{\frac{p}{2}} \, dx\Big)^{\Theta q\chi/p} \Big(\int_{T_{R,r}} (1+|\nabla^2 u_m|^2)^s \, dx\Big)^{(1-\Theta)\frac{q\chi}{2s}} + \overline{c}\Big].$$

Now, from (1.3) it follows that $(1-\Theta)\frac{q\chi}{2s} < 1$, and we may therefore apply Young's inequality on the r.h.s. of (2.8) with the result

$$\int_{B_r} (1+|\nabla^2 u_m|^2)^s \, dx \le \tag{2.9}$$

$$\int_{T_{R,r}} (1+|\nabla^2 u_m|^2)^s \, dx + c(R-r)^{-\beta_1}\Big[\Big(\int_{\Omega_2} (1+|\nabla^2 u_m|^2)^{\frac{p}{2}} \, dx\Big)^{\beta_2} + \overline{c}\Big],$$

β_1, β_2 denoting positive exponents. Adding $\int_{B_r} (1 + |\nabla^2 u_m|^2)^s \, dx$ on both sides of (2.9) this inequality turns into

$$\text{(2.10)} \qquad \int_{B_r} (1 + |\nabla^2 u_m|^2)^s \, dx \le \frac{1}{2} \int_{B_R} (1 + |\nabla^2 u_m|^2)^s \, dx + K(R - r)^{-\beta_1},$$

where the constant K on the r.h.s. of (2.9) also depends on $\sup_m \int_{\Omega_2} |\nabla^2 u_m|^p \, dx$. If we use [Gi], Lemma 5.1, p. 81, inequality (2.10) implies the following

LEMMA 2.1. *Under the hypothesis of Theorem 1.1 and with the notation introduced before we have that* $\{u_m\}$ *is uniformly bounded in the space* $W^2_{2s,\text{loc}}(\Omega_2)$, $s := \frac{p}{2} \frac{n}{n-2}$. *In particular we have that* u *belongs to* $W^2_{q,\text{loc}}(\Omega_2)$. *Moreover, the functions* $h_m = (1 + |\nabla^2 u_m|^2)^{p/4}$ *are uniformly bounded in* $W^1_{2,\text{loc}}(\Omega_2)$.

Note that the last statement follows from (2.2) together with $\sup_m \|u_m\|_{W^2_{q,\text{loc}}(\Omega_2)} < \infty$. We return to (2.1) and choose $\varphi = \partial_i(\eta^6 \partial_i [u_m - P_m])$ where $\eta \in C^\infty_\circ(\Omega_2)$, $0 \le \eta \le 1$, and P_m denotes a polynomial function of degree ≤ 2. Similar to (2.2) we get (using difference quotients)

$$\begin{aligned} &\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_i \nabla^2 u_m, \partial_i \nabla^2 u_m) \, dx \\ &\le - \int_{\operatorname{spt} \nabla\eta} D^2 f_m(\nabla^2 u_m) \Big(\partial_i \nabla^2 u_m, \nabla^2 \eta^6 \partial_i [u_m - P_m] \\ &\qquad + 2 \nabla \eta^6 \odot \nabla \partial_i (u_m - P_m) \Big) \, dx, \end{aligned}$$

where the sum is taken w.r.t. $i = 1, \ldots, n$. We apply the Cauchy–Schwarz inequality to the bilinear form $D^2 f_m(\nabla^2 u_m)$ with the result

$$\begin{aligned} &\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_i \nabla^2 u_m, \partial_i \nabla^2 u_m) \, dx \\ \text{(2.11)} \qquad &\le c \Big\{ \Big(\|\nabla^2 \eta\|^2_\infty + \|\nabla \eta\|^4_\infty \Big) \int_{\operatorname{spt} \nabla\eta} |D^2 f_m(\nabla^2 u_m)| \, |\nabla (u_m - P_m)|^2 \, dx \\ &\qquad + \|\nabla \eta\|^2_\infty \int_{\operatorname{spt} \nabla\eta} |D^2 f_m(\nabla^2 u_m)| \, |\nabla^2 (u_m - P_m)|^2 \, dx \Big\} \end{aligned}$$

in particular $\int_{\Omega_2} \eta^6 |\nabla h_m|^2 \, dx$ is bounded by the right–hand side of (2.11). We claim

LEMMA 2.2. *Let* $h := (1 + |\nabla^2 u|^2)^{p/4}$ *Then the following statements hold:*

i) $h \in W^1_{2,\text{loc}}(\Omega_2)$;

ii) $h_m \rightharpoondown h$ *in* $W^1_{2,\text{loc}}(\Omega_2)$;

iii) $\nabla^\ell u_m \longrightarrow \nabla^\ell u$ *a.e. on* Ω_2, $\ell \le 2$.

If P is a polynomial function of degree ≤ 2, then

$$
\begin{aligned}
&\int_{\Omega_2} \eta^6 |\nabla h|^2 \, dx \\
(2.12) \qquad &\leq c \Big\{ \Big(\|\nabla^2 \eta\|_\infty^2 + \|\nabla \eta\|_\infty^4 \Big) \int_{\mathrm{spt}\, \nabla \eta} |D^2 f(\nabla^2 u)| \, |\nabla (u - P)|^2 \, dx \\
&\qquad + \|\nabla \eta\|_\infty^2 \int_{\mathrm{spt}\, \nabla \eta} |D^2 f(\nabla^2 u)| \, |\nabla^2 (u - P)|^2 \, dx \Big\}
\end{aligned}
$$

is true for any $\eta \in C_\circ^\infty(\Omega_2)$, $0 \leq \eta \leq 1$.

Proof: From Lemma 2.1 we deduce that there exists a function $\hat{h} \in W^1_{2,\mathrm{loc}}(\Omega_2)$ such that $h_m \rightharpoondown \hat{h}$ in $W^1_{2,\mathrm{loc}}(\Omega_2)$ and almost everywhere. Suppose that we already have iii). Then i), ii) are trivial. Moreover, if we choose $P_m \equiv P$ in (2.11), Fatou's lemma implies that

$$
\int_{\Omega_2} \eta^6 |\nabla h|^2 \, dx \leq \liminf_{m \to \infty} \int_{\Omega_2} \eta^6 |\nabla h_m|^2 \, dx,
$$

and we may control the quantities $\int_{\Omega_2} \eta^6 |\nabla h_m|^2 \, dx$ with the help of (2.11) in terms of the integrals $\int_{\mathrm{spt}\, \nabla \eta} |D^2 f_m(\nabla^2 u_m)| \, |\nabla^2 u_m - \nabla^2 P|^2 \, dx =: \int_{\mathrm{spt}\, \nabla \eta} \Phi_m \, dx$ and $\int_{\mathrm{spt}\, \nabla \eta} |D^2 f_m(\nabla^2 u_m)| \, |\nabla u_m - \nabla P|^2 \, dx =: \int_{\mathrm{spt}\, \nabla \eta} \Psi_m \, dx$. By Lemma 2.1 the integrand Φ_m is uniformly bounded in $L^{1+\varepsilon}(\mathrm{spt}\, \nabla \eta)$ for some $\varepsilon > 0$, thus $\Phi_m \rightharpoondown: \Phi$ in $L^{1+\varepsilon}(\mathrm{spt}\, \nabla \eta)$ and therefore $\int_{\mathrm{spt}\, \nabla \eta} \Phi_m \, dx \to \int_{\mathrm{spt}} \Phi \, dx$. But with the pointwise convergence iii) we see that $\Phi = |D^2 f(\nabla^2 u)| \, |\nabla^2 u - \nabla^2 P|$. Obviously a similar argument applies to $\int_{\mathrm{spt}\, \nabla \eta} \Psi_m \, dx$ which proves (2.12), and it remains to show iii) just for $\ell = 2$, the other cases are obvious. To this purpose we recall that in fact we have shown that u is in the space $W^2_{q,\mathrm{loc}}(\Omega)$ (due to the arbitrariness of Ω_2) and that by definition u_m is of class $\overline{u}_m + \overset{\circ}{W}{}^2_q(\Omega_2)$. Therefore the following calculations are justified: we have

$$
\begin{aligned}
&\int_{\Omega_2} \Big(f(\nabla^2 u_m) - f(\nabla^2 u) \Big) \, dx \; = \\
(2.13) \qquad &\int_{\Omega_2} Df(\nabla^2 u) : (\nabla^2 u_m - \nabla^2 u) \, dx \; + \\
&\int_{\Omega_2} \int_0^1 D^2 f\Big(\nabla^2 u + t[\nabla^2 u_m - \nabla^2 u]\Big)(\nabla^2 u_m - \nabla^2 u, \nabla^2 u_m - \nabla^2 u)(1 - t) dt \, dx.
\end{aligned}
$$

Note that $\|u - \overline{u}_m\|_{W^2_q(\tilde{\Omega})} \longrightarrow 0$ for all $\tilde{\Omega} \subset\subset \Omega$, moreover the Euler equation for u implies

$$
\int_{\Omega_2} Df(\nabla^2 u) : (\nabla^2 u_m - \nabla^2 u) \, dx = \int_{\Omega_2} Df(\nabla^2 u) : (\nabla^2 \overline{u}_m - \nabla^2 u) \, dx,
$$

thus the first term on the r.h.s. of (2.13) vanishes as $m \to \infty$. The same is true for the l.h.s. of (2.13) as it was remarked at the beginning of this section. This implies

$\lim\limits_{m\to\infty} \int_{\Omega_2} \int_0^1 D^2 f(\nabla^2 u + t[\nabla^2 u_m - \nabla^2 u])(\nabla^2 u_m - \nabla^2 u, \nabla^2 u_m - \nabla^2 u)\, dt\, dx = 0$ and in the case $p \geq 2$ the claim follows from (1.1). Suppose now that $p < 2$. Then again by (1.1)

$$\int_0^1 \dots dt \geq \lambda \int_0^1 (1 + |\nabla^2 u + t(\nabla^2 u_m - \nabla^2 u)|^2\Big)^{\frac{p-2}{2}} |\nabla^2 u_m - \nabla^2 u|^2 (1-t)\, dt$$
$$\geq c\Big(1 + [|\nabla^2 u| + |\nabla^2 u_m|]^2\Big)^{\frac{p-2}{2}} |\nabla^2 u_m - \nabla^2 u|^2.$$

For almost all $x \in \Omega_2$ we have

$$h_m(x) \to \hat{h}(x) < \infty,$$

therefore $\lim\limits_{m\to\infty} |\nabla^2 u_m(x)|$ exists and is finite for almost all $x \in \Omega_2$ (by the definition of h_m). If we consider such points $x \in \Omega_2$ and observe that by the above estimate

$$\Big(1 + [|\nabla^2 u| + |\nabla^2 u_m|]^2\Big)^{\frac{p-2}{2}} |\nabla^2 u_m - \nabla^2 u|^2 \longrightarrow 0 \quad \text{a.e.},$$

then it is immendiate that $|\nabla^2 u_m - \nabla^2 u|^2 \longrightarrow 0$ a.e., and the claim follows. □

3 Blow–up and partial regularity

In this section we give a proof of Theorem 1.1 where for technical simplicity we restrict ourselves to the case that $p \geq 2$. The necessary adjustments concerning exponents $p \in (1, 2)$ can be found in [CFM], [BF1] or [Bi]. So let the hypothesis of Theorem 1.1 hold. Then we have the following excess–decay lemma which is the key to partial regularity.

LEMMA 3.1. *Given a positive number L, define the constant $C^*(L)$ according to (3.11) below and let $C_* := C_*(L) := 2C^*(L)$. Then, for any $\tau \in (0, 1/2)$ there exists $\varepsilon = \varepsilon(\tau, L)$ such that the validity of*

$$|(\nabla^2 u)_{x,r}| \leq L \text{ and } E(x, r) \leq \varepsilon(L, \tau) \tag{3.1}$$

for some ball $B_r(x) \subset\subset \Omega$ implies the estimate

$$E(x, \tau r) \leq \tau^2 C_*(L) E(x, r). \tag{3.2}$$

Here we have set

$$E(x, \rho) := \fint_{B_\rho(x)} |\nabla^2 u - (\nabla^2 u)_{x,\rho}|^2\, dy + \fint_{B_\rho(x)} |\nabla^2 u - (\nabla^2 u)_{x,\rho}|^q\, dy$$

for balls $B_\rho(x)$ compactly contained in Ω, and $\fint_{B_\rho(x)} g\, dy$ or $(g)_{x,\rho}$ denote the mean value of a function g w.r.t. $B_\rho(x)$. Let us recall that we consider the case $p \geq 2$, thus $q > 2$. If $p < 2$ is allowed, then $q < 2$ is possible but the statement of Lemma 3.1 (and thereby partial regularity) remains true if the excess function E then is defined according to [CFM].

REMARK 3.1. *i) It is well known how to iterate the result of Lemma 3.1 leading to the result that the set of points* $x_\circ \in \Omega$ *such that*

$$\limsup_{r\searrow 0} |(\nabla^2 u)_{x_\circ,r}| < \infty$$

together with $\liminf_{r\searrow 0} E(x_\circ, r) = 0$ *is an open set (of full Lebesgue–measure) on which the local minimizer* u *is of class* $C^{2,\nu}$ *for any* $0 < \nu < 1$. *We refer the reader to Giaquinta's text book [Gia] and mention the papers [GiuMi] of Giusti and Miranda, [Ev] of Evans or the contribution [FH] of Fusco and Hutchinson.*
ii) We will give an indirect proof of Lemma 3.1 using the blow–up technique following more or less the ideas of Evans and Gariepy outlined in [Ev] and [EG].

Proof of Lemma 3.1:
To argue by contradiction we assume that for $L > 0$ fixed and for some $\tau \in (0, 1/2)$ there exists a sequence of balls $B_{r_m}(x_m) \subset\subset \Omega$ such that

$$|(\nabla^2 u)_{x_m,r_m}| \;\le\; L,\ E(x_m, r_m) =: \lambda_m^2 \underset{m\to\infty}{\longrightarrow} 0, \tag{3.3}$$

$$E(x_m, \tau\, r_m) \;>\; C_* \tau^2 \lambda_m^2. \tag{3.4}$$

Now a sequence of rescaled functions is introduced by letting

$$\begin{aligned} a_m &:= (u)_{x_m,r_m},\ A_m := (\nabla u)_{x_m,r_m},\ \Theta_m := (\nabla^2 u)_{x_m,r_m},\\ \hat{u}_m(z) &:= \frac{1}{\lambda_m r_m^2}\Big[u_m(x_m + r_m z) - a_m - r_m A_m z\\ &\qquad -\frac{1}{2}\, r_m^2 \Theta_m(z, z) + \frac{1}{2}\, r_m^2 ⨍_{B_1} \Theta_m(\tilde{z}, \tilde{z}) d\tilde{z}\Big],\ |z| < 1. \end{aligned}$$

Direct calculations show that

$$\begin{aligned} \nabla \hat{u}_m(z) &= \tfrac{1}{\lambda_m r_m}\Big[\nabla u(x_m + r_m z) - A_m - \tfrac{1}{2}\, r_m \nabla(\Theta_m^{\alpha\beta} z_\alpha z_\beta)\Big],\\ \nabla^2 \hat{u}_m(z) &= \tfrac{1}{\lambda_m}\Big[\nabla^2 u(x_m + r_m z) - \Theta_m\Big], \end{aligned}$$

moreover, the quantities $(\hat{u}_m)_{0,1}$, $(\nabla \hat{u}_m)_{0,1}$, $(\nabla^2 \hat{u}_m)_{0,1}$ vanish for all m. From our assumptions (3.3) we get

$$⨍_{B_1} |\nabla^2 \hat{u}_m|^2 dz + \lambda_m^{q-2} ⨍_{B_1} |\nabla^2 \hat{u}_m|^q dz = \lambda_m^{-2} E(x_m, r_m) = 1, \tag{3.5}$$

and after passing to subsequences which are not relabeled we find (using Poincaré's inequality for deriving (3.7) from (3.5))

$$\Theta_m \longrightarrow: \Theta, \tag{3.6}$$

$$\hat{u}_m \rightharpoondown: \hat{u} \quad \text{in} \quad W_2^2(B_1), \tag{3.7}$$

$$\lambda_m \nabla^2 \hat{u}_m \longrightarrow 0 \quad \text{in} \quad L^2(B_1) \text{ and a.e.}, \tag{3.8}$$

$$\lambda_m^{1-2/q} \nabla^2 \hat{u}_m \rightharpoondown 0 \quad \text{in} \quad L^q(B_1). \tag{3.9}$$

After these preparations we claim that the limit function $\hat{u}$ satisfies

$$\int_{B_1} D^2 f(\Theta)(\nabla^2 \hat{u}, \nabla^2 \varphi)\, dz = 0 \quad \forall\, \varphi \in C_\circ^\infty(B_1). \tag{3.10}$$

To prove (3.10) we proceed exactly as in [Ev] (see also [BF1] and [Bi], Proposition 3.33) taking into account (3.6), (3.7) and (3.9).
Moreover, the application of Poincaré's inequality in combination with estimate (3.2) from [GiaMo1] and Lemma 7 of [Kr1] (see also [Ca1,2]) give the existence of a constant C^*, only depending on n, L, p, q, λ and Λ, such that

$$\fint_{B_\tau} |\nabla^2 \hat{u} - (\nabla^2 \hat{u})_\tau|^2 dz \le C^* \tau^2. \tag{3.11}$$

To be precise, we have

$$\fint_{B_\tau} |\nabla^2 \hat{u} - (\nabla^2 \hat{u})_\tau|^2 dz \le c\, \tau^2 \fint_{B_\tau} |\nabla^3 \hat{u}|^2\, dz \le c\, \tau^2 \fint_{B_{1/2}} |\nabla^3 \hat{u}|^2\, dz,$$

which follows from [GiaMo1], (3.2), applied to the function $v := \partial_\gamma \hat{u}$, $\gamma = 1, \ldots, n$. Moreover

$$\fint_{B_{1/2}} |\nabla^3 \hat{u}|^2\, dz \le c \sup_{B_{1/2}} |\nabla^3 \hat{u}|^2 \le c \fint_{B_1} |\nabla^2 \hat{u}|^2\, dz \le \liminf_{m\to\infty} c \fint_{B_1} |\nabla^2 \hat{u}_m|^2\, dz \le c,$$

where we used (3.5), (3.7) and [Kr1], Lemma 7. This proves (3.11) for a suitable constant C^*. Clearly (3.11) is in contradiction to (3.4), if we can improve the convergences stated in (3.8) and (3.9) to the strong convergences

$$\nabla^2 \hat{u}_m \longrightarrow \nabla^2 \hat{u} \quad \text{in} \quad L^2_{\mathrm{loc}}(B_1), \tag{3.12}$$

$$\lambda_m^{1-2/q} \nabla^2 \hat{u}_m \longrightarrow 0 \quad \text{in} \quad L^q_{\mathrm{loc}}(B_1). \tag{3.13}$$

To verify (3.12) and (3.13) we want to show first for any $0 < \rho < 1$ the identity

$$\lim_{m\to\infty} \int_{B_\rho} \left(1 + |\Theta_m + \lambda_m \nabla^2 \hat{u} + \lambda_m \nabla^2 w_m|^2\right)^{\frac{p-2}{2}} |\nabla^2 w_m|^2\, dz = 0, \tag{3.14}$$

where $w_m := \hat{u}_m - \hat{u}$. Following the basic ideas given in [EG] (see also [BF1] or [Bi], Proposition 3.34) we observe that for all $\varphi \in C^\infty_\circ(B_1), 0 \le \varphi \le 1$,

$$\lambda_m^{-2}\int_{B_1}\varphi\Big[f(\Theta_m+\lambda_m\nabla^2\hat{u}_m)-f(\Theta_m+\lambda_m\nabla^2\hat{u})\Big]\,dz$$

$$-\lambda_m^{-1}\int_{B_1}\varphi Df\Big(\Theta_m+\lambda_m\nabla^2\hat{u}\Big):\nabla^2 w_m dz \tag{3.15}$$

$$=\int_{B_1}\int_0^1 \varphi D^2 f\Big(\Theta_m+\lambda_m\nabla^2\hat{u}+s\lambda_m\nabla^2 w_m\Big)\Big(\nabla^2 w_m,\nabla^2 w_m\Big)(1-s)ds\,dz.$$

Obviously (3.14) will follow from the ellipticity of $D^2 f$, if we can show that the left–hand side of (3.15) tends to zero as $m\to\infty$. Using the minimality of u as well as the convexity of f we can estimate

$$\begin{aligned}
\text{l.h.s. of (3.15)} &\le \lambda_m^{-2}\int_{B_1} f\Big(\Theta_m+\lambda_m\nabla^2[\hat{u}_m+\varphi(\hat{u}-\hat{u}_m)]\Big)dz\\
&- \lambda_m^{-2}\int_{B_1} f\Big(\Theta_m+\lambda_m\Big[(1-\varphi)\nabla^2\hat{u}_m+\varphi\nabla^2\hat{u}\Big]\Big)dz\\
&- \lambda_m^{-1}\int_{B_1}\varphi Df(\Theta_m+\lambda_m\nabla^2\hat{u}):\nabla^2 w_m dz\\
&=: I_1-I_2-I_3.
\end{aligned}$$

Setting
$X_m := \Theta_m+\lambda_m\Big[(1-\varphi)\nabla^2\hat{u}_m+\varphi\nabla^2\hat{u}\Big]$, $Z_m := 2\nabla\varphi\otimes\nabla(\hat{u}-\hat{u}_m)+\nabla^2\varphi(\hat{u}-\hat{u}_m)$
we obtain

$$\begin{aligned}
I_1-I_2 &= \lambda_m^{-1}\int_{B_1} Df(X_m):Z_m\,dz\\
&\quad+\int_{B_1}\int_0^1 D^2 f\Big(X_m+s\lambda_m Z_m\Big)(Z_m,Z_m)(1-s)ds\,dz\\
&\le \lambda_m^{-1}\int_{B_1} Df(X_m):Z_m\,dz\\
&\quad+c\int_{B_1}\Big(1+\Big\{|\Theta_m|+\lambda_m|\nabla^2\hat{u}_m|+\lambda_m|\nabla^2\hat{u}|+\lambda_m|Z_m|\Big\}^2\Big)^{\frac{q-2}{2}}|Z_m|^2 dz.
\end{aligned}$$

With the notation $\epsilon(m)\to 0$ as $m\to\infty$ we get on account of (3.7) that the last integral can be estimated from above by

$$c\int_{B_1}\lambda_m^{q-2}|\nabla\hat{u}_m|^{q-2}|Z_m|^2dz+c\int_{B_1}\lambda_m^{q-2}|Z_m|^q dz+\epsilon(m).$$

Furthermore,

$$\begin{aligned}
J_1 &:= c\int_{B_1}\lambda_m^{q-2}|\nabla\hat{u}_m|^{q-2}|Z_m|^2dz\\
&\le c\int_{\operatorname{spt}\varphi}\lambda_m^{q-2}|\nabla^2\hat{u}_m|^{q-2}\Big\{|\nabla\hat{u}-\nabla\hat{u}_m|+|\hat{u}-\hat{u}_m|\Big\}^2dz\\
&\le c\Big\{\int_{\operatorname{spt}\varphi}\lambda_m^{q-2}|\nabla^2\hat{u}_m|^q dz\Big\}^{1-2/q}\Big\{\lambda_m^{q-2}\int_{\operatorname{spt}\varphi}|\nabla\hat{u}-\nabla\hat{u}_m|^q dz\\
&\quad+\lambda_m^{q-2}\int_{\operatorname{spt}\varphi}|\hat{u}-\hat{u}_m|^q dz\Big\}^{2/q}\\
&\le c\Big\{\lambda_m^{q-2}\int_{\operatorname{spt}\varphi}|\nabla\hat{u}-\nabla\hat{u}_m|^q dz+\lambda_m^{q-2}\int_{\operatorname{spt}\varphi}|\hat{u}-\hat{u}_m|^q dz\Big\}^{2/q},
\end{aligned}$$

where the last inequality follows from (3.9). We also note that due to (3.9) $\lambda_m^{1-2/q}\nabla^k\hat{u}_m\underset{m\to\infty}{\longrightarrow}0$ in $L^q(B_1)$ for $k=0,1$. This immediately implies

$$J_1\le\epsilon(m)\to 0\quad\text{as}\quad m\to\infty.$$

Analogous arguments applied to

$$J_2:=c\int_{B_1}\lambda_m^{q-2}|Z_m|^q dz$$

guarantee that

$$J_2\le\epsilon(m)\to 0\quad\text{as}\quad m\to\infty.$$

Thus, we arrive at

$$\text{l.h.s. of (3.15)}\le\epsilon(m)+\lambda_m^{-1}\Big[\int_{B_1}Df(X_m):Z_m\,dz-\int_{B_1}Df(\Theta_m+\lambda_m\nabla^2\hat{u}):\nabla^2 w_m\varphi\,dz\Big]. \tag{3.16}$$

Next we are going to discuss the last two integrals in (3.16). Since

$$\nabla^2(\varphi w_m)=\nabla^2 w_m\varphi-Z_m,$$

we have that

$$[\dots] = \int_{B_1} \Big(Df(X_m) - Df(\Theta_m + \lambda_m \nabla^2 \hat{u}) \Big) : Z_m \, dz$$

$$- \int_{B_1} Df\Big(\Theta_m + \lambda_m \nabla^2 \hat{u}) : \nabla^2(\varphi w_m) dz =: I_4 - I_5.$$

From (1.1) and from the requirement that $0 \le \varphi \le 1$ we obtain by recalling the definition of Z_m

$$\begin{aligned}
I_4 &= \int_{B_1} \Big(Df\Big(\Theta_m + \lambda_m[(1-\varphi)\nabla^2 \hat{u}_m + \varphi \nabla^2 \hat{u}]\Big) - Df(\Theta_m + \lambda_m \nabla^2 \hat{u})\Big) : Z_m \, dz \\
&= \int_{B_1} \int_0^1 \frac{d}{ds} Df\Big(\Theta_m + \lambda_m \nabla^2 \hat{u} + s\lambda_m (1-\varphi)\nabla^2(\hat{u}_m - \hat{u})\Big) ds : Z_m \, dz \\
&= \lambda_m \int_{B_1} \int_0^1 D^2 f\Big(\Theta_m + \lambda_m \nabla^2 \hat{u} + s\lambda_m(1-\varphi)\nabla^2 w_m)(\nabla^2 w_m, Z_m)(1-\varphi) ds \, dz \\
&\le \lambda_m c \int_{B_1} \Big(1 + (|\Theta_m| + \lambda_m |\nabla^2 \hat{u}| + \lambda_m |\nabla^2 w_m|)^2\Big)^{\frac{q-2}{2}} \\
&\quad \cdot |\nabla^2 w_m| \Big[|\nabla \varphi| \, |\nabla w_m| + |\nabla^2 \varphi| \, |w_m| \Big] dz,
\end{aligned}$$

and similar to the previous discussion of J_1 we get

$$\lambda_m^{-1} I_4 \to 0 \quad \text{as} \quad m \to \infty.$$

Finally, we observe that

$$\begin{aligned}
\lambda_m^{-1} I_5 &= \lambda_m^{-1} \int_{B_1} \Big(Df(\Theta_m + \lambda_m \nabla^2 \hat{u}) - Df(\Theta_m) \Big) : \nabla^2(\varphi w_m) dz \\
&= \lambda_m^{-1} \int_{B_1} \int_0^1 D^2 f(\Theta_m + s\lambda_m \nabla^2 \hat{u}) \Big(\lambda_m \nabla^2 \hat{u}, \nabla^2(\varphi w_m) \Big) ds \, dz,
\end{aligned}$$

and, consequently, $\lambda_m^{-1} I_5$ vanishes after passing to the limit $m \to \infty$ on account of the weak convergence (3.7). Summarizing these results we have shown that $\lim_{m \to \infty}$ (l.h.s. of (3.15)) = 0.
Therefore, identity (3.14) is proved, and (3.12) immediately follows from (3.14) since we

assume that $p \geq 2$. To proceed further, i.e. to prove the strong convergence stated in (3.13), we introduce the auxiliary functions

$$\Psi_m(z) := \lambda_m^{-1}\Big[(1+|\Theta_m+\lambda_m\nabla^2\hat{u}_m(z)|^2)^{p/4}-(1+|\Theta_m|^2)^{p/4}\Big].$$

For any $\rho < 1$ Lemma 2.2 implies

$$\int_{B_\rho}|\nabla\Psi_m|^2 dz = \lambda_m^{-2}\, r_m^{2-n}\int_{B_{\rho r_m}(x_m)}|\nabla h|^2\, dx$$

$$\leq\ c\,(\rho)\lambda_m^{-2}r_m^{2-n}\int_{B_{r_m}(x_m)}|D^2 f(\nabla^2 u)|\cdot\Big\{r_m^{-2}|\nabla^2(u-P)|^2+r_m^{-4}|\nabla(u-P)|^2\Big\}\, dx.$$

For the last estimate we used inequality (2.12), h being defined in Lemma 2.2 and P representing a polynomial function of degree ≤ 2. If we choose

$$P(x) := A_m\, x+\frac{1}{2}\Theta_m(x-x_m,\, x-x_m)\quad\text{for}\quad x\in B_{r_m}(x_m)$$

we get

$$\begin{aligned}\nabla(u(x)-P(x)) &= \lambda_m r_m\nabla\hat{u}_m\Big(\tfrac{x-x_m}{r_m}\Big),\\ \nabla^2(u(x)-P(x)) &= \lambda_m\nabla^2\hat{u}_m\Big(\tfrac{x-x_m}{r_m}\Big).\end{aligned}$$

So, taking into account (3.7) and (3.9) we obtain for any $\rho < 1$ the inequality

$$\begin{aligned}\int_{B_1}|\nabla\Psi_m|^2 dz &\leq c(\rho)\int_{B_1}|D^2 f(\Theta_m+\lambda_m\nabla^2\hat{u}_m)|\cdot\Big\{|\nabla^2\hat{u}_m|^2+|\nabla\hat{u}_m|^2\Big\}\, dz\\ &\leq c(\rho)<\infty.\end{aligned}\tag{3.17}$$

In addition, one can write

$$\begin{aligned}|\Psi_m| &\leq c\int_0^1|\nabla^2\hat{u}_m|\Big(1+|\Theta_m+s\lambda_m\nabla^2\hat{u}_m|^2\Big)^{\frac{p-2}{4}}\, ds\\ &\leq c\Big\{|\nabla^2\hat{u}_m|+\lambda_m^{\frac{p-2}{2}}|\nabla^2\hat{u}_m|^{p/2}+1\Big\}.\end{aligned}\tag{3.18}$$

It follows from (3.14) that

$$\int_{B_\rho}\lambda_m^{p-2}|\nabla^2\hat{u}_m|^p\, dx\leq c\,(\rho)<\infty.$$

Combining the last estimate with (3.17) and (3.18) we can conclude that the sequence Ψ_m is bounded in $W^1_{2,\mathrm{loc}}(B_1)$. Now we proceed as follows: consider a number $M >> 1$ and let

$$U_m := \{z\in B_\rho : \lambda_m|\nabla^2\hat{u}_m|\leq M\}.$$

Then

$$\int_{U_m} \lambda_m^{q-2} |\nabla^2 \hat{u}_m|^q \, dz \le c \Big\{ \int_{U_m} \lambda_m^{q-2} |\nabla^2 w_m|^q \, dz + \int_{U_m} \lambda_m^{q-2} |\nabla^2 \hat{u}|^q \, dz \Big\}$$

$$\le \quad c \Big\{ \int_{U_m} \lambda_m^{q-2} \Big(|\nabla^2 \hat{u}_m|^{q-2} + |\nabla^2 \hat{u}|^{q-2} \Big)$$

$$\cdot \, |\nabla^2 w_m|^2 \, dz + \int_{U_m} \lambda_m^{q-2} |\nabla^2 \hat{u}|^q \, dz \Big\} \tag{3.19}$$

$$\le \quad c \Big\{ \int_{B_\rho} (M^{q-2} + |\nabla^2 \hat{u}|^{q-2}) |\nabla^2 w_m|^2 \, dz + \int_{B_\rho} \lambda_m^{q-2} |\nabla^2 \hat{u}|^q \, dz \Big\}$$

$$\to 0 \quad \text{as} \quad m \to \infty$$

on account of $\nabla^2 w_m \to 0$ in $L^2(B_\rho)$ and $\nabla^2 \hat{u} \in L^\infty(B_\rho)$. On the other hand, if we choose M sufficiently large, then on $B_\rho - U_m$ we get

$$\Psi_m(z) \ge c \, \lambda_m^{-1+p/2} \, |\nabla^2 \hat{u}_m|^{p/2}$$

and, consequently

$$|\nabla^2 \hat{u}_m|^q \, \lambda_m^{q-2} \le c \lambda_m^{2\frac{q}{p}-2} \, \Psi_m^{\frac{2q}{p}} .$$

Since (1.3) guarantees $\frac{2q}{p} < \frac{2n}{n-2}$ and since Ψ_m is uniformly bounded in $W^1_{2,\mathrm{loc}}(B_1)$, we can conclude

$$\int_{B_\rho - U_m} \lambda_m^{q-2} |\nabla^2 \hat{u}_m|^q \, dz \to 0 \text{ as } m \to \infty \text{ for any } \rho < 1. \tag{3.20}$$

It only remains to note that obviously the results (3.19) and (3.20) provide (3.13), which completes the proof. □

References

[AF] Acerbi, E., Fusco, N., Partial regularity under anisotropic (p, q) growth conditions. J. Diff. Equ. 107, no.1 (1994), 46–67.

[Ad] Adams, R.A., Sobolev spaces. Academic Press, New York–San Francisco–London, 1975.

[Bi] Bildhauer, M., Convex variational problems. Linear, nearly linear and anisotropic growth conditions. Lecture Notes in Mathematics 1818, Springer, Berlin-Heidelberg-New York, 2003.

[BF1] Bildhauer, M., Fuchs, M., Partial regularity for variational integrals with (s, μ, q)–growth. Calc. Var. 13 (2001), 537–560.

[BF2] Bildhauer, M., Fuchs, M., Higher order variational problems on two–dimensional domains. to appear in Ann. Acad. Sci. Fenn. Math.

[BF3] Bildhauer, M., Fuchs, M., $C^{1,\alpha}$–solutions to non–autonomous anisotropic variational problems. Calc. Var. 24 (2005), 309–340.

[Ca1] Campanato, S., Equazioni ellitichi del secundo ordine e spazi $\mathcal{L}^{2,\lambda}$. Ann. Mat. Pura Appl. 69 (1965), 321–381.

[Ca2] Campanato, S., Alcune osservazioni relative alle soluzioni di equazioni ellitiche di ordine $2m$, in: Atti Convegno Equaz. Der. Parz., Bologna 1967, pp. 17–25.

[CFM] Carozza, M., Fusco, N., Mingione, G., Partial regularity of minimizers of quasiconvex integrals with subquadratic growth. Ann. Mat. Pura e Appl. 4 (1998), 141–164.

[DGG] Duzaar, F., Gastel, A., Grotowski, J.F., Optimal partial regularity for nonlinear elliptic systems of higher order. J. Math. Sci. Univ. Tokyo 8 (2001), 463–499.

[ELM1] Esposito, L., Leonetti, F., Mingione, G., Regularity results for minimizers of irregular integrals with (p,q)–growth. Forum Math. 14 (2002), 245–272.

[ELM2] Esposito, L., Leonetti, F., Mingione, G., Regularity for minimizers of functionals with p-q growth. Nonlinear Diff. Equ. Appl. 6 (1999), 133-148.

[ELM3] Esposito, L., Leonetti, F., Mingione, G., Sharp regularity for functionals with (p, q) growth. J. Diff. Equ. 204 (2004), 5–55.

[Ev] Evans, L.C., Quasiconvexity and partial regularity in the calculus of variations. Arch. Rat. Mech. Anal. 95 (1986), 227–252.

[EG] Evans, L.C., Gariepy, R.F., Blow–up, compactness and partial regularity in the calculus of variations. Ind. Univ. Math. J. 36 no.2 (1987), 361–371.

[FH] Fusco, N., Hutchinson, J., $C^{1,\alpha}$–partial regularity of functions minimizing quasiconvex integrals. Manus. Math. 54 (1985), 121–143.

[Gi] Giaquinta, M., Multiple integrals in the calculus of variations and non-linear elliptic systems. Ann. of Math. Studies 105, Princeton UP, Princeton 1983.

[GiaMo1] Giaquinta, M., Modica, G., Regularity results for some classes of higher order nonlinear elliptic systems. J. Reine Angew. Math. 311 (1979), 145–169.

[GiaMo2] Giaquinta, M., Modica, G., Almost everywhere regularity results for solutions of nonlinear elliptic systems. Manus.Math. 28 (1979), 109–158.

[GiuMi] Giusti, E., Miranda, M., Sulla regolarità delle soluzioni deboli di una classe di sistemi ellitici quasi–lineari. Arch. Rat. Mech. Anal. 31 (1968), 173–184.

[Kr1] Kronz, M., Partial regularity results for minimizers of quasiconvex functionals of higher order. Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002), no.1, 81–112.

[Kr2] Kronz, M., Quasimonotone systems of higher order. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.(8) 6 (2003), no.2, 459–480.

[Ma1] Marcellini, P., Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. Arch. Rat. Mech. Anal. 105 (1989), 267–284.

[Ma2] Marcellini, P., Regularity and existence of solutions of elliptic equations with (p, q) growth conditions. J. Diff. Equ. 90 (1991), 1–30.

[Ma3] Marcellini, P., Regularity for elliptic equations with general growth conditions. J. Diff. Equ. 105 (1993), 296–333.

[Mo] Morrey, C.B., Multiple integrals in the calculus of variations. Grundlehren der math. Wiss. in Einzeldarstellungen 130, Springer, Berlin–Heidelberg–New York 1966.

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