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Michael Didas and Jörg Eschmeier

Let $D \subset \mathbb{C}^n$ be a relatively compact strictly pseudoconvex open set or a bounded symmetric and circled domain, and let S denote the Shilov boundary of D. Given Hilbert $A(D)$-modules H, J and K, we prove that if the $A(D)$-module structure on H or K extends to a Hilbert $C(S)$-module structure, then each short exact sequence $0 \to H \to J \to K \to 0$ splits in the category of Hilbert $A(D)$-modules.

In their book [5] from 1989, Douglas and Paulsen presented a first systematic study of Hilbert modules over function algebras. One of the main obstacles in using standard methods from homological algebra in this setting is that Hilbert module categories may not have enough projective and injective objects. At the early stage of the theory it was not even clear whether there is any function algebra A allowing projective Hilbert modules other than $A = C(X)$ in which case every Hilbert module is projective (see Problem 4.6 in [5]). In 1994 Carlson, Clark, Foias and Williams succeeded to show that Hilbert modules with a unitary module action are projective objects in the category of all Hilbert modules over the disc algebra $A(\mathbb{D})$. In other words, a sequence of Hilbert $A(\mathbb{D})$-modules

$$0 \longrightarrow H \longrightarrow J \longrightarrow K \longrightarrow 0$$

splits under the condition that K extends to a Hilbert $C(\partial\mathbb{D})$-module. Imposing an additional weak* continuity assumption on the module action, Guo [6] was able to prove a multi-variable analogue of this result in the category of the so-called *normal* Hilbert $A(\mathbb{B})$-modules (see Section 1 below for a precise definition of normality) over the open Euclidean unit ball $\mathbb{B}$ in $\mathbb{C}^n$, $n \geq 1$. It is the aim of this work to show that the normality condition in Guo's result can be dropped from the hypotheses. The idea is to use a decomposition theorem for $A(\mathbb{B})$-functional calculi in order to separate each short exact sequence of Hilbert $A(\mathbb{B})$-modules into a discrete and a continuous part. The continuous part, consisting of normal Hilbert modules, can be treated by the methods of Guo. On the discrete part, the module action is given by the multiplication with complex scalars and therefore it splits trivially. Along the way we replace (as indicated by Guo in [6]) the unit ball $\mathbb{B}$ by an arbitrary strictly pseudoconvex set $D \subset \mathbb{C}^n$. Finally we show that $\mathbb{B}$ may also be replaced by a bounded symmetric and circled domain D, using the fact that each such domain D is contained in a suitably chosen Euclidean ball B in such a way that the Shilov boundary of D is contained in the corresponding sphere ∂B.

1 Notations and preliminaries

Let H be a separable Hilbert space and A a unital Banach algebra. Recall that a representation of A (or an A-functional calculus) $\Phi : A \to L(H)$ is a norm continuous unital algebra homomorphism from A to the C^*-algebra $L(H)$ of all bounded linear operators on H. The Hilbert space H is said to be a *Hilbert A-module* if it is an A-module (in the algebraic sense) with the additional property that the module multiplication $A \times H \to H$ is norm-continuous. By assigning with each representation $\Phi : A \to L(H)$ a module multiplication via the formula

$$A \times H \to H, \quad (f, x) \mapsto f \cdot x = \Phi(f)x \qquad (f \in A,\ x \in H)$$

one obtains a one-to-one correspondence between the representations of A and the Hilbert A-module structures on H. A *module homomorphism* $L \in \mathrm{Hom}_A(H, K)$ between two Hilbert A-modules H and K is a continuous linear map $L : H \to K$ satisfying $L(f \cdot x) = f \cdot L(x)$ for all $f \in A$ and $x \in H$. The category of all Hilbert A-modules with the corresponding homomorphisms will be abbreviated by $\mathscr{H}(A)$ in the sequel.

We say that a Hilbert A-module H is *contractive* (*isometric*) if the underlying representation Φ is a contraction (an isometry, respectively), while H is *cramped* if there exists a contractive Hilbert A-module K which is *similar* to H in the sense that there is a bijective module homomorphism (*similarity*) $L : H \to K$. The *cramped category* $\mathscr{C}(A)$ consists of all cramped Hilbert A-modules as objects and all (not necessarily contractive) A-module homomorphisms between any two such objects as morphisms.

Let A be a *dual algebra*, that is, a Banach algebra which carries a natural weak* topology as the dual space of a Banach space such that the multiplication on A is separately weak* continuous. A Hilbert A-module H is called *normal* if, for each $x \in H$, the mapping $A \to H, \quad f \mapsto f \cdot x$ is weak*-weak continuous. In the case that A has a separable predual it is not hard show that this is equivalent to the underlying representation $\Phi : A \to L(H)$ being weak* continuous (where the weak* topology on $L(H)$ is induced by the trace duality). Again the normal Hilbert A-modules together with the set of (ordinary) A-module homomorphisms form a category, called $\mathscr{N}(A)$.

Let $\mathscr{X}$ be any one of the Hilbert A-module categories defined above. Two short exact sequences

$$E : 0 \to H \overset{\alpha}{\to} J \overset{\beta}{\to} K \to 0 \quad \text{and} \quad E' : 0 \to H \overset{\alpha'}{\to} J' \overset{\beta'}{\to} K \to 0$$

in $\mathscr{X}$ are called *equivalent* if there exists a map $\theta \in \mathrm{Hom}_A(J, J')$ making the diagram

$$\begin{array}{ccccccccc} E:0 & \longrightarrow & H & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & K & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\theta} & & \| & & \\ E':0 & \longrightarrow & H & \xrightarrow{\alpha'} & J' & \xrightarrow{\beta'} & K & \longrightarrow & 0 \end{array}$$

commutative. The first cohomology group is defined by

$$\mathrm{Ext}^1_{\mathscr{X}}(K, H) = \Big\{[E],\ E : 0 \to H \overset{\alpha}{\to} J \overset{\beta}{\to} K \to 0 \text{ exact sequence in } \mathscr{X}\Big\}$$

where $[E]$ stands for the corresponding equivalence class of the short exact sequence E. The zero element of $\mathrm{Ext}^1_{\mathscr{X}}(K, H)$ is the equivalence class of the split extension

$$0 \longrightarrow H \xrightarrow{\iota} H \oplus K \xrightarrow{\pi} K \longrightarrow 0$$

where $\iota(h) = h \oplus 0$ and $\pi(h \oplus k) = k$ for $h \in H, k \in K$.

In the three cases $\mathscr{X} = \mathscr{H}(A)$, $\mathscr{C}(A)$ or $\mathscr{N}(A)$ one can show that $\mathrm{Ext}^1_{\mathscr{X}}(\cdot, \cdot)$ is a bi-functor from the category $\mathscr{X}$ to the category of A-modules (cp. [2], [3], [6]).

A simple description of $\mathrm{Ext}^1_{\mathscr{X}}(K, H)$ is known for $\mathscr{X} = \mathscr{H}(A)$ or $\mathscr{X} = \mathscr{N}(A)$. To point this out let, with the notations from above, the sequence E be exact. Then J possesses a decomposition as orthogonal direct sum

$$J = \alpha(H) \oplus \alpha(H)^{\perp} \cong H \oplus K$$

of Hilbert *spaces*, but since in contrast to the image $\alpha(H)$ of an A-module map the orthogonal complement $\alpha(H)^{\perp}$ may not be invariant under the module multiplication of J, the above decomposition is in general not a sum of Hilbert A-modules. Identifying $J \cong H \oplus K$ as Hilbert spaces, the module multiplication on J can be represented as

$$f \cdot \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} f \cdot h + \sigma(f, k) \\ f \cdot k \end{pmatrix} \qquad (f \in A,\ h \in H,\ k \in K),$$

where $\sigma : A \times K \to H$ is easily seen to be a continuous bilinear map satisfying the so-called cocycle identity

$$f \cdot \sigma(g,k) = \sigma(fg,k) - \sigma(f, g \cdot k) \qquad (f,g \in A,\ k \in K).$$

Such a map $\sigma : A \times K \to H$ is called a 1-*cocycle*. Note that the 1-cocycles arising in this way in the category $\mathscr{X} = \mathscr{N}(A)$ are also *normal*, which means that $\sigma(\cdot, k) : A \to H$ is weak*-weak continuous for each $k \in K$. We write

$$C_{\mathscr{H}(A)}(K,H) \qquad \text{and} \qquad C_{\mathscr{N}(A)}(K,H)$$

for the vector space of all 1-cocycles (normal 1-cocycles, respectively). Given any bounded linear operator $T \in L(K,H)$ we obtain a 1-cocycle σ_T (even being normal in the case $\mathscr{X} = \mathscr{N}(A)$) by setting

$$\sigma_T : A \times K \to H, \quad (f,k) \mapsto f \cdot T(k) - T(f \cdot k).$$

Writing $B(K,H) = \{\sigma_T : T \in L(K,H)\}$ for the vector space of all these so-called 1-*coboundaries* we are ready to state the announced description of the first cohomology group. For a proof of the following result, compare Theorem 2.2.2 in [2] and Proposition 2.3 in [6].

1.1 Proposition. *In the categories $\mathscr{X} = \mathscr{H}(A)$ and $\mathscr{X} = \mathscr{N}(A)$, the assignment*

$$\mathrm{Ext}^1_{\mathscr{X}}(K,H) \to C_{\mathscr{X}}(K,H)/B(K,H), \quad [E] \mapsto [\sigma],$$

where σ is the 1-cocycle induced by E in the way pointed out above, is a bijection. $\square$

For further reference we remark that if there are similarities $H \xrightarrow{R} H'$ and $K \xrightarrow{S} K'$, then $\mathrm{Ext}^1_{\mathscr{X}}(K,H)$ and $\mathrm{Ext}^1_{\mathscr{X}}(K',H')$ are isomorphic. To prove this quickly in the situation of the preceding proposition, we define a map

$$\gamma : C_{\mathscr{X}}(K,H) \to C_{\mathscr{X}}(K',H'), \quad \sigma \mapsto \sigma',$$

where $\sigma'(f,k') = R\sigma(f, S^{-1}k')$ for $f \in A$ and $k' \in K'$. It is easy to check that γ is well-defined and bijective (γ^{-1} has the same structure), and maps $B_{\mathscr{X}}(K,H)$ onto $B_{\mathscr{X}}(K',H')$ since $\sigma'_T = \sigma_{RTS^{-1}}$. Hence the induced map between the quotient spaces is the desired isomorphism.

Let us now turn to some general results on Hilbert modules over algebras of continuous and bounded measurable functions. Let $C(K)$ denote the C^*-algebra of all complex-valued continuous functions on a compact set $K \subset \mathbb{C}^n$, and let $M^+(K)$ be the set of all finite positive regular Borel measures on K. The structure theory of Hilbert $C(K)$-modules seems to be completely understood and can be found in detail in [5]. For our purposes, we only need a few basic results of the theory. The first one says that, as far as Ext^1-groups are concerned, we can restrict ourselves to contractive $C(K)$-modules (see Theorem 1.9 in [5]):

1.2 Proposition. *Each Hilbert $C(K)$-module is similar to a contractive Hilbert $C(K)$-module.* □

By the definition of a contractive Hilbert $C(K)$-module, the underlying representation $\Phi : C(K) \to L(H)$ is contractive and hence a $*$-homomorphism. This implies that the tuple $Z = (\Phi(z_1), \ldots, \Phi(z_n)) \in L(H)^n$ of module multiplication with the coordinate functions is a commuting tuple of normal operators with Taylor spectrum $\sigma(Z) \subset K$. If $\nu \in M^+(K)$ denotes a scalar-valued spectral measure for Z, then Φ possesses an extension to an isometric and weak*-continuous functional calculus $\Psi : L^\infty(\nu) \to L(H)$. In the language of modules, this fact reads as follows.

1.3 Proposition. *Every contractive Hilbert $C(K)$-module extends to a normal and isometric Hilbert $L^\infty(\nu)$-module for a suitably chosen measure $\nu \in M^+(K)$.* □

In our context K will be the boundary ∂D of a relatively compact strictly pseudoconvex open set $D \subset \mathbb{C}^n$ in the sense that there exist an open neighborhood U of ∂D and a strictly plurisubharmonic C^2-function $\rho : U \to \mathbb{R}$ such that $D \cap U = \{z \in U : \rho(z) < 0\}$. Note that the boundary ∂D is not assumed to be smooth. The objects we are interested in are Hilbert $A(D)$-modules, where $A(D)$ denotes the algebra of all continuous functions $f : \bar{D} \to \mathbb{C}$ that are holomorphic on D. The supremum norm on $\bar{D}$ turns $A(D)$ into a Banach algebra. By the maximum modulus principle, the restriction to boundary values yields an isometric embedding $A(D) \hookrightarrow C(\partial D)$, hence each Hilbert $C(\partial D)$-module induces a Hilbert $A(D)$-module in a natural way.

2 Decomposition of Hilbert $A(D)$-modules

Our first aim is to establish an orthogonal decomposition of a given Hilbert $A(D)$-module into a discrete and a continuous part where the latter one has a nice extension property. To be more specific, a Hilbert $A(D)$-module H will be called ζ-*atomic* for some $\zeta \in \partial D$ if

$$f \cdot x = f(\zeta)x \qquad (f \in A(D))$$

holds for each $x \in H$, while we call H *continuous* if it does not contain any ζ-atomic $A(D)$-submodule at all. An orthogonal direct sum of atomic $A(D)$-modules is called *discrete*. If H is a ζ-atomic $A(D)$-module for some $\zeta \in \partial D$ and $\alpha \in \mathrm{Hom}_{A(D)}(H, K)$ is a homomorphism, then, given $x \in H$ and $f \in A(D)$, we have $f \cdot \alpha(x) = \alpha(f \cdot x) = \alpha(f(\zeta)x) = f(\zeta)\alpha(x)$. Hence images of ζ-atomic $A(D)$-modules are again ζ-atomic.

Before we give a precise formulation of the announced decomposition result we have to provide the measure theoretical framework the proof is based on. Given an arbitrary regular complex Borel measure $\mu \in M(\bar{D})$, we define the dual algebra $H^\infty(\mu)$ to be the weak* closure of the image of the contraction $A(D) \to L^\infty(\mu)$, $f \mapsto [f]$. We say that μ is a *Henkin measure* if the latter map extends to a weak* continuous contraction

$$r_\mu : H^\infty(D) \to H^\infty(\mu),$$

where $H^\infty(D)$ stands for the dual algebra of all bounded holomorphic functions on D. Recall that the dual algebra structure on $H^\infty(D)$ is inherited from the inclusion $H^\infty(D) \subset L^\infty(\lambda)$, λ denoting the Lebesgue measure of $\mathbb{C}^n$ restricted to D. The set of all Henkin measures on $\bar{D}$ will be denoted by $HM(\bar{D})$. We say that μ is a *faithful Henkin measure* if the induced map r_μ is an isomorphism of dual algebras (i.e. a weak* continuous isometric isomorphism). If the boundary ∂D is smooth, then the surface measure is a faithful Henkin measure. The fact that faithful Henkin measures supported by ∂D do also exist in the case of non-smooth boundary can be shown by using operator theoretical methods from dilation theory (see [4], Proposition 5.2.1).

2.1 Proposition. *For each relatively compact strictly pseudoconvex open set $D \subset \mathbb{C}^n$ there exists a faithful Henkin probability measure $\sigma \in HM(\bar{D})$ satisfying $\sigma(D) = 0$.* □

A *band of measures* is a closed subspace $\mathcal{B} \subset M(\bar{D})$ such that each measure $\nu \in M(\bar{D})$ with $\nu \ll \mu$ for some $\mu \in \mathcal{B}$ also belongs to $\mathcal{B}$. By the Theorem of Henkin (see Theorem 2.2.2 in [4]), $HM(\bar{D})$ is a band of measures. Collecting all the measures that are singular to each measure in $HM(\bar{D})$, we obtain the band $S(\bar{D})$ which allows a decomposition $M(\bar{D}) = S(\bar{D}) \oplus_1 HM(\bar{D})$ defined in the obvious way. The dual space of a band $\mathcal{B}$ can be identified with the von Neumann algebra

$$L^\infty(\mathcal{B}) = \big\{(f_\mu) \in \prod_{\mu\in\mathcal{B}} L^\infty(\mu) : f_\nu = f_\mu \;\; (\mu - \text{a.e.}) \;\; \forall \mu,\nu \in \mathcal{B} \text{ with } \mu \ll \nu\big\},$$

carrying the norm $\|f\| = \sup_{\mu\in\mathcal{B}} \|f_\mu\|_{\infty,\mu} \quad (f = (f_\mu) \in L^\infty(\mathcal{B}))$. Given a band $\mathcal{B} \subset M(\bar{D})$ we define a dual subalgebra of $L^\infty(\mathcal{B})$ by

$$H^\infty(\mathcal{B}) = \overline{A(D)}^{(w^*,L^\infty(\mathcal{B}))}.$$

Using the fact that there are faithful Henkin measures in $HM(\bar{D})$ it is elementary to check that the map

$$r : H^\infty(D) \to H^\infty(HM(\bar{D})), \quad f \mapsto (r_\mu(f))_\mu$$

is a dual algebra isomorphism. As carried out in the proof of Lemma 2.2.9 in [4], the identification

$$H^\infty(M(\bar{D})) = L^\infty(S(\bar{D})) \oplus_\infty H^\infty(HM(\bar{D}))$$

arises by dualizing the identity $M(\bar{D}) = S(\bar{D}) \oplus_1 HM(\bar{D})$. A detailed discussion of these aspects of measure theory and the underlying function theory on strictly pseudoconvex sets can be found, for instance, in Section 2 of [4].

Finally, we call an arbitrary regular complex Borel measure $\mu \in M(\bar{D})$ *continuous*, if one-point sets have μ-measure zero. Note that there is an at most countable set $A_\mu \subset \bar{D}$ such that $\mu(\{a\}) > 0$ for each $a \in A_\mu$. The elements of A_μ are called *atoms* of μ. Defining μ_a and μ_c to be the trivial extensions of $\mu|A_\mu$ and $\mu|\bar{D} \setminus A_\mu$ to measures in $M(\bar{D})$ we obtain a decomposition $\mu = \mu_a + \mu_c$ of μ into a *purely atomic* part μ_a and a continuous part μ_c being clearly singular to each other.

2.2 Theorem. *Let H be a Hilbert $A(D)$-module.*

(a) *There exists a unique countable subset $A_H \subset \partial D$ and a family H_d^ζ of non-zero ζ-atomic Hilbert $A(D)$-modules ($\zeta \in A_H$) as well as a continuous Hilbert $A(D)$-module H_c such that H is similar to the orthogonal direct sum*

$$H \cong \bigoplus_{\zeta\in\partial D} H_d^\zeta \;\oplus\; H_c \qquad (\text{where } H_d^\zeta = 0 \text{ for } \zeta \notin A_H)$$

This representation is unique up to similarity.

(b) *There exists a continuous measure $\mu_H \in M^+(\partial D)$ with the property that the $A(D)$-module structure of H_c can be extended to a normal $H^\infty(\mu_H)$-module structure.*

(c) *If $M \subset H$ is any $A(D)$-submodule of H extending to a normal $H^\infty(\mu)$-module for some continuous measure $\mu \in M^+(\partial D)$, then $M \subset H_c$ via the identification of part (a).*

Proof. Let $\Phi : A(D) \to L(H)$ be the underlying representation of the $A(D)$-module structure on H, and let $i : C^1(H) \to L(H)'$ be the canonical embedding of the trace class $C^1(H)$ into its second dual. The composition

$$\hat{\Phi} : A(D)'' \xrightarrow{\Phi''} L(H)'' \xrightarrow{i'} L(H)$$

is easily seen to be a weak* continuous linear extension of Φ. General duality theory yields the identification

$$A(D)'' = \overline{A(D)}^{(w^*, L^\infty(M(\bar{D}))} = H^\infty(M(\bar{D}))$$

where the dual algebra on the right possesses the decomposition

$$H^\infty(M(\bar{D})) = L^\infty(S(\bar{D})) \oplus_\infty H^\infty(HM(\bar{D})).$$

Thus Φ extends to a weak* continuous representation (multiplicativity follows by a density argument)

$$\hat{\Phi} : L^\infty(S(\bar{D})) \oplus H^\infty(HM(\bar{D})) \to L(H)$$

which induces a (*not necessarily orthogonal*) direct sum decomposition

$$H = H_s + H_a,\ H_s \cap H_a = 0 \quad \text{where } H_s = \hat{\Phi}(1 \oplus 0)H,\ H_a = \hat{\Phi}(0 \oplus 1)H.$$

Now let $\sigma \in M^+(\bar{D})$ be a faithful Henkin measure with $\sigma(D) = 0$. Then the restriction of $\hat{\Phi}$ to the H^∞-part induces a weak* continuous representation

$$\Psi_a : H^\infty(\sigma) \xrightarrow{r_\sigma^{-1}} H^\infty(D) \xrightarrow{r} H^\infty(HM(\bar{D})) \xrightarrow{\hat{\Phi}(0\oplus\cdot)} L(H_a),$$

while the L^∞-part $L^\infty(S(\bar{D})) \to L(H_s)$, $f \mapsto \hat{\Phi}(f \oplus 0)|H_s$ is a bounded homomorphism from a unital commutative C^*-algebra to $L(H)$ and thus is similar to a contractive representation

$$\hat{\Psi}_s : L^\infty(S(\bar{D})) \to L(H_s), \quad \hat{\Psi}_s(f) = S \circ \hat{\Phi}(f \oplus 0)|H_s \circ S^{-1}$$

with an invertible bounded linear map $S : H_s \to H_s$. Since $\hat{\Psi}_s$ maps orthogonal projections onto orthogonal projections it is a $*$-homomorphism. Exactly as in [4], Lemma 3.2.4, it can be shown that there exists a measure $\nu \in S(\bar{D})$ such that the tuple $(\hat{\Psi}_s(z_1), \cdots, \hat{\Psi}_s(z_n))$ is a commuting tuple of normal operators on H_s possessing an isometric and weak* continuous functional calculus

$$\Psi_s : L^\infty(\nu) \to L(H_s).$$

The exterior orthogonal direct sum $K = H_s \oplus H_a$ equipped with the norm

$$\|(x_s, x_a)\|^2 = \|x_s\|^2 + \|x_a\|^2 \qquad (x_s \in H_s,\ x_a \in H_a)$$

is a Hilbert space which is similar to H as can be seen from the estimate $\|x\|^2 \leq \|x_s\|^2 + \|x_a\|^2 + 2\|x_s\|\|x_a\| \leq 2(\|x_s\|^2 + \|x_a\|^2)$ $(x = x_s + x_a \in H)$ and the open mapping theorem. If $K = H_s \oplus H_a$ is turned into a normal Hilbert $L^\infty(\nu) \oplus H^\infty(\sigma)$-module via the representation

$$\Psi : L^\infty(\nu) \oplus H^\infty(\sigma) \to L(K), \quad f \oplus g \mapsto \Psi_s(f) \oplus \Psi_a(g),$$

then, by construction, the map $S \oplus 1_{H_a} : H = H_s + H_a \to K$ is a similarity of the underlying $A(D)$-modules. Thus to prove the theorem, we are allowed to assume that $H = K$.

Let $\zeta \in \partial D$. Since one-point sets have σ-measure zero (see Lemma 2.2.3 in [4]), the equivalence class $\chi_\zeta \in L^\infty(\nu) \oplus H^\infty(\sigma)$ of the characteristic function of $\{\zeta\}$ is non-trivial if and only if $\{\zeta\}$ is an atom of ν. In this case, the multiplication operator $P_\zeta = M_{\chi_\zeta} \in L(H)$, being clearly an $A(D)$-module homomorphism, is also an orthogonal projection. Since, for $x \in P_\zeta H$, we have

$$f \cdot x = f\chi_\zeta \cdot x = f(\zeta)x \qquad (f \in A(D)),$$

the $A(D)$-submodule $H_d^\zeta = P_\zeta H \subset H$ is ζ-atomic. Clearly, $\chi_{\zeta_1} \cdot \chi_{\zeta_2} = 0$ implies that $H_d^{\zeta_1} \perp H_d^{\zeta_2}$ whenever $\zeta_1 \neq \zeta_2$.

In order to isolate the discrete part of H, we we declare A_H to be the set of all one-point atoms of ν and define the discrete and continuous part of ν as $\nu_d = \nu | A_H$ and $\nu_c = \nu | \bar{D} \setminus A_H$, trivially extended to measures on $\bar{D}$. (Note that A_H is countable, since ν is finite.) Since the measures ν_d, ν_c and σ are pairwise singular to each other, we have the inclusion

$$L^\infty(\nu_d) \oplus H^\infty(\nu_c + \sigma) \subset L^\infty(\nu) \oplus H^\infty(\sigma).$$

By restriction of the module multiplication, we therefore obtain a normal and contractive $L^\infty(\nu_d) \oplus H^\infty(\mu_H)$-module structure on H with the continuous measure $\mu_H = \nu_c + \sigma$.

Let $\chi_{A_H} \in L^\infty(\nu_d) \oplus H^\infty(\mu_H)$ be the equivalence class of the characteristic function of the set A_H. Then the module multiplication $P_d = M_{\chi_{A_H}}$ is the orthogonal projection from H onto the discrete part $H_d = \oplus_{\lambda \in A_H} H_d^\zeta$. Since $P_c = 1 - P_d = M_{1-\chi_{A_H}}$ is a module homomorphism as well, the set $H_c = P_c H$ is an $A(D)$-submodule of H for which

$$H = \bigoplus_{\zeta \in A_H} H_d^\zeta \quad \oplus \quad H_c$$

holds. To see that the $A(D)$-submodule H_c defined in this way is continuous, let $x \in H$ be an arbitrary vector satisfying

$$f \cdot x = f(\zeta)x \qquad (f \in A(D)) \quad \text{for some } \zeta \in \partial D.$$

We fix a peaking function $f \in A(D)$ with $f(\zeta) = 1$ and $|f| < 1$ on $\bar{D} \setminus \{\zeta\}$. Since the sequence of powers $(f^k)_{k \geq 1}$ converges pointwise to the characteristic function of $\{\zeta\}$, we deduce that

$$x = f^k \cdot x = f^k \cdot P_d x \oplus f^k \cdot P_c x \overset{k \to \infty}{\longrightarrow} \chi_{\{\zeta\}} P_d x \oplus 0 = P_d x.$$

Hence $x \in H_d^\zeta$ and therefore H_c is continuous. To finish the proof of part (a) we have to consider uniqueness. For this purpose, let α be a similarity between two $A(D)$-modules with the structure under consideration $\oplus_{\lambda \in A_H} H_d^\zeta \oplus H_c \overset{\alpha}{\longrightarrow} \oplus_{\lambda \in A_K} K_d^\zeta \oplus K_c$. The remark preceding the theorem guarantees that $\alpha(H_d^\zeta) \subset K_d^\zeta$ and $\alpha^{-1}(K_d^\zeta) \subset H_d^\zeta$. Hence $A_H = A_K$, $\alpha(H_d^\zeta) = K_d^\zeta$ for each $\zeta \in A_H$ and consequently $\alpha(H_c) = K_c$.

To prove part (b) it suffices to observe that the normal $H^\infty(\mu_H)$-module structure on H_c is inherited from the $L^\infty(\nu_d) \oplus H^\infty(\mu_H)$-module structure on H described above.

Towards a proof of the assertion (c) suppose that M is an $A(D)$-submodule of H extending to a normal $H^\infty(\mu)$-module with a continuous measure μ. An arbitrary $x \in M$ can be decomposed as $x = P_d x \oplus P_c x$. If $P_d x \neq 0$, then there exists at least one $\zeta \in A_H$ such that $\chi_\zeta \cdot x = \chi_\zeta \cdot P_d x \neq 0$. Choosing a peaking function $f \in A(D)$ for ζ we deduce that, on the one hand, $f^k \cdot P_d x \overset{k}{\to} \chi_\zeta \cdot x \neq 0$ (by normality), and on the other hand $f^k \cdot P_d x = f^k \cdot x - f^k \cdot P_c x \overset{k}{\to} 0$ by the continuity of the module structures of M and H_c. From this contradiction it follows that $P_d M = 0$ and hence $M \subset H_c$, as desired. □

If in the situation of the above theorem H is a Hilbert $C(\partial D)$-module, then, modulo similarity, it extends to a normal and isometric Hilbert $L^\infty(\eta)$-module K for some $\eta \in M^+(\partial D)$ by Proposition 1.2 and 1.3. Writing $\eta = \nu + \omega$

with $\nu \in S(\bar{D})$ and $\omega \in HM(\bar{D})$, we may replace the map Ψ occuring in the above proof by the functional calculus

$$\Psi : L^\infty(\nu) \oplus L^\infty(\omega) \to L(K)$$

induced by the normal and isometric $L^\infty(\eta)$-module K. Along this way, we obtain the following completion of the above theorem:

2.3 Remark. *If H is a Hilbert $C(\partial D)$-module, then the module H_c occuring in the decomposition of H in the above theorem can be chosen in such a way that it extends to a normal isometric $L^\infty(\mu_H)$-module for some continuous measure $\mu_H \in M^+(\partial D)$.*

In the next section, the following simple observation will be applied to obtain a decomposition of short exact sequences of Hilbert $A(D)$-modules into atomic and continuous parts.

2.4 Lemma. *Given a homomorphism*

$$\bigoplus_{\zeta \in \partial D} H_d^\zeta \ \oplus \ H_c \xrightarrow{\alpha} \bigoplus_{\zeta \in \partial D} K_d^\zeta \ \oplus \ K_c$$

between Hilbert $A(D)$-modules as described in part (a) of the preceding theorem we have

$$\alpha(H_d^\zeta) \subset K_d^\zeta \qquad \text{and} \qquad \alpha(H_c) \subset H_c.$$

Proof. The first assertion follows from the remarks preceding the cited theorem. To verify the second one observe that the range $\alpha(H_c)$ inherits a normal $H^\infty(\mu_H)$-module structure from H_c. Thus part (c) of the preceding theorem guarantees that $\alpha(H_c) \subset K_c$. $\square$

3 Projectivity of Hilbert $C(\partial D)$-modules

Applying the decomposition theorem established in the last section we are now able to prove the announced vanishing result for Ext^1. As a main tool we use the existence of abstact inner functions relative to an arbitrary continuous measure $\mu \in M^+(\partial D)$ which has been settled by Aleksandrov [1].

3.1 Theorem. *If H is a Hilbert $C(\partial D)$-module, then*

$$\mathrm{Ext}^1_{\mathscr{H}(A(D))}(K, H) = 0 \qquad \text{and} \qquad \mathrm{Ext}^1_{\mathscr{H}(A(D))}(H, K) = 0$$

for every Hilbert $A(D)$-module K.

Proof. Step (1): The reduction to the continuous case.

Since similarities do not change the Ext^1-group, we may replace H and K by their decompositions

$$H' = \bigoplus_{\lambda \in \partial D} H_d^\zeta \;\oplus\; H_c \qquad \text{and} \qquad K' = \bigoplus_{\lambda \in \partial D} K_d^\zeta \;\oplus\; K_c$$

established in Theorem 2.2 and the subsequent remark. Let $E : 0 \to H' \xrightarrow{\alpha} J \xrightarrow{\beta} K' \to 0$ be an exact sequence of Hilbert $A(D)$-module maps and let $\theta : J \to J' = \oplus_{\lambda \in \partial D} J_d^\zeta \;\oplus\; J_c$ be the similarity identifying J with its canonical decomposition. Since the diagram

$$\begin{array}{ccccccccc} E : 0 & \longrightarrow & H' & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & K' & \longrightarrow & 0 \\ & & \Big\| & & {\scriptstyle\theta}\big\downarrow & & \Big\| & & \\ E' : 0 & \longrightarrow & H' & \xrightarrow{\alpha'} & J' & \xrightarrow{\beta'} & K' & \longrightarrow & 0 \end{array}$$

where $\alpha' = \theta \circ \alpha$ and $\beta' = \beta \circ \theta^{-1}$ commutes, we do not change the Ext^1-equivalence class if we replace E by E'. Applying Lemma 2.4 we can decompose the sequence E' into a direct sum of the induced exact sequences between the atomic components of the underlying modules

$$E_d^\zeta : 0 \longrightarrow H_d^\zeta \xrightarrow{\alpha'} J_d^\zeta \xrightarrow{\beta'} K_d^\zeta \longrightarrow 0$$

for $\zeta \in \partial D$ and a sequence between the continuous parts

$$E_c : 0 \longrightarrow H_c \xrightarrow{\alpha'} J_c \xrightarrow{\beta'} K_c \longrightarrow 0.$$

Note that, since the module action on J_d^ζ is ζ-atomic, each direct sum decomposition of J_d^ζ in the category of Hilbert spaces is also a direct sum in the $A(D)$-module sense. Therefore, the sequences E_d^ζ $(\zeta \in \partial D)$ split. To finish the proof of the theorem, it suffices to check that E_c splits.

Let $\mu_H, \mu_J, \mu_K \in M^+(\partial D)$ denote continuous measures allowing a normal extension of the module structure on H, K, J in the sense of Theorem 2.2 and Remark 2.3. The sum $\mu = \mu_H + \mu_J + \mu_K \in M^+(\partial D)$ remains continuous, and via the canonical maps

$$L^\infty(\mu) \to L^\infty(\mu_H), \qquad H^\infty(\mu) \to H^\infty(\mu_J), \qquad H^\infty(\mu) \to H^\infty(\mu_K),$$

we can reagrd the sequence E_c as a short exact sequence of normal $H^\infty(\mu)$-modules whose first term H extends to an isometric normal $L^\infty(\mu)$-module.

To finish the proof of the theorem we prove that each such sequence splits. This will be done in the next step.

Step (2): The continuous case.

Let $\mu \in M^+(\partial D)$ be a continuous measure, K a normal Hilbert $H^\infty(\mu)$-module and H a normal and isometric Hilbert $L^\infty(\mu)$-module. We use an idea of Guo (see [6], Theorem 3.2) to prove that $\mathrm{Ext}^1_{\mathscr{N}(H^\infty(\mu)))}(K,H) = 0$. According to the identification

$$\mathrm{Ext}^1_{\mathscr{N}(H^\infty(\mu))}(K,H) \to C_{\mathscr{N}(H^\infty(\mu))}(K,H)/B(K,H), \quad [E] \mapsto [\sigma],$$

described in Section 1 we have to show that, for each normal 1-cocycle $\sigma \in C_{\mathscr{N}(H^\infty(\mu))}(K,H)$, there exists a bounded linear operator $T \in L(K,H)$ satisfying $\sigma = \sigma_T$.

To do this, we consider the multiplicative semigroup

$$I = \{\theta \in H^\infty(\mu) : |\theta| = 1 \text{ in } L^\infty(\mu)\}$$

of all μ-inner functions on ∂D and choose an invariant mean $m : \ell^\infty(I) \to \mathbb{C}$, i.e. a linear form $m \in \ell^\infty(I)'$ with $\|m\| = m(1) = 1$ and

$$m(f) = m(f_\omega) \quad (f \in \ell^\infty(I),\ \omega \in I) \quad \text{where } f_\omega(\theta) = f(\omega \cdot \theta) \quad (\theta \in I).$$

Via the dual pairing

$$\langle A, B\rangle = \mathrm{tr}(AB) \qquad (A \in C^1(H,K),\ B \in L(K,H))$$

we identify $L(K,H)$ with the dual space of the nuclear operators $C^1(H,K)$. Given $f \in L^\infty(\mu)$ and $g \in H^\infty(\mu)$ we denote the corresponding multiplication operators by $M_f^H : H \to H$ and $M_g^K : K \to K$, respectively.

Now, fix a normal 1-cocycle $\sigma : H^\infty(\mu) \times K \to H$. Since the linear form

$$C^1(H,K) \to \mathbb{C}, \quad A \mapsto m_\theta\big(\langle A, M_{\bar\theta} \circ \sigma(\theta,\cdot)\rangle\big)$$

is continuous it has a unique representation of the form $\langle \cdot, T\rangle$ with an operator $T \in L(K,H)$. We claim that $\sigma = \sigma_T$. Towards this end, let $\theta_0 \in I$ and $C \in C^1(H,K)$ be arbitrary elements. Then we have

$$\begin{aligned} \langle A, M_{\theta_0}^H T - T M_{\theta_0}^K\rangle &= \langle A M_{\theta_0}^H - M_{\theta_0}^K A, T\rangle \\ &= m_\theta\big(\langle A M_{\theta_0}^H - M_{\theta_0}^K A, M_{\bar\theta}^H \circ \sigma(\theta,\cdot)\rangle\big) \\ &= m_\theta\big(\langle A, M_{\bar\theta\theta_0}^H \circ \sigma(\theta,\cdot) - M_{\bar\theta}^H \circ \sigma(\theta,\cdot) \circ M_{\theta_0}^K\rangle\big) \end{aligned}$$

Applying the cocycle identity $\sigma(\theta, M^K_{\theta_0}(\cdot)) = \sigma(\theta\theta_0, \cdot) - M^H_\theta \circ \sigma(\theta_0, \cdot)$ to the composition on the right, we can write the above as

$$\begin{aligned} \ldots \;=\; & m_\theta\big(\langle A, M^H_{\bar\theta\theta_0} \circ \sigma(\theta, \cdot)\rangle\big) - m_\theta\big(\langle A, M^H_{\bar\theta} \circ \sigma(\theta\theta_0, \cdot)\rangle\big) \\ & + m_\theta\big(\langle A, M^H_{\bar\theta\theta} \circ \sigma(\theta_0, \cdot)\rangle\big). \end{aligned}$$

Using the invariance of $m = m_\theta$ we may replace θ by $\theta\theta_0$ in the argument of the first m_θ-term. But then the first line vanishes, and since $M^H_{\bar\theta\theta} = 1_H$ we have finally shown that

$$\langle A, M^H_{\theta_0} T - T M^K_{\theta_0}\rangle = \langle A, \sigma(\theta_0, \cdot)\rangle \qquad (A \in C^1(H, K),\ \theta_0 \in I).$$

A result of Aleksandrov ([1], Corollary 29) guarantees that the weak*-closure of I contains the unit ball of $A(D)$. Hence the above equality extends to all $\theta_0 \in A(D)$ and then, by continuity, to all $\theta_0 \in H^\infty(\mu)$. Thus we can state that

$$\sigma(\theta_0, k) = \theta_0 \cdot T(k) - T(\theta_0 \cdot k) \qquad (\theta_0 \in H^\infty(\mu),\ k \in K),$$

as desired. This proves that $\mathrm{Ext}^1_{\mathscr{H}(A(D))}(K, H) = 0$. The second part of the assertion can be derived from this by standard duality arguments (see e.g. the proof of Theorem 3.2 in [6]). □

Since in the cramped category $\mathscr{C}(A(\mathbb{D}))$ over the disc algebra any isometric Hilbert module is projective (see Theorem 3.1 and Corollary 3.3 in [2]), it seems natural to conjecture that each spherically isometric Hilbert $A(\mathbb{B})$-module is projective in the cramped category $\mathscr{C}(A(\mathbb{B}))$ over the ball algebra. However, the Hardy module $H^2(\sigma)$ over the unit ball $\mathbb{B} \subset \mathbb{C}^n$ with respect to the surface measure $\sigma \in M^+(\partial\mathbb{B})$ is spherically isometric and, by Remark 4.3 in Guo [6], it can be shown that $\mathrm{Ext}^1_{\mathscr{N}(H^\infty(\sigma))}(H^2(\sigma), H^2(\sigma)) \neq 0$ for $n > 1$. Since furthermore, by the multi-variable analogue of Corollary 4.2 in [2], $\mathrm{Ext}^1_{\mathscr{C}(A(\mathbb{B}))}(H^2(\sigma), H^2(\sigma)) = \mathrm{Ext}^1_{\mathscr{H}(A(\mathbb{B}))}(H^2(\sigma), H^2(\sigma))$ the Hardy module $H^2(\sigma)$ yields a counter-example to the above conjecture.

Finally we want to show that our main theorem possesses an analogue in the situation that $D \subset \mathbb{C}^n$ is a bounded symmetric domain. By definition this means that, for each $z \in D$, there exists a biholomorphic map $s_z : D \to D$ possessing z as an isolated fixed point and such that $s_z \circ s_z$ is the identity on D. We shall further assume that D is circled at the origin, that is, $0 \in D$ and $e^{it} D \subset D$ for all $t \in \mathbb{R}$. It is well known that every bounded symmetric domain is isomorphic to a circled one. By Corollary 4.6 in [7] a set D of this type is convex. Hence D is the open unit ball in the norm given by its Minkowski functional. The Shilov boundary S of $A(D)$ is known to consist precisely of those points in $\bar{D}$ with maximal Euclidean distance from the origin $0 \in \mathbb{C}^n$ (Theorem 6.5 in [7]).

Let us denote by r this maximal Euclidean distance and let $B = B_r(0)$ be the open Euclidean ball of radius r at 0 in $\mathbb{C}^n$. Then the inclusions

$$\bar{D} \subset \bar{B} \quad \text{and} \quad \mathrm{S} \subset \partial \mathrm{B}$$

hold. Hence via restriction every Hilbert $A(D)$-module H becomes a Hilbert $A(B)$-module. Furthermore, if the $A(D)$-module structure of H extends to a Hilbert $C(S)$-module structure, then the associated Hilbert $A(B)$-module structure extends to the Hilbert $C(\partial B)$-module structure defined by restriction.

In this way every short exact sequence

$$0 \to H \xrightarrow{\alpha} J \xrightarrow{\beta} K \to 0$$

of Hilbert $A(D)$-modules becomes a short exact sequence of Hilbert $A(B)$-modules. If H is supposed to be a Hilbert $C(S)$-module, then the above sequence splits as a sequence of Hilbert $A(B)$-modules. But, since $A(B)|_{\bar{D}}$ is dense in $A(D)$, it follows that every bounded $A(B)$-module homomorphism acting as a right inverse for β will also be a right inverse in the category of Hilbert $A(D)$-modules.

Thus the projectivity result proved above for strictly pseudoconvex domains immediately implies a corresponding result for symmetric domains.

3.2 Corollary. *Let $D \subset \mathbb{C}^n$ be a bounded symmetric and circled domain with Shilov boundary S. If H is a Hilbert $C(S)$-module, then*

$$\mathrm{Ext}^1_{\mathscr{H}(A(D))}(K,H) = 0 \quad \textit{and} \quad \mathrm{Ext}^1_{\mathscr{H}(A(D))}(H,K) = 0$$

for every Hilbert $A(D)$-module K. □

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