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Thomas Brox and Joachim Weickert

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Thomas Brox

Mathematical Image Analysis Group,
Faculty of Mathematics and Computer Science,
Saarland University, Building 27.1,
66041 Saarbrücken, Germany
brox@mia.uni-saarland.de

Joachim Weickert

Mathematical Image Analysis Group,
Faculty of Mathematics and Computer Science,
Saarland University, Building 27.1,
66041 Saarbrücken, Germany
weickert@mia.uni-saarland.de

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

Abstract

The popularity of level sets for segmentation is mainly based on the sound and convenient treatment of regions and their boundaries. Unfortunately, this convenience is so far not known from level set methods when applied to images with more than two regions. This paper introduces a comparatively simple way how to extend active contours to multiple regions keeping the familiar quality of the two-phase case. We further suggest a strategy to determine the optimum number of regions as well as initializations for the contours.

Key Words: Image segmentation, Active-contour and level-set-based methods

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1 Introduction

Image segmentation has a long tradition as one of the fundamental tasks in image processing. Quite early, the problem has been formalized by Mumford and Shah as the minimization of a functional [18]

$$E(u,\Gamma) = \int_\Omega (u-I)^2 \, dx + \mu \int_{\Omega-\Gamma} |\nabla u| \, dx + \nu \int_\Gamma ds \tag{1}$$

that supports the creation of a piecewise smooth approximation of an image $I : \Omega \to \mathbb{R}$. Its first term ensures the solution u to stay close to the input image, whereas the second term penalizes deviations from smoothness within regions separated from each other by a boundary set Γ. Finally, the third term minimizes the total length of Γ. The tuning parameters $\mu > 0$ and $\nu > 0$ weight the relative importance of the three terms.
Later in [35], Zhu and Yuille related this formulation to the *minimum description length* [15] and the Bayesian restoration of images [11]. They presented a new energy functional that unified many existing approaches on image segmentation:

$$E(\Omega_i, \Gamma_i, p_i, N) = \sum_{i=1}^{N} \left(- \int_{\Omega_i} \log p_i \, dx + \frac{\nu}{2} \int_{\Gamma_i} ds + \varrho \right). \tag{2}$$

Here the first term maximizes the total a-posteriori probability of pixels being assigned to the correct region. The regions Ω_i are modelled by probability density functions p_i. The second term minimizes the boundary length like the Mumford-Shah functional. The third term introduces a fixed penalty for each additional region. $\nu > 0$ and $\varrho > 0$ are again tuning parameters.
Although the suggested energy functionals describe the segmentation problem quite accurately, their minimization is very difficult. Firstly, one has to deal with a highly non-convex problem yielding lots of local optima. Secondly, the functionals contain terms acting on different domains: once on the two-dimensional image domain and once on a one-dimensional curve.
A very nice tool to deal with the latter problem appeared with the introduction of level sets [10, 21, 30, 19, 20]. In the level set framework, the one-dimensional contour is embedded into the image domain as the zero-level line of an artificial level set function $\Phi : \Omega \to \mathbb{R}$. It therefore allows for connecting region and contour based constraints and for solving for a single unknown representing both the regions and the separating contour. Interestingly, the first application of level sets in the scope of image segmentation by active contour models [5, 16, 6, 14] did not make use of this property, as these models comprised only a constraint on the contour taking edge information into account. Only some time later, level sets have also been applied to region based segmentation. The methods in [7, 8] and [22, 24] minimize variants of the functionals of Mumford-Shah and Zhu-Yuille that are restricted to two regions. In the level set formulation this reads:

$$E(\Phi, p_1, p_2) = - \int_\Omega H(\Phi) \log p_1 \, dx - \int_\Omega (1 - H(\Phi)) \log p_2 \, dx + \nu \int_\Omega |\nabla H(\Phi)| \, dx \tag{3}$$

where Φ is the level set function representing the regions Ω_1 for $\Phi > 0$ and Ω_2 for $\Phi < 0$, as well as the contour Γ for $\Phi = 0$. The Heaviside function $H(\Phi) = 0$ for $\Phi < 0$ and $H(\Phi) = 1$ for $\Phi > 0$ is used to distinguish the two regions. Replacing H by a regularized version H_ϵ ensures the functional to be differentiable; for details see [8].
Using level sets for image segmentation has many advantages. Firstly, level sets yield a convenient representation of regions and their boundaries on the pixel grid without the need of

complex data structures. This considerably simplifies optimization, as variational methods and mature numerical techniques can be employed. Moreover, level sets can describe topological changes in the segmentation, i.e., parts of a region need not necessarily be connected; regions may split and merge.

Unfortunately, this image of a perfect segmentation framework is disturbed by an inherent drawback: the restriction of the embedding function to separate only two regions. Images with more than one object region can therefore not be captured by the model. Thereby, the ability of the level set framework to change the topology of regions should not be confused with multi-region segmentation. Although parts of a region may be disconnected, they still share the same region model given by p_i and the same representation in the level set framework.

The restriction to two regions has already been addressed by a few works in the past. The immediate idea to overcome the problem is to use more than one embedding function and to assign a separate embedding function Φ_i to each region Ω_i

$$E(\Phi_i, p_i) = \int_\Omega \sum_{i=1}^{N} \Big(-H(\Phi_i) \log p_i + \frac{\nu}{2} |\nabla H(\Phi_i)| \Big) \; dx \tag{4}$$

where N is the fixed number of regions. This concept has been proposed in [17, 34] and has been adopted for segmentation in [29, 23].

While the basic idea is very simple, assigning a level set function to each region entails a new difficulty that is not easy to address: the level set functions need a coupling in order to respect the constraint of disjoint regions, i.e., regions must not overlap and there must not be pixels that are not assigned to any region. Note that in the two-region case this constraint is automatically satisfied by the regions' representation.

Two different coupling concepts have been suggested. In [34, 29] the constraint of disjoint regions is integrated by means of a Lagrangian multiplier λ:

$$E(\Phi_i, p_i) = \int_\Omega \sum_{i=1}^{N} \Big(-H(\Phi_i) \log p_i + \frac{\nu}{2} |\nabla H(\Phi_i)| \Big) \; dx + \frac{\lambda}{2} \int_\Omega \sum_{i=1}^{N} (H(\Phi_i) - 1)^2 \; dx. \tag{5}$$

This is a sound strategy to integrate the constraint, though it leads to a considerably more complex optimization problem than in the two-phase case. Alternatively, it was proposed in [23] to supplement an artificial coupling force to the functional by means of an additional parameter. This methodology is easier to implement than the concept in [34]. On the other hand it induces additional numerical parameters which can be difficult to choose. In particular, for a fixed parameter setting disjoint regions cannot be ensured for arbitrary images.

A completely different approach has been introduced in [33]. Instead of assigning a separate level set function to each region, the level set functions recursively split the domain into two subdomains. This way, n level set functions can represent $N = 2^n$ regions. Indeed, if the number of regions is a power of 2, this multi-region model adopts from the two-region model the nice property of implicitly respecting the constraint of disjoint regions, so no further coupling forces are necessary. However, the model looses parts of its attractiveness when the number of regions is not a power of 2. In such cases, parts of the region boundaries are weighted twice, and the model must introduce empty regions, which does not go well together with region models beyond the piecewise constant model $\log p_i = (I - \mu_i)^2$.

In a recent work, it has further been suggested to use also level lines besides the zero-level line for representing contours [9]. In this setting, a single embedding function Φ can be sufficient for representing N regions. For dealing with triple junctions, however, the authors in [9] suggest a combination with the model from [33].

In this paper, we present a new coupled curve evolution for an arbitrary number of regions. We embark on the concept of using one level set function for each region. Despite the larger memory requirements than in [33] and [9], it can be quite convenient to have direct access to a region by its level set function. In contrast to [34] and [24] the constraint of disjoint regions is enforced directly by the coupling of the evolution equations without touching the energy functional or adding further parameters. The evolution is comparatively easy to implement and works also in case of triple junctions and with arbitrary many regions. Since there are no empty regions involved, the region model can be chosen freely.
We further propose to automatically estimate the optimum number of regions according to the energy in (2) by applying hierarchical splitting based on the two-phase case. Besides the region number, the splitting also yields good initializations for the level set functions evolved with the proposed coupled scheme. With such a combination, the concept of level sets can be fully exploited, which leads to excellent segmentation results.
This paper involves and extends ideas that have been presented earlier at a conference [3]. In contrast to the coupling proposed in the present paper, however, the scheme in [3] could not fully avoid region overlapping and vacuum since the length constraint has not been subject to the coupling.
In the next section we will briefly review the two-phase case. It will help to derive the new coupled scheme in Section 3 The method's description is followed by an experiment. In Section 4, the method will be used together with a hierarchical approach to minimize the energy of Zhu-Yuille. This section contains as well some experiments. The paper is concluded by a brief summary.

2 The basic two-phase case

Contrary to the segmentation with multiple regions, two-region segmentation with level sets is well understood. Reducing the functional of Zhu-Yuille in (2) to two regions and applying a level set function Φ for representing the two regions and the separating contour leads to the functional in (3). For minimization with respect to the level set function, one derives the Euler-Lagrange equation, which has to be satisfied in an optimum. Introducing the artificial time variable t and initializing Φ at $t = 0$ with some contour, one can then perform a gradient descent according to

$$\partial_t \Phi = H'(\Phi) \left(\log \frac{p_1}{p_2} + \nu \operatorname{div} \left(\frac{\nabla \Phi}{|\nabla \Phi|} \right) \right) \tag{6}$$

where $H'(s)$ is the derivative of $H(s)$ with respect to its argument. Since the probability densities p_1 and p_2 of the two regions depend on Φ, they have to be updated after each iteration according to the expectation-maximization principle. In case of Gaussian densities this means, one has to recompute the means and standard deviations of the regions. It has been shown in [28] that for Gaussian densities, expectation-maximization comes down to the full gradient descent considering also the dependency of p_i on Φ. In [12] it has been shown empirically for Laplacian densities that higher order terms of the gradient descent can be neglected in the evolution. For the experiments in this paper, we determine p_i by nonparametric Parzen density estimates [26], which are computed by means of smoothed histograms of the regions [27, 2], and apply the texture feature space proposed in [4].
This iterative process converges to the next local minimum, i.e., the initialization matters. In order to attenuate this dependency on the initialization, it is recommendable to apply a coarse-to-fine strategy, also known as graduated non-convexity (GNC) [1]. Starting with a

downsampled image, there are less local minima, so the segmentation is more robust. The resulting segmentation can then be used as initialization for a finer scale, until the original optimization problem is solved.
Under the assumption of two regions in the image, this framework works very well. For results obtained in the two-phase case we refer to [8, 32, 25, 27, 2, 4, 12]. The inherent shortcoming of this method is the fact that the assumption of exactly two regions in an image is mostly not true.

3 Coupled curve evolution by means of competing regions

As a remedy, the segmentation framework is extended by employing N level set functions $\Phi_i, i = 1, ..., N$, each representing one region Ω_i, by setting $x \in \Omega_i$ if and only if $\Phi(x) > 0$. We then seek to minimize (4) under the constraint $\bigcup_i \Omega_i = \Omega$ and $\bigcap_i \Omega_i = \emptyset$. Neglecting this constraint, the gradient descent equations to (4) read:

$$\partial_t \Phi_i = H'(\Phi_i) \left(\log p_i + \frac{\nu}{2} \operatorname{div} \left(\frac{\nabla \Phi_i}{|\nabla \Phi_i|} \right) \right). \tag{7}$$

Since $\log p_i$ is always negative, the curve evolution according to these equations will cause the level set functions to quickly become negative everywhere, i.e., all regions Ω_i disappear. To avoid this, we regard the concept that prevents this situation in the classical two-region case. In this case, where both regions are represented by a single level set function, there is always a competition between the two regions. Thus the negative value of $\log p_1$ is balanced by subtracting the negative value $\log p_2$ of the competing region; cf. (6). This balance is missing in (7). The same is true for the term resulting from the length constraint: in the two-region model, a shrinking region due to the length constraint automatically induces the competing region to capture the released area. This recapturing is missing in (7) as well.
With these considerations in mind, it is possible to prevent overlap and vacuum in (7): one has to add a competitor to the evolution equation. If this competitor is chosen to be the most severe competitor, there will be a balanced competition between two regions as in the classical two-region setting. The enhanced evolution equation reads:

$$\begin{aligned} e_k &:= \log p_k + \frac{\nu}{2} \operatorname{div} \left(\frac{\nabla \Phi_k}{|\nabla \Phi_k|} \right) \\ \partial_t \Phi_i &= H'(\Phi_i) \left(e_i - \max_{\substack{H'(\Phi_j)>0 \\ j \neq i}} (e_j, e_i - 1) \right). \end{aligned} \tag{8}$$

The additional term $e_i - 1$ in the maximum operation ensures a balancing term if there is no competing region in the vicinity, i.e., there is a vacuum region. In such situations, the region extends with constant speed towards the vacuum. One can verify easily that the constraint of disjoint regions is satisfied, as each pixel is captured by one of the competing regions. This is also true for triple or quadruple junctions. In such cases, each involved region competes with the best fitting region, which itself competes with the second best fitting region. Moreover, for $t \to \infty$ the constraint of disjoint regions is even ensured, if there has been vacuum and/or overlap in the initialization.

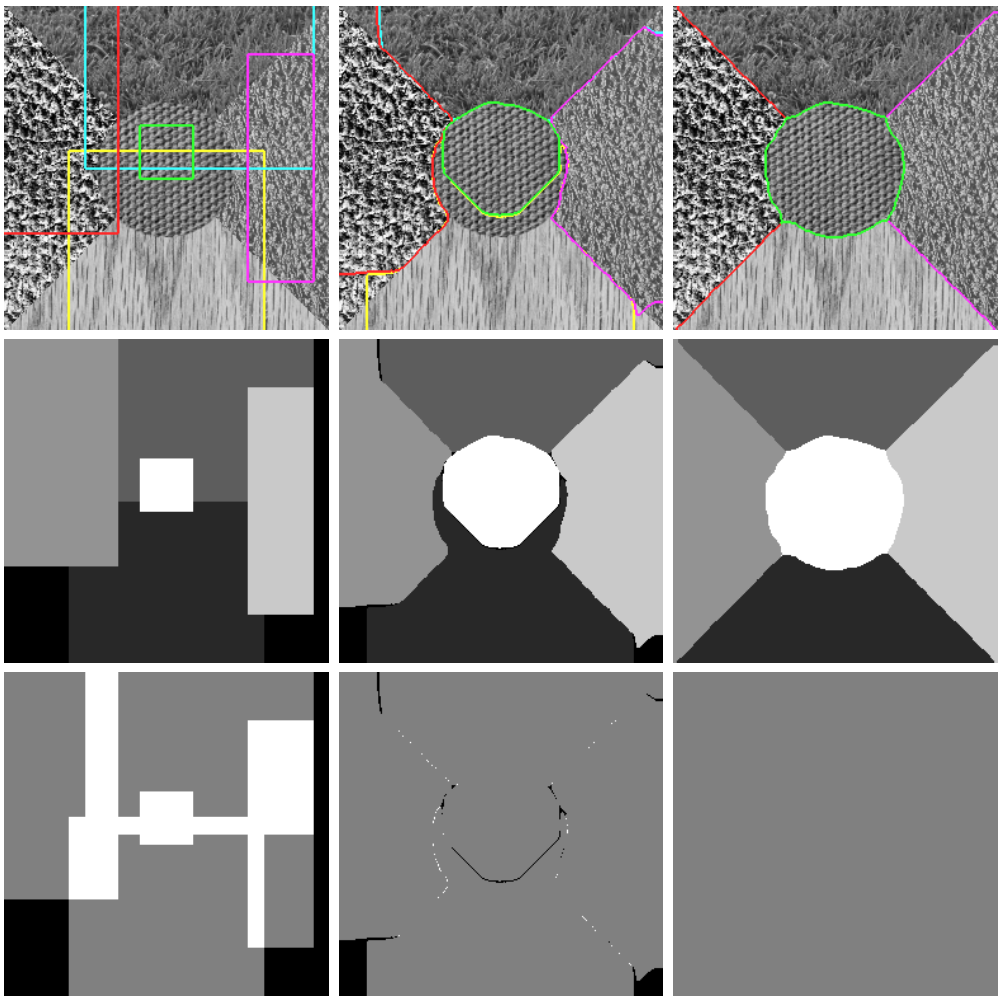

Figure 1: Coupled evolution of 5 curves in a textured image (256×256). **From Left to Right:** $t = 0$, $t = 200$, $t = 2000$. **Top Row:** Evolving curves. **Center Row:** Evolving regions shown in different gray scales. Black represents vacuum. **Bottom Row:** Vacuum (black) and overlap (white) versus pixels assigned to a single region (gray).

This evolution model has several advantages when compared to previous approaches:

- In contrast to the coupling by means of a Lagrange multiplier, the evolution equations stay simple and are comparatively easy to implement.
- In contrast to the method suggested in [23] the coupling does not involve any further numerical parameters that might influence the segmentation result.
- In contrast to the model in [33], each region, including its boundary, is directly accessible by its assigned level set function. Furthermore, the evolution is independent from the number of regions involved. There are no empty regions and no varying weights for the length constraint if the number of regions is not a power of 2. Since empty regions are not an issue anymore, the evolution method can be employed together with more sophisticated statistical region models than the piecewise constant model in [33].

Fig. 1 shows the evolution of 5 regions in a texture image. The initialization of the level set function at $t = 0$ does not satisfy the constraint of disjoint regions; there are large areas

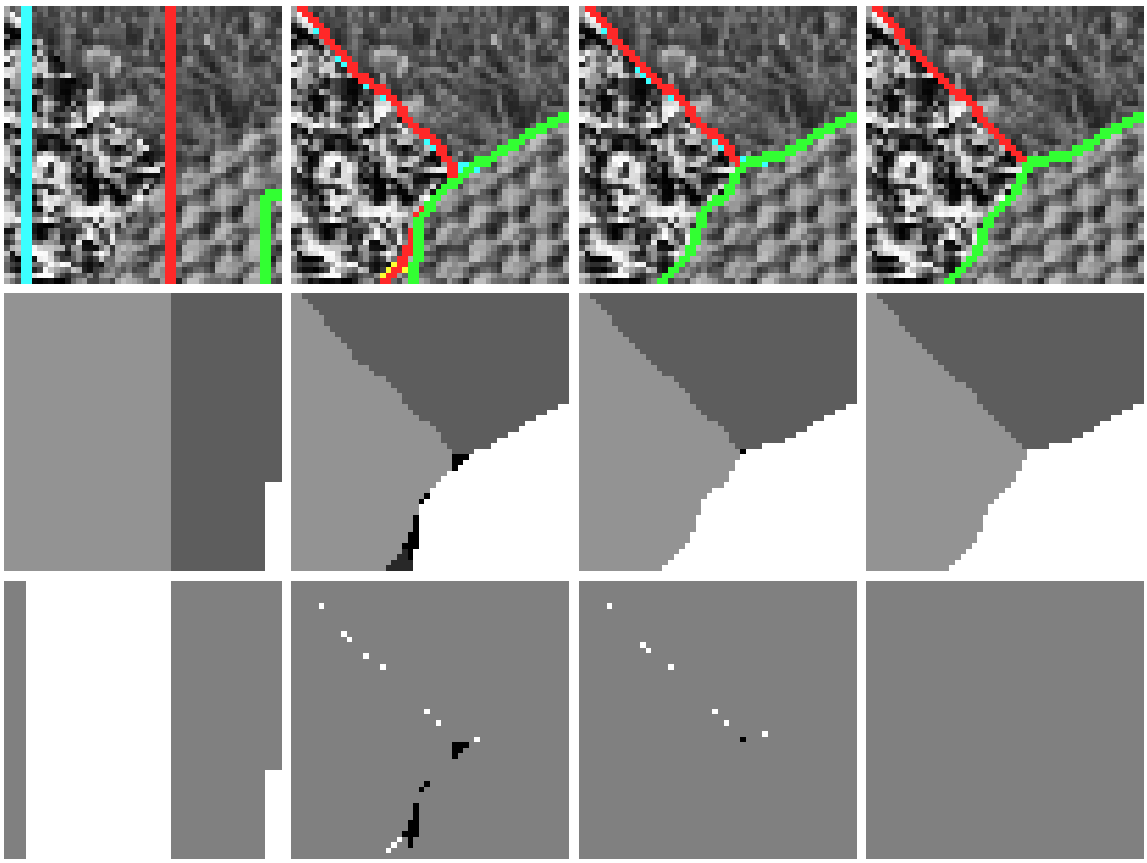

Figure 2: Enlarged cutout of a triple junction. **From Left to Right:** $t = 0$, $t = 200$, $t = 400$, $t = 2000$. **Top Row:** Evolving regions shown in different gray scales. Black represents vacuum. **Bottom Row:** Vacuum (black) and overlap (white) versus pixels assigned to a single region (gray).

of vacuum as well as overlap. During the evolution, the regions successively evolve towards the vacuum. In areas of overlap and at region boundaries, competition between the regions resolves the ambiguities such that the best fitting region captures the area.
The evolution can also deal with triple junctions as demonstrated in Fig. 2 that shows an enlarged cutout of a triple junction from Fig. 1. Obviously, the complex region model, which takes texture and higher order statistics into account, also causes no problems with the coupled equations.

4 Estimating the initialization and number of regions

So far, the number of regions N has been presumed to be known in advance. However, keeping the number of regions fixed from the beginning introduces a considerable degree of supervision and actually solves only a simplified version of the segmentation problem stated in (2). For a full unsupervised partitioning of the image, also the number of regions is a free variable and has to be determined automatically by the segmentation model.
Especially in the scope of active contours, it has usually been avoided to raise the segmentation model to this more general case. An exception is the work in [23] where the number of regions is estimated in a preliminary stage by means of a Gaussian mixture estimate of the image histogram. This way, the optimum number of mixture coefficients determines the number of

regions. Also the model in [33] can up to a certain degree deal with a variable number of regions by relying on the consideration that dispensable regions may shrink to empty regions. However, both approaches do generally not minimize the energy in (2) where the sought number of regions N is incorporated into the functional

$$E(\Omega_i, \Gamma_i, p_i, N) = \sum_{i=1}^{N} \left(- \int_{\Omega_i} \log p_i \, dx + \frac{\nu}{2} \int_{\Gamma_i} ds + \varrho \right).$$

and optimized together with the shape of the regions. The constant term weighted with the parameter $\varrho > 0$ is supplemented in order to prevent N from becoming too large. The necessity of this additional term becomes obvious, if one regards the change of energy when a region is split into two regions. Since the region model for two separated regions can adopt much better to the underlying data than for a joint region, the energy evoked by the statistical term always decreases by a split. The energy increase due to the length constraint on the boundary is in general much too small to compensate for the reduced energy in the statistical term. For an acceptable balance, the weighting parameter ν has to be chosen very large, thus the evolving contour gets much too smooth. The supplement of a fixed penalty for each region allows for setting ν to reasonable values while keeping the number of regions small.
For minimizing the energy in (2), a mixture of seeded region growing, region merging, and explicit active contours has been suggested in the original work. In contrast, it is sought here to minimize the energy by exploiting the advantages of the level set framework. Furthermore, the coarse-to-fine strategy coupled with a hierarchical splitting substitutes the placement of seeds. It should be noted here that N is an integer variable. Consequently, only minimization with respect to the contour can be performed in the classical variational framework. For optimization also with respect to the number of regions, a different strategy has to be employed. To this end, consider the following two special cases for level set segmentation:

- The image or a subdomain of the image can be split into two parts by a two-phase segmentation as described in Section 2. Due to the usage of a coarse-to-fine strategy, the outcome of this partitioning is often independent from the initialization.

- By means of the coupled evolution presented in Section 3, an arbitrary number of region contours can evolve, minimizing the energy in Eq. 2, provided the number of regions N is fixed and reasonable initializations for the regions are available.

By combining both methods, one can efficiently minimize the energy in (2). Starting with the whole image domain Ω being a single region, so $N = 1$, the two-region segmentation can be applied in order to find the optimum split of this domain. If the energy decreases by the split, this results in two regions. On each of these regions, again the two-phase split can be applied, and so on, until the energy does not decrease by further splits anymore. With this proceeding, not only the optimum number of regions is determined, but also suitable initializations for the regions are obtained.
Such hierarchical splits have already been proposed in combination with other segmentation techniques, e.g. the normalized cut [31] and semi-definite programming [13]. As a nice property of such a procedure, each split completely ignores the cluttering rest of the image and can therefore succeed in establishing new contours without knowing good initializations.
On the other hand, ignoring parts of the image will likely not minimize the global energy, since possibilities of a region to evolve have been ignored by restricting the regions to a subdomain of the image. However, as the region number and the initialization are known, the energy

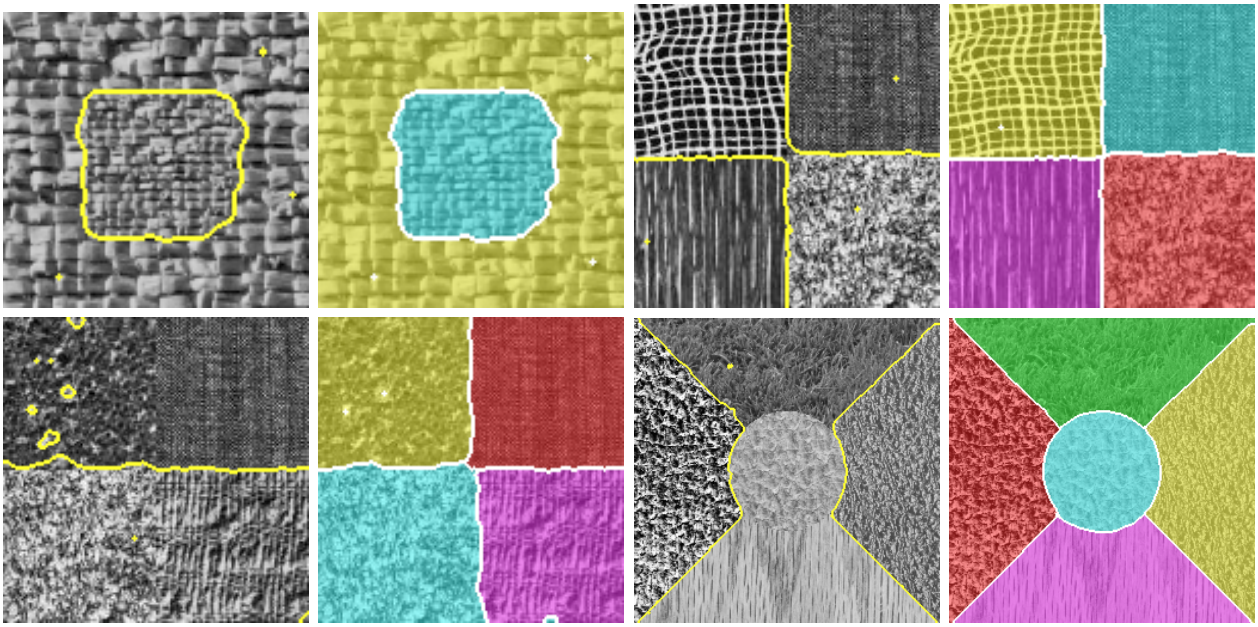

Figure 3: Segmentation results with synthetic images. **Left:** Result with two-phase segmentation (curve is shown). **Right:** Result with multiple regions (regions are shown).

can now be further optimized in the global scope by applying (8). This evolution adapts the regions to the new situation where they may have more than one competitor.
The combined procedure is applied in a multi-scale setting. Starting the procedure as described on the coarsest scale, with every refinement step on the next finer scale it is checked whether any further split or merge decreases the energy before the evolution according to (8) is applied. So for each scale the optimum N as well as the region boundaries and the region statistics are updated.
In contrast to other splitting based segmentation algorithms, the method proposed here has the advantage that the contour is not fixed by an early split. It is allowed to evolve after the number and coarse location of regions is determined and thus all competitors are known. In contrast to other curve evolution techniques, the method is less dependent on good initializations, since the problem's complexity is temporarily reduced to two-phase problems.
The experiments depicted in Fig. 3 and Fig. 4 demonstrate the performance of this method. All parameters have been kept fixed during the experiments, except an automatic adaptation to the image size $|\Omega|$. In particular, $\nu := 0.001|\Omega|^{0.7}$ and $\varrho := 0.075|\Omega|$. The value of 0.7 has been determined empirically in a number of experiments and can be motivated by the fact that the weight has to mediate 1-D and 2-D constraints.
Despite the fixed set of parameters, a variety of region numbers reaching from 2 to 5 regions is detected. Even for several natural images, where the homogeneity of regions is often not ensured, one obtains reasonable results. It can be observed, particularly in case of the koala, that the curve evolution may change the region shape considerably after the hierarchical split. In classical splitting based methods, the region boundary found by the two-phase segmentation would be kept in the final result.

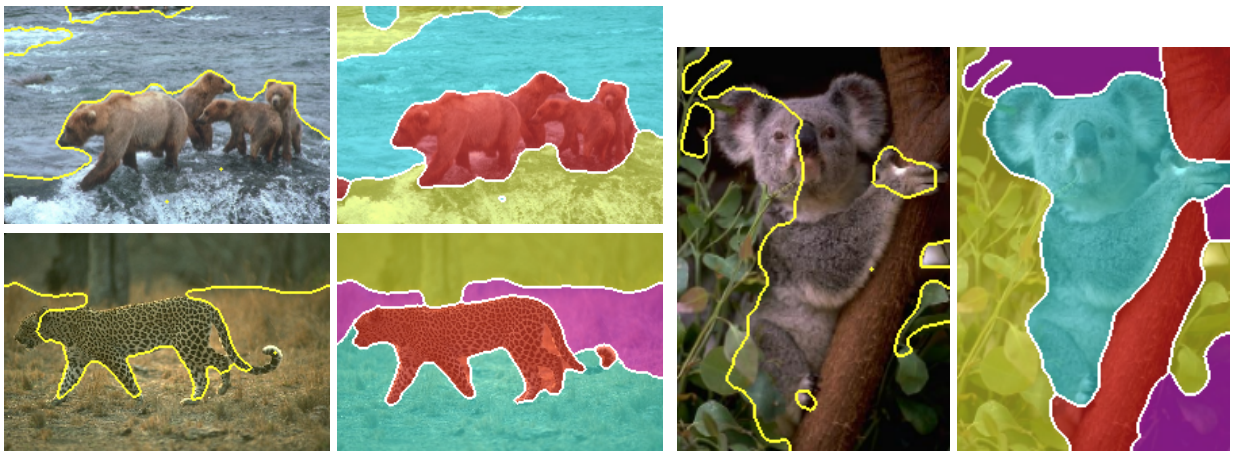

Figure 4: Segmentation results with natural images. **Left:** Result with two-phase segmentation. **Right:** Result with multiple regions.

5 Conclusion

This paper has presented an easy to implement curve evolution by means of coupled level set functions. The coupling ensures the requirement of disjoint regions to be satisfied and allows for triple junctions. It does not introduce further parameters and has no restrictions concerning the number of regions. In combination with hierarchical splitting, the new technique can be employed to minimize the energy of Zhu-Yuille in the level set framework.

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