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# Variational integrals with a wide range of anisotropy

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**Abstract**

We consider anisotropic variational integrals of $(p, q)$-growth and prove for the scalar case interior $C^{1,\alpha}$-regularity of bounded local minimizers under the assumption that $q \leq 2p$ by the way discussing a famous counterexample of Giaquinta. In the vector case we obtain some higher integrability result for the gradient.

# 1 Introduction

Roughly speaking, an anisotropic variational integral of the type $J[u] = \int_\Omega f(\nabla u)\,\mathrm{d}x$ defined for functions $u$: $\Omega \to \mathbb{R}^N$ on some bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is characterized through a growth condition like

$$a|Q|^p - b \leq f(Q) \leq A|Q|^q + B\,, \tag{1.1}$$

where $a$, $b$, $A$, $B$ denote positive constants, $1 < p < q < \infty$ are given exponents, and $Q$ is an arbitrary matrix from $\mathbb{R}^{nN}$. A natural space for local minimizers is the class of functions $u$ from $W^1_p(\Omega;\mathbb{R}^{nN})$ such that $\int_{\Omega'} f(\nabla u)\,\mathrm{d}x < \infty$ for each subregion $\Omega' \Subset \Omega$, and so one is interested in the regularity properties of local minimizers $u$ which means that one asks for higher integrability of $\nabla u$, Hölder-continuity of $u$ or even Hölder-continuity of $\nabla u$ provided that $f$ satisfies additional smoothness and convexity assumptions. In general, the hope for positive results increases in the scalar case but counterexamples of [Gi2] and (later) of [Ho] show that even for $N = 1$ unbounded minimizers exist, when $q$ is too big with respect to $p$.

On the contrary, there is a long list of authors investigating the different aspects of the regularity theory, we mention (without being complete) the works of Acerbi and Fucso ([AF]), Fusco and Sbordone ([FS]), Marcellini ([Ma2]), Choe ([Ch]) and the papers [ELM1], [ELM2] of Esposito, Leonetti and Mingione, where the interested reader can also find further references.

Typically, in the above mentioned works either a bound of the form

$$q < c(n)p \tag{1.2}$$

with $c(n) > 1$, but $c(n) \to 1$ as $n \to \infty$ is required, or a dimensionless restriction like

$$q < p + 2 \tag{1.3}$$

occurs together with the assumption that $u$ is a locally bounded function. Then, in a first step, it is shown that actually $\nabla u$ is in the space $L^q_{loc}(\Omega;\mathbb{R}^{nN})$. This result in turn is used in a second step to prove $C^{1,\alpha}$-regularity in the scalar case or in the vector case with an additional structure condition, whereas in the general vectorial setting partial $C^{1,\alpha}$-regularity is established. Of course, to do so, (1.1) has to be replaced by a stronger condition: for example, one may assume that $f$: $\mathbb{R}^{nN} \to [0,\infty)$ is of class $C^2$ together with

$$\lambda(1+|Q|^2)^{\frac{p-2}{2}}|Z|^2 \leq D^2 f(Q)(Z,Z) \leq \Lambda(1+|Q|^2)^{\frac{q-2}{2}}|Z|^2\,, \quad Q,\,Z \in \mathbb{R}^{nN}\,, \tag{1.4}$$

where $\lambda$, $\Lambda$ denote positive constants. Clearly (1.4) implies (1.1), moreover, the first inequality in (1.4) shows that $f$ is strictly convex.

In this note we have a closer look on the counterexamples mentioned above. Giaquinta's example works with the choice $p = 2$, $q = 4$, $n \geq 6$: he considers the variational integral

$$J[u] = \int \Big[\sum_{i=1}^{n-1}(\partial_i u)^2 + \frac{1}{2}(\partial_n u)^4\Big]\,\mathrm{d}x\,,$$

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for which in case $n \geq 6$

$$u(x) := \sqrt{\frac{n-4}{24}} \frac{x_n^2}{\left(\sum_{i=1}^{n-1} x_i^2\right)^{\frac{1}{2}}}$$

is of finite energy and satisfies the Euler-Lagrange equation, i.e. $u$ is a local $J$-minimizer. This singular minimizer however is not bounded, in fact in [FS] a sharp condition is proved under which we have to expect unbounded minimizers. This condition reads in the case $p = 2$ as

$$n - 1 > \frac{2q}{q-2},$$

in particular for $q = 4$ we have singular unbounded solutions for $n \geq 6$. Note that the question of unbounded solutions is strongly related to the dimension $n$. On the other hand, as mentioned above, the smoothness of bounded solutions should follow from a dimensionless condition. For instance, up to now it is not clear, if an anisotropic energy exists which satisfies a condition like (1.4) with $p = 2$, $q = 4$ and for which locally bounded but nonsmooth local minimizers can be constructed. Note that for the choice $p = 2$, $q = 4$ we have reached the limit case $q = p + 2$ of condition (1.3), and according to [Bi], Theorem 5.4, the smoothness of locally bounded minimizers is only known under the stronger hypothesis that $q < p + 2$. Here we are going to include the limit case of (1.3) in our considerations and to weaken (in case $p \geq 2$) condition (1.3) to

$$2 \leq p \leq q \leq 2p \tag{1.5}$$

by making use of the particular properties of the functionals under consideration. More precisely we will show that (1.5) together with some structural hypotheses imposed on $f$ actually is strong enough to obtain the usual smoothness properties of locally bounded solutions at least in the scalar case.
To discuss some details, let us split an element $Q$ of $\mathbb{R}^{nN}$ in the form $Q = (\tilde{Q}, Q_n)$, where

$$\tilde{Q} := (Q_1, \dots, Q_{n-1}), \quad Q_i \in \mathbb{R}^N, \quad i = 1, \dots, n.$$

Then, a typical example we have in mind is given by

$$f(Q) = (1 + |\tilde{Q}|^2)^{\frac{p}{2}} + (1 + |Q_n|^2)^{\frac{q}{2}}. \tag{1.6}$$

In fact we could also look at any decomposition

$$Q = (Q^{(1)}, Q^{(2)})$$

of the matrix $Q$ into two submatrices $Q^{(i)}$ and consider $f$ growing of order $p$ with respect to $Q^{(1)}$ and of order $q$ with respect to $Q^{(2)}$. Another model we could discuss is

$$f(Q) = \sum_{i=1}^{n} (1 + |Q_i|^2)^{\frac{p_i}{2}},$$

where now $p := \min p_i$, $q := \max p_i$.
In order to keep our exposition simple, we assume from now on that $f \in C^2(\mathbb{R}^{nN})$ can be written as (compare (1.6))

$$f(Q) = f_1(\tilde{Q}) + f_2(Q_n) \tag{1.7}$$

with

$$\left.\begin{array}{rcccl} \lambda(1 + |\tilde{Q}|^2)^{\frac{p-2}{2}} |\tilde{Z}|^2 & \leq & D^2 f_1(\tilde{Q})(\tilde{Z}, \tilde{Z}) & \leq & \Lambda(1 + |\tilde{Q}|^2)^{\frac{p-2}{2}} |\tilde{Z}|^2, \\ \lambda(1 + |Q_n|^2)^{\frac{q-2}{2}} |Z_n|^2 & \leq & D^2 f_2(Q_n)(Z_n, Z_n) & \leq & \Lambda(1 + |Q_n|^2)^{\frac{q-2}{2}} |Z_n|^2, \end{array}\right\} \tag{1.8}$$

moreover, we assume that

$$\left.\begin{array}{rclcl} f_1(\tilde{Q}) & = & f_1(Q_1,\dots,Q_{n-1}) & = & g_1(|Q_1|,\dots,|Q_{n-1}|), \\ f_2(Q_n) & = & g_2(|Q_n|) & & \end{array}\right\} \tag{1.9}$$

with $g_2$ increasing and $g_1$ increasing with respect to each argument. The assumption (1.9) ensures the convex hull property (see, e.g. [BF3]), i.e. the global minimizer w.r.t. Dirichlet boundary data $u_0 \in L^\infty(\Omega;\mathbb{R}^N)$ also is a bounded function. Therefore, it is reasonable to look at locally bounded local minimizers, where the notion of a local minimizer $u$ means that

$$J[u,\Omega'] := \int_{\Omega'} f(\nabla u)\,\mathrm{d}x < \infty$$

together with $J[u,\Omega'] \le J[v,\Omega']$ for all $v$ such that $\mathrm{spt}(u-v) \subset \Omega'$, and for all subdomains $\Omega' \Subset \Omega$. Let us now state our results:

**Theorem 1.1.** *Suppose that we are given exponents $2 \le p \le q < \infty$ with (1.5). Let $f$ satisfy (1.7), (1.8), (1.9) and let $u \in L^\infty_{loc}(\Omega;\mathbb{R}^N)$ denote a local minimizer. Then we have that $\tilde{\nabla} u := (\partial_1 u,\dots,\partial_{n-1}u) \in L^{p+1}_{loc}(\Omega;\mathbb{R}^{(n-1)N})$. Moreover, the function $\partial_n u$ is in the class $L^{q+1}_{loc}(\Omega;\mathbb{R}^N)$.*

**Corollary 1.1.** *Let the assumptions of Theorem 1.1 hold and assume in addition that $n-1 < p \le n$, $q > n$. Then we have that $u \in C^{0,\alpha}(\Omega;\mathbb{R}^N)$ for some $0 < \alpha < 1$.*

**Remark 1.1.** i) *As already noted similar results can be obtained for decompositions different from the one considered here.*

ii) *We may also consider functionals as discussed above with an additional $x$-dependence (compare [BF4]). Note that in the situation at hand we do not have to expect a Lavrentiev phenomenon.*

iii) *Modifying the proof according to [BFM], for instance, we can also handle the case of degenerate ellipticity, which means that the 1 is dropped in the left-hand sides of the inequalities stated in (1.8).*

iv) *In [Bi], Corollary 5.6, a partial regularity result is shown provided that $q < 2+p$ and $q < pn/(n-2)$. It would be interesting to see, whether partial regularity holds in the above examples under a dimension free condition on the exponents. This will be investigated in a separate paper.*

v) *If $p < 2$, then we obtain better integrability results under the weaker assumption $q < p+2$ valid even for a more general class of functionals, we refer to [Bi], Theorem 5.4. Also Theorem 1.2 and Corollary 1.2 below continue to hold for $p < 2$ together with $q < p+2$.*

Next we turn our attention to the scalar case for which we can improve the integrability of $\nabla u$, more precisely we have the following result:

**Theorem 1.2.** *Let $N = 1$, let $f$ satisfy (1.7) and (1.8) together with $2 \le p \le q \le 2p$. Then, if $u \in L^\infty_{loc}(\Omega)$ denotes a local minimizer, we have*

i) *$\tilde{\nabla} u \in L^s_{loc}(\Omega;\mathbb{R}^{n-1})$ for any finite exponent $s$, in particular $u$ actually is in the space $W^1_{q,loc}(\Omega)$;*

ii) *$\nabla u$ is in the space $L^t_{loc}(\Omega;\mathbb{R}^n)$ for any finite $t$.*

**Remark 1.2.** i) *Note that the structural condition (1.9) is not required in the scalar case.*

ii) *According to [Ma2], $u \in W^1_{q,loc}(\Omega)$ implies $u \in C^{1,\alpha}(\Omega)$ if we additionaly require $q < p\frac{n}{n-2}$.*

iii) *It would be desirable to extend Theorem 1.2 including Corollary 1.2 to the vector-situation studied in Theorem 1.1. We think that this is possible if in addition to (1.9) $f_1$ also just depends on the modulus of the matrix, i.e. $f_1(\tilde{Q}) = g_1(|\tilde{Q}|)$. We leave the details to the reader.*

**Corollary 1.2.** *In the scalar case locally bounded local minimizers of the variational integral $J[u,\Omega] = \int_\Omega f(\nabla u)\,dx$ are of class $C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$, provided that $f$ satisfies (1.7), (1.8) and $2 \le p \le q \le 2p$ holds.*

**Remark 1.3.** i) *We conjecture that the bound $q \le 2p$ is optimal, and so it would be interesting to find bounded solutions which are not of class $C^{1,\alpha}$ for a functional $J$ satisfying the hypotheses of Corollary 1.2 but with $q > 2p$, where $q$ can be chosen arbitrarily close to $2p$.*

ii) *We like to remark explicitely that sufficient conditions for regularity of the form (1.2) in general give better results for low dimensions $n$, for example, we mention the paper [FS] where for $n = 2$, $3$ the bounds on $p$ and $q$ are less restrictive.*

iii) *As mentioned above, in this note we do not touch the question of (partial) regularity in the vector case. We just remark that for two-dimensional vectorial problems (i.e. $n = 2$ and $N > 1$) the condition $q < 2p$ is sufficient for interior $C^{1,\alpha}$-regularity even for a priori unbounded local minimizers of an energy $\int_\Omega f(\nabla u)\,\mathrm{d}x$ with $f$ just satisfying (1.4). We refer the reader to [BF5].*

Our paper is organized as follows: in Section 2 we introduce an appropriate local regularization, i.e. we replace the integrand $f$ and the minimizer $u$ by suitable sequences $f_k$ and $u_k$, and prove a Caccioppoli-type inequality for the approximation. In the vector-case this procedure is rather delicate since it is not clear if the test-functions one likes to use are really admissible. The Caccioppoli-type inequality then is used to prove that $\partial_1 u, \dots, \partial_{n-1} u$ are in the space $L^{p+1}_{loc}(\Omega;\mathbb{R}^N)$. In Section 3 we study the scalar case and show by iteration the first part of Theorem 1.2. ¿From this we deduce in Section 4 that $u \in W^1_{t,\mathrm{loc}}(\Omega)$ for any finite $t$, and we use this to get $u \in C^{1,\alpha}(\Omega)$.

## 2 Approximation and proof of Theorem 1.1

Let the hypotheses of Theorem 1.1 be satisfied. Following familiar arguments we introduce an appropriate local regularization: given $\epsilon > 0$, we let $(u)_\epsilon$ denote the mollification of the local minimizer $u$ with radius $\epsilon$. Let us fix $x_0 \in \Omega$ and a ball $B_R(x_0) \Subset \Omega$. Moreover, define

$$\delta := \delta(\epsilon) = \frac{1}{1 + \epsilon^{-1} + \|\tilde{\nabla}(u)_\varepsilon\|^{2q}_{L^q(B_R(x_0);\mathbb{R}^{nN})}}$$

and let $f_\delta(Q) = \delta(1 + |\tilde{Q}|^2)^{q/2} + f(Q)$. Finally, let $u_\delta \in (u)_\varepsilon + \overset{\circ}{W}{}^1_q(B_R(x_0);\mathbb{R}^N)$ denote the unique solution of the problem

$$J_\delta[w, B_R(x_0)] := \int_{B_R(x_0)} f_\delta(\nabla w)\,\mathrm{d}x \to \min \ \text{ in } \ (u)_\epsilon + \overset{\circ}{W}{}^1_q(B_R(x_0);\mathbb{R}^N)\,.$$

**Lemma 2.1.** *We have as $\epsilon \to 0$:*

i) $u_\delta \rightharpoondown u$ *in* $W^1_p(B_R(x_0);\mathbb{R}^N)$,

ii) $\delta\int_{B_R(x_0)}(1+|\tilde\nabla u_\delta|^2)^{q/2}\,\mathrm{d}x\to 0$,

iii) $\int_{B_R(x_0)} f(\nabla u_\delta)\,\mathrm{d}x\to\int_{B_R(x_0)} f(\nabla u)\,\mathrm{d}x$.

*Moreover, $\|u_\delta\|_{L^\infty(B_R(x_0),\mathbb{R}^N)}$ is uniformly bounded.*

*Proof. i)–iii)*: compare e.g. [BF1] with minor adjustments; the last statement follows from the convex hull property established in [BF3]. ☐

**Lemma 2.2.** *(Caccioppoli-type inequality) For any $\eta\in C_0^\infty(B_R(x_0))$ we have*

$$\begin{aligned}&\int_{B_R(x_0)}\eta^2 D^2 f_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma\nabla u_\delta)\,\mathrm{d}x\\ &\quad\leq\; c\int_{B_R(x_0)} D^2 f_\delta(\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta,\nabla\eta\otimes\partial_\gamma u_\delta)\,\mathrm{d}x\,.\end{aligned}\tag{2.1}$$

*Here $\gamma\in\{1,\dots,n\}$ is arbitrary (**no summation in (2.1)**), $c$ is a constant independent of $\delta$ and $\otimes$ denotes the tensor product.*

*Proof.* Compare, for example, [BF1], Lemma 3.1 ; it is easy to check that the proof given in [BF1] actually produces inequality (2.1). ☐

Let us now have a closer look at inequality (2.1) for our special integrand $f$. Using (1.7) and (1.8) we deduce from (2.1)

$$\begin{aligned}&\delta\int_{B_R(x_0)}\eta^2(1+|\tilde\nabla u_\delta|^2)^{\frac{q-2}{2}}|\partial_\gamma\tilde\nabla u_\delta|^2\,\mathrm{d}x+\int_{B_R(x_0)}\eta^2(1+|\tilde\nabla u_\delta|^2)^{\frac{p-2}{2}}|\partial_\gamma\tilde\nabla u_\delta|^2\,dx\\ &\qquad+\int_{B_R(x_0)}\eta^2(1+|\partial_n u_\delta|^2)^{\frac{q-2}{2}}|\partial_\gamma\partial_n u|^2\,\mathrm{d}x\\ &\quad\leq\; c\Bigg[\delta\int_{B_R(x_0)}|\nabla\eta|^2(1+|\tilde\nabla u_\delta|^2)^{\frac{q-2}{2}}|\partial_\gamma u_\delta|^2\,\mathrm{d}x+\int_{B_R(x_0)}|\nabla\eta|^2(1+|\tilde\nabla u_\delta|^2)^{\frac{p-2}{2}}|\partial_\gamma u_\delta|^2\,\mathrm{d}x\\ &\qquad+\int_{B_R(x_0)}|\nabla\eta|^2(1+|\partial_n u|^2)^{\frac{q-2}{2}}|\partial_\gamma u_\delta|^2\,\mathrm{d}x\Bigg]\,.\end{aligned}$$

Now, taking the sum w.r.t. $\gamma$ from 1 to $n-1$ on both sides, we get with an obvious meaning of $\tilde\nabla^2$:

$$\begin{aligned}&\delta\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{q-2}{2}}|\tilde\nabla^2 u_\delta|^2\,\mathrm{d}x+\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{p-2}{2}}|\tilde\nabla^2 u_\delta|^2\,\mathrm{d}x\\ &\qquad+\int_{B_R(x_0)}\eta^2\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n\tilde\nabla u_\delta|^2\,\mathrm{d}x\\ &\quad\leq\; c\Bigg[\delta\int_{B_R(x_0)}|\nabla\eta|^2\tilde\Gamma_\delta^{\frac{q-2}{2}}|\tilde\nabla u_\delta|^2\,\mathrm{d}x+\int_{B_R(x_0)}|\nabla\eta|^2\tilde\Gamma_\delta^{\frac{p-2}{2}}|\tilde\nabla u_\delta|^2\,\mathrm{d}x\\ &\qquad+\int_{B_R(x_0)}|\nabla\eta|^2\Gamma_{n,\delta}^{\frac{q-2}{2}}|\tilde\nabla u_\delta|^2\,\mathrm{d}x\Bigg]\,,\end{aligned}\tag{2.2}$$

where $\tilde\Gamma_\delta:=1+|\tilde\nabla u_\delta|^2$, $\Gamma_{n,\delta}:=1+|\partial_n u_\delta|^2$.

Before we prove Theorem 1.1, let us recall the following auxiliary result, a proof can be found in [GM]

**Proposition 2.1.** *Consider a function $g\colon \mathbb{R}^L \to \mathbb{R}$ of class $C^2$ such that for some $s \geq 2$ we have with a positive constant $c_1$*

$$c_1 \left(1 + |A|^2\right)^{\frac{s-2}{2}} |B|^2 \leq D^2 g(A)(B,B) \quad \text{for all } A,\, B \in \mathbb{R}^L\,.$$

*Then there is another positive constant $c_2$, just depending on $s$ and $c_1$ such that*

$$\int_0^1 D^2 g(A + tB)(X,X)\,\mathrm{d}t \geq c_2 D^2 g(A)(X,X) \quad \text{for all } A,\, B,\, X \in \mathbb{R}^L\,. \tag{2.3}$$

We proceed by showing a version of the Caccioppoli-type inequality involving difference quotients, which will be an essential ingredient in the proof of Theorem 1.1.

**Lemma 2.3.** *Fix a direction $e_\gamma$, $\gamma \leq n-1$, and let $v := \Delta_h u_\delta$ denote the difference quotient of $u_\delta$ in this direction. Then we have for any $\eta \in C_0^\infty(B_R(x_0))$*

$$\begin{aligned}
\int_{B_R(x_0)} \eta^2 D^2 f_\delta(\nabla u_\delta)(\nabla v, \nabla v)\,\mathrm{d}x \quad &\leq \quad c\|\nabla\eta\|_\infty^2 \Bigg[\delta \int_{\operatorname{spt}\eta} \int_0^1 (1 + |\tilde{U}|^2)^{\frac{q-2}{2}} |v|^2 \,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_{\operatorname{spt}\eta} \int_0^1 (1 + |\tilde{U}|^2)^{\frac{p-2}{2}} |v|^2 \,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_{\operatorname{spt}\eta} \int_0^1 (1 + |U_n|^2)^{\frac{q-2}{2}} |v|^2 \,\mathrm{d}t\,\mathrm{d}x \Bigg] \\
&=: \quad c\|\nabla\eta\|_\infty^2 [\delta \cdot I_1 + I_2 + I_3]\,.
\end{aligned}$$

*Here we have set $U := \nabla u_\delta + th\Delta_h \nabla u_\delta$.*

*Proof.* Let us introduce the bilinear form $\mathcal{B} := \int_0^1 D^2 f_\delta(U)\,\mathrm{d}t$. If we write

$$\begin{aligned}
\mathcal{B}(X,X) &= \int_0^1 D^2 f_\delta\left(\nabla u_\delta + th\Delta_h \nabla u_\delta\right)(X,X)\,\mathrm{d}t \\
&= \int_0^1 D^2 f_\delta\left(A + tB\right)(X,X)\,\mathrm{d}t \\
&= \int_0^1 D^2 g_\delta(\tilde{A} + t\tilde{B})(\tilde{X},\tilde{X})\,\mathrm{d}t + \int_0^1 D^2 f_2(A_n + tB_n)(X_n, X_n)\,\mathrm{d}t
\end{aligned}$$

with $A = \nabla u_\delta$, $B = h\Delta_h \nabla u_\delta$, $g_\delta(\epsilon) := \delta(1+|\epsilon|^2)^q + f_1(\epsilon)$, then – due to the ellipticity conditions for $f_1$ and $f_2$ – the inequality (2.3) can be applied to both terms on the r.h.s. of the above equation leading to the estimate

$$D^2 f_\delta(\nabla u_\delta)(X,X) \leq \mathcal{B}(X,X) \quad \text{for all } X \in \mathbb{R}^N\,.$$

This together with (3.2) of [BF1] gives

$$\int_{B_R(x_0)} \eta^2 D^2 f_\delta(\nabla u_\delta)(\nabla v, \nabla v)\,\mathrm{d}x \leq c \int_{B_R(x_0)} \mathcal{B}\left(\nabla v, \nabla v\right) \eta^2 \,\mathrm{d}x \leq c \int_{B_R(x_0)} \eta \mathcal{B}\left(\nabla v, \nabla\eta \otimes v\right)\,\mathrm{d}x\,.$$

Using now the Cauchy-Schwarz inequality for the bilinear form $\mathcal{B}$ we get

$$\begin{aligned}
\int_{B_R(x_0)} \eta^2 D^2 f_\delta(\nabla u_\delta)(\nabla v, \nabla v)\,\mathrm{d}x \quad &\leq \quad c \int_{B_R(x_0)} \mathcal{B}\left(\nabla\eta \otimes v, \nabla\eta \otimes v\right)\,\mathrm{d}x \\
&\leq \quad c \Bigg[\int_{B_R(x_0)} \int_0^1 D^2 g_\delta(\tilde{U})(\tilde{\nabla}\eta \otimes v, \tilde{\nabla}\eta \otimes v)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_{B_R(x_0)} \int_0^1 D^2 f_2(U_n)\left(\partial_n \eta v, \partial_n \eta v\right)\,\mathrm{d}t\,\mathrm{d}x \Bigg]\,,
\end{aligned}$$

which immediately gives the lemma on account on our ellipticity assumption (1.8). $\square$

*Proof of Theorem 1.1.* ¿From the minimality of $u_\delta$ it follows that

$$\int_{B_R(x_0)} Df_\delta(\nabla u_\delta) : \nabla\varphi \,\mathrm{d}x = 0 \tag{2.4}$$

for any $\varphi \in \overset{\circ}{W}{}^1_q(B_R(x_0);\mathbb{R}^N)$. As a test vector in (2.4) we like to choose $\varphi = \eta^2|\tilde{\nabla}u_\delta|u_\delta$. Using the standard difference quotient procedure, see e.g. [Mo] or [Ca], we get that $\tilde{\nabla}u_\delta$ is just of local Sobolev class $W^1_2$ so that the admissibility of $\varphi$ ist not immediate. To overcome this problem, we fix a direction $e_\gamma, \gamma \leq n-1$, and as above we let $v := \Delta_h u_\delta$ denote the corresponding difference quotient of $u_\delta$. Then (2.4) gives choosing $\varphi = \eta^2 u_\delta |v|$, $\eta \in C^\infty_0(B_R(x_0))$,

$$\begin{aligned}\int_{B_R(x_0)} Df_\delta(\nabla u_\delta) : \nabla u_\delta \eta^2 |v| \,\mathrm{d}x &= -2\int_{B_R(x_0)} Df_\delta(\nabla u_\delta) : (\nabla\eta \otimes u_\delta)\,\eta|v|\,\mathrm{d}x \\ &\quad - \int_{B_R(x_0)} Df_\delta(\nabla u_\delta) : (u_\delta \otimes \nabla|v|)\,\eta^2 \,\mathrm{d}x\,. \end{aligned}\tag{2.5}$$

For any matrices $X$, $Z$ we have (assuming w.l.o.g. $Df(0) = 0$) $Df_\delta(X) : Z = \int_0^1 D^2 f_\delta(tX)(X,Z)\,dt$, so that by(1.7), (1.8) we get the estimates

$$\begin{aligned} Df_\delta(X) : X &\geq c\Big[\delta(1+|\tilde{X}|^2)^{\frac{q-2}{2}}|\tilde{X}|^2 \\ &\quad +(1+|\tilde{X}|^2)^{\frac{p-2}{2}}|\tilde{X}|^2 + (1+|X_n|^2)^{\frac{q-2}{2}}|X_n|^2\Big]\,, \end{aligned}\tag{2.6}$$

$$\begin{aligned} |Df_\delta(X)| &\leq c\Big[\delta(1+|\tilde{X}|^2)^{\frac{q-2}{2}}|\tilde{X}| \\ &\quad +(1+|\tilde{X}|^2)^{\frac{p-2}{2}}|\tilde{X}| + (1+|X_n|^2)^{\frac{q-2}{2}}|X_n|\Big] \end{aligned}\tag{2.7}$$

with positive constants independent of $\epsilon$. ¿From (2.6) we immediately deduce that

$$\begin{aligned} \text{l.h.s. of (2.5)} &\geq c\Bigg[\delta\int_{B_R(x_0)} |v|\eta^2\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla}u_\delta|^2\,\mathrm{d}x + \int_{B_R(x_0)} |v|\eta^2\tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla}u_\delta|^2\,\mathrm{d}x \\ &\quad + \int_{B_R(x_0)} |v|\eta^2\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta|^2\,\mathrm{d}x\Bigg]\,. \end{aligned}\tag{2.8}$$

For the r.h.s. of (2.5) we observe that (2.7) together with the uniform $L^\infty$-bound of $u_\delta$ implies:

$$\begin{aligned} &|1^{\text{st}} \text{ term on the r.h.s. of (2.5)}| \\ &\leq c\Bigg[\|\nabla\eta\|_\infty\Bigg(\delta\int_{\operatorname{spt}\eta}|v|\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla}u_\delta|\,\mathrm{d}x + \int_{\operatorname{spt}\eta}|v|\tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla}u_\delta|\,\mathrm{d}x\Bigg) \\ &\quad + \int_{\operatorname{spt}\eta}\eta|\nabla\eta||v|\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta|\,\mathrm{d}x\Bigg] \\ &=: c\Big[\|\nabla\eta\|_\infty(T_1+T_2)+T_3\Big]\,. \end{aligned}$$

By Hölder's inequality and elementary properties of difference quotients we see by Lemma 2.1, *i*), that

$$T_1 \leq c\delta\int_{B_R(x_0)} \tilde{\Gamma}_\delta^{\frac{q}{2}}\,\mathrm{d}x \to 0 \quad \text{as } \epsilon\to 0\,,$$

whereas

$$T_2 \le c \int_{B_R(x_0)} \tilde{\Gamma}_\delta^{\frac{p}{2}} \le c\,.$$

The last inequality follows from the minimality of the functions $u_\delta$, i.e.

$$J_\delta[u_\delta, B_R(x_0)] \le J_\delta[(u)_\epsilon, B_R(x_0)] \le c\,.$$

We further have (for any $\tau \in (0,1)$)

$$\begin{aligned}
T_3 &\le \int_{\operatorname{spt}\eta} \Big[\tau\eta^2|\partial_n u_\delta|^2 + \frac{1}{\tau}\|\nabla\eta\|_\infty^2\Big]|v|\Gamma_{n,\delta}^{\frac{q-2}{2}}\,\mathrm{d}x \\
&\le \tau\int_{\operatorname{spt}\eta} \eta^2|v||\partial_n u_\delta|^2\Gamma_{n,\delta}^{\frac{q-2}{2}}\,\mathrm{d}x + \frac{c}{\tau}\|\nabla\eta\|_\infty^2\int_{\operatorname{spt}\eta}\Big\{\Gamma_{n,\delta}^{\frac{q}{2}} + |v|^{\frac{q}{2}}\Big\}\,\mathrm{d}x\,. \qquad (2.9)
\end{aligned}$$

Since we assume $q \le 2p$, we get that

$$\int_{\operatorname{spt}\eta} |v|^{\frac{q}{2}}\,\mathrm{d}x$$

is bounded in terms of $\int_{B_R(x_0)} |\tilde{\nabla} u_\delta|^p\,\mathrm{d}x$. Moreover, if $\tau$ is sufficiently small, then the first integral on the right-hand side of (2.9) can be absorbed in the last integral on the right-hand side of (2.8) with the result (w.l.o.g. $\|\nabla\eta\|_\infty \ge 1$)

$$\begin{aligned}
&\delta\int_{B_R(x_0)} |v|\eta^2\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla} u_\delta|^2\,\mathrm{d}x + \int_{B_R(x_0)} |v|\eta^2\tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla} u_\delta|^2\,\mathrm{d}x + \int_{B_R(x_0)} |v|\eta^2\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta|^2\,\mathrm{d}x \\
&\qquad \le c\|\nabla\eta\|_\infty^2 + c\left|\int_{B_R(x_0)} Df_\delta(\nabla u_\delta) : (u_\delta \otimes \nabla|v|)\eta^2\,\mathrm{d}x\right|. \qquad (2.10)
\end{aligned}$$

Let us now discuss the remaining integral on the right-hand side of (2.10): we observe that

$$Df_\delta(X) : Y = \delta\frac{q}{2}\Big(1+|\tilde{X}|^2\Big)^{\frac{q-2}{2}}\tilde{X} : \tilde{Y} + Df_1(\tilde{X}) : \tilde{Y} + Df_2(X_n)\cdot Y_n\,, \qquad (2.11)$$

which implies (using the uniform boundedness of $u_\delta$) for any $0 < \tau < 1$

$$\begin{aligned}
|Df_\delta(\nabla u_\delta) : (\nabla|v| \otimes u)| &\le c\Big[\delta\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla} u_\delta||\tilde{\nabla} v| + \tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla} u_\delta||\tilde{\nabla} v| + \Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta||\partial_n v|\Big] \\
&\le c\Big[\delta\tau\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla} v|^2 + \delta\frac{1}{\tau}\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla} u_\delta|^2 + \tau\tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla} v|^2 + \frac{1}{\tau}\tilde{\Gamma}_\delta^{\frac{p}{2}} \\
&\qquad + \tau\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n v|^2 + \frac{1}{\tau}\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta|^2\Big].
\end{aligned}$$

This gives the following upper bound for the integral under consideration:

$$\begin{aligned}
&c\Big[\delta\tau\int_{B_R(x_0)} \eta^2\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla} v|^2\,\mathrm{d}x + \tau\int_{B_R(x_0)} \tilde{\Gamma}_\delta^{\frac{p-2}{2}}\eta^2|\tilde{\nabla} v|^2\,\mathrm{d}x + \tau\int_{B_R(x_0)} \eta^2\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n v|^2\,dx \\
&+\frac{1}{\tau}\int_{B_R(x_0)} \eta^2\tilde{\Gamma}_\delta^{\frac{p}{2}}\,\mathrm{d}x + \frac{1}{\tau}\delta\int_{B_R(x_0)} \eta^2\tilde{\Gamma}_\delta^{\frac{q}{2}}\,\mathrm{d}x + \frac{1}{\tau}\int_{B_R(x_0)} \Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x\Big], \qquad (2.12)
\end{aligned}$$

where all the quantities which are multiplied by $\frac{1}{\tau}$ stay bounded uniformly w.r.t $\epsilon$ (recall $\delta = \delta(\epsilon)$). In order to continue we look at the sum of the first three items of (2.12) and observe that

$$\begin{aligned}\int_{B_R(x_0)} &\eta^2 \Big[\delta\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla} v|^2 + \tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla} v|^2 + \Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n v|^2\Big]\,\mathrm{d}x\\ &\leq c\int_{B_R(x_0)} \eta^2 D^2 f_\delta(\nabla u_\delta)(\nabla v, \nabla v)\,\mathrm{d}x\,.\end{aligned}$$

Note that the right hand side can be estimated with the help of Lemma 2.3, i.e. (with the notation of Lemma 2.3) we get from (2.10) (for any $0 < \tau < 1$)

$$\begin{aligned}\delta\int_{B_R(x_0)} |v|\eta^2\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla} u_\delta|^2\,\mathrm{d}x + \int_{B_R(x_0)} |v|\eta^2\tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla} u_\delta|^2\,\mathrm{d}x + \int_{B_R(x_0)} |v|\eta^2\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta|^2\,\mathrm{d}x\\ \leq c\|\nabla\eta\|_\infty^2 + c\tau\|\nabla\eta\|_\infty^2[\delta I_1 + I_2 + I_3] + \frac{c}{\tau}\,.\end{aligned} \tag{2.13}$$

We have

$$\begin{aligned}I_1 &\leq \int_{\operatorname{spt}\eta} |v|^q\,\mathrm{d}x + \int_{\operatorname{spt}\eta}\int_0^1 (1+|\tilde{U}|^2)^{\frac{q}{2}}\,\mathrm{d}t\,\mathrm{d}x\\ &\leq c\int_{B_R(x_0)} |\tilde{\nabla} u_\delta|^q\,\mathrm{d}x + c\int_{\operatorname{spt}\eta}\int_0^1 (|(1-t)\tilde{\nabla} u_\delta(x) + t\tilde{\nabla} u_\delta(x+he_\gamma)|^2 + 1)^{\frac{q}{2}}\,\mathrm{d}t\,\mathrm{d}x\,,\end{aligned}$$

and if from now on we assume that $\operatorname{spt}\eta \subset B_{R/2}(x_0)$, then of course (for all $|h| \ll 1$ uniform in $\epsilon$)

$$I_1 \leq c\int_{B_R(x_0)} \tilde{\Gamma}_\delta^{\frac{q}{2}}\,\mathrm{d}x\,,$$

and we know $\delta I_1 \to 0$ as $\epsilon \to 0$, thus $\delta I_1$ is uniformly bounded for all small $\epsilon$ and $|h|$. Since we also know that $\int_{B_R(x_0)} \tilde{\Gamma}_\delta^{p/2}\,\mathrm{d}x \leq c$, the same argument applies to $I_2$. To handle $I_3$ we observe that in the limit $h \to 0$

$$I_3 \leq \int_{\operatorname{spt}\eta}\int_0^1 (1+|U_n|^2)^{\frac{q-2}{2}}|v|^2\,\mathrm{d}t\,\mathrm{d}x \to \int_{\operatorname{spt}\eta} \Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_\gamma u_\delta|^2\,\mathrm{d}x\,. \tag{2.14}$$

To prove (2.14) we note that according to [Gi], Theorem 3.1, p. 159, there exists an exponent $s > q$ (depending on $\epsilon$) such that $\nabla u_\delta \in L^s_{loc}(B_R(x_0);\mathbb{R}^{nN})$. This implies (as $h \to 0$)

$$\partial_n u_\delta(x+he_\gamma) \to \partial_n u_\delta\,, \qquad v \to \partial_\gamma u_\delta$$

in $L^s_{loc}(B_R(x_0);\mathbb{R}^N)$ and a.e. Clearly (as $h \to 0$)

$$\zeta_h := \int_0^1 (1+|U_n|^2)^{\frac{q-2}{2}}\,\mathrm{d}t|v|^2 \to \Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_\gamma u_\delta|^2$$

a.e. and, using the local $L^s$-convergences, the equi-integrability of $\zeta_h$ follows. Then (2.14) follows from Vitali's convergence theorem. Returning to (2.13), using (2.14) and the foregoing estimates and applying Fatou's lemma on the l.h.s. of (2.13), we find in the limit $h \to 0$

$$\begin{aligned}\delta\int_{B_s(\bar{x})} |\partial_\gamma u_\delta|\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla} u_\delta|^2\,\mathrm{d}x + \int_{B_s(\bar{x})} |\partial_\gamma u_\delta|\tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla} u_\delta|^2\,\mathrm{d}x + \int_{B_s(\bar{x})} |\partial_\gamma u_\delta|\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta|^2\,\mathrm{d}x\\ \leq c(t-s)^{-2} + \frac{c}{\tau} + c\tau(t-s)^{-2}\int_{B_t(\bar{x})} \Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_\gamma u_\delta|^2\,\mathrm{d}x\,.\end{aligned} \tag{2.15}$$

where $\eta \in C_0^\infty(B_t(\bar{x}))$ $(0 < s < t < T,\ B_T(\bar{x}) \Subset B_{R/2}(x_0))$ has been chosen according to $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_s(\bar{x})$, $|\nabla \eta| \le c/(t-s)$. We estimate the last integral of (2.15) in the following way. By Young's inequality

$$\begin{aligned}\int_{B_t(\bar{x})} \Gamma_{n,\delta}^{\frac{q-2}{2}} |\partial_\gamma u_\delta|^2 \,\mathrm{d}x &\le c \int_{B_t(\bar{x})} |\partial_n u_\delta|^{q-2} |\partial_\gamma u_\delta|^2 \,\mathrm{d}x + c \int_{B_t(\bar{x})} |\partial_\gamma u_\delta|^2 \,\mathrm{d}x \\ &\le c \int_{B_t(\bar{x})} |\partial_\gamma u_\delta| |\partial_n u_\delta|^q \,\mathrm{d}x + c \int_{B_t(\bar{x})} (|\partial_\gamma u_\delta|^{\frac{q}{2}+1} + |\partial_\gamma u_\delta|^2|) \,\mathrm{d}x \,.\end{aligned}$$

Moreover, since $q \le 2p$,

$$\begin{aligned}|\partial_\gamma u_\delta|^{\frac{q}{2}+1} &\le \tilde{\Gamma}_\delta^{\frac{p-2}{2}} |\partial_\gamma u_\delta|^{\frac{q}{2}+1-p+2} \le \tilde{\Gamma}_\delta^{\frac{p-2}{2}} |\tilde{\nabla} u_\delta|^2 |\partial_\gamma u_\delta|^{\frac{q}{2}-p+1} \\ &\le \tilde{\Gamma}_\delta^{\frac{p-2}{2}} |\tilde{\nabla} u_\delta|^2 \big[|\partial_\gamma u_\delta| + 1\big] \,.\end{aligned}$$

This gives

$$\int_{B_t(\bar{x})} \Gamma_{n,\delta}^{\frac{q-2}{2}} |\partial_\gamma u_\delta|^2 \,\mathrm{d}x \le c \Bigg[ \int_{B_t(\bar{x})} |\partial_\gamma u_\delta| |\partial_n u_\delta|^q \,\mathrm{d}x + \int_{B_t(\bar{x})} \tilde{\Gamma}_\delta^{\frac{p-2}{2}} |\tilde{\nabla} u_\delta|^2 |\partial_\gamma u_\delta| dx + 1 \Bigg] \,.$$

Inserting this estimate into (2.15) we obtain (by neglecting the first term on the l.h.s. of (2.15))

$$\begin{aligned} h(s) :=& \int_{B_s(\bar{x})} |\partial_\gamma u_\delta| \tilde{\Gamma}_\delta^{\frac{p-2}{2}} |\tilde{\nabla} u_\delta|^2 \,\mathrm{d}x + \int_{B_s(\bar{x})} |\partial_\gamma u_\delta| \Gamma_{n,\delta}^{\frac{q-2}{2}} |\partial_n u_\delta|^2 \,\mathrm{d}x \\ \le&\ c(t-s)^{-2} + \frac{c}{\tau} + c\tau (t-s)^{-2} \\ &+ c\tau(t-s)^{-2} \Bigg[ \int_{B_t(\bar{x})} \tilde{\Gamma}_\delta^{\frac{p-2}{2}} |\tilde{\nabla} u_\delta|^2 |\partial_\gamma u_\delta| dx + \int_{B_t(\bar{x})} |\partial_\gamma u_\delta| \Gamma_{n,\delta}^{\frac{q-2}{2}} |\partial_n u_\delta|^2 \,\mathrm{d}x \Bigg] \,, \end{aligned} \tag{2.16}$$

i.e. with the choice $\tau = (t-s)^2/(2c)$

$$h(s) \le c(t-s)^{-2} + c + \frac{1}{2} h(t)$$

for any $s$, $t$ as above. Lemma 3.1, p. 161, of [Gi] finally shows that

$$\int_{B_s(\bar{x})} |\partial_\gamma u_\delta| \tilde{\Gamma}_\delta^{\frac{p-2}{2}} |\tilde{\nabla} u_\delta|^2 \,\mathrm{d}x + \int_{B_s(\bar{x})} |\partial_\gamma u_\delta| \Gamma_{n,\delta}^{\frac{q-2}{2}} |\partial_n u_\delta|^2 \,\mathrm{d}x \le c\big[(T-s)^{-2} + 1\big]$$

with a constant $c$ independent of $\epsilon$ being valid for all $s < T$. Recalling that $\gamma \le n-1$ we get the uniform bound

$$\int_{B_{R/4}(x_0)} |\tilde{\nabla} u_\delta|^{p+1} \,\mathrm{d}x + \int_{B_{R/4}(x_0)} |\tilde{\nabla} u_\delta| \ |\partial_n u_\delta|^q \,dx \le c(R) \,.$$

Since by Lemma 2.1 we already know $u_\delta \rightharpoondown u$ as $\epsilon \to 0$ in $W^1_p(B_R(x_0);\mathbb{R}^N)$, the first claim of Theorem 1.1 follows.
The second statement is obtained by a similar calculation replacing the function $v$ by the difference quotient of $u$ in the $n^{th}$ coordinate direction. We also refer to the last section where this calculation is carried out for the scalar case (avoiding the difference-quotient technique). $\square$

The statement of the corollary is an immediate consequence of Sobolev's embedding theorem. $\square$

## 3 Higher integrability in the scalar case: proof of the first part of Theorem 1.2

We will use the notation introduced in Section 2. From [LU] (see the discussion in Remark 2.3 of [BF4]) we deduce that $u_\delta$ is in the space $W^2_{t,\mathrm{loc}}(B_R(x_0))$ for any $t < \infty$, therefore we may test the differentiated Euler equation valid for $u_\delta$ with the function $\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{\beta/2}$, where $\beta \geq 0$, $\eta \in C_0^\infty(B_R(x_0))$ and $\gamma$ runs from 1 to $n-1$. Since we consider the scalar case, it is easy to check that (**from now on summation w.r.t. $\gamma$ from 1 to $n-1$**)

$$0 \leq \int_{B_R(x_0)} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \eta^2 \partial_\gamma u_\delta \nabla \tilde{\Gamma}_\delta^{\frac{\beta}{2}}) \,\mathrm{d}x\,.$$

In fact, this is the only place where $N = 1$ is needed. Thus (2.1) is replaced by the inequality

$$\begin{aligned}\int_{B_R(x_0)} & \tilde{\Gamma}_\delta^{\frac{\beta}{2}} \eta^2 D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \,\mathrm{d}x \\ & \leq c \int_{B_R(x_0)} \eta^2 \tilde{\Gamma}_\delta^{\frac{\beta}{2}} D^2 f_\delta(\nabla u_\delta)(\nabla\eta \partial_\gamma u_\delta, \nabla\eta\partial_\gamma u_\delta) \,\mathrm{d}x\,,\end{aligned}$$

which means that we get (2.2) with factor $\tilde{\Gamma}_\delta^{\beta/2}$ on both sides:

$$\begin{aligned}\delta \int_{B_R(x_0)} & \tilde{\Gamma}^{\frac{\beta}{2}} \eta^2 \tilde{\Gamma}_\delta^{\frac{q-2}{2}} |\tilde{\nabla}^2 u_\delta|^2 \,dx + \int_{B_R(x_0)} \tilde{\Gamma}^{\frac{\beta}{2}} \eta^2 \tilde{\Gamma}_\delta^{\frac{p-2}{2}} |\tilde{\nabla}^2 u_\delta|^2 \,\mathrm{d}x \\ & + \int_{B_R(x_0)} \tilde{\Gamma}^{\frac{\beta}{2}} \eta^2 \Gamma_{n,\delta}^{\frac{q-2}{2}} |\partial_n \tilde{\nabla} u_\delta|^2 \,\mathrm{d}x \\ \leq\ & c\Bigg[\delta \int_{B_R(x_0)} \tilde{\Gamma}^{\frac{\beta}{2}} |\nabla\eta|^2 \tilde{\Gamma}_\delta^{\frac{q-2}{2}} |\tilde{\nabla} u_\delta|^2 \,\mathrm{d}x + \int_{B_R(x_0)} \tilde{\Gamma}^{\frac{\beta}{2}} |\nabla\eta|^2 \Gamma_\delta^{\frac{p-2}{2}} |\tilde{\nabla} u_\delta|^2 \,\mathrm{d}x \\ & + \int_{B_R(x_0)} \tilde{\Gamma}^{\frac{\beta}{2}} |\nabla\eta|^2 \Gamma_{n,\delta}^{\frac{q-2}{2}} |\tilde{\nabla} u_\delta|^2 \,\mathrm{d}x\Bigg]\,. \end{aligned} \tag{2.2$_\beta$}$$

We apply (2.4) with $\varphi := \eta^2 u_\delta \tilde{\Gamma}_\delta^{(1+\alpha)/2}$ as admissible test function, $\alpha \geq 0$ being some number specified below. As a result we get (2.5) with $|v|$ replaced by $\tilde{\Gamma}_\delta^{(1+\alpha)/2}$:

$$\begin{aligned}\int_{B_R(x_0)} D f_\delta(\nabla u_\delta) \cdot \nabla u_\delta \eta^2 \tilde{\Gamma}_\delta^{\frac{1+\alpha}{2}} \,\mathrm{d}x \ &=\ -2\int_{B_R(x_0)} D f_\delta(\nabla u_\delta) \cdot \nabla\eta u_\delta \eta \tilde{\Gamma}_\delta^{\frac{1+\alpha}{2}} \,\mathrm{d}x \\ & \quad - \int_{B_R(x_0)} D f_\delta(\nabla u_\delta) \cdot u_\delta \nabla \tilde{\Gamma}_\delta^{\frac{1+\alpha}{2}} \eta^2 \,\mathrm{d}x\,. \end{aligned} \tag{2.5$_\alpha$}$$

We observe (compare (2.8))

$$\begin{aligned}\text{l.h.s.of (2.5}_\alpha) \ \geq\ & c\Bigg[\delta \int_{B_R(x_0)} \tilde{\Gamma}_\delta^{\frac{q-2}{2}} \tilde{\Gamma}_\delta^{\frac{1+\alpha}{2}} \eta^2 |\tilde{\nabla} u_\delta|^2 \,\mathrm{d}x + \int_{B_R(x_0)} \tilde{\Gamma}_\delta^{\frac{p-2}{2}} \tilde{\Gamma}_\delta^{\frac{1+\alpha}{2}} \eta^2 |\tilde{\nabla} u_\delta|^2 \,\mathrm{d}x \\ & + \int_{B_R(x_0)} \tilde{\Gamma}_\delta^{\frac{1+\alpha}{2}} \Gamma_{n,\delta}^{\frac{q-2}{2}} \eta^2 |\partial_n u_\delta|^2 \,\mathrm{d}x\Bigg]\,. \end{aligned} \tag{3.1}$$

Next we estimate the first term on the r.h.s. of $(2.5_\alpha)$ (compare the inequality stated after (2.10))

$$\begin{aligned}
&\left|2\int_{B_R(x_0)} Df_\delta(\nabla u_\delta)\cdot\nabla\eta u_\delta\eta\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\,\mathrm{d}x\right|\\
&\quad\leq\ c\|\nabla\eta\|_\infty\Bigg[\delta\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\tilde\Gamma_\delta^{\frac{q-2}{2}}|\tilde\nabla u_\delta|\,\mathrm{d}x+\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\tilde\Gamma_\delta^{\frac{p-2}{2}}|\tilde\nabla u_\delta|\,\mathrm{d}x\\
&\qquad+\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta|\,\mathrm{d}x\Bigg]\\
&\quad\leq\ c\|\nabla\eta\|_\infty\Bigg[\delta\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{q+\alpha}{2}}\,\mathrm{d}x+\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{p+\alpha}{2}}\,\mathrm{d}x+\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\Gamma_{n,\delta}^{\frac{q-1}{2}}\,\mathrm{d}x\Bigg]\\
&\quad\leq\ c\|\nabla\eta\|_\infty\Bigg[\tau\delta\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{q+1+\alpha}{2}}\,\mathrm{d}x+c(\tau)\delta\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{q+\alpha-1}{2}}\,\mathrm{d}x+\tau\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{p+\alpha+1}{2}}\,\mathrm{d}x\\
&\qquad+c(\tau)\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{p+\alpha-1}{2}}\,\mathrm{d}x+\tau\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x+c(\tau)\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\,\mathrm{d}x\Bigg]\,,
\end{aligned}\tag{3.2}$$

where $0<\tau<1$ is arbitrary. In order to handle the second term on the r.h.s. of $(2.5_\alpha)$ we recall (2.11) and get

$$\begin{aligned}
\left|Df_\delta(\nabla u_\delta)\cdot\nabla\tilde\Gamma_\delta^{\frac{\alpha+1}{2}}\right|\ &\leq\ c\frac{\alpha+1}{2}\tilde\Gamma_\delta^{\frac{\alpha-1}{2}}\Bigg[\delta\tilde\Gamma_\delta^{\frac{q-2}{2}}|\tilde\nabla u_\delta||\tilde\nabla\tilde\Gamma_\delta|+\tilde\Gamma_\delta^{\frac{p-2}{2}}|\tilde\nabla u_\delta||\tilde\nabla\tilde\Gamma_\delta|+\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta||\partial_n\tilde\Gamma_\delta|\Bigg]\\
&\leq\ c(\alpha)\Bigg[\delta\tilde\Gamma_\delta^{\frac{q+\alpha-1}{2}}|\tilde\nabla^2 u_\delta|+\tilde\Gamma_\delta^{\frac{p+\alpha-1}{2}}|\tilde\nabla^2 u_\delta|+\tilde\Gamma_\delta^{\frac{\alpha}{2}}\Gamma_{n,\delta}^{\frac{q-1}{2}}|\partial_n\tilde\nabla u_\delta|\Bigg]\,.
\end{aligned}\tag{3.3}$$

We have

$$\begin{aligned}
\tilde\Gamma_\delta^{\frac{p+\alpha-1}{2}}|\tilde\nabla^2 u_\delta|\ &\leq\ \tilde\Gamma_\delta^{\frac{\beta}{2}}\tilde\Gamma_\delta^{\frac{p-2}{2}}|\tilde\nabla^2 u_\delta|^2+\tilde\Gamma_\delta^{-\frac{\beta}{2}+\frac{p+2\alpha}{2}}\,,\\
\delta\tilde\Gamma_\delta^{\frac{q+\alpha-1}{2}}|\tilde\nabla^2 u_\delta|\ &\leq\ \delta\tilde\Gamma_\delta^{\frac{\beta}{2}}\tilde\Gamma_\delta^{\frac{q-2}{2}}|\tilde\nabla^2 u_\delta|^2+\delta\tilde\Gamma_\delta^{-\frac{\beta}{2}+\frac{q+2\alpha}{2}}
\end{aligned}$$

as well as

$$\begin{aligned}
\tilde\Gamma_\delta^{\frac{\alpha}{2}}\Gamma_{n,\delta}^{\frac{q-1}{2}}|\partial_n\tilde\nabla u_\delta|\ &=\ \tilde\Gamma_\delta^{\frac{\beta}{4}}\Gamma_{n\delta}^{\frac{q-2}{4}}|\partial_n\tilde\nabla u_\delta|\tilde\Gamma_\delta^{-\frac{\beta}{4}+\frac{\alpha}{2}}\Gamma_{n,\delta}^{\frac{q}{4}}\\
&\leq\ \tilde\Gamma_\delta^{\frac{\beta}{2}}\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n\tilde\nabla u_\delta|^2+\Gamma_{n,\delta}^{\frac{q}{2}}\tilde\Gamma_\delta^{\frac{2\alpha-\beta}{2}}\,.
\end{aligned}$$

We insert these inequalities into (3.3) and get

$$
\begin{aligned}
&\left|\int_{B_R(x_0)} Df_\delta(\nabla u_\delta)\cdot\nabla\Gamma_\delta^{\frac{\alpha+1}{2}} u_\delta\eta^2\,\mathrm{d}x\right| \\
&\quad\le\; c(\alpha)\left[\delta\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{\beta}{2}}\tilde\Gamma_\delta^{\frac{q-2}{2}}|\tilde\nabla^2 u_\delta|^2\,\mathrm{d}x+\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{\beta}{2}}\tilde\Gamma_\delta^{\frac{p-2}{2}}|\tilde\nabla^2 u_\delta|^2\,\mathrm{d}x\right.\\
&\qquad\left.+\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{\beta}{2}}\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n\tilde\nabla u_\delta|^2\,\mathrm{d}x\right]\\
&\qquad+c(\alpha)\left[\delta\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{q+2\alpha-\beta}{2}}\,\mathrm{d}x+\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{p+2\alpha-\beta}{2}}\,\mathrm{d}x+\int_{B_R(x_0)}\eta^2\Gamma_{n,\delta}^{\frac{q}{2}}\tilde\Gamma_\delta^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x\right]\\
&\quad\le\; c(\alpha)\|\nabla\eta\|_\infty^2\left[\delta\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{q+\beta}{2}}\,\mathrm{d}x+\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{p+\beta}{2}}\,\mathrm{d}x+\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{\beta+2}{2}}\Gamma_{n,\delta}^{\frac{q-2}{2}}\,dx\right]\\
&\qquad+c(\alpha)\left[\delta\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{q+2\alpha-\beta}{2}}\,\mathrm{d}x+\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{p+2\alpha-\beta}{2}}\,\mathrm{d}x\right.\\
&\qquad\left.+\int_{B_R(x_0)}\eta^2\Gamma_{n,\delta}^{\frac{q}{2}}\tilde\Gamma_\delta^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x\right],
\end{aligned}\tag{3.4}
$$

where the last inequality follows from ($2.2_\beta$).
In a next step we combine (3.1), (3.2) and (3.4) with the result that

$$
\begin{aligned}
&\delta\int_{B_R(x_0)}\eta^2\tilde\Gamma_\delta^{\frac{q+1+\alpha}{2}}\,\mathrm{d}x+\int_{B_R(x_0)}\tilde\Gamma_\delta^{\frac{p+1+\alpha}{2}}\,\mathrm{d}x+\int_{B_R(x_0)}\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\Gamma_{n,\delta}^{\frac{q}{2}}\eta^2\,\mathrm{d}x\\
&\le\; c(\alpha)\left[\tau\delta\|\nabla\eta\|_\infty\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{q+1+\alpha}{2}}\,\mathrm{d}x+c(\tau)\delta\|\nabla\eta\|_\infty\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{q+\alpha-1}{2}}\,\mathrm{d}x+\tau\|\nabla\eta\|_\infty\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{p+\alpha+1}{2}}\,\mathrm{d}x\right.\\
&\quad+c(\tau)\|\nabla\eta\|_\infty\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{p+\alpha-1}{2}}\,\mathrm{d}x+\tau\|\nabla\eta\|_\infty\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x+c(\tau)\|\nabla\eta\|_\infty\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{1+\alpha}{2}}\,\mathrm{d}x\\
&\quad+\|\nabla\eta\|_\infty^2\delta\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{q+\beta}{2}}\,\mathrm{d}x+\|\nabla\eta\|_\infty^2\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{p+\beta}{2}}\,\mathrm{d}x+\|\nabla\eta\|_\infty^2\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{\beta+2}{2}}\cdot\Gamma_{n,\delta}^{\frac{q-2}{2}}\,\mathrm{d}x\\
&\quad\left.+\delta\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{q+2\alpha-\beta}{2}}\,\mathrm{d}x+\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{\frac{p+2\alpha-\beta}{2}}\,\mathrm{d}x+\int_{\operatorname{spt}\eta}\Gamma_{n,\delta}^{\frac{q}{2}}\tilde\Gamma_\delta^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x\right]\\
&=:\; c(\alpha)\sum_{l=1}^{12}T_l\,.
\end{aligned}\tag{3.5}
$$

We recall that from the proof of Theorem 1.1 we already know that the quantities

$$
\tilde\Gamma_\delta^{\frac{p+1}{2}}\,,\qquad \tilde\Gamma_\delta^{\frac{1}{2}}\Gamma_{n,\delta}^{\frac{q}{2}}
$$

are uniformly bounded in the space $L^1_{loc}(B_R(x_0))$, and (2.15) and the uniform boundedness of the r.h.s. of (2.16) immediately gives the same result for $\delta\tilde\Gamma_\delta^{(q+1)/2}$. We now define

$$
\alpha_0=\beta_0=0\,,\qquad \alpha_i=\frac{1}{2}+\alpha_{i-1}\,,\ \ \beta_i=\alpha_{i-1}\,,\ \ i\in\mathbb{N}\,,
$$

i.e. $\alpha_i=i/2$, $\beta_i=(i-1)/2$. Then we suppose that for a suitable constant $c(\rho)$ (also depending on $i$)

$$
\int_{B_\rho(x_0)}\delta\tilde\Gamma_\delta^{\frac{q+1+\alpha_{i-1}}{2}}\,\mathrm{d}x+\int_{B_\rho(x_0)}\left[\tilde\Gamma_\delta^{\frac{p+1+\alpha_{i-1}}{2}}+\tilde\Gamma_\delta^{\frac{1+\alpha_{i-1}}{2}}\Gamma_{n,\delta}^{\frac{q}{2}}\right]\mathrm{d}x\le c(\rho)\tag{3.6}
$$

for any $\rho < R$, and we like to prove (3.6) with $\alpha_{i-1}$ being replaced by $\alpha_i$. (Note that by the remarks after (3.5) we already know the validity of (3.6) for $i = 1$.)
To do so we choose $s < t < R$ and let $\eta \in C_0^\infty(B_t(x_0))$ satisfy $\eta \equiv 1$ on $B_s(x_0)$, $|\nabla\eta| \le c/(t-s)$. We then apply (3.5) with $\alpha = \alpha_i$, $\beta = \beta_i$. The terms $T_1$, $T_3$, $T_5$ can be handled easily (see Section 2) by requiring that

$$\tau \|\nabla\eta\|_\infty c(\alpha_i) = \frac{1}{2}\,.$$

Note that with this choice the constants $c(\tau)$ occurring in (3.5) are bounded from above by $c(t-s)^{-\kappa}$ with some suitable power $-\kappa$. For $T_2$ we observe that clearly

$$\tilde\Gamma_\delta^{\frac{q+\alpha_i-1}{2}} \le \tilde\Gamma_\delta^{\frac{q+\alpha_{i-1}+1}{2}}\,,$$

hence $T_2$ is bounded by a local constant on account of (3.6). The same is true for $T_4$ and $T_6$.
Since $q + \beta_i = q + \alpha_{i-1} < q + \alpha_{i-1} + 1$, there is no problem with $T_7$, and $p + \beta_i < p + 1 + \alpha_{i-1}$ shows a nice behaviour of $T_8$, i.e. we just replace $\delta\int_{B_t(x_0)} \tilde\Gamma_\delta^{(q+\beta_i)/2} dx$ and $\int_{B_t(x_0)} \tilde\Gamma_\delta^{(p+\beta_i)/2}\,\mathrm{d}x$ by the quantities $\delta\int_{B_t(x_0)} \tilde\Gamma_\delta^{(q+1+\alpha_{i-1})/2}\,\mathrm{d}x$ and $\int_{B_t(x_0)} \tilde\Gamma_\delta^{(p+1+\alpha_{i-1})/2}\,\mathrm{d}x$, respectively. In order to control $T_9$ we estimate

$$\begin{aligned}
\tilde\Gamma_\delta^{\frac{\beta_i+2}{2}} \Gamma_{n,\delta}^{\frac{q-2}{2}} &= \Gamma_{n,\delta}^{\frac{q-2}{2}} \tilde\Gamma_\delta^{\frac{q-2}{q}\frac{1+\alpha_{i-1}}{2}} \tilde\Gamma_\delta^{\frac{\beta_i+2}{2} - \frac{q-2}{q}\frac{1+\alpha_{i-1}}{2}} \\
&\le \Gamma_{n,\delta}^{\frac{q}{2}} \tilde\Gamma_\delta^{\frac{1+\alpha_{i-1}}{2}} + \tilde\Gamma_\delta^{\frac{q}{2}\left[\frac{\beta_i+2}{2} - \frac{q-2}{q}\frac{1+\alpha_{i-1}}{2}\right]}\,.
\end{aligned}$$

Obviously

$$\frac{q}{2}\Big[\frac{\beta_i+2}{2} - \frac{q-2}{q}\frac{1+\alpha_{i-1}}{2}\Big] \le \frac{p+1+\alpha_{i-1}}{2} \Leftrightarrow \frac{q}{2}\Big[\frac{q+2+2\alpha_{i-1}}{2q}\Big] \le \frac{p+1+\alpha_{i-1}}{2}\,,$$

and the latter inequality holds on account of our requirement $q \le 2p$. The calculation further shows that $T_9$ is bounded due to (3.6).
We further have $q + 2\alpha_i - \beta_i = q + 1 + \alpha_{i-1}$, $p + 2\alpha_i - \beta_i = p + 1 + \alpha_{i-1}$, hence $T_{10}, T_{11}$ are bounded by (3.6).
Quoting (3.6) for a last time, we also get a bound for $\int_{B_t(x_0)} \Gamma_{n,\delta}^{q/2} \tilde\Gamma_\delta^{(2\alpha_i-\beta_i)/2}\,\mathrm{d}x$. Collecting these estimates and going back to (3.5) we get

$$\int_{B_s(x_0)} \Big[\delta\tilde\Gamma_\delta^{\frac{q+1+\alpha_i}{2}} + \tilde\Gamma_\delta^{\frac{p+1+\alpha_i}{2}} + \Gamma_{n,\delta}^{\frac{q}{2}} \tilde\Gamma_\delta^{\frac{1+\alpha_i}{2}}\Big]\,\mathrm{d}x \le \frac{1}{2}\int_{B_t(x_0)} [\ldots]\,\mathrm{d}x + A(t-s)^{-\gamma} + B$$

being valid for $0 < s < t \le \rho < R$, where $A$ and $B$ are local constants depending in particular on $\rho$ and the bounds for the quantity [...], when $\alpha_i$ is replaced by $\alpha_{i-1}$, but being independent of $\delta = \delta(\epsilon)$. As in Section 2 the above inequality immediately implies the desired version of (3.6). Since $\alpha_i \to \infty$ as $i \to \infty$, we have shown that

$$\int_{B_\rho(x_0)} \tilde\Gamma_\delta^{\frac{t}{2}}\,\mathrm{d}x + \int_{B_\rho(x_0)} \tilde\Gamma_\delta^{\frac{t}{2}} \Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x \le c(t,\rho) \tag{3.7}$$

for any $t > 1$ and all radii $\rho < R$, where the constant is independent of $\delta(\epsilon)$. Using $u_\delta \rightharpoondown u$ in $W_p^1(B_R(x_0))$ as $\epsilon \to 0$ it is immediate that $\tilde\nabla u \in L^s_{loc}(\Omega;\mathbb{R}^{n-1})$ for any $s < \infty$, thus $\nabla u \in L^q_{loc}(\Omega;\mathbb{R}^n)$. □

# 4 Improvement of the initial higher integrability in the scalar case: proof of the second part of Theorem 1.2

Using the method of iteration introduced in the previous section and also using (3.7) we will show that

$$\int_{B_\rho(x_0)} |\partial_n u_\delta|^t \, dx \le c(t,\rho) < \infty \tag{4.1}$$

for any $t < \infty$ and all $\rho < R$. Let $\beta \ge 0$ and $\eta \in C_0^\infty(B_R(x_0))$. We have

$$0 = \int_{B_R(x_0)} D^2 f_\delta(\nabla u_\delta)\big(\partial_n \nabla u_\delta, \nabla[\eta^2 \partial_n u_\delta \Gamma_{n,\delta}^{\frac{\beta}{2}}]\big)\, \mathrm{d}x\,,$$

and since we are in the scalar case this implies

$$\begin{aligned}&\int_{B_R(x_0)} D^2 f_\delta(\nabla u_\delta)(\partial_n \nabla u_\delta, \partial_n \nabla u_\delta)\eta^2 \Gamma_{n,\delta}^{\frac{\beta}{2}}\, \mathrm{d}x\\ &\quad\le\; c\int_{B_R(x_0)} D^2 f_\delta(\nabla u_\delta)(\nabla\eta^2, \nabla\eta^2)|\partial_n u_\delta|^2 \Gamma_{n,\delta}^{\frac{\beta}{2}}\, \mathrm{d}x\,.\end{aligned}$$

Here, as before, we used the inequality of Cauchy-Schwarz and Young's inequality. The structure of $D^2 f_\delta$ gives the estimate

$$\begin{aligned}&\delta\int_{B_R(x_0)} \eta^2 \tilde\Gamma_\delta^{\frac{q-2}{2}} \Gamma_{n,\delta}^{\frac{\beta}{2}} |\partial_n \tilde\nabla u_\delta|^2\, \mathrm{d}x + \int_{B_R(x_0)} \eta^2 \tilde\Gamma_\delta^{\frac{p-2}{2}} \Gamma_{n,\delta}^{\frac{\beta}{2}} |\partial_n \tilde\nabla u_\delta|^2\, \mathrm{d}x\\ &\quad+\int_{B_R(x_0)} \eta^2 \Gamma_{n,\delta}^{\frac{q-2}{2}+\frac{\beta}{2}} |\partial_n\partial_n u_\delta|^2\, \mathrm{d}x\\ &\quad\le c\|\nabla\eta\|_\infty^2 \Bigg[\int_{\mathrm{spt}\,\eta} \delta\tilde\Gamma_\delta^{\frac{q-2}{2}} \Gamma_{n,\delta}^{\frac{\beta}{2}+1}\, \mathrm{d}x + \int_{\mathrm{spt}\,\eta} \tilde\Gamma_\delta^{\frac{p-2}{2}} \Gamma_{n,\delta}^{\frac{\beta}{2}+1}\, \mathrm{d}x + \int_{\mathrm{spt}\,\eta} \Gamma_{n,\delta}^{\frac{q+\beta}{2}}\, \mathrm{d}x\Bigg]\,.\end{aligned} \tag{4.2}$$

Next we return to (2.4) and choose $\varphi = \eta^2 u_\delta \Gamma_{n,\delta}^{(1+\alpha)/2}$, $\alpha \ge 0$. We get

$$\begin{aligned}&\int_{B_R(x_0)} \eta^2 \Gamma_{n,\delta}^{\frac{1+\alpha}{2}} Df_\delta(\nabla u_\delta)\cdot\nabla u_\delta\, \mathrm{d}x\\ &\quad=\; -2\int_{B_R(x_0)} \eta\nabla\eta\cdot Df_\delta(\nabla u_\delta) u_\delta \Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\, \mathrm{d}x\\ &\qquad -(1+\alpha)\int_{B_R(x_0)} u_\delta \eta^2 Df_\delta(\nabla u_\delta)\cdot\nabla(\partial_n u_\delta)\Gamma_{n,\delta}^{\frac{\alpha-1}{2}} \partial_n u_\delta\, \mathrm{d}x\,.\end{aligned} \tag{4.3}$$

¿From (2.6) we deduce

$$\begin{aligned}\text{l.h.s.of (4.3)} \;\ge\; c\Bigg[&\int_{B_R(x_0)} \delta\eta^2 \tilde\Gamma_\delta^{\frac{q-2}{2}} |\tilde\nabla u_\delta|^2 \Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\, \mathrm{d}x + \int_{B_R(x_0)} \eta^2 \tilde\Gamma_\delta^{\frac{p-2}{2}} |\tilde\nabla u_\delta|^2 \Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\, \mathrm{d}x\\ &+\int_{B_R(x_0)} \eta^2 \Gamma_{n,\delta}^{\frac{q-2}{2}+\frac{1+\alpha}{2}} |\partial_n u_\delta|^2\, \mathrm{d}x\Bigg]\,,\end{aligned} \tag{4.4}$$

moreover, (2.7) shows

$$
\begin{aligned}
&|1^{st} \text{ term on the r.h.s. of (4.3)}| \\
\le\ & c\Big[\int_{B_R(x_0)} \eta|\nabla\eta|\delta\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla}u_\delta|\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta|\nabla\eta|\tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla}u_\delta|\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x \\
&+\int_{B_R(x_0)} \eta|\nabla\eta|\Gamma_{n,\delta}^{\frac{q-2}{2}+\frac{1+\alpha}{2}}|\partial_n u_\delta|\,\mathrm{d}x\Big] \\
\le\ & c\Big[\tau\int_{B_R(x_0)} \delta\eta^2\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla}u_\delta|^2\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \frac{1}{\tau}\int_{B_R(x_0)} \delta|\nabla\eta|^2\tilde{\Gamma}_\delta^{\frac{q-2}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x \\
&+\tau\int_{B_R(x_0)} \eta^2\tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla}u_\delta|^2\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \frac{1}{\tau}\int_{B_R(x_0)} |\nabla\eta|^2\tilde{\Gamma}_\delta^{\frac{p-2}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x \\
&+\tau\int_{B_R(x_0)} \eta^2\Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta|^2\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \frac{1}{\tau}\int_{B_R(x_0)} |\nabla\eta|^2\Gamma_{n,\delta}^{\frac{q-2}{2}+\frac{1+\alpha}{2}}\,\mathrm{d}x\Big]
\end{aligned}
$$

for $\tau \in (0,1)$. If we use this estimate and choose $\tau$ small enough, then the $\tau$-terms can be absorbed in the l.h.s., more precisely, they can be absorbed in the terms giving the lower bound stated in (4.4). This implies

$$
\begin{aligned}
&\int_{B_R(x_0)} \delta\eta^2\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla}u_\delta|^2\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla}u_\delta|^2\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\Gamma_{n,\delta}^{\frac{q-1+\alpha}{2}}|\partial_n u_\delta|^2\,\mathrm{d}x \\
&\quad\le c\|\nabla\eta\|_\infty^2\Big[\int_{\operatorname{spt}\eta} \delta\tilde{\Gamma}_\delta^{\frac{q-2}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \tilde{\Gamma}_\delta^{\frac{p-2}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \Gamma_{n,\delta}^{\frac{q-1+\alpha}{2}}\,\mathrm{d}x\Big] \\
&\qquad + c|2^{nd} \text{ term on the r.h.s. of (4.3) }| \,. \qquad (4.5)
\end{aligned}
$$

To estimate the second term on the r.h.s. of (4.3), we observe that (compare (2.11))

$$
|Df_\delta(\nabla u_\delta)\cdot\nabla(\partial_n u_\delta)| \le c\Big[\delta\tilde{\Gamma}_\delta^{\frac{q-2}{2}}|\tilde{\nabla}u_\delta||\partial_n\tilde{\nabla}u_\delta| + \tilde{\Gamma}_\delta^{\frac{p-2}{2}}|\tilde{\nabla}u_\delta||\partial_n\tilde{\nabla}u_\delta| + \Gamma_{n,\delta}^{\frac{q-2}{2}}|\partial_n u_\delta||\partial_n\partial_n u_\delta|\Big]\,.
$$

Thus (using (4.2))

$$
\begin{aligned}
&|2^{nd} \text{ term on the r.h.s. of (4.3)}| \\
\le\ & c(\alpha)\Big[\int_{B_R(x_0)} \delta\eta^2\tilde{\Gamma}_\delta^{\frac{q-1}{2}}|\partial_n\tilde{\nabla}u_\delta|\Gamma_{n,\delta}^{\frac{\alpha}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\tilde{\Gamma}_\delta^{\frac{p-1}{2}}|\partial_n\tilde{\nabla}u_\delta|\Gamma_{n,\delta}^{\frac{\alpha}{2}}\,\mathrm{d}x \\
&+\int_{B_R(x_0)} \eta^2\Gamma_{n,\delta}^{\frac{q-1}{2}}|\partial_n\partial_n u_\delta|\Gamma_{n,\delta}^{\frac{\alpha}{2}}\,\mathrm{d}x\Big] \\
\le\ & c(\alpha)\Big[\int_{B_R(x_0)} \delta\eta^2\tilde{\Gamma}_\delta^{\frac{q-2}{2}}\Gamma_{n,\delta}^{\frac{\beta}{2}}|\partial_n\tilde{\nabla}u_\delta|^2\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\tilde{\Gamma}_\delta^{\frac{p-2}{2}}\Gamma_\delta^{\frac{\beta}{2}}|\partial_n\tilde{\nabla}u_\delta|^2\,\mathrm{d}x \\
&+\int_{B_R(x_0)} \eta^2\Gamma_{n,\delta}^{\frac{q-2}{2}+\frac{\beta}{2}}|\partial_n\partial_n u_\delta|^2\,\mathrm{d}x \\
&+\int_{B_R(x_0)} \delta\eta^2\tilde{\Gamma}_\delta^{\frac{q}{2}}\Gamma_{n,\delta}^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\tilde{\Gamma}_\delta^{\frac{p}{2}}\Gamma_{n,\delta}^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\Gamma_{n,\delta}^{\frac{q}{2}}\Gamma_{n,\delta}^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x\Big] \\
\le\ & c(\alpha)\|\nabla\eta\|_\infty^2\Big[\int_{\operatorname{spt}\eta} \delta\tilde{\Gamma}_\delta^{\frac{q-2}{2}}\Gamma_{n,\delta}^{\frac{\beta}{2}+1}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \tilde{\Gamma}_\delta^{\frac{p-2}{2}}\Gamma_{n,\delta}^{\frac{\beta}{2}+1}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \Gamma_{n,\delta}^{\frac{q+\beta}{2}}\,\mathrm{d}x\Big] \\
&+c(\alpha)\Big[\int_{\operatorname{spt}\eta} \delta\tilde{\Gamma}_\delta^{\frac{q}{2}}\Gamma_{n,\delta}^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \tilde{\Gamma}_\delta^{\frac{p}{2}}\Gamma_{n,\delta}^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \Gamma_{n,\delta}^{\frac{q+2\alpha-\beta}{2}}\,\mathrm{d}x\Big]\,.
\end{aligned}
$$

We insert this estimate into (4.5) observing at the same time that quantities like $\tilde\Gamma_\delta^{(q-2)/2}|\tilde\nabla u|^2$ occuring on the l.h.s. of (4.5) can be replaced by $\tilde\Gamma_\delta^{q/2}$ since the resulting difference already appears on the r.h.s. of (4.5), therefore we get:

$$\begin{aligned}
&\int_{B_R(x_0)} \delta\eta^2\tilde\Gamma_\delta^{\frac{q}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\tilde\Gamma_\delta^{\frac{p}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\Gamma_{n,\delta}^{\frac{q+1+\alpha}{2}}\,\mathrm{d}x \\
\leq\ & c(\alpha)\|\nabla\eta\|_\infty^2\Bigg[\int_{\operatorname{spt}\eta} \delta\tilde\Gamma_\delta^{\frac{q-2}{2}}\Big[\Gamma_{n,\delta}^{\frac{1+\alpha}{2}} + \Gamma_{n,\delta}^{\frac{2+\beta}{2}}\Big]\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \tilde\Gamma_\delta^{\frac{p-2}{2}}\Big[\Gamma_{n,\delta}^{\frac{1+\alpha}{2}} + \Gamma_{n,\delta}^{\frac{2+\beta}{2}}\Big]\,\mathrm{d}x \\
&+ \int_{\operatorname{spt}\eta}\Big[\Gamma_{n,\delta}^{\frac{q-1+\alpha}{2}} + \Gamma_{n,\delta}^{\frac{q+\beta}{2}}\Big]\,\mathrm{d}x\Bigg] \\
&+c(\alpha)\Bigg[\int_{\operatorname{spt}\eta} \delta\tilde\Gamma_\delta^{\frac{q}{2}}\Gamma_{n,\delta}^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \tilde\Gamma_\delta^{\frac{p}{2}}\Gamma_{n,\delta}^{\frac{2\alpha-\beta}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \Gamma_{n,\delta}^{\frac{q+2\alpha-\beta}{2}}\,\mathrm{d}x\Bigg]\,. \qquad (4.6)
\end{aligned}$$

Now we make use of this inequality with the choices $\alpha_0 = 0$, $\beta_0 = 0$, $\alpha_i = \alpha_{i-1} + 1/2$, $\beta_i = \alpha_{i-1}$, $i \geq 1$, in particular we have $1 + \beta_i/2 = \frac{1}{2}(\alpha_i + \frac{3}{2})$, and obtain for $i \geq 1$

$$\begin{aligned}
&\int_{B_R(x_0)} \delta\eta^2\tilde\Gamma_\delta^{\frac{q}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha_i}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\tilde\Gamma_\delta^{\frac{p}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha_i}{2}}\,\mathrm{d}x + \int_{B_R(x_0)} \eta^2\Gamma_{n,\delta}^{\frac{q+1+\alpha_i}{2}}\,\mathrm{d}x \\
\leq\ & c(i)\|\nabla\eta\|_\infty^2\Bigg[\int_{\operatorname{spt}\eta} \delta\tilde\Gamma_\delta^{\frac{q-2}{2}}\Gamma_{n,\delta}^{\frac{\alpha_i+3/2}{2}} + \int_{\operatorname{spt}\eta} \tilde\Gamma_\delta^{\frac{p-2}{2}}\Gamma_{n,\delta}^{\frac{\alpha_i+3/2}{2}} + \int_{\operatorname{spt}\eta}\Gamma_{n,\delta}^{\frac{q+\beta_i}{2}}\,\mathrm{d}x\Bigg] \\
&+c(i)\Bigg[\int_{\operatorname{spt}\eta} \delta\tilde\Gamma_\delta^{\frac{q}{2}}\Gamma_{n,\delta}^{\frac{2\alpha_i-\beta_i}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \tilde\Gamma_\delta^{\frac{p}{2}}\Gamma_{n,\delta}^{\frac{2\alpha_i-\beta_i}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta} \Gamma_{n,\delta}^{\frac{q+2\alpha_i-\beta_i}{2}}\,\mathrm{d}x\Bigg]\,. \qquad (4.6_i)
\end{aligned}$$

We claim that we have for all $i \in \mathbb{N}_0$ and for any radius $\rho < R$

$$\int_{B_\rho(x_0)} \delta\tilde\Gamma_\delta^{\frac{q}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha_i}{2}}\,\mathrm{d}x + \int_{B_\rho(x_0)} \tilde\Gamma_\delta^{\frac{p}{2}}\Gamma_{n,\delta}^{\frac{1+\alpha_i}{2}}\,\mathrm{d}x + \int_{B_\rho(x_0)} \Gamma_{n,\delta}^{\frac{q+1+\alpha_i}{2}}\,\mathrm{d}x \leq c(i,\rho) < \infty\,. \qquad (4.7_i)$$

In fact, for $i = 0$ this is an immediate consequence of (4.6) with $\alpha = \beta = 0$ and the estimate (3.7) from which we get finiteness of the r.h.s. of (4.6) together with a local bound independent of $\delta$.

Suppose that $(4.7_{i-1})$, $i \geq 1$, is true. We look at the r.h.s. of $(4.6_i)$ and observe that by asssumption

$$\int_{B_\rho(x_0)} \Gamma_{n,\delta}^{\frac{q+1+\alpha_{i-1}}{2}}\,\mathrm{d}x \leq c(i,\rho)\,, \ \text{ i.e. } \int_{B_\rho(x_0)} \Gamma_{n,\delta}^{\frac{q+\alpha_i+1/2}{2}}\,\mathrm{d}x \leq c(i,\rho)\,.$$

Using Young's inequality with $s$ very large we get

$$\int_{\operatorname{spt}\eta} \tilde\Gamma_\delta^{\frac{q-2}{2}}\Gamma_{n,\delta}^{\frac{\alpha_i+3/2}{2}}\,\mathrm{d}x \leq c\Bigg[\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{s\frac{q-2}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta}\Gamma_{n,\delta}^{\frac{s}{s-1}\frac{\alpha_i+3/2}{2}}\,\mathrm{d}x\Bigg]$$

and obviously the exponent $\frac{s}{s-1}\frac{\alpha_i+3/2}{2}$ is below $(q+\frac{1}{2}+\alpha_i)/2$. In the same way (recall (3.7)) we can bound the quantity $\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{(p-2)/2}\Gamma_{n,\delta}^{(\alpha_i+3/2)/2}\,\mathrm{d}x$. The finiteness of $\int_{\operatorname{spt}\eta}\Gamma_{n,\delta}^{(q+\beta_i)/2}\,\mathrm{d}x$ follows from $(4.7_{i-1})$. We have $2\alpha_i - \beta_i = \alpha_{i-1} + 1$ and $\int_{B_\rho(x_0)}\Gamma_{n,\delta}^{(q+1+\alpha_{i-1})/2}\,\mathrm{d}x \leq c(i,\rho)$, hence

$$\int_{\operatorname{spt}\eta} \tilde\Gamma_\delta^{\frac{q}{2}}\Gamma_{n,\delta}^{\frac{2\alpha_i-\beta_i}{2}} \leq c\Bigg[\int_{\operatorname{spt}\eta}\tilde\Gamma_\delta^{s\cdot\frac{q}{2}}\,\mathrm{d}x + \int_{\operatorname{spt}\eta}\Gamma_{n,\delta}^{\frac{s}{s-1}\frac{\alpha_{i-1}+1}{2}}\,\mathrm{d}x\Bigg]\,,$$

and $\frac{s}{s-1}(\alpha_{i-1}+1)\frac{1}{2} < (q+1+\alpha_{i-1})/2$ for sufficiently large $s$. For $\int_{\mathrm{spt}\,\eta} \tilde{\Gamma}_\delta^{p/2} \Gamma_{n,\delta}^{(2\alpha_i-\beta_i)/2}\,\mathrm{d}x$ we argue in the same way. Finally

$$\int_{\mathrm{spt}\,\eta} \Gamma_{n,\delta}^{\frac{q+2\alpha_i-\beta_i}{2}}\,\mathrm{d}x = \int_{\mathrm{spt}\,\eta} \Gamma_{n,\delta}^{\frac{q+\alpha_{i-1}+1}{2}}\,\mathrm{d}x$$

stays bounded by $(4.7_{i-1})$. Thus we have proved that all of the quantities on the r.h.s. of $(4.6_i)$ are bounded in an appropriate way which gives $(4.7_i)$, in particular we have that

$$\int_{B_\rho(x_0)} \Gamma_{n,\delta}^{\frac{q+1+\alpha_i}{2}}\,\mathrm{d}x \le c(i,\rho) < \infty$$

for any $i$ and any $\rho < R$. Since $\alpha_i \to \infty$ as $i \to \infty$, the claim follows since now we know

$$\|u_\delta\|_{W_t^1(B_\rho(x_0))} \le c(\rho,t) \tag{4.8}$$

for all $t < \infty, \rho < R$. □

Having established (4.8), the proof of $C^{1,\alpha}$-regularity can be obtained following for example [Bi], proof of Theorem 5.22, or [BF4] Lemma 2.9, where it is shown that from (4.8) we can deduce $\|\nabla u_\delta\|_{L^\infty(B_\rho(x_0))} \le c(\rho) < \infty$. Uniform Hölder continuity of $\nabla u_\delta$ then follows as outlined in [BF4], end of Section 2.1. Alternatively we may quote [LU], Chap.4, Sec.6, or [Ma1], Theorem D, as references for the step from Lipschitz regularity to Hölder continuity of the gradient.

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