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Abstract

We study local minimizers of anisotropic variational integrals of the form $J[u] = \int_\Omega f(\cdot, \nabla u)\,\mathrm{d}x$ with integrand f satisfying a $(p, \overline{q})$-growth condition w.r.t. ∇u and with $D_P f(x,P)$ satisfying a Lipschitz condition w.r.t. $x \in \Omega$. If the Lavrentiev gap functional $\mathcal{L}$ relative to J vanishes for all balls $B_R \Subset \Omega$ and if $\overline{q} < p(1 + 1/n)$, then (partial) $C^{1,\alpha}$-regularity holds. Moreover, the bound on the exponents can be replaced by $\overline{q} < p + 1$ provided we study locally bounded minimizers.

We also investigate the relaxation of global minimization problems and discuss the regularity of the corresponding solutions. The importance of the condition $\mathcal{L} \equiv 0$ was recently discovered by Esposito, Leonetti and Mingione in [ELM], where besides other results the higher integrability of the gradient is proved even under weaker assumptions than used here.

1 Introduction

In a recent paper [ELM] Esposito, Leonetti and Mingione discuss regularity theorems for minimizers of functionals of the form $J[u] = \int_\Omega f(\cdot, \nabla u)\,\mathrm{d}x$, where the integrand f is of anisotropic (p,q)-growth with respect to the second argument. Let us summarize some of their results: suppose that the function $D_P f(x,P)$ is α-Hölder continuous with respect to the variable x and that certain natural growth and ellipticity assumptions are satisfied. Then one is interested in the following question: do (local) minimizers u actually belong to the space $W^1_{q,loc}(\Omega; \mathbb{R}^N)$? As shown in Section 3 of [ELM] one can only hope for a positive answer if ($\Omega \subset \mathbb{R}^n$)

$$\frac{q}{p} < \frac{1}{n}(n + \alpha) \tag{1.1}$$

is satisfied. Assuming (1.1) they then exhibit in Section 4 of their paper a sufficient condition for higher integrability, precisely: if

$$\mathcal{L}(u, B_R) = 0 \quad \text{for all balls } B_R \Subset \Omega\,, \tag{1.2}$$

where $\mathcal{L}$ denotes the Lavrentiev gap functional relative to the energy J (see [ELM], Section 2.1), then Theorem 4 of [ELM] gives local integrability of ∇u for exponents even bigger than q. At least to our opinion it seems to be a very delicate problem to decide in a general way if (1.2) holds or not. Esposito, Leonetti and Mingione present a list of explicit examples and prove the validity of (1.2) in these concrete cases. This in turn has a very nice application: if the energy density $f(x,P)$ can be bounded by one of these explicit examples, then (1.2) holds and by the way higher integrability of local minimizers is true (see Theorem 6 of [ELM]). If (1.2) is violated, then minimizers cannot be regular, and it makes sense to pass to the relaxed functional $\overline{J}$ and to discuss the integrability properties of local minimizers of $\overline{J}$. This is done with a positive result in Theorem 8 of [ELM]. Finally we like to mention that some further extensions of [ELM] are given in [MM].

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The purpose of our note is to investigate the higher regularity properties of local minimizers u, precisely we ask, if ∇u belongs to some Hölder class. Obviously such a result can only hold under condition (1.2). Moreover, we will need a condition comparable to (1.1), and the growth and ellipticity of f are now stated in terms of the second derivatives of f. Let us give a detailed formulation:

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, denote a bounded domain and consider an energy density $f = f(x, P) \geq 0$, $x \in \overline{\Omega}$, $P \in \mathbb{R}^{nN}$, which satisfies with exponents $1 < p \leq \overline{q} < \infty$

ASSUMPTION 1.1 *There are positive constants λ, Λ, c_1 such that for any $x \in \overline{\Omega}$ and all U, $P \in \mathbb{R}^{nN}$:*

$$\begin{aligned} \lambda(1+|P|^2)^{\frac{p-2}{2}}|U|^2 &\leq D_P^2 f(x,P)(U,U) \leq \Lambda(1+|P|^2)^{\frac{\overline{q}-2}{2}}|U|^2\,, && (1.3)\\ |D_x D_P f(x,P)| &\leq c_1(1+|P|^2)^{\frac{\overline{q}-1}{2}}\,. && (1.4) \end{aligned}$$

We remark that (1.4) implies condition (H5) of [ELM] with $\alpha = 1$. Here f is assumed to be sufficiently smooth which means that we require the partial derivatives $D_P^2 f$ and $D_x D_P f$ to be at least continuous. Note that (1.3) implies the anisotropic growth condition (a, $A > 0$, $b \in \mathbb{R}$)

$$a|P|^p - b \leq f(x,P) \leq A(|P|^{\overline{q}} + 1)\,.$$

For open subsets Ω' of Ω let us define the energy of a function u: $\Omega' \to \mathbb{R}^N$ via

$$J[u,\Omega'] := \int_{\Omega'} f(\cdot, \nabla u)\,\mathrm{d}x\,.$$

The following definition seems to be natural in our setting.

DEFINITION 1.1 *A function $u \in W^1_{1,loc}(\Omega;\mathbb{R}^N)$ is termed a local J-minimizer iff*

i) *$J[u,\Omega'] < \infty$ for any domain $\Omega' \Subset \Omega$ and*

ii) *$J[u,\Omega'] \leq J[v,\Omega']$ for any $\Omega' \Subset \Omega$ and all $v \in W^1_{1,loc}(\Omega;\mathbb{R}^N)$ with* $\operatorname{spt}(u-v) \subset \Omega'$.

Our first result is in the spirit of [ELM], i.e. we get differentiability of local minimizers under the hypothesis (1.2) together with a version of (1.1):

THEOREM 1.1 *Let Assumption 1.1 hold together with*

$$\overline{q} < p\frac{n+1}{n}\,. \tag{1.5}$$

Suppose further that u is a local J-minimizer with the property that $\mathcal{L}(u, B_R) = 0$ for any ball $B_R \Subset \Omega$. Then we have

i) *There exists an open subset $\Omega_0 \subset \Omega$ such that $|\Omega - \Omega_0| = 0$ and $u \in C^{1,\alpha}(\Omega_0;\mathbb{R}^N)$ for any $\alpha \in (0,1)$.*

ii) *If $n = 2$, then $\Omega_0 = \Omega$.*

iii) *If* $N = 1$ *or if* f *is of special structure, i.e.* $f(x, P) = g(x, |P|^2)$, *and if in addition for* $N > 1$

$$|D_P^2 f(x, P) - D_P^2 f(x, Q)| \le c(1 + |P|^2 + |Q|^2)^{\frac{\overline{q}-2-\gamma}{2}} |P - Q|^\gamma \tag{1.6}$$

holds with some $0 < \gamma < 1$ *and for all* $x \in \overline{\Omega}$, P, $Q \in \mathbb{R}^{nN}$, *then* u *is of class* $C^{1,\alpha}$ *in the interior of* Ω.

REMARK 1.1 *Note again that from (1.4) and (1.5) it follows that (H2) and (H5) of [ELM] are satisfied with* $\alpha = 1$. *Thus, according to Theorem 4 of [ELM], the hypothesis* $\mathcal{L}(u, B_R) \equiv 0$ *implies that* $\nabla u \in L^{\overline{q}}_{loc}(\Omega; \mathbb{R}^{nN})$. *For higher integrability we can also quote Theorem 6 of [ELM]: if (1.3), (1.4), (1.5) and also (83) of [ELM] hold for our energy density* f, *then* $\nabla u \in L^{\overline{q}}_{loc}(\Omega; \mathbb{R}^{nN})$ *is satisfied.*

REMARK 1.2 *The reader should note that Marcellini (see [Ma]) investigates the existence and the regularity of solutions of elliptic equations under a* (p, q)*-growth condition. If a weak solution is in the space* $W^1_{q,loc}(\Omega)$ *and if* $q < pn/(n-2)$, *then Marcellini proves Lipschitz regularity (and even higher regularity), whereas the existence of a weak solution of class* $W^1_{q,loc}(\Omega)$ *is established under the restriction that* $q < p(n+2)/n$.

Note that here it is not possible to argue with the same relations between p *and* q *as done in [Ma] since our hypothesis on* $D_x D_P f$ *are weaker.*

Finally we conjecture that we may replace (1.5) by the weaker condition $\overline{q} < p(n-1)/(n-2)$ *provided* $\mathcal{L}(u, B_R) \equiv 0$ *is replaced by the requirement* $\nabla u \in L^{\overline{q}}_{loc}(\Omega; \mathbb{R}^{nN})$. *Note that the upper bound* $p(1 + 1/n)$ *occurring in (1.5) is the mean value of* p *and the upper bound in the autonomous case,* $p(n-1)/(n-2)$ *is the mean value of* p *and* $pn/(n-2)$. *An analogous observation holds for the condition* $\overline{q} < p + 1$ *dicussed in Theorem 1.2. To be a little bit more precise remark that the above conjecture gives (compare (2.21))* $2\overline{q} - p < pn/(n-2)$.

Our second theorem deals with locally bounded minimizers. As a consequence, condition (1.5) can be weakend if $p < n$. Note that on account of Sobolev's embedding theorem, we cannot expect to improve (1.5) in the case $p > n$ since then the boundedness of minimizers is no additional assumption at all (compare Remark 5.5 of [Bi]).

THEOREM 1.2 *Let* u *denote a local* J*-minimizer such that* $\mathcal{L}(u, B_R) = 0$ *for any ball* $B_R \Subset \Omega$ *and let Assumption 1.1 hold. If* $N = 1$ *or if* f *is of special structure, i.e.* $f(x, P) = g(x, |P|^2)$ *and if in addition in the case* $N > 1$ *we have (1.6), then* u *has Hölder continuous first derivatives in the interior of* Ω, *provided we assume*

$$u \in L^\infty_{loc}(\Omega; \mathbb{R}^n) \tag{1.7}$$

together with

$$\overline{q} < p + 1 \, . \tag{1.8}$$

REMARK 1.3 *Now* $\nabla u \in L^{\overline{q}}_{loc}(\Omega; \mathbb{R}^{nN})$ *is a consequence of [Mi].*

REMARK 1.4 *The counterexamples of [ELM] and [FMM] satisfy* $\overline{q} > p + 1$. *Since the solutions constructed there are locally bounded, we see that (1.8) is a rather natural condition for regularity.*

Let us now look at the global minimization problem: consider $u_0 \in W^1_\infty(\Omega;\mathbb{R}^N)$, let Assumption 1.1 hold and let u denote the unique solution of

$$J[v,\Omega] = \int_\Omega f(\cdot,\nabla v)\,\mathrm{d}x \to \min$$

in the energy class

$$\mathbb{K} = \{v \in W^1_p(\Omega;\mathbb{R}^N) : \, v - u_0 \in \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N),\ J[v,\Omega] < \infty\}\,.$$

If $\mathcal{L}(u,B_R) = 0$ or if higher integrability of ∇u is assumed, then clearly Theorem 1 or 2 can be applied to our global minimizer. Otherwise it is reasonable to introduce the relaxed problem, i.e. the problem of minimizing

$$\overline{J}[v,\Omega] := \inf\Big\{\liminf_{m\to\infty}\int_\Omega f(\cdot,\nabla u_m)\,\mathrm{d}x : \, u_m \in u_0 + \overset{\circ}{W}{}^1_{\overline{q}}(\Omega;\mathbb{R}^N),\ u_m \rightharpoondown w \ \ in\, W^1_p(\Omega;\mathbb{R}^N)\Big\}$$

in $u_0 + \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N)$. Note that we have $\overline{J} = J$ on $u_0 + \overset{\circ}{W}{}^1_{\overline{q}}(\Omega;\mathbb{R}^N)$. We then can show

THEOREM 1.3 *Let Assumption 1.1 together with (1.5) hold. Then there exists a solution $\bar{u}$ of the relaxed problem such that the claims ii) ($n = 2$) and iii) ($N = 1$ or $N > 1$ plus additional structure) of Theorem 1.1 are valid for $\bar{u}$. If (1.5) is replaced by $\overline{q} < p+1$, then we have the regularity result of Theorem 1.2 for $\bar{u}$.*

REMARK 1.5 *As we shall see below, the relaxed problem admits a unique solution which on account of Theorem 1.3 is of class $C^{1,\alpha}$ under appropriate assumptions on the data. As we will prove in Section 4, $\bar{u}$ also is a solution of the Dirichlet problem*

$$\left.\begin{aligned}&\int_\Omega D_P f(\cdot,\nabla w) : \nabla\varphi\,\mathrm{d}x = 0 \quad \textit{for all}\ \ \varphi \in C^\infty_0(\Omega;\mathbb{R}^N)\,,\\ &w - u_0 \in \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N)\,.\end{aligned}\right\} \tag{1.9}$$

Of course we could replace (1.9) by a more general system of the form

$$\left.\begin{aligned}&\int_\Omega T(\cdot,\nabla w) : \nabla\varphi\,\mathrm{d}x = 0 \quad \textit{for all}\ \ \varphi \in C^\infty_0(\Omega;\mathbb{R}^N)\,,\\ &w - u_0 \in \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N)\,,\end{aligned}\right\} \tag{1.10}$$

where the tensor T satisfies appropriate growth and ellipticity conditions. Then a variant of the global regularization procedure outlined in Section 4 would lead to a smooth solution v of (1.10) provided the hypotheses of Theorem 1.3 are formulated in the corresponding way. Such a result would be in the spirit of Marcellini's work discussed in Remark 1.2, we leave the details to the reader.

REMARK 1.6 *Existence and uniqueness of the $\overline{J}$-minimizer $\bar{u}$ can be proved by just using (1.3) of Assumption 1.1 (even a weaker condition would be enough), and we ask what can be said about $\bar{u}$ if we add (1.4) together with $\overline{q} < p(1+1/n)$, the situation which is closest to the one studied in [ELM]. We have the following results briefly sketched at the end of the last section:*

i) $\nabla\bar{u} \in L^{p\chi}_{loc}(\Omega;\mathbb{R}^{nN})$, *where* $\chi = n/(n-2)$, *if* $n \geq 3$, *any finite number, if* $n = 2$.

ii) $\bar{u}$ *still solves (1.9) from Remark 1.5.*

iii) $\bar{u} \in C^{1,\alpha}(\Omega_0;\mathbb{R}^N)$ *for an open set* $\Omega_0 \subset \Omega$ *such that* $|\Omega - \Omega_0| = 0$.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 follows a well-known line (compare, for instance, [Se], [Ma], [MS], [BF1], [Bi] and the references quoted therein), we give a short summary of the necessary steps and emphasizing the modifications which are needed to handle the non-autonomous case.

Step 1. Approximation.
We fix a ball $B_{2R} = B_{2R}(x_0) \Subset \Omega$ and define for $0 < \delta < 1$

$$f_\delta(x,P) = \delta(1+|P|^2)^{\frac{q}{2}} + f(x,P)\,, \qquad x \in \overline{\Omega}\,, \quad P \in \mathbb{R}^{nN}\,,$$

where the exponent q is chosen according to

$$\overline{q} < q < p\Big(1+\frac{2}{n}\Big)\,. \tag{2.1}$$

Note that f_δ still satisfies (1.4), whereas (1.3) holds with exponent $\overline{q}$ replaced by q. Define u_ε as the mollification of u with parameter $\varepsilon > 0$ and let $v_{\varepsilon,\delta}$ denote the unique solution of the minimization problem

$$J_\delta[w,B_{2R}] := \int_{B_{2R}} f_\delta(\cdot,\nabla w)\,\mathrm{d}x \to \min \qquad \text{in } u_\varepsilon + \overset{\circ}{W}{}^1_q(B_{2R};\mathbb{R}^N)\,.$$

We have the following convergence results:

LEMMA 2.1 *Suppose that the hypotheses of Theorem 1.1 hold. If ε and δ are related via*

$$\delta \;=\; \delta(\varepsilon) \;:=\; \frac{1}{1+\varepsilon^{-1}+\|Du_\varepsilon\|^{2q}_{L^q(B_{2R})}}$$

and if we abbreviate $v_\varepsilon = v_{\varepsilon,\delta(\varepsilon)}$, $f_\varepsilon = f_{\delta(\varepsilon)}$, then we have as $\varepsilon \to 0$:

i) $$v_\varepsilon \rightharpoondown u \quad \textit{in } W^1_1\big(B_{2R},\mathbb{R}^N\big)\,,$$

ii) $$\delta(\varepsilon)\int_{B_{2R}}(1+|Dv_\varepsilon|^2)^{\frac{q}{2}}\,\mathrm{d}x \;\to\; 0\,,$$

iii) $$\int_{B_{2R}} f(\cdot,Dv_\varepsilon)\,\mathrm{d}x \;\to\; \int_{B_{2R}} f(\cdot,Du)\,\mathrm{d}x\,,$$

iv) $$\int_{B_{2R}} f_\varepsilon(\cdot,Dv_\varepsilon)\,\mathrm{d}x \;\to\; \int_{B_{2R}} f(\cdot,Du)\,\mathrm{d}x\,.$$

Proof. We have by the minimality of v_ε

$$\begin{aligned}\int_{B_{2R}} f(\cdot,\nabla v_\varepsilon)\,\mathrm{d}x \;&\le\; \int_{B_{2R}} f_\varepsilon(\cdot,\nabla v_\varepsilon)\,\mathrm{d}x \;\le\; \int_{B_{2R}} f_\varepsilon(\cdot,\nabla u_\varepsilon)\,\mathrm{d}x\\ &=\; \delta(\varepsilon)\int_{B_{2R}}(1+|\nabla u_\varepsilon|^2)^{\frac{q}{2}}\,\mathrm{d}x + \int_{B_{2R}} f(\cdot,\nabla u_\varepsilon)\,\mathrm{d}x\,.\end{aligned} \tag{2.2}$$

Here the choice of $\delta(\varepsilon)$ implies that the first term on the r.h.s. converges to 0 as $\varepsilon \to 0$. Next we recall that f is at most of growth order $\overline{q}$, moreover we have (see Remark 1.1) that ∇u is of class $L^{\overline{q}}_{loc}$, hence

$$\nabla u_\varepsilon \overset{\varepsilon \to 0}{\to} \nabla u \quad \text{in} \ L^{\overline{q}}(B_{2R}; \mathbb{R}^{nN}) .$$

This in turn gives

$$\int_{B_{2R}} f(\cdot, \nabla u_\varepsilon) \, \mathrm{d}x \overset{\varepsilon \to 0}{\to} \int_{B_{2R}} f(\cdot, \nabla u) \, \mathrm{d}x . \tag{2.3}$$

In fact, to verify (2.3), we may consider the convex function

$$H : W^1_{\overline{q}}(B_{2R}, \mathbb{R}^N) \ni v \mapsto \int_{B_{2R}} f(\cdot, \nabla v) \, \mathrm{d}x$$

which is locally bounded from above, hence locally Lipschitz (compare, for instance, [Da], Theorem 2.3, p. 29). This gives (2.3). Then we conclude from (2.2) that $\int_{B_{2R}} f(\cdot, \nabla v_\varepsilon) \, \mathrm{d}x \leq$ *const*, hence

$$v_\varepsilon \overset{\varepsilon \to 0}{\rightharpoondown} : v \quad \text{in} \ W^1_p(B_{2R}; \mathbb{R}^N) , \qquad v = u \quad \text{on} \ \partial B_{2R} .$$

The lower semicontinuity of J and the uniqueness of minimizers finally prove $v = u$ on B_{2R}, i.e. the lemma is established. □

Step 2. Caccioppoli-type inequalities and higher integrability.

In the following we use the notation from above and observe that v_ε solves the Euler equation

$$\int_{B_{2R}} D_P f_\varepsilon(\cdot, \nabla v_\varepsilon) : \nabla \varphi \, \mathrm{d}x = 0 \quad \text{for all} \ \varphi \in \overset{\circ}{W}{}^1_q(B_{2R}; \mathbb{R}^N) . \tag{2.4}$$

This implies

LEMMA 2.2 *There is a real number $c > 0$ such that for all $\eta \in C^1_0(B_{2R})$, $0 \leq \eta \leq 1$, and for all $Q \in \mathbb{R}^{nN}$*

$$\begin{aligned} \int_{B_{2R}} \eta^2 \Gamma_\varepsilon^{\frac{p-2}{2}} |\nabla^2 v_\varepsilon|^2 \, \mathrm{d}x \ &\leq \ c \Bigg[\|\nabla \eta\|^2_\infty \int_{\operatorname{spt} \nabla \eta} \Gamma_\varepsilon^{\frac{q-2}{2}} |\nabla v_\varepsilon - Q|^2 \, \mathrm{d}x \\ &\qquad + \int_{\operatorname{spt} \eta} \Gamma_\varepsilon^{\overline{q} - \frac{p}{2}} \, \mathrm{d}x \Bigg] , \end{aligned} \tag{2.5}$$

where $\Gamma_\varepsilon := 1 + |v_\varepsilon|^2$.

Proof. Using the method of difference quotients in equation (2.4) (see e.g. [AF], Proposition 2.4 and Lemma 2.5, [GM], [Ca] or [To] for further details in a related setting; note that Lemma 4.1 of [To] works under our hypotheses) we obtain weak differentiability of ∇v_ε together with

$$\Gamma_\varepsilon^{\frac{p-2}{4}} |\partial_\gamma \nabla v_\varepsilon| \in L^2_{loc}(B_{2R}) .$$

Then, as outlined in the proof of Lemma 3.1 in [BF1], we deduce from the above integrability property (again using the method of difference quotients and passing to the limit) the inequality

$$
\begin{aligned}
&\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma \nabla v_\varepsilon)\eta^2 \,\mathrm{d}x \\
\leq\ & -2\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma \nabla v_\varepsilon, \partial_\gamma (v_\varepsilon - Qx)\otimes \nabla\eta)\eta \,\mathrm{d}x \\
& -2\int_{B_{2R}} (\partial_\gamma D_P f_\varepsilon)(\cdot,\nabla v_\varepsilon) : \partial_\gamma (v_\varepsilon - Qx)\otimes \nabla\eta\eta \,\mathrm{d}x \\
& -\int_{B_{2R}} (\partial_\gamma D_p f_\varepsilon)(\cdot,\nabla v_\varepsilon) : \partial_\gamma \nabla v_\varepsilon \eta^2 \,\mathrm{d}x\,, \qquad (2.6)
\end{aligned}
$$

being valid for any matrix $Q \in \mathbb{R}^{nN}$. With the help of Young's inequality we get (2.5) by absorbing terms after suitable application of (1.3) and (1.4). Note that (2.5) just follows from our assumptions (1.3) and (1.4), the hypotheses (1.5) and (2.1) do not enter. □

REMARK 2.1 *We can arrange that*

$$\overline{q} - \frac{p}{2} < \frac{q}{2}\,. \qquad (2.7)$$

In fact, up to now q was chosen according to $q > \overline{q}$ and $q < p(1+2/n)$. Here we observe that (1.5) gives

$$2\Big(\overline{q} - \frac{p}{2}\Big) < 2\Big(\frac{n+1}{n}p - \frac{p}{2}\Big) = p\frac{n+2}{n}$$

which means that it is possible to choose q in $(2(\overline{q}-p/2), p(n+2)/n)$ by the way satisfying (2.7) which will be assumed from now on.

As already remarked local higher integrability of $\nabla\bar{u}$ up to a certain exponent is established in Theorem 4 of [ELM]. We give a slight improvement which in particular is needed to discuss the case $n = 2$.

LEMMA 2.3 *(compare [BF1], Lemma 3.4) Let $\chi := n/(n-2)$, if $n > 2$, for $n = 2$ let $\chi > 2p/(2p-q)$. Then we have*

$$\nabla v_\varepsilon \in L^{p\chi}_{loc}(B_{2R};\mathbb{R}^{nN})$$

uniformly w.r.t. ε, in particular we find

$$\nabla u \in \begin{cases} L^{pn/(n-2)}_{loc}(\Omega;\mathbb{R}^{nN})\,, & \textit{if } n \geq 3\,, \\ \textit{any } L^s_{loc}(\Omega;\mathbb{R}^{nN})\,,\quad s<\infty\,, & \textit{if } n = 2\,. \end{cases}$$

Proof of Lemma 2.3. We consider the case $n \geq 3$, the calculations for $n = 2$ have to be adjusted according to [BF1] or [Bi]. Let

$$\alpha := \frac{p}{2}\frac{n}{n-2} = \frac{p}{2}\chi$$

and observe that by (1.5) we have

$$\overline{q} - \frac{p}{2} < \alpha\,. \qquad (2.8)$$

Let us fix radii r and ρ such that $R < r < \frac{3}{2}R$ and $0 < \rho < \frac{R}{2}$. Moreover, let $\eta \in C_0^1(B_{r+\rho/2})$, $\eta = 1$ on B_r, $|\nabla\eta| \leq c/\rho$. Using (2.5), the calculations from the proof of [BF1], Lemma 3.4, lead to the inequality (compare [Bi], second inequality on p. 60)

$$\int_{B_r} \Gamma_\varepsilon^\alpha \, dx \leq c\rho^{-\beta}\left[\int_{B_{2R}} \Gamma_\varepsilon^{\frac{p}{2}}\right]^{\bar\beta} + c\left[\int_{B_{r+\rho}} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}} \, dx\right]^\chi + \vartheta \int_{B_{r+\rho}} \Gamma_\varepsilon^\alpha \, dx \tag{2.9}$$

with positive constants β, $\bar\beta$, a positive constant c and another constant $\vartheta < 1$ being all independent of ε. The second term on the r.h.s. of (2.9) is new but can be handled via interpolation: note that (2.8) implies that $2\overline{q} - p < 2\alpha = p\chi$, and since $2\overline{q} - p > p$ we have with $\mu \in (0,1)$

$$\frac{1}{2\overline{q}-p} = \frac{\mu}{p} + \frac{1-\mu}{p\chi},$$

hence

$$\|\nabla v_\varepsilon\|_{L^{2\overline{q}-p}} \leq \|\nabla v_\varepsilon\|_{L^p}^\mu \|\nabla v_\varepsilon\|_{L^{p\chi}}^{1-\mu},$$

where the norms are taken w.r.t. $B_{r+\rho}$. Recalling the boundedness of ∇v_ε in $L^p(B_{2R})$, we get

$$\left[\int_{B_{r+\rho}} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}} \, dx\right]^\chi \leq c\left[\int_{B_{r+\rho}} \Gamma_\varepsilon^\alpha \, dx\right]^{(1-\mu)\frac{1}{p}(2\overline{q}-p)}.$$

The definition of μ together with (1.5) implies

$$(1-\mu)\frac{1}{p}(2\overline{q}-p) < 1,$$

thus Young's inequality gives

$$\left[\int_{B_{r+\rho}} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}} \, dx\right]^\chi \leq \tau \int_{B_{r+\rho}} \Gamma_\varepsilon^\alpha \, dx + c(\tau).$$

Inserting this into (2.9) and choosing τ small enough we find

$$\int_{B_r} \Gamma_\varepsilon^\alpha \, dx \leq c\rho^{-\beta}\left[\int_{B_{2R}} \Gamma_\varepsilon^{\frac{p}{2}} \, dx\right]^{\bar\beta} + \tilde\vartheta \int_{B_{r+\rho}} \Gamma_\varepsilon^\alpha \, dx$$

with $\tilde\vartheta \in (0,1)$. Now the proof of Lemma 2.3 can be completed along well known lines using Lemma 5.1, p. 81, from [Gi1]. The last claim of Lemma 2.3 follows from Lemma 2.1 and a covering argument combined with the first part of Lemma 2.3. □

The next result can be established as in [BF1], Proposition 3.5, or as in [Bi], Proposition 3.29.

LEMMA 2.4 *Let $h_{(\varepsilon)} := \Gamma_{(\varepsilon)}^{\frac{p}{4}}$, where $\Gamma := 1 + |\nabla u|^2$. Then we have*

i) *$h \in W^1_{2,loc}(B_{2R})$;*

ii) *$h_\varepsilon \rightharpoonup h$ in $W^1_{2,loc}(B_{2R})$;*

iii) *$\nabla v_\varepsilon \to \nabla u$ a.e. on B_{2R} as $\varepsilon \to 0$.*

Together with the higher integrability result from Lemma 2.3, part iii) of Lemma 2.4 is essential for proving a limit version of the ε-Caccioppoli inequality stated in Lemma 2.2.

LEMMA 2.5 *There exists a constant (depending on R) such that for all balls $B_{2r}(\bar{x}) \subset B_R$ we have*

$$\int_{B_r(\bar{x})} |\nabla h|^2 \,\mathrm{d}x \le c\left[r^{-2} \int_{B_{2r}(\bar{x})-B_r(\bar{x})} \Gamma^{\frac{q-2}{2}} |\nabla u - Q|^2 \,\mathrm{d}x + \int_{B_{2r}(\bar{x})} \Gamma^{\overline{q}-\frac{p}{2}} \right],$$

where $Q \in \mathbb{R}^{nN}$ is arbitrary.

REMARK 2.2 *On the l.h.s. $|\nabla h|^2$ may be replaced by $\Gamma^{\frac{p-2}{2}}|\nabla^2 u|^2$.*

Proof of Lemma 2.5. In (2.5) we choose $\eta \in C_0^1(B_{2r}(\bar{x}))$ such that $\eta \equiv 1$ on $B_r(\bar{x})$, $0 \le \eta \le 1$, and $|\nabla \eta| \le 2/r$. Then, on the l.h.s. we use lower semicontinuity, the first term on the r.h.s. is handled as in the proof of Lemma 3.6, [BF1]. By (2.7), the second term from the r.h.s. of (2.5) is dominated by $\int_{B_{2r}(\bar{x})} \Gamma_\varepsilon^{q/2}\,\mathrm{d}x$ and on account of $\Gamma_\varepsilon \to \Gamma$ a.e. together with the higher integrability of Γ_ε we may pass to the limit as well. $\square$

Step 3. Blow up and proof of Theorem 1.1 i).
Once having established Lemma 2.5, we can follow the arguments of [BF1], Section 4, (compare also [Bi]) by introducing the excess function for balls $B_r(x) \subset B_R$. With

$$\begin{aligned} E(x,r) &:= \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2 \,\mathrm{d}y + \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^q \,\mathrm{d}y\,, \quad \text{if } q \ge 2\,, \\ E(x,r) &:= \fint_{B_r(x)} |V(\nabla u) - V((\nabla u)_{x,r})|^2 \,\mathrm{d}y\,, \quad V(\xi) := (1+|\xi|^2)^{\frac{q-2}{4}}\xi\,, \quad \text{if } q < 2\,, \end{aligned}$$

we have to formulate the blow-up Lemma 4.1 from [BF1] in the following way:

LEMMA 2.6 *Fix $L > 0$. Then there exists a constant $C_*(L)$ such that for every $0 < \tau < 1/4$ there is an $\varepsilon = \varepsilon(L,\tau)$ satisfying: if $B_r(x) \Subset B_R$ and if we have*

$$|(\nabla u)_{x,r}| \le L\,, \qquad E(x,r) + r^{\gamma^*} < \varepsilon(L,\tau)\,,$$

then

$$E(x,\tau r) \le C_*(L)\tau^2\big[E(x,r) + r^{\gamma^*}\big]\,.$$

Here γ^* denotes some arbitrary number in $(0,2)$.

Let us give a short comment: if we follow the arguments from [BF1], Section 4, and introduce the function ψ_m as done there, then we have to bound the quantity $\int_{B_\rho} |\nabla \psi_m|^2\,\mathrm{d}z$ for $\rho < 1$ which can be done with the scaled version of Lemma 2.5 leading to the inequality (recall (2.7))

$$\begin{aligned} \int_{B_\rho} |\nabla \psi_m|^2 \,\mathrm{d}z &\le c(\rho)\left[1 + \lambda_m^{-2} r_m^2 \fint_{B_{r_m}(x_m)} \Gamma^{\overline{q}-\frac{p}{2}}\,\mathrm{d}x\right] \\ &\le c(\rho)\left[1 + \lambda_m^{-2} r_m^2 \fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}}\,\mathrm{d}x\right]. \qquad (2.10) \end{aligned}$$

We now let for any $1 < t < \infty$

$$V_t(\xi) := (1 + |\xi|^2)^{\frac{t-2}{4}} \xi\,, \qquad H_t(\xi) := (1 + |\xi|^2)^{\frac{t}{2}}\,.$$

By Lemma 2.3 of [Ha] we then have

$$\Big|\sqrt{H_t(\xi)} - \sqrt{H_t(\bar\xi)}\,\Big| \le c\big|V_t(\xi) - V_t(\bar\xi)\big|\,. \tag{2.11}$$

By assumption, $|(\nabla u)_{x_m,r_m}| \le L$, hence we obtain from (2.11)

$$\begin{aligned}
\fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}}\,\mathrm{d}x &= \fint_{B_{r_m}(x_m)} \Big[\Gamma^{\frac{q}{4}}\Big]^2\,\mathrm{d}x \\
&\le c \fint_{B_{r_m}(x_m)} \Big[\Big|\sqrt{H_q(\nabla u)} - \sqrt{H_q((\nabla u)_{x_m,r_m})}\,\Big| + \sqrt{H_q((\nabla u)_{x_m,r_m})}\,\Big]^2\,\mathrm{d}x \\
&\le c \fint_{B_{r_m}(x_m)} \Big|\sqrt{H_q(\nabla u)} - \sqrt{H_q((\nabla u)_{x_m,r_m})}\,\Big|^2\,\mathrm{d}x + c(L) \\
&\le c \fint_{B_{r_m}(x_m)} |V_q(\nabla u) - V_q((\nabla u)_{x_m,r_m})|^2\,\mathrm{d}x + c(L) = cE(x_m,r_m) + c(L)\,,
\end{aligned}$$

where the last identity follows from the definition of E in the case $q \le 2$. If $q > 2$, then we simply estimate

$$\begin{aligned}
\fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}}\,\mathrm{d}x &\le c\Bigg[1 + \fint_{B_{r_m}(x_m)} |\nabla u|^q\,\mathrm{d}x\Bigg] \\
&\le c\Bigg[1 + \fint_{B_{r_m}(x_m)} |\nabla u - (\nabla u)_{x_m,r_m}|^q\,\mathrm{d}x + c(L)\Bigg] \\
&\le cE(x_m,r_m) + c(L)\,,
\end{aligned}$$

thus (2.10) gives in both cases

$$\int_{B_\rho} |\nabla\psi_m|^2\,\mathrm{d}z \le c(\rho)\Big[1 + r_m^2 + \lambda_m^{-2} r_m^2 c(L)\Big]\,.$$

Recalling the choice of γ^* we observe that as $m \to \infty$

$$\lambda_m^{-2} r_m^2 \to 0\,,$$

hence the boundedness of $\int_{B_\rho} |\nabla\psi_m|^2\,\mathrm{d}z$ follows, and the proof can be completed as in [BF1].

Step 4. Proof of Theorem 1.1 ii).
If $n = 2$, then we know by Lemma 2.3 that $\nabla v_\varepsilon \in L^t_{loc}(B_{2R};\mathbb{R}^{nN})$ for any $t < \infty$ uniform w.r.t. ε. Now we quote [BF2], proof of Theorem 1: on the r.h.s. of (9) from [BF2] we have to add

$$-\int D_{x_s} D_P f_\varepsilon(\cdot,\nabla v_\varepsilon) : \nabla(\eta^2 \partial_s[v_\varepsilon - Qx])\,\mathrm{d}x$$

and by using the growth properties of D_xD_Pf together with Young's inequality and the higher integrability of ∇v_ε it is easy to see that we have (14) of [BF2] with an extra additive term of the form $const\, r^\beta$, $0 < \beta < 1$, on the r.h.s. But as outlined in [BF3] or [ABF] this term does not affect the application of the Frehse-Seregin Lemma (see [FS]) and the claim follows as before with the help of Frehse's variant of the Dirichlet-Growth Theorem (see [Fr]).

Step 5. Proof of Theorem 1.1 iii).
We are first going to prove the following auxiliary lemma which gives good initial regularity for our regularizing sequence in the vector case $N > 1$ together with the special structure $f = g(x, |P|^2)$.

LEMMA 2.7 *Assume that $F(x,P)$ satisfies with some given $1 < t < \infty$ for all $x \in \overline{\Omega}$, P, $U \in \mathbb{R}^{nN}$ and with positive constants λ, Λ, c*

$$\begin{aligned} \lambda(1+|P|^2)^{\frac{t-2}{2}}|U|^2 &\le D_P^2F(x,P)(U,U) \le \Lambda(1+|P|^2)^{\frac{t-2}{2}}|U|^2\,; && (2.12)\\ |D_xD_PF(x,P)| &\le c(1+|P|^2)^{\frac{t-1}{2}}\,; && (2.13)\\ F(x,P) &= G(x,|P|^2)\,. && (2.14) \end{aligned}$$

Here $G\colon \overline{\Omega}\times\mathbb{R} \to [0,\infty)$ is a function of class C^2. Moreover we assume that for some $\gamma > 0$

$$|D_P^2F(x,P) - D_P^2F(x,Q)| \le c(1+|P|^2+|Q|^2)^{\frac{t-2-\gamma}{2}}|P-Q|^\gamma\,.$$

Then, if $u \in W^1_{t,loc}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\int_\Omega F(x,\nabla u)\,\mathrm{d}x$, u is of class $C^{1,\kappa}(\Omega;\mathbb{R}^N)$ for any $0 < \kappa < 1$.

REMARK 2.3 *If $N = 1$, then the statement of course holds without (2.14), see [LU]. Once having established the $C^{1,\kappa}$-regularity of the solution u studied in Lemma 2.7, we immediately obtain $u \in W^2_{2,loc}(\Omega;\mathbb{R}^N)$. Combining both facts and using potential theory for linear elliptic systems with continuous coefficients we arrive at $u \in W^2_{\tilde{t},loc}(\Omega;\mathbb{R}^N)$ for any finite $\tilde{t}$.*

Proof of Lemma 2.7. We concentrate on the case $t \ge 2$. In the case $1 < t < 2$ the following arguments have to be modified using Proposition 2.11 in [AF]. Note that for both cases the above Hölder condition for $D_p^2F(x,\cdot)$ implies the corresponding ones in [AF] and [GM], respectively, if x is considered as fixed. Let $B_R(x_0) \Subset \Omega$, $R \le R_0$, where R_0 is fixed later on. We denote by v the unique solution of the variational problem

$$\int_{B_R(x_0)} F_0(\nabla w)\,\mathrm{d}x \to \min \quad \text{in } u_{|B_R(x_0)} + \overset{\circ}{W}{}^1_t(B_R(x_0);\mathbb{R}^N)\,,$$

where $F_0 := F(x_0,\cdot)$. Then inequality (3.1) of Theorem 3.1 in [GM] gives together with the minimality of v and the growth of F_0:

$$\|\nabla v\|^t_{L^\infty(B_{r/2})} \le c \fint_{B_R} (1+|\nabla v|^2)^{\frac{t}{2}}\,\mathrm{d}x \le c \fint_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x\,. \qquad (2.15)$$

We define $V(\xi) = V_t(\xi)$ as in the third step and recall Lemma 2.3 of [Ha] to obtain for $\rho \le R/2$

$$\begin{aligned}\int_{B_\rho} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x &\le c\Big[\int_{B_\rho} (1+|\nabla v|^2)^{\frac{t}{2}}\,\mathrm{d}x + \int_{B_\rho} \Big|(1+|\nabla u|^2)^{\frac{t}{4}} - (1+|\nabla v|^2)^{\frac{t}{4}}\Big|^2\,\mathrm{d}x \\ &\le c\int_{B_\rho} (1+|\nabla v|^2)^{\frac{t}{2}}\,\mathrm{d}x + c\int_{B_\rho} |V(\nabla u) - V(\nabla v)|^2\,\mathrm{d}x\,.\end{aligned}$$

Hence, (2.15) implies

$$\int_{B_\rho} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x \le c\Big(\frac{\rho}{R}\Big)^n \int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x + c\int_{B_\rho} |V(\nabla u) - V(\nabla v)|^2\,\mathrm{d}x\,. \quad (2.16)$$

Then (2.3) of [Ha] and (2.1) of [GM] yield

$$\begin{aligned}\int_{B_\rho} |V(\nabla u) - V(\nabla v)|^2\,\mathrm{d}x &\le c\int_{B_R} (1+|\nabla u|^2+|\nabla v|^2)^{\frac{t-2}{2}}|\nabla u - \nabla v|^2\,\mathrm{d}x \\ &\le c\int_{B_R} \underbrace{\int_0^1 (1+|\nabla v + t(\nabla u - \nabla v)|^2)^{\frac{t-2}{2}}|\nabla u - \nabla v|^2\,\mathrm{d}t}_{=:(*)}\,\mathrm{d}x\,.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\big(DF_0(\nabla u) - DF_0(\nabla v)\big) : (\nabla u - \nabla v) \\ &= \int_0^1 D^2F_0(\nabla v + t(\nabla u - \nabla v))(\nabla u - \nabla v, \nabla u - \nabla v)\,\mathrm{d}t \ge \lambda(*)\,.\end{aligned}$$

Putting together these two inequalities, using the equations for u, v and recalling the growth condition (2.13) one has (again see [Gi1], p. 151)

$$\begin{aligned}\int_{B_R} |V(\nabla u) - V(\nabla v)|^2\,\mathrm{d}x &\le c\int_{B_R} \big(DF_0(\nabla u) - DF_0(\nabla v)\big) : (\nabla u - \nabla v)\,\mathrm{d}x \\ &= c\int_{B_R} \big(DF_0(\nabla u) - D_PF(x,\nabla u)\big) : (\nabla u - \nabla v)\,\mathrm{d}x \\ &\le cR\int_{B_R} (1+|\nabla u|^2)^{\frac{t-1}{2}}|\nabla u - \nabla v|\,\mathrm{d}x \\ &\le \varepsilon\int_{B_R} (1+|\nabla u|^2)^{\frac{t-2}{2}}|\nabla u - \nabla v|^2\,\mathrm{d}x \\ &\quad + c(\varepsilon)R^2\int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x \\ &\le c\varepsilon\int_{B_R} |V(\nabla u) - V(\nabla v)|^2\,\mathrm{d}x \\ &\quad + c(\varepsilon)R^2\int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x\,.\end{aligned}$$

Now, if $\varepsilon > 0$ is sufficiently small, then it is shown that

$$\int_{B_R} |V(\nabla u) - V(\nabla v)|^2\,\mathrm{d}x \le cR^2\int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\,\mathrm{d}x\,. \quad (2.17)$$

Inserting this in (2.16) we arrive at

$$\int_{B_\rho} (1+|\nabla u|^2)^{\frac{t}{2}}\, \mathrm{d}x \le c\Big[\Big(\frac{\rho}{R}\Big)^n + R^2\Big] \int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\, \mathrm{d}x\,. \tag{2.18}$$

Note that (2.18) was just shown in case $\rho \le R/2$, for $R/2 < \rho < R$ the estimate is trivial.

Next we choose $\beta < n$ which may be arbitrarily close to n. With a suitable choice of R_0 we may apply Lemma 2.1 from [Gi1] to (2.18). As a consequence, for all radii $\rho^* \le R^* \le R_0$ which are sufficiently small we have

$$\int_{B_{\rho^*}} (1+|\nabla u|^2)^{\frac{t}{2}}\, \mathrm{d}x \le c\Big(\frac{\rho^*}{R^*}\Big)^\beta \int_{B_{R^*}} (1+|\nabla u|^2)^{\frac{t}{2}}\, \mathrm{d}x\,.$$

Choosing $\rho^* = R$ and $R^* = R_0$ it is shown in particular that

$$\int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\, \mathrm{d}x \le c\Big(\frac{R}{R_0}\Big)^\beta \int_{B_{R_0}} (1+|\nabla u|^2)^{\frac{t}{2}}\, \mathrm{d}x\,. \tag{2.19}$$

Finally we make use of [GM], formula (3.2), i.e. for some exponent $\sigma > 0$ it holds

$$\fint_{B_\rho} |V(\nabla v) - (V(\nabla v))_{x_0,\rho}|^2\, \mathrm{d}x \le c\Big(\frac{\rho}{R}\Big)^\sigma \fint_{B_R} |V(\nabla v) - (V(\nabla v))_{x_0,R}|^2\, \mathrm{d}x\,. \tag{2.20}$$

Note that (2.20) implies as in [GM], (5.6),

$$\begin{aligned}\fint_{B_\rho} |V(\nabla u) - (V(\nabla u))_{x_0,\rho}|^2\, \mathrm{d}x \quad &\le \quad c\Big(\frac{\rho}{R}\Big)^\sigma \fint_{B_R} |V(\nabla u) - (V(\nabla u))_{x_0,R}|^2\, \mathrm{d}x \\ &\qquad + \Big(\frac{R}{\rho}\Big)^n \fint_{B_R} |V(\nabla u) - V(\nabla v)|^2\, \mathrm{d}x\,,\end{aligned}$$

hence (2.17) and (2.19) imply

$$\begin{aligned}\int_{B_\rho} |V(\nabla u) - (V(\nabla u))_{x_0,\rho}|^2\, \mathrm{d}x \quad &\le \quad c\Bigg[\Big(\frac{\rho}{R}\Big)^{n+\sigma} \int_{B_R} |V(\nabla u) - (V(\nabla u))_{x_0,R}|^2\, \mathrm{d}x \\ &\qquad + R^2 \int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}}\, \mathrm{d}x\Bigg] \\ &\le \quad c\Bigg[\Big(\frac{\rho}{R}\Big)^{n+\sigma} \int_{B_R} |V(\nabla u) - (V(\nabla u))_{x_0,R}|^2\, \mathrm{d}x + R^{2+\beta}\Bigg].\end{aligned}$$

Now

$$\Psi:\quad \rho \mapsto \Psi(\rho) := \int_{B_\rho} |V(\nabla u) - (V(\nabla u))_{x_0,\rho}|^2\, \mathrm{d}x$$

clearly is an increasing function. From [Gi1], p. 86, we infer (choosing $n < 2+\beta < n+\sigma$) that Ψ growth like $\rho^{2+\beta}$. Since $2+\beta > n$, this gives Hölder continuity of $V(\nabla u)$, in particular ∇u is of class C^0. We then let $w = \partial_s u$ and observe that w solves an elliptic system with continuous coefficients. Theorem 3.1 of [Gi1], p. 87, then proves our claim. $\Box$

For the proof of Theorem 1.1 iii) we will now use DeGiorgi type arguments as done in the proof of Theorem 3.16 in [Bi] which has to be adjusted to the situation at hand. W.l.o.g. we may assume that $n \geq 3$, since by the second part of the theorem regularity in the two-dimensional case holds without structure condition. We still work on the ball B_{2R} and choose $B_r(\bar{x}) \subset B_R$ and $\eta \in C_0^1(B_r(\bar{x}), [0,1])$. We further let $\omega_\varepsilon = \ln(\Gamma_\varepsilon)$, $\Gamma_\varepsilon = 1 + |\nabla v_\varepsilon|^2$, and consider the sets

$$A(h,r) := \{x \in B_r(\bar{x}) : \omega_\varepsilon > h\}\,.$$

From Lemma 2.7 we deduce $v_\varepsilon \in W^1_{\infty,loc}(B_{2R};\mathbb{R}^N)$ (and therefore $\nabla v_\varepsilon \in W^1_{2,loc}(B_{2R};\mathbb{R}^{nN})$) which enables us to use the same test functions as in [Bi]. Thus we have (30), p. 62, of [Bi], where on the r.h.s. we have to add the quantity

$$I := \int_{A(k,r)} |D_x D_P f_\varepsilon(\cdot, \nabla v_\varepsilon)|\, |\nabla(\eta^2 \nabla v_\varepsilon(\omega_\varepsilon - k))|\,\mathrm{d}x\,.$$

I itself splits into a sum of three integrals, one of them being

$$\begin{aligned}\int_{A(k,r)} |D_x D_P f_\varepsilon(\cdot,\nabla v_\varepsilon)|\eta^2(\omega_\varepsilon - k)|\nabla^2 v_\varepsilon|\,\mathrm{d}x \;&\leq\; \gamma \int_{A(k,r)} \Gamma_\varepsilon^{\frac{p-2}{2}} \eta^2 |\nabla^2 v_\varepsilon|^2 (\omega_\varepsilon - k)\,\mathrm{d}x\\ &\quad + c(\gamma)\int_{A(k,r)} \Gamma_\varepsilon^{\frac{2-p}{2}+\overline{q}-1}(\omega_\varepsilon - k)\,\mathrm{d}x\,,\end{aligned}$$

where we used condition (1.4) and Young's inequality. If γ is small enough, then the first integral on the r.h.s. can be absorbed in the first integral on the l.h.s of (30), p. 62, in [Bi]. Then (34), p. 63, of [Bi] reads:

$$\begin{aligned}\int_{A(k,r)} \Gamma_\varepsilon^{\frac{p}{2}} \eta^2 |\nabla \omega_\varepsilon|^2\,\mathrm{d}x \;&\leq\; c\left[\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} (\omega_\varepsilon - k)^2 |\nabla\eta|^2\,\mathrm{d}x + \xi\right], \\ \xi \;&:=\; \int_{A(k,r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}}\,\mathrm{d}x + \int_{A(k,r)} \Gamma_\varepsilon^{\frac{\overline{q}}{2}}\,\mathrm{d}x + \int_{A(k,r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}}(\omega_\varepsilon - k)\,\mathrm{d}x\,.\end{aligned} \tag{2.21}$$

In the same way we use (35), p. 63, of [Bi] with the extra term

$$\int_{A(k,r)} |D_x D_P f_\varepsilon(\cdot,\nabla v_\varepsilon)|\,|\nabla(\eta^2 \nabla v_\varepsilon(\omega_\varepsilon - k)^2|\,\mathrm{d}x$$

on the right-hand side, this time we get

$$\begin{aligned}\int_{A(k,r)} \Gamma_\varepsilon^{\frac{p-2}{2}} (\omega_\varepsilon - k)^2 |\nabla^2 v_\varepsilon|^2 \eta^2\,\mathrm{d}x \;&\leq\; c\left[\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} (\omega_\varepsilon - k)^2 |\nabla\eta|^2\,\mathrm{d}x + \bar{\xi}\right], \\ \bar{\xi} \;&:=\; \int_{A(k,r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}}\,\mathrm{d}x + \int_{A(k,r)} \Gamma_\varepsilon^{\frac{\overline{q}}{2}}(\omega_\varepsilon - k)^2\,\mathrm{d}x \\ &\quad + \int_{A(k,r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}}(\omega_\varepsilon - k)^2\,\mathrm{d}x\,.\end{aligned} \tag{2.22}$$

By combining (2.21) and (2.22) we obtain the following version of (27), p. 61, in [Bi]:

$$\begin{aligned}\int_{A(k,r)} \Gamma_\varepsilon^{\frac{p}{2}} \eta^2 |\nabla \omega_\varepsilon|^2 \, dx &+ \int_{A(k,r)} \Gamma_\varepsilon^{\frac{p-2}{2}} (\omega_\varepsilon - k)^2 \eta^2 |\nabla v_\varepsilon|^2 \, dx \\ &\le \; c\left[\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} |\nabla \eta|^2 (\omega_\varepsilon - k)^2 \, dx + \xi + \bar{\xi} \right] . \end{aligned} \tag{2.23}$$

For handling $\xi + \bar{\xi}$ we use (2.7). If we let

$$a(k,r) := \int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} \, dx \, ,$$

then we have

$$\xi + \bar{\xi} \le c a(k,r) \, . \tag{2.24}$$

Let us further set

$$\tau(k,r) := \int_{A(k,r)} \Gamma_\varepsilon^{\frac{q}{2}} (\omega_\varepsilon - k)^2 \, dx \, .$$

Next we fix numbers $h > k > 0$ and radii $r < \bar{r}$ such that $B_{\bar{r}}(\bar{x}) \subset B_R$. Then, as in [Bi], we deduce from (2.21) – (2.24):

$$\tau(h,r) \le c\Big[(h-k)^{-2\frac{\chi-1}{\chi}-2} + (h-k)^{-2\frac{\chi-1}{\chi}} \Big] (r'-r)^{-2} \tau(k,r')^{1+\frac{\chi-1}{\chi}}$$

provided we assume w.l.o.g. that $R \le 1$. For the application of the Stamppachia Lemma it is sufficient to study the case $h - k \le 1$, thus we can replace the quantity $[\dots]$ by $(h-k)^{-2-2(\chi-1)/\chi}$ and argue as in [Bi] with the result that the functions v_ε are locally Lipschitz on B_R uniform w.r.t. ε. As a consequence we get $u \in W^1_{\infty,loc}(\Omega; \mathbb{R}^N)$. Let us fix $\Omega' \Subset \Omega$ and a constant $M > 0$ s.t. $|\nabla u(x)| \le M$ for a.a. $x \in \Omega'$. Then, as outlined in [MS], we can construct an integrand F on $\overline{\Omega'} \times \mathbb{R}^{nN}$ satisfying (2.12)–(2.14) for a suitable t and s.t.

$$F(x,P) = f(x,P)$$

for $x \in \overline{\Omega'}$ and $P \in \mathbb{R}^{nN}$, $|P| \le 2M$. But then u is a local minimizer of $\int_{\Omega'} F(\cdot, \nabla v) \, dx$ on Ω', hence of class $C^{1,\alpha}$ by Lemma 2.7. The reader should note that the Hölder condition for $D_P^2 F(x,\cdot)$ required for the application of Lemma 2.7 is a consequence of the corresponding condition for $D_P^2 f(x,\cdot)$ as stated in the hypotheses of Theorem 1.1 iii) if the vector case is considered. □

3 Proof of Theorem 1.2

We use the same regularization as in Step 1 of Section 2 where the exponent q is now chosen in $(\overline{q}, p+2)$ sufficiently close to $p+2$ s.t.

$$\overline{q} \le \frac{1}{2}(p+q) \, . \tag{3.1}$$

Note that such a choice is possible on account of (1.8). Note also that Lemma 2.1 continues to hold since again we have that $\nabla u \in L^{\overline{q}}_{loc}(\Omega; \mathbb{R}^{nN})$ on account of [Mi]. From (1.7) together with the maximum principle it follows that

$$\sup_{0<\varepsilon<1} \|v_\varepsilon\|_{L^\infty(B_{2R})} \le \sup_{B_{2R}} |u| < \infty \, . \tag{3.2}$$

Step 1. Higher integrability.
We follow [Bi], proof of Theorem 5.21, and show

LEMMA 3.1 *There is a constant c independent of ε such that*

$$\int_{B_r(\bar{x})} |\nabla v_\varepsilon|^s \le c$$

for any ball $B_r(\bar{x}) \Subset B_{2R}$ and any $s \in (1, \infty)$. The constant c depends on the location of the ball, the constants appearing in (1.3) and (1.4), on s and on $\sup_{B_{2R}} |u|$.

Proof. Let $\alpha \ge 0$ denote a fixed real number and define the quantities $\beta := 2 + p - q$,

$$0 < \sigma := \frac{\alpha}{2} + \frac{q}{2} < 1 + \frac{\alpha}{2} + \frac{p}{2} =: \sigma' .$$

For $k \in \mathbb{N}$ large enough we have

$$2k\frac{\sigma}{\sigma'} < 2k - 2 .$$

Finally, we consider $\eta \in C_0^\infty(B_{2R})$, $0 \le \eta \le 1$, and obtain with exactly the same arguments as in [Bi], inequality (19) on p. 155 (by letting $h \equiv 1$ during this calculation and by using (3.2))

$$\begin{aligned} \int_{B_{2R}} \eta^{2k} \Gamma_\varepsilon^{\frac{\alpha+p}{2}+1} \, dx &\le c\left[1 + \int_{B_{2R}} |\nabla^2 v_\varepsilon|^2 \Gamma_\varepsilon^{\frac{\alpha+p}{2}} \eta^{2k} \, dx + \int_{B_{2R}} \eta^{2k-1} |\nabla \eta| \Gamma_\varepsilon^{\frac{1+\alpha+p}{2}} \, dx\right] \\ &=: c[1 + I + II] . \end{aligned} \tag{3.3}$$

If $\operatorname{spt}\eta \subset B_{\rho'} = B_{\rho'}(x_0)$, $\eta = 1$ on $B_\rho = B_\rho(x_0)$ and $|\nabla\eta| \le c/(\rho' - \rho)$, then we can use (20), p. 155 in [Bi] to handle II, i.e. we have

$$II \le \tau \int_{B_{2R}} \eta^{2k} \Gamma_\varepsilon^{1+\frac{\alpha+p}{2}} \, dx + c(\rho' - \rho)^{-2} \tau^{-1} \int_{B_{2R}} \Gamma_\varepsilon^{\frac{\alpha+p}{2}} \eta^{2k-2} \, dx \tag{3.4}$$

valid for any $\tau \in (0, 1)$, where for τ small enough the first term on the r.h.s. of (3.4) can be absorbed on the l.h.s. of (3.3). For I we observe

$$\begin{aligned} I &\le \tau \int_{B_{2R}} \eta^{2k+2} \Gamma_\varepsilon^{\frac{p-2}{2}+\frac{\alpha+\beta}{2}} |\nabla^2 v_\varepsilon|^2 \, dx + \tau^{-1} \int_{B_{2R}} \eta^{2k-2} \Gamma_\varepsilon^{\frac{\alpha+q}{2}} \, dx \\ &=: \tau I_1 + \tau^{-1} I_2 . \end{aligned} \tag{3.5}$$

As we shall prove below the quantity I_1 can be bounded in the following form:

$$I_1 \le c(\rho' - \rho)^{-2} \int_{B_{2R}} \eta^{2k} \Gamma_\varepsilon^{\frac{\alpha+\beta}{2}} \Gamma_\varepsilon^{\frac{q}{2}} \, dx , \tag{3.6}$$

where c also depends on α. We insert (3.6) into (3.5) and replace τ in (3.5) by $\tau'(\rho' - \rho)^2$ for some $\tau' > 0$. Since

$$\frac{\alpha + \beta}{2} + \frac{q}{2} = \frac{\alpha + p}{2} + 1 ,$$

we see that for $\tau' \ll 1$ the term corresponding to τ' can be absorbed on the l.h.s. of (3.3). Moreover, we have with Young's inequality

$$\begin{aligned}
(\tau')^{-1}(\rho'-\rho)^{-2}I_2 &= (\tau')^{-1}(\rho'-\rho)^{-2}\int_{B_{2R}} \eta^{2k-2}\Gamma_\varepsilon^{\frac{\alpha+q}{2}}\,\mathrm{d}x \\
&\le (\tau')^{-1}(\rho'-\rho)^{-2}\left[\tau''\int_{B_{2R}}\left[\eta^{2k-2}\Gamma_\varepsilon^{\frac{\alpha+q}{2}}\right]^{\frac{\sigma'}{\sigma}}\mathrm{d}x + (\tau'')^{-\frac{\sigma}{\sigma'-\sigma}}|B_{2R}|\right] \\
&\le (\tau')^{-1}(\rho'-\rho)^{-2}\left[\tau''\int_{B_{2R}}\eta^{2k}\Gamma_\varepsilon^{\frac{\alpha+p+2}{2}} + (\tau'')^{-\frac{\sigma}{\sigma'-\sigma}}|B_{2R}|\right].
\end{aligned}$$

If we let $\tau'' = \tau'(\rho'-\rho)^2\tau^*$ and if τ^* is small enough, the first term on the r.h.s. of the above inequality can be absorbed on the l.h.s. of (3.3). Putting together our results we have inequality (23), p. 156, of [Bi], i.e.

$$\int_{B_{2R}}\eta^{2k}\Gamma_\varepsilon^{\frac{\alpha+p+2}{2}}\,\mathrm{d}x \le c\left[1+\int_{B_{2R}}\eta^{2k-2}\Gamma_\varepsilon^{\frac{\alpha+p}{2}}\,\mathrm{d}x\right]$$

with c also depending on α, ρ and ρ' but independent of ε. Now the same iteration as in [Bi] gives

$$\int_{B_r(x_0)}|\nabla v_\varepsilon|^s\,\mathrm{d}x \le \mathit{const}$$

for any $s<\infty$ and $r<2R$. It remains to prove the inequality (3.6). But this follows from an appropriate version of Lemma 5.20 i) of [Bi]. Note that (3.6) is the only place where we use the fact that v_ε solves a variational problem. To be more precise, we take

$$\varphi = \eta^{2k+2}\partial_\gamma v_\varepsilon\Gamma_\varepsilon^s$$

as test function in

$$\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\nabla\varphi)\,\mathrm{d}x = -\int_{B_{2R}} D_{x_\gamma}D_P f_\varepsilon(\cdot,\nabla v_\varepsilon) : \nabla\varphi\,\mathrm{d}x\,,$$

where s is some exponent ≥ 0 and k denotes some integer ≥ 1. The admissibility of φ follows from Lemma 2.7 and Remark 2.3. We get

$$\begin{aligned}
&\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\partial_\gamma\nabla v_\varepsilon)\eta^{2k+2}\Gamma_\varepsilon^s\,\mathrm{d}x \\
&\quad + \int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\partial_\gamma v_\varepsilon\otimes\nabla\Gamma_\varepsilon^s)\eta^{2k+2}\,\mathrm{d}x \\
&= -(2k+2)\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\nabla\eta\otimes\partial_\gamma v_\varepsilon)\eta^{2k+1}\Gamma_\varepsilon^s\,\mathrm{d}x \\
&\quad -\int_{B_{2R}} D_{x_\gamma}D_P f_\varepsilon(\cdot,\nabla v_\varepsilon) : \nabla(\eta^{2k+2}\partial_\gamma v_\varepsilon\Gamma_\varepsilon^s)\,\mathrm{d}x\,. \qquad (3.7)
\end{aligned}$$

To the first integral on the r.h.s. we apply the Cauchy-Schwarz inequality (for the bilinear form $D_P^2(x,\nabla v_\varepsilon(x))$) and then use Young's inequality to get the bound

$$\begin{aligned}
&\tau\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon,\partial_\gamma\nabla v_\varepsilon)\eta^{2k+2}\Gamma_\varepsilon^s\,\mathrm{d}x \\
&\quad +c(\tau)\int_{B_{2R}}|\nabla\eta|^2\eta^{2k}|D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)|\Gamma_\varepsilon^{1+s}\,\mathrm{d}x\,, \qquad (3.8)
\end{aligned}$$

and for τ small the first term can be absorbed on the l.h.s. of (3.7). For the second integral on the r.h.s. of (3.7) we use (1.4), thus

$$\begin{aligned}
\text{l.h.s. of (3.7)} &\le c\Bigg[\int_{B_{2R}} \Gamma_\varepsilon^{\frac{\overline{q}-1}{2}}\Gamma_\varepsilon^s|\nabla\eta|\eta^{2k+1}|\nabla v_\varepsilon|\,\mathrm{d}x + \int_{B_{2R}} \Gamma_\varepsilon^{\frac{\overline{q}-1}{2}}\Gamma_\varepsilon^s\eta^{2k+2}|\nabla^2 v_\varepsilon|\,\mathrm{d}x \\
&\quad + \int_{B_{2R}} \Gamma_\varepsilon^{\frac{\overline{q}-1}{2}}\Gamma_\varepsilon^{s-1}\eta^{2k+2}|\nabla v_\varepsilon||\nabla\Gamma_\varepsilon|\,\mathrm{d}x\Bigg] \\
&=: c[J_1+J_2+J_3]\,.
\end{aligned}$$

We have (since $0\le\eta\le 1$, $|\nabla\eta|\le(\rho'-\rho)^{-1}$)

$$\begin{aligned}
J_1 &\le c(\rho'-\rho)^{-2}\int_{B_{2R}} \eta^{2k}\Gamma_\varepsilon^{\frac{\overline{q}}{2}+s}\,\mathrm{d}x \\
&= c(\rho'-\rho)^{-2}\int_{B_{2R}} \eta^{2k}\Gamma_\varepsilon^{\frac{\overline{q}-2}{2}+1+s}\,\mathrm{d}x
\end{aligned}$$

which means that we obtain the same bound as for the second term in (3.8). With $\kappa>0$ arbitrary we have

$$J_2 \le \kappa\int_{B_{2R}} \Gamma_\varepsilon^{\frac{p-2}{2}}\Gamma_\varepsilon^s\eta^{2k+2}|\nabla^2 v_\varepsilon|^2\,\mathrm{d}x + c(\kappa)\int_{B_{2R}} \Gamma_\varepsilon^{s+1}\Gamma_\varepsilon^{-\frac{p}{2}+\overline{q}-1}\eta^{2k+2}\,\mathrm{d}x\,.$$

By (1.3) and by choosing κ small enough the first term can be absorbed in the first integral on the l.h.s. of (3.7). For the second term we use $\eta^{2k+2}\le\eta^{2k}$ and observe $-\frac{p}{2}+\overline{q}-1\le\frac{q-2}{2}$ which is a consequence of (3.1). In order to handle J_3 we observe that the second integral on the l.h.s. of (3.7) can be written as

$$\frac{1}{2}\int_{B_{2R}} D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(e_\gamma\otimes\nabla\Gamma_\varepsilon^s, e_\gamma\otimes\nabla\Gamma_\varepsilon^s)\eta^{2k+2}\,\mathrm{d}x$$

which is obvious if $N=1$, whereas in the vector-case we use the special structure. By ellipticity we therefore obtain the lower bound

$$J_4 := c\int_{B_{2R}} \Gamma_\varepsilon^{s-1}\eta^{2k+2}\Gamma_\varepsilon^{\frac{p-2}{2}}\big|\nabla|\nabla v_\varepsilon|^2\big|^2\,\mathrm{d}x$$

for this term. On the other hand

$$\begin{aligned}
J_3 &\le c\int_{B_{2R}} \eta^{2k+2}\Gamma_\varepsilon^{\frac{\overline{q}}{2}}\Gamma_\varepsilon^{s-1}\big|\nabla|\nabla v_\varepsilon|^2\big|^2\,\mathrm{d}x \\
&\le \kappa\int_{B_{2R}} \eta^{2k+2}\Gamma_\varepsilon^{s-1}\Gamma_\varepsilon^{\frac{p-2}{2}}\big|\nabla|\nabla v_\varepsilon|^2\big|^2\,\mathrm{d}x \\
&\quad + c(\kappa)\int_{B_{2R}} \eta^{2k+2}\Gamma_\varepsilon^{s-1+\overline{q}+\frac{2-p}{2}}\,\mathrm{d}x\,,
\end{aligned}$$

and for all κ small enough the first term is absorbed in J_4. For the second one we use $\eta^{2k+2}\le\eta^{2k}$ and observe that by (3.1)

$$s-1+\overline{q}+\frac{2-p}{2} = s+1+\overline{q}-\frac{p}{2}-1 \le s+1+\frac{q-2}{2}\,.$$

Altogether we have shown that

$$
\begin{aligned}
\int_{B_{2R}} \eta^{2k+2}|\nabla^2 v_\varepsilon|^2 \Gamma_\varepsilon^{s+\frac{p-2}{2}}\,\mathrm{d}x &\le c(\rho'-\rho)^{-2}\int_{B_{2R}} \Gamma_\varepsilon^{\frac{q-2}{2}}\Gamma_\varepsilon^{1+s}\eta^{2k}\,\mathrm{d}x\\
&= c(\rho'-\rho)^{-2}\int_{B_{2R}} \Gamma_\varepsilon^{\frac{q}{2}}\Gamma_\varepsilon^{s}\eta^{2k}\,\mathrm{d}x\,,
\end{aligned}
$$

and (3.6) is established by choosing $s=\frac{1}{2}(\alpha+\beta)$. □

Step 2. Uniform local gradient bounds

LEMMA 3.2 *There is a finite local constant independent of ε s.t.*

$$|\nabla v_\varepsilon| \le c \quad \text{on } B_r \Subset B_{2R}\,.$$

Proof. We modify the proof of Theorem 5.22 in [Bi]. To this purpose let us fix radii $0<r<\hat r<2R$ and consider $\eta\in C_0^\infty(B_{\hat r})$ with the usual properties where all balls are centered at x_0. Moreover, for $k>0$ we let

$$A(k,r)=\{x\in B_r:\,\Gamma_\varepsilon\ge k\}\,.$$

By elementary calculations (see [Bi], p. 157) we obtain

$$\int_{A(k,r)} (\Gamma_\varepsilon-k)^{\frac{n}{n-1}}\,\mathrm{d}x \le c[I_1^{\frac{n}{n-1}}+I_2^{\frac{n}{n-1}}]\,, \tag{3.9}$$

where

$$
\begin{aligned}
I_1^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat r)} |\nabla\eta|(\Gamma_\varepsilon-k)\,\mathrm{d}x\right]^{\frac{n}{n-1}}\\
&\le c(\hat r-r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}\\
&\quad\cdot\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{2-p}{p}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}, \qquad (3.10)\\
I_2^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat r)} \eta|\nabla\Gamma_\varepsilon|\,\mathrm{d}x\right]^{\frac{n}{n-1}}\\
&\le c\left[\int_{A(k,\hat r)} \eta^2|\nabla\Gamma_\varepsilon|^2\Gamma_\varepsilon^{\frac{p-2}{2}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{2-p}{2}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}. \qquad (3.11)
\end{aligned}
$$

We claim the validity of

$$
\begin{aligned}
\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla\Gamma_\varepsilon|^2\eta^2\,\mathrm{d}x &\le c\left[\int_{A(k,\hat r)} |\nabla\eta|^2\Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon-k)^2\,\mathrm{d}x\right.\\
&\quad\left.+\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q}{2}-\frac{p}{2}+1}\eta^2\,\mathrm{d}x\right]. \qquad (3.12)
\end{aligned}
$$

Accepting (3.12) for the moment, we get by combining (3.9)–(3.12)

$$\int_{A(k,r)} (\Gamma_\varepsilon - k)^{\frac{n}{n-1}}\,\mathrm{d}x \;\le\; c(\hat r - r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon - k)^2\,\mathrm{d}x + \int_{A(k,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}$$
$$\cdot\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{2-p}{2}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}}, \tag{3.13}$$

which corresponds to the inequality (24) on p. 157 of [Bi]. Let s and t denote real numbers > 1. With Hölder's inequality we deduce from Lemma 3.1

$$\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon - k)^2\,\mathrm{d}x \le c\left[\int_{A(k,r)} (\Gamma_\varepsilon - k)^{\frac{n}{n-1}}\,\mathrm{d}x\right]^{\frac{1}{s}}$$

and

$$\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{2-p}{2}}\,\mathrm{d}x \le c\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x\right]^{\frac{1}{t}},$$

where c now is a local constant and we assume $\hat r \le R_0$ for some $R_0 < R$. Inserting the above inequalities into (3.13) we end up with

$$\int_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon - k)^2\,\mathrm{d}x$$
$$\le\; c(\hat r - r)^{-\frac{1}{s}\frac{n}{n-1}}\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon - k)^2\,\mathrm{d}x + \int_{A(k,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}\,\mathrm{d}x\right]^{\frac{1}{2s}\frac{n}{n-1}}$$
$$\cdot\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x\right]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}}. \tag{3.14}$$

Let $h > k$ and define

$$\tau(k,r) := \int_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}}(\Gamma_\varepsilon - k)^2\,\mathrm{d}x\,,$$
$$a(k,r) := \int_{A(k,r)} \Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x\,.$$

Clearly $a(h,r) \le (h-k)^{-2}\tau(k,r)$ and from (3.14) (with k replaced by h) it follows

$$\tau(h,r) \;\le\; c(\hat r - r)^{-\gamma}\left[\tau(h,\hat r) + \int_{A(h,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}\,\mathrm{d}x\right]^{\frac{1}{2s}\frac{n}{n-1}} a(h,\hat r)^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}}$$
$$\le\; c(\hat r - r)^{-\gamma}(h-k)^{-\alpha}\tau(k,\hat r)^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}}$$
$$\left[\tau(h,\hat r) + \int_{A(h,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}\,\mathrm{d}x\right]^{\frac{1}{2s}\frac{n}{n-1}} \tag{3.15}$$

with positive exponents γ and α. By (3.1) we have $\overline{q} \le \frac{1}{2}(p+q)$, i.e. $\overline{q} - \frac{p}{2} + 1 \le \frac{1}{2}(q+2)$. If we choose $m > 1$, quote Lemma 3.1 and use Hölder's inequality we therefore get

$$\begin{aligned}
\int_{A(h,\hat r)} \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}\,\mathrm{d}x &\le \int_{A(h,\hat r)} \Gamma_\varepsilon^{\frac{q+2}{2}}\,\mathrm{d}x \\
&= \int_{A(h,\hat r)} \Gamma_\varepsilon^{\frac{1}{m}\frac{q-2}{2}} \Gamma_\varepsilon^{\frac{q+2}{2}-\frac{1}{m}\frac{q-2}{2}}\,\mathrm{d}x \\
&\le c\Bigg[\int_{A(h,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x\Bigg]^{\frac{1}{m}} = ca(h,\hat r)^{\frac{1}{m}} \\
&\le c(h-k)^{-\frac{2}{m}}\tau(k,\hat r)^{\frac{1}{m}}\,.
\end{aligned}$$

W.l.o.g. we may assume $h - k \le 1$. Then, with some suitable new positive exponent α (depending on the parameters!) we obtain from (3.15)

$$\begin{aligned}
\tau(h,r) &\le c(\hat r - r)^{-\gamma}(h-k)^{-\alpha}\tau(k,\hat r)^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}} \\
&\quad \cdot\big[\tau(k,\hat r) + \tau(k,\hat r)^{\frac{1}{m}}\big]^{\frac{1}{2s}\frac{n}{n-1}}\,.
\end{aligned}$$

Let us finally assume that R_0 is chosen in such a way that

$$\int_{B_{R_0}} \Gamma_\varepsilon^{\frac{q}{2}+1}\,\mathrm{d}x \le 1$$

which is possible by Lemma 3.1. Then $\tau(k,\hat r) \le 1$ and therefore

$$\tau(h,r) \le c(\hat r - r)^{-\gamma}(h-k)^{-\alpha}\tau(k,\hat r)^{\frac{1}{2}\frac{n}{n-1}\frac{1}{st}+\frac{1}{m}\frac{1}{2s}\frac{n}{n-1}}\,. \tag{3.16}$$

Obviously

$$\beta := \frac{1}{2}\frac{n}{n-1}\frac{1}{st} + \frac{1}{m}\frac{1}{2s}\frac{n}{n-1} = \frac{1}{2}\frac{n}{n-1}\frac{1}{s}\Big[\frac{1}{t} + \frac{1}{m}\Big] > 1$$

if the parameters m, s and t are close to 1. Thus we may apply a lemma of Stampacchia [St] to inequality (3.16) to get the claim of Lemma 3.2 (see also [Bi], p. 122, for further details).

It remains to prove (3.12) which means that we have to give a variant of Lemma 5.20 ii) of [Bi]. This time we test the differentiated Euler equation valid for v_ε with $\eta^2\partial_\gamma v_\varepsilon \max[\Gamma_\varepsilon - k, 0]$ being admissible on account of Lemma 2.7. We get

$$\begin{aligned}
&\int_{A(k,\hat r)} \eta^2(\Gamma_\varepsilon - k)D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon, \partial_\gamma\nabla v_\varepsilon)\,\mathrm{d}x \\
&+\; 2\int_{A(k,\hat r)} \eta(\Gamma_\varepsilon - k)D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon, \nabla\eta\otimes\partial_\gamma v_\varepsilon)\,\mathrm{d}x \\
&+\; \int_{A(k,\hat r)} \eta^2 D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(\partial_\gamma\nabla v_\varepsilon, \partial_\gamma v_\varepsilon\otimes\nabla\Gamma_\varepsilon)\,\mathrm{d}x \\
&=:\; T_1 + 2T_2 + T_3 \\
&=\; -\int_{A(k,\hat r)} D_{x_\gamma} D_P f_\varepsilon(\cdot,\nabla v_\varepsilon) : \nabla(\eta^2\partial_\gamma v_\varepsilon(\Gamma_\varepsilon - k))\,\mathrm{d}x\,.
\end{aligned} \tag{3.17}$$

If $N > 1$ we make use of the special structure and of (1.3) to see

$$\begin{aligned} T_3 &= \frac{1}{2}\int_{A(k,\hat r)} \eta^2 D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(e_\gamma\otimes\nabla\Gamma_\varepsilon, e_\gamma\otimes\nabla\Gamma_\varepsilon)\,\mathrm{d}x \\ &\geq c\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla\Gamma_\varepsilon|^2\eta^2\,\mathrm{d}x\,. \end{aligned} \tag{3.18}$$

Also by the special structure we find

$$T_2 = \frac{1}{2}\int_{A(k,\hat r)} \eta(\Gamma_\varepsilon - k) D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(e_\gamma\otimes\nabla\eta, e_\gamma\otimes\nabla\Gamma_\varepsilon)\,\mathrm{d}x\,,$$

hence

$$\begin{aligned} T_2 &\leq \tau\int_{A(k,\hat r)} \eta^2 D_P^2 f_\varepsilon(\cdot,\nabla v_\varepsilon)(e_\gamma\otimes\nabla\Gamma_\varepsilon, e_\gamma\otimes\nabla\Gamma_\varepsilon)\,\mathrm{d}x \\ &\quad + c(\tau)\int_{A(k,\hat r)} |\nabla\eta|^2(\Gamma_\varepsilon - k)^2\Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x\,, \end{aligned}$$

where we used the Cauchy-Schwarz inequality for $D_P^2 f_\varepsilon(x,\nabla v_\varepsilon)$, Young's inequality and (1.3). Note that the "τ-term" can be absorbed in T_3. Using the ellipticity for T_1, we deduce from (3.17), (3.18) and the latter estimates:

$$\begin{aligned} &\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla\Gamma_\varepsilon|^2\eta^2\,\mathrm{d}x + \int_{A(k,\hat r)} \eta^2(\Gamma_\varepsilon - k)\Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla^2 v_\varepsilon|^2\,\mathrm{d}x \\ &\quad \leq c\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{q-2}{2}}|\nabla\eta|^2(\Gamma_\varepsilon - k)^2\,\mathrm{d}x + |\text{r.h.s. of (3.17)}|\right]. \end{aligned} \tag{3.19}$$

On account of (1.4) we have

$$\begin{aligned} |\text{r.h.s. of (3.17)}| &\leq c\left[\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{\bar q-1}{2}}\eta^2(\Gamma_\varepsilon - k)|\nabla^2 v_\varepsilon|\,\mathrm{d}x \right.\\ &\quad + \int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{\bar q-1}{2}}\eta^2|\nabla v_\varepsilon||\nabla\Gamma_\varepsilon|\,\mathrm{d}x \\ &\quad \left.+ \int_{A(k,\hat r)} \eta|\nabla\eta||\nabla v_\varepsilon|(\Gamma_\varepsilon - k)\Gamma_\varepsilon^{\frac{\bar q-1}{2}}\,\mathrm{d}x\right] \\ &=: c[S_1 + S_2 + S_3]\,, \end{aligned}$$

and with Young's inequality we get $(0 < \tau < 1)$

$$S_1 \leq \tau\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}\eta^2(\Gamma_\varepsilon - k)|\nabla^2 v_\varepsilon|^2\,\mathrm{d}x + c(\tau)\int_{A(k,\hat r)} \Gamma_\varepsilon^{\bar q-1-\frac{p-2}{2}}\eta^2(\Gamma_\varepsilon - k)\,\mathrm{d}x\,,$$

and for τ small the first integral on the r.h.s. can be absorbed in the second integral on the l.h.s. of (3.19). In the same way we handle S_2, i.e.

$$S_2 \leq \tau\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}\eta^2|\nabla\Gamma_\varepsilon|^2\,\mathrm{d}x + c(\tau)\int_{A(k,\hat r)} \Gamma_\varepsilon^{\bar q-1-\frac{p-2}{2}}\eta^2|\nabla v_\varepsilon|^2\,\mathrm{d}x\,.$$

Finally we have

$$S_3 \le c\int_{A(k,\hat r)} |\nabla\eta|^2(\Gamma_\varepsilon - k)^2\Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x + c\int_{A(k,\hat r)} \eta^2|\nabla v_\varepsilon|^2\Gamma_\varepsilon^{\overline{q}-1-\frac{q-2}{2}}\,\mathrm{d}x\,.$$

Collecting terms and dropping the second term on the l.h.s. of (3.19) we end up with

$$\begin{aligned}\int_{A(k,\hat r)} \Gamma_\varepsilon^{\frac{p-2}{2}}|\nabla\Gamma_\varepsilon|^2\eta^2\,\mathrm{d}x \;\le\; c\Bigg[&\int_{A(k,\hat r)} \eta^2(\Gamma_\varepsilon - k)\Gamma_\varepsilon^{\overline{q}-1-\frac{p-2}{2}}\,\mathrm{d}x\\ &+\int_{A(k,\hat r)} \eta^2|\nabla v_\varepsilon|^2\Gamma_\varepsilon^{\overline{q}-1-\frac{q-2}{2}}\,\mathrm{d}x\\ &+\int_{A(k,\hat r)} |\nabla\eta|^2(\Gamma_\varepsilon - k)^2\Gamma_\varepsilon^{\frac{q-2}{2}}\,\mathrm{d}x\Bigg]\,.\end{aligned}$$

Observing

$$(\Gamma_\varepsilon - k)\Gamma_\varepsilon^{\overline{q}-1-\frac{p-2}{2}} \le \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}$$

and

$$|\nabla v_\varepsilon|^2\Gamma_\varepsilon^{\overline{q}-1-\frac{q-2}{2}} \le \Gamma_\varepsilon^{\overline{q}-\frac{q}{2}+1} \le \Gamma_\varepsilon^{\overline{q}-\frac{p}{2}+1}\,,$$

inequality (3.12) is established. □

From Lemma 3.2 the claim of Theorem 1.1 follows as outlined at the end of Step 5 of Section 2.

4 Proof of Theorem 1.3

For $0 < \delta < 1$ let us now introduce the global regularization

$$J_\delta[v] = \int_\Omega f_\delta(\cdot,\nabla v)\,\mathrm{d}x \to \min \qquad \text{in } u_0 + \overset{\circ}{W}{}^1_q(\Omega;\mathbb{R}^N)\,, \tag{4.1}$$

where $f_\delta(x,P) = \delta(1+|P|^2)^{q/2} + f(x,P)$ with exponent q to be specified later. Assume for the moment that f satisfies (1.3) and (1.4) from Assumption 1.1. (Of course the following considerations are true under weaker hypotheses.) If u_δ denotes the unique solution of (4.1), then obviously

$$\sup_{0<\delta<1}\int_\Omega f(\cdot,\nabla u_\delta)\,\mathrm{d}x < \infty\,, \tag{4.2}$$

hence there exists a function $\bar u \in u_0 + \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N)$ such that as $\delta \to 0$ we have $u_\delta \rightharpoondown \bar u$ in $W^1_p(\Omega;\mathbb{R}^N)$ (at least for a suitable subsequence) and $\int_\Omega f(\cdot,\nabla\bar u)\,\mathrm{d}x < \infty$. This follows from (4.2), the growth of f and from lower semicontinuity arguments. For the readers convenience we first like to show that $\{u_\delta\}$ forms a $\overline{J}$-minimizing sequence and that $\bar u$ is the unique $\overline{J}$-minimizer within the class $u_0 + \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N)$. To this purpose we recall the definition of the relaxed functional $\overline{J}$ given before Theorem 1.3 and observe that $\overline{J}$ is lower semicontinuous w.r.t. weak convergence on the space $W^1_p(\Omega;\mathbb{R}^N)$ and that $J = \overline{J}$ on $u_0 + \overset{\circ}{W}{}^1_{\overline{q}}(\Omega;\mathbb{R}^N)$, moreover, we have

$$\inf\{J[w] : \; w \in u_0 + \overset{\circ}{W}{}^1_{\overline{q}}(\Omega;\mathbb{R}^N)\} = \inf\{\overline{J}[w] : \; w \in u_0 + \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N)\}\,. \tag{4.3}$$

Lower semicontinuity of $\overline{J}$ immediately implies that

$$\overline{J}[\bar{u}] \le \liminf_{\delta \to 0} \overline{J}[u_\delta] = \liminf_{\delta \to 0} J[u_\delta] \,. \tag{4.4}$$

Consider next a sequence $u_m \in u_0 + \overset{\circ}{W}{}^1_q(\Omega;\mathbb{R}^N)$ s.t.

$$J[u_m] \overset{m\to\infty}{\to} \inf\{J[w] :\ w \in u_0 + \overset{\circ}{W}{}^1_{\overline{q}}(\Omega;\mathbb{R}^N)\} =: \alpha \,. \tag{4.5}$$

In order to find $\{u_m\}$ we first consider a sequence $u'_m \in u_0 + \overset{\circ}{W}{}^1_{\overline{q}}(\Omega;\mathbb{R}^N)$ s.t. $J[u'_m] \to \alpha$ as $m \to \infty$. With m fixed pick $\varphi_k \in C^\infty_0(\Omega;\mathbb{R}^N)$ s.t.

$$\|\varphi_k - (u'_m - u_0)\|_{W^1_{\overline{q}}(\Omega;\mathbb{R}^N)} \overset{k\to\infty}{\to} 0 \,, \tag{4.6}$$

and let $\psi_k := u_0 + \varphi_k$. Clearly J is locally bounded on the space $W^1_{\overline{q}}(\Omega;\mathbb{R}^N)$ and convex, thus locally Lipschitz. This implies by (4.6) that

$$J[\psi_k] - J[u'_m] \overset{k\to\infty}{\to} 0 \,,$$

and we may choose $u_m = \psi_k$ with $k = k(m)$ such that $|J[u_m] - J[u'_m]| \le 1/m$. Now (4.5) is immediate. From the J_δ-minimality of u_δ and the admissibility of the functions u_m we deduce $J_\delta[u_\delta] \le J_\delta[u_m]$, hence $\liminf_{\delta\to 0} J[u_\delta] \le J[u_m]$ which together with (4.4) implies that $\overline{J}[\bar{u}] \le J[u_m]$, and the $\overline{J}$-minimality of $\bar{u}$ follows from (4.5) and (4.3).

Let us briefly indicate how to prove the unique solvability of the relaxed problem following [ELM]. Let $v \in u_0 + \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N)$ such that $\overline{J}[v] < \infty$. By the definition of $\overline{J}$ there exists for each $m \in \mathbb{N}$ a sequence $\{v^{(m)}_k\}_{k\in\mathbb{N}}$ in $u_0 + \overset{\circ}{W}{}^1_{\overline{q}}(\Omega;\mathbb{R}^N)$ such that

$$\left.\begin{aligned} &\alpha_m := \lim_{k\to\infty} J[v^{(m)}_k] \overset{m\to\infty}{\to} \overline{J}[v] \,, \\ &v^{(m)}_k \overset{k\to\infty}{\rightharpoondown} v \quad \text{in } L^p(\Omega;\mathbb{R}^N) \,, \\ &\nabla v^{(m)}_k \overset{k\to\infty}{\rightharpoondown} \nabla v \quad \text{in } L^p(\Omega;\mathbb{R}^{nN}) \,. \end{aligned}\right\} \tag{4.7}$$

We choose dense subsets $\{\varphi_l\}_{l\in\mathbb{N}}$ and $\{\psi_l\}_{l\in\mathbb{N}}$ of $L^{p'}(\Omega;\mathbb{R}^N)$ and $L^{p'}(\Omega;\mathbb{R}^{nN})$, respectively, $p' := p/(p-1)$, and choose for each $m \in \mathbb{N}$ an index $k = k_m$ such that

$$\left.\begin{aligned} &\left|\int_\Omega (v^{(m)}_k - v)\cdot \varphi_l \,\mathrm{d}x\right| \le \frac{1}{m}\,, \qquad l = 1,\dots,m\,, \\ &\left|\int_\Omega (\nabla v^{(m)}_k - \nabla v) : \psi_l \,\mathrm{d}x\right| \le \frac{1}{m}\,, \quad l = 1,\dots,m\,, \end{aligned}\right\} \tag{4.8}$$

which is possible on account of (4.7). If necessary, we also increase k_m such that

$$|\alpha_m - J[v^{(m)}_{k_m}]| \le \frac{1}{m} \tag{4.9}$$

holds for all m. Finally we let $v_m := v^{(m)}_{k_m} \in u_0 + \overset{\circ}{W}{}^1_{\overline{q}}(\Omega;\mathbb{R}^N)$. From (4.8) we get

$$\left.\begin{aligned} \lim_{m\to\infty} \int_\Omega v_m \cdot \varphi_l \,\mathrm{d}x &= \int_\Omega v \cdot \varphi_l \,\mathrm{d}x \,, \\ \lim_{m\to\infty} \int_\Omega \nabla v_m : \psi_l \,\mathrm{d}x &= \int_\Omega \nabla v : \psi_l \,\mathrm{d}x \end{aligned}\right\} \tag{4.10}$$

and (4.9) implies

$$\lim_{m \to \infty} J[v_m] = \overline{J}[v]\,, \tag{4.11}$$

in particular we get from (4.11) the bound

$$\sup_m \|v_m\|_{W^1_p(\Omega;\mathbb{R}^N)} < \infty\,.$$

This fact combined with (4.10) finally implies $v_m \rightharpoondown v$ as $m \to \infty$ in $W^1_p(\Omega;\mathbb{R}^N)$. Thus we have found a sequence $\{v_m\}$ in $u_0 + \overset{\circ}{W}{}^1_{\overline{q}}(\Omega;\mathbb{R}^N)$ whose weak W^1_p-limit is v and which has the property (4.11). Consider a second function $u \in u_0 + \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^N)$ such that $\overline{J}[u] < \infty$ and define $\{u_m\}$ as before. Then we claim that $\overline{J}[u] = \overline{J}[v]$ implies $u = v$, in particular we get the uniqueness for our $\overline{J}$-minimizer $\bar{u}$. Following [ELM] it is easy to check that under our Assumption 1.1 inequality (94) of [ELM] holds with f_ε replaced by f, and with the choices $\mu = 1$, $\varepsilon = 0$. We therefore get (97) of [ELM] and we can adopt their arguments by replacing y_m, y, w_m, w by v_m, v, u_m, u, respectively, with the result

$$\overline{J}\Big[\frac{1}{2}u + \frac{1}{2}v\Big] < \frac{1}{2}\overline{J}[u] + \frac{1}{2}\overline{J}[v]$$

(see (101) of [ELM]) if $\nabla u \neq \nabla v$ on a set of positive measure. This proves our claim.

Up to now all our considerations just used the first part of Assumption 1.1. Now we like to investigate the regularity of $\bar{u}$ under the various additional hypotheses.

Case 1. $\overline{q} < p(1+1/n)$ together with Assumption 1.1. We proceed exactly as in Section 2 with f_ε, v_ε, B_{2R} being replaced by f_δ, u_δ, Ω, respectively, where the exponent q is chosen according to (2.1). Then our first result is higher integrability of $\nabla\bar{u}$, more precisely (see Lemma 2.3):

$$\nabla\bar{u} \in L^{p\chi}_{loc}(\Omega;\mathbb{R}^{nN})$$

with the former choice of χ.

Suppose now that $n = 2$ or that we are in the situation of Theorem 1.1, iii). In the first case we can argue exactly as in Step 4 of Section 2, in the second case we deduce $\nabla u_\delta \in L^\infty_{loc}(\Omega;\mathbb{R}^{nN})$ uniformly w.r.t. δ. Going back to Lemma 2.2 and using the local boundedness of ∇u_δ we get from (2.5) that u_δ is uniformly bounded w.r.t. δ in the space $W^2_{2,loc}(\Omega;\mathbb{R}^N)$, in particular we have $\nabla u_\delta \to \nabla\bar{u}$ a.e. on Ω as $\delta \to 0$. Consider $\varphi \in C^\infty_0(\Omega;\mathbb{R}^N)$ and recall

$$\int_\Omega D_P f_\delta(\cdot,\nabla u_\delta) : \nabla\varphi\,\mathrm{d}x = 0\,. \tag{4.12}$$

By dominated convergence we may pass to the limit $\delta \to 0$ with the result

$$\int_\Omega D_P f(\cdot,\nabla\bar{u}) : \nabla\varphi\,\mathrm{d}x = 0\,.$$

Since $\nabla\bar{u}$ is locally bounded, we deduce that $\bar{u}$ is a local minimizer of the functional J (in the sense of Definition 1.1), and the arguments outlined at the end of Section 2 imply the Hölder continuity of $\nabla\bar{u}$.

Case 2. The hypotheses of Theorem 1.2 hold. This time we choose q according to (3.1) and argue exactly as in Section 3 to get the appropriate version of Lemma 3.2. Then we proceed as in Case 1.

Let us finally look at the unstructured situation, i.e. we discuss the case

$$\overline{q} < p(1 + 1/n)$$

just under Assumption 1.1, where the regularization is done as before. From (2.5) we deduce

$$\int_{\Omega'} (1 + |\nabla u_\delta|^2)^{\frac{p-2}{2}} |\nabla^2 u_\delta|^2 \, \mathrm{d}x \le c(\Omega')$$

for any subdomain $\Omega' \Subset \Omega$, thus

$$\int_{\Omega'} |\nabla^2 u_\delta|^2 \, \mathrm{d}x \le c(\Omega')$$

provided that $p \ge 2$. Otherwise consider $s \in (1,2)$ and $r \in \mathbb{R}$ to be specified later and use Young's inequality to get ($\Gamma_\delta := 1 + |\nabla u_\delta|^2$)

$$\begin{aligned} \int_{\Omega'} |\nabla^2 u_\delta|^s \, \mathrm{d}x &= \int_{\Omega'} \Gamma_\delta^r |\nabla^2 u_\delta|^s \Gamma_\delta^{-r} \, \mathrm{d}x \\ &\le c \left[\int_{\Omega'} \Gamma_\delta^{\frac{2r}{s}} |\nabla^2 u_\delta|^2 \, \mathrm{d}x + \int_{\Omega'} \Gamma_\delta^{-r\frac{2}{2-s}} \, \mathrm{d}x \right]. \end{aligned}$$

Now we relate r and s through the condition $r2/s = (p-2)/2$, i.e. $r = s(p-2)/4$. If $s \downarrow 1$, then

$$-r\frac{2}{2-s} \to \frac{2-p}{2},$$

which means that for s close to 1 u_δ belongs to $W^2_{s,loc}(\Omega;\mathbb{R}^N)$ uniformly w.r.t. δ, moreover, as a consequence, we get $\nabla u_\delta \to \nabla \bar{u}$ a.e. on Ω. By Lemma 2.3 we have uniform local integrability of ∇u_δ up to the exponent $p\chi$ which is even bigger than q (compare the choice of q after (2.7)), thus we may pass to the limit $\delta \to 0$ in equation (4.12) including "testfunctions" φ of class $W^1_q(\Omega;\mathbb{R}^N)$ with compact support. Thus we have shown that $\bar{u}$ is a solution of

$$\left.\begin{array}{l} \operatorname{div} D_P f(\cdot, \nabla \bar{u}) = 0 \quad \text{a.e. on } \Omega\,, \\ \bar{u} = u_0 \quad \text{on } \partial\Omega\,, \end{array}\right\}$$

where the boundary condition has to be understood in the W^1_p-sense. Clearly

$$\int_{\Omega'} f(\cdot, \nabla \bar{u}) \, \mathrm{d}x \le \int_{\Omega'} f(\cdot, \nabla v) \, \mathrm{d}x$$

for all domains $\Omega' \Subset \Omega$ and any function $v \in W^1_{\overline{q},loc}(\Omega;\mathbb{R}^N)$ such that $v = \bar{u}$ outside of Ω'. But this type of local minimality is enough to prove partial $C^{1,\alpha}$-regularity of $\bar{u}$ in the interior of Ω since all comparison functions used during the proof belong to this class. For details we refer to [BF1] and leave the necessary adjustments of Step 3 in Section 2 to the reader.

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