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Abstract

Let $\Omega \subset \mathbb{R}^2$ denote a bounded Lipschitz domain and consider some portion Γ_0 of $\partial\Omega$ representing the austenite-twinned martensite interface which is not assumed to be a straight segment. We prove

$$\inf_{u \in \mathcal{W}(\Omega)} \int_\Omega \varphi(\nabla u(x,y))dxdy = 0 \qquad (*)$$

for an elastic energy density $\varphi : \mathbb{R}^2 \to [0,\infty)$ such that $\varphi(0,\pm 1) = 0$. Here $\mathcal{W}(\Omega)$ consists of all functions u from the Sobolev class $W^{1,\infty}(\Omega)$ such that $|u_y| = 1$ a.e. on Ω together with $u = 0$ on Γ_0. We will first show that for Γ_0 having a vertical tangent one cannot always expect a finite surface energy, i.e. in the above problem the condition

$$u_{yy} \text{ is a Radon measure such that } \int_\Omega |u_{yy}(x,y)|dxdy < +\infty$$

in general cannot be included. This generalizes a result of [W.] where Γ_0 is a vertical straight line. Property (*) is established by constructing some minimizing sequences vanishing on the whole boundary $\partial\Omega$, that is, one can even take $\Gamma_0 = \partial\Omega$. We also show that the existence or nonexistence of minimizers depends on the shape of the austenite-twinned martensite interface Γ_0.

AMS classification: 49, 74

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1 Introduction.

In solid-solid phase transformations one often observes certain characteristical microstructural features involving fine mixtures of the phases. If we consider martensitic phase transformations, then one usually has a plane interface which separates one homogeneous phase called austenite from a very fine mixture of twins of the other phase termed martensite. We now consider a two-dimensional section and assume that for some physical reasons the interface which seperates the two phases is not a segment but a curve not necessarily being smooth.
For instance, it is known that some applied small loads easily change the austenite-martensite interface. For further details concerning the physical background of martensitic phase transformation and also the mathematical

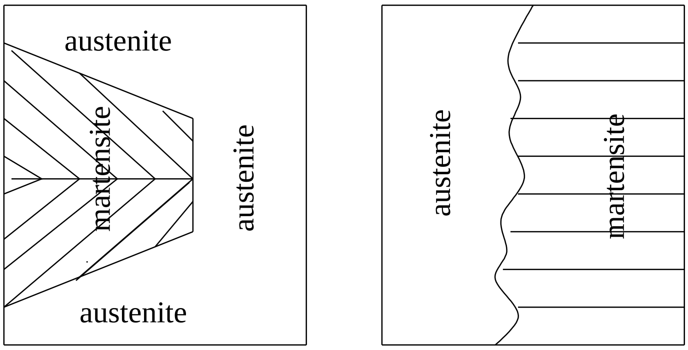

Figure 1. The austenite−twinned martensite interface

modelling we refer the reader to the papers [B.J.$_1$] and [B.J.$_2$] and the references quoted therein. To give a more precise formulation of the problem we like to investigate, let us consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ representing the martensitic configuration, and let Γ_0 denote a part of $\partial\Omega$ with positive measure having the meaning of the austenite-twinned martensite interface. Let $\varphi : \mathbb{R}^2 \to [0, \infty)$ denote a Borel function such that

$$\varphi(0,1) = \varphi(0,-1) = 0. \tag{1.1}$$

For example, φ could be the elastic energy density of the martensite with wells in $(0, \pm 1)$ corresponding to the stress-free states of two possible variants of the martensite. We then would like to consider the problem

$$I^\infty := \inf_{u \in \mathcal{W}(\Omega)} \int_\Omega \varphi(\nabla u(x,y))dxdy \tag{1.2}$$

in the class of admissible comparison functions

$$\mathcal{W} := \mathcal{W}(\Omega) := \{u \in W^{1,\infty}(\Omega) \ : \ |u_y| = 1 \text{ a.e. in } \Omega \text{ and } u = 0 \text{ on } \Gamma_0\}.$$

Here $W^{1,\infty}(\Omega)$ is the Sobolev space of all weakly differentiable functions $u : \Omega \to \mathbb{R}$ such that u, $|\nabla u| \in L^\infty(\Omega)$. Since Ω is a bounded Lipschitz domain, Sobolev's embedding theorem implies $W^{1,\infty}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, and the requirement $u = 0$ on Γ_0 has to be understood in the pointwise sense. If $u = 0$ on the whole of $\partial\Omega$, we just say that u is of class $W_0^{1,\infty}(\Omega)$. For a further discussion of Sobolev spaces we refer the reader to [A.].
We remark that the boundary condition occurring in $\mathcal{W}$ refers to elastic compatibility with the austenitic phase in the extreme case of complete rigidity

of the austenite (see [B.J.$_1$], [B.J.$_2$] and [Ko.]). Problems of the type (1.2) have been investigated by Chipot and Collins (compare [C.] and [C.C.]) but without the constraint $|u_y| = 1$. This constraint was introduced by Kohn and Müller (see [K.M.$_1$] and [K.M.$_2$]): they considered a functional consisting of an elastic energy plus a surface energy term for the case that the martensitic configuration is a rectangle like $(0, L) \times (0, 1)$ and the austenite-martensite interface is the segment $\{0\} \times (0, 1)$.

Problem (1.2) was studied in [E.F.] for the case when no loads are applied, i.e. the austenite-martensite interface is given by a segment Γ_0. We proved that the value of I^∞ is zero by constructing suitable minimizing sequences from the class $\mathcal{W}(\Omega)$ which represent, according to the Ball-James theory, the microstructure. The minimizing sequences discussed in [E.F.] differ for the case when the segment Γ_0 is vertical and for the case when Γ_0 is oblique. In particular, for non-vertical segments we could even replace the set $\mathcal{W}(\Omega)$ by a smaller class by adding the additional constraint

$$u_{yy} \text{ is a Radon measure of finite mass} \tag{1.3}$$

which is not possible in the vertical case (see [W.]). In what follows the term

$$\int_\Omega |u_{yy}(x, y)| dx dy$$

is adressed as the surface energy and it should be understood as the mass $|u_{yy}|(\Omega)$ of the Radon measure u_{yy}. Here our terminology follows the paper $[K.M._1]$, i.e. we use the same formula for the surface energy as Kohn and Müller did for the case when Ω is a rectangle.

In the present note we want to extend the results of [E.F.] and [W.] to the general case of curved boundary portions, precisely we have:

THEOREM 1.1 *There exist domains Ω and boundary portions Γ_0 which do not contain a vertical straight line for which condition (1.3) cannot be included into problem (1.2), i.e. there is no function u in $\mathcal{W}(\Omega)$ satisfying (1.3).*

Indeed, we will exhibit in section 2 some "bad" curved boundaries for which one cannot incorporate the condition (1.3). Namely, they only have a vertical tangent forcing the surface energy to tend to infinity. This generalizes a result of [W.] where the domain Ω is chosen to be a rectangle and Γ_0 is a vertical straight line. Now, if the condition of finite surface energy is dropped, then one has for any curved boundary :

THEOREM 1.2 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and consider a non empty portion Γ_0 of $\partial\Omega$ having positive measure . If φ satisfies (1.1), then we have*

$$I^\infty := \inf_{u \in \mathcal{W}(\Omega)} \int_\Omega \varphi(\nabla u(x,y))dxdy = 0.$$

Moreover, we can find a minimizing sequence $(u_n)_n \subset \mathcal{W}(\Omega)$ such that $u_n = 0$ on the whole boundary $\partial\Omega$.

For the proof of this result we will first discuss the case when the Lipschitz domain Ω is replaced by some elementary domain, e.g. the domain enclosed by a triangle or a square. Then, we consider the general situation by covering every bounded open set with a countable number of such elementary domains (Vitali's covering lemma).

2 Some curved boundaries leading to infinite surface energies: proof of theorem 1.1.

In this section we are going to prove that if the boundary part Γ_0 of Ω has a vertical tangent one cannot in general add the constraint (1.3) to the definition of the class $\mathcal{W}(\Omega)$. Without loss of generality we assume that the origin lies on Γ_0 and the tangent at this point is vertical. To be more precise, we assume that there exists a continuous function $f : I := [0,T] \to \mathbb{R}$ of class C^1 on $(0,T)$ such that

$$f(0) = 0, \ \lim_{t \to 0^+} f'(t) = +\infty \tag{2.1}$$

$$\{(t, f(t)) \mid t \in I\} \subset \Gamma_0, \tag{2.2}$$

$$\{(t,y) \in [0,T] \times \mathbb{R} \mid 0 \le y \le f(t)\} \subset \Omega. \tag{2.3}$$

Notice that by (2.1) and by eventually reducing the interval I we can assume that f is strictly increasing in I. The function f is a bijective mapping $[0,T] \to [0, f(T)]$ and its inverse f^{-1} is also strictly increasing, continuous on $[0, f(T)]$ and of class $C^1(0, f(T))$. Moreover one has

$$(f^{-1})'(0) \text{ exists and is equal to } 0.$$

In order to prove that the surface energy in general cannot expected to be finite, we are first going to bound it from below. We have the following estimate :

LEMMA 2.1 *For every $u \in \mathcal{W}(\Omega)$ with (1.3) one has*

$$\int_0^T \int_0^{f(t)} |u_{yy}(t,y)|dydt \geq \int_0^{f(T)} \Big\{ \frac{s^{\frac{3}{2}}}{\sqrt{3K}} \frac{(f^{-1})'(s)}{\left(\int_0^s (f^{-1}(s) - f^{-1}(y))^2 dy\right)^{\frac{1}{2}}} - 2 \Big\} ds \tag{2.4}$$

where $\sqrt{K}$ is the Lipschitz constant of u.

Proof : Let $u \in \mathcal{W}(\Omega)$ satisfying (1.3), $t \in I$ and $y \in (0, f(t))$; since $(f^{-1}(y), y) \in \Gamma_0$ and $u = 0$ on Γ_0, one has

$$u(t,y) = u(t,y) - u(f^{-1}(y), y) = \int_{f^{-1}(y)}^t u_x(s,y)ds.$$

Hence

$$|u(t,y)| \leq \int_{f^{-1}(y)}^t |u_x(s,y)|ds \leq \sqrt{K}(t - f^{-1}(y))$$

and we get :

$$\int_0^{f(t)} u(t,y)^2 dy \leq K \int_0^{f(t)} (t - f^{-1}(y))^2 dy. \tag{2.5}$$

Then we use the following lemma

LEMMA 2.2 *Let $g \in W^{1,\infty}(0,l)$ such that $|g'| = 1$ a.e. and such that g' changes sign N times on the open interval $(0,l)$. Then one has*

$$\int_0^l g^2(x)dx \geq \frac{l^3}{12}(N+1)^{-2} = \frac{l^3}{12}\Big(\frac{1}{2}\int_0^l |g''(x)|dx + 1\Big)^{-2}$$

Proof : The above inequality was proved by Kohn and Müller for $l = 1$ (lemma 2.7 in [K.M.$_1$]). One can easily derive the general case by scaling.

■

Lemma 2.2 then yields

$$\int_0^{f(t)} u(t,y)^2 dy \geq \frac{f(t)^3}{12}\Big(\frac{1}{2}\int_0^{f(t)} |u_{yy}(t,y)|dy + 1\Big)^{-2}. \tag{2.6}$$

Combining (2.5) and (2.6) one gets

$$\int_0^{f(t)} |u_{yy}(t,y)|dy \geq \frac{1}{\sqrt{3K}} \frac{f(t)^{\frac{3}{2}}}{\left(\int_0^{f(t)} (t - f^{-1}(y))^2 dy\right)^{\frac{1}{2}}} - 2.$$

Therefore one obtains

$$\int_0^T \int_0^{f(t)} |u_{yy}(t,y)|dydt \geq \int_0^T \Big\{ \frac{1}{\sqrt{3K}} \frac{f(t)^{\frac{3}{2}}}{\left(\int_0^{f(t)} \left\{(t - f^{-1}(y))^2 dy\right)^{\frac{1}{2}}} - 2 \Big\} dt,$$

from which the claim of lemma 2.1 follows by the change of variables $s = f(t)$.

■

REMARK 2.1 *One should observe that by lemma 2.1 the surface energy which is bounded from below by*

$$\int_0^T \int_0^{f(t)} |u_{yy}(t,y)|dydt$$

is infinite whenever f is chosen in such a way that

$$\int_0^{f(T)} s^{\frac{3}{2}} \frac{(f^{-1})'(s)}{\left(\int_0^s (f^{-1}(s) - f^{-1}(y))^2 dy\right)^{\frac{1}{2}}} ds = +\infty.$$

Therefore Theorem 1.1 will be a consequence of

THEOREM 2.1 *Assume that the function f satisfies in addition*

$$s\frac{(f^{-1})'(s)}{f^{-1}(s)} \geq \frac{c}{s^{\alpha}} \ \text{ on } (0, f(T)) \tag{2.7}$$

for a positive constant c and some exponent $\alpha \geq 1$. Then one has

$$\int_0^{f(T)} s^{\frac{3}{2}} \frac{(f^{-1})'(s)}{\left(\int_0^s (f^{-1}(s) - f^{-1}(y))^2 dy\right)^{\frac{1}{2}}} ds = +\infty.$$

Proof : Since

$$0 < f^{-1}(s) - f^{-1}(y) \leq f^{-1}(s) \ \forall y \in (0,s)$$

one has

$$\int_0^{f(T)} s^{\frac{3}{2}} \frac{(f^{-1})'(s)}{\left(\int_0^s (f^{-1}(s) - f^{-1}(y))^2 dy\right)^{\frac{1}{2}}} ds \geq \int_0^{f(T)} s\frac{(f^{-1})'(s)}{f^{-1}(s)} ds.$$

Using (2.7) one deduces

$$\int_0^{f(T)} s\frac{(f^{-1})'(s)}{f^{-1}(s)} ds = \int_0^{f(T)} \frac{c}{s^{\alpha}} ds = +\infty$$

and the theorem is proved. Notice that for example

$$s\frac{(f^{-1})'(s)}{f^{-1}(s)} = \frac{1}{s^{\alpha}},\ \alpha \geq 1,\ f^{-1}(s) > 0 \text{ on } (0, f(T)),\ f^{-1}(0) = 0$$

if and only if

$$f^{-1}(s) = \begin{cases} c\exp(-\frac{1}{\alpha s^{\alpha}}) \text{ if } s \neq 0 \ (c > 0 \text{ is a constant}), \\ \\ 0 \text{ if } s = 0, \end{cases}$$

or equivalently

$$f(t) = \begin{cases} \left(-\frac{1}{\alpha \ln(\frac{t}{c})}\right)^{\frac{1}{\alpha}} \text{ if } s \neq 0, \\ \\ 0 \text{ if } s = 0. \end{cases}$$

■

REMARK 2.2 *Notice that condition (2.7) is equivalent to*

$$f^{-1}(s)exp(\frac{c}{\alpha s^{\alpha}}) \text{ is increasing on } (0, f(T)).$$

Recall that one has

$$f^{-1}(0) = (f^{-1})'(0) = 0.$$

Now if in addition f^{-1} is of class $C^n([0, f(T)])$ one also has

$$(f^{-1})^{(m)}(0) = 0,\ \forall m \in \{0, 1, 2, \ldots, n\}.$$

This result which is not a priori evident follows from theorem 2.1 and the following lemma

LEMMA 2.3 *Let $h > 0$ and consider a function g of class $C^n([0, h])$ satisfying*

$$g(0) = g'(0) = \ldots = g^{(n-1)}(0) = 0,\ g^{(n)}(0) \neq 0. \tag{2.8}$$

Then one has

$$\int_0^h s^{\frac{3}{2}} \frac{|g'(s)|}{\left(\int_0^s (g(s) - g(y))^2 dy\right)^{\frac{1}{2}}} ds < +\infty.$$

Proof : One has

$$\int_0^s (g(s)-g(y))^2dy = sg(s)^2 - 2g(s)\int_0^s g(y)dy + \int_0^s g(y)^2dy. \qquad (2.9)$$

From the assumption on $g^{(k)}(0)$ it follows that

$$\tilde{g}^{(k)}(0) = 0 \text{ for all } 0 \leq k \leq n,$$

where we have set

$$\tilde{g}(t) := g(t) - \frac{1}{n!}g^{(n)}(0)t^n.$$

If we write

$$\tilde{g}(t) = \int_0^t \tilde{g}'(t_1)dt_1 = \int_0^t\int_0^{t_1} \tilde{g}''(t_2)dt_2dt_1 = \dots$$

$$\dots = \int_0^t\int_0^{t_1}\dots\int_0^{t_{n-1}} \tilde{g}^{(n)}(t_n)dt_n\dots dt_1,$$

then it easily follows that

$$|\tilde{g}(t)| \leq t^n \max_{[0,T]} |\tilde{g}^{(n)}|$$

and in conclusion

$$g(s) = \frac{1}{n!}g^{(n)}(0) + o(s^n).$$

Using this formula on the right hand side of (2.9) we deduce

$$\int_0^s (g(s)-g(y))^2dy = \Big(\frac{g^{(n)}(0)}{n!}\Big)^2\Big[1-\frac{2}{n+1}+\frac{1}{2n+1}\Big]s^{2n+1} + o(s^{2n+1}).$$

This implies that

$$\lim_{s\to 0^+} s^{\frac{3}{2}}\frac{g'(s)}{\left(\int_0^s (g(s)-g(y))^2dy\right)^{\frac{1}{2}}} = n\frac{g^{(n)}(0)}{|g^{(n)}(0)|}(1-\frac{2}{n+1}+\frac{1}{2n+1})^{-\frac{1}{2}}$$

and lemma is proved.

∎

3 Proof of theorem 1.2.

First we prove Theorem 1.2 for some special domains having "nice" boundaries. Let Δ denote the interior of the triangle with vertices in $(-1,0)$, $(1,0)$ and $(0,1)$.

THEOREM 3.1 *Assume that φ satisfies (1.1). Then there exists a sequence $v_n \in W_0^{1,\infty}(\Delta)$ satisfying $|\partial_y v_n| = 1$ a.e. for each n and such that*

$$\lim_{n\to\infty} \int_\Delta \varphi(\nabla v_n(x,y))dxdy = 0.$$

Proof. Given $N \in \mathbb{N}$ we will define $u \in W_0^{1,\infty}(\Delta)$, $|u_y| = 1$, such that

$$\int_\Delta \varphi(\nabla u(x,y))dxdy$$

is of order $\frac{1}{N}$. Let $\delta := \frac{1}{N}$ and consider the δ-periodic extension to the whole line of

$$h(t) := \begin{cases} t & \text{if } 0 \le t \le \frac{\delta}{2}, \\ \\ \delta - t & \text{if } \frac{\delta}{2} \le t \le \delta. \end{cases}$$

We then let

$$u(x,y) := \begin{cases} (x+1-y) \wedge h(y) & \text{if } (x,y) \in \Delta, -1 \le x \le 0, \\ \\ (1-x-y) \wedge h(y) & \text{if } (x,y) \in \Delta, 0 \le x \le 1. \end{cases}$$

Here we write $\alpha \wedge \beta$ for the minimum of two numbers α, $\beta \in \mathbb{R}$. Figure 2 below shows the situation for $N = 3$.
Clearly $u \in W_0^{1,\infty}(\Delta)$ and

$$\nabla u(x,y) = (0, \pm 1)$$

for points (x,y) not belonging to the $2N$ triangles Δ_i and Δ_i', $i = 1, \ldots, N$. It is easy to check that

$$\nabla u(x,y) = (1,-1) \text{ on } \Delta_i$$

whereas

$$\nabla u(x,y) = (-1,-1) \text{ on } \Delta_i'.$$

Therefore $|u_y| = 1$ a.e. on Δ and (1.1) implies

$$\int_\Delta \varphi(\nabla u(x,y))dxdy = \sum_{i=1}^N \Big[\int_{\Delta_i} \varphi(\nabla u(x,y))dxdy + \int_{\Delta_i'} \varphi(\nabla u(x,y))dxdy\Big]$$

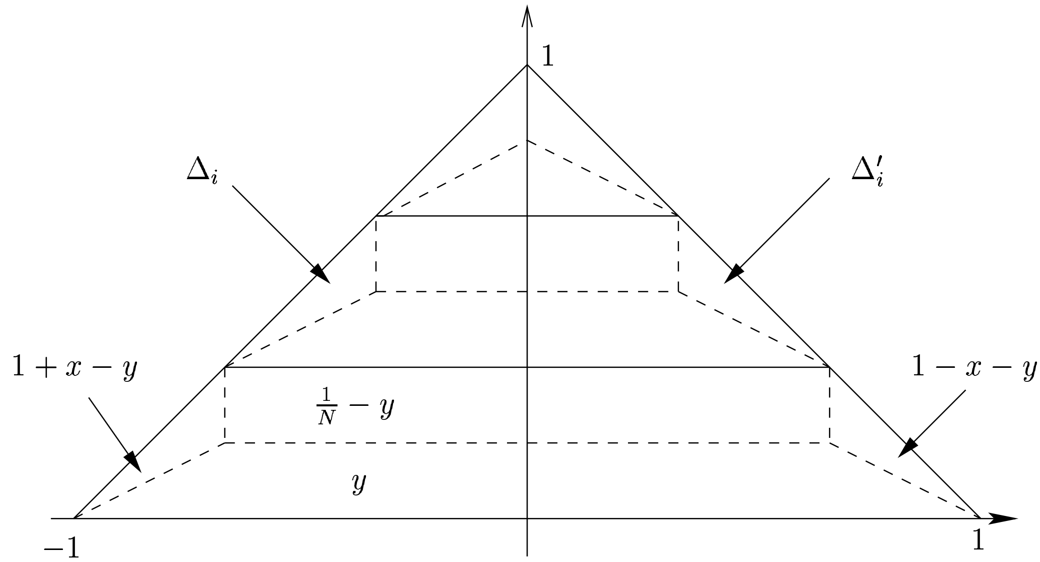

Figure 2: the function u for $N=3$

$$= \sum_{i=1}^{N} \Big[\mathcal{L}^2(\Delta_i)\varphi(1,-1) + \mathcal{L}^2(\Delta'_i)\varphi(-1,-1)\Big]$$

$$= N\frac{\delta^2}{4}[\varphi(1,-1)+\varphi(-1,-1)],$$

thus

$$0 \le I^\infty \le \int_\Delta \varphi(\nabla u(x,y))dxdy = \frac{1}{4N}[\varphi(1,-1)+\varphi(-1,-1)],$$

and Theorem 3.1 is established.

■

Let S now denote the set of points (x,y) such that $(x,y) \in \overline{\Delta}$ or $(x,-y) \in \overline{\Delta}$, i.e. S is the closed square with vertices in $(\pm 1, 0)$ and $(0, \pm 1)$. Then we have the following

Corollary 3.1 *Assume that φ satisfies (1.1). Then there exists a sequence $v_n \in W_0^{1,\infty}(\mathring{S})$ satisfying $|\partial_y v_n| = 1$ a.e. for each n and such that*

$$\lim_{n\to\infty} \int_S \varphi(\nabla v_n(x,y))dxdy = 0.$$

Proof. Let us define on S the following function

$$v(x,y) := \begin{cases} u(x,y) & \text{if } (x,y) \in \Delta, \\ u(x,-y) & \text{if } (x,y) \in S\backslash\Delta \end{cases}$$

where the function $u : \Delta \to \mathbb{R}$ is defined in the proof of Theorem 3.1. One can easily check that

$$\int_S \varphi(\nabla v(x,y))dxdy = \int_\Delta \varphi(\nabla u(x,y))dxdy + \int_\Delta \tilde{\varphi}(\nabla u(x,y))dxdy$$

where

$$\tilde{\varphi}(x,y) = \varphi(x,-y).$$

Thus

$$\int_S \varphi(\nabla v(x,y))dxdy = \frac{1}{4N}[\varphi(1,-1) + \varphi(-1,-1) + \tilde{\varphi}(1,-1) + \tilde{\varphi}(-1,-1)]$$

$$= \frac{1}{4N}[\varphi(1,-1) + \varphi(-1,-1) + \varphi(1,1) + \varphi(-1,1)],$$

and Corollary 3.1 is proved.

■

REMARK 3.1 *Notice that for the elementary domains we considered above one can add the constraint*

$$|u_{yy}| \text{ is a Radon measure of finite mass.}$$

Recall that for domains like squares with sides parallel to the x and y axis or domains as studied in section 2 (with function f satisfying (2.7)) it is not possible to incorporate the above constraint. For domains with f^{-1} satisfying (2.8) like discs the question is still open since by lemma 2.3 the right hand side of (2.4) is finite; we cannot conclude that the surface energy is $+\infty$ for any $u \in \mathcal{W}(\Omega)$.

In order to prove Theorem 1.2 for general domains we need the following lemmas

LEMMA 3.1 *(Vitali's covering lemma) Let Ω denote a bounded open subset of $\mathbb{R}^2$. Then there exist points $(x_n, y_n) \in \Omega$ and positive numbers r_n such that*

$$S_n := r_n S + (x_n, y_n) \subset \Omega \text{ and } \mathring{S}_l \cap \mathring{S}_k = \emptyset \text{ for } l \neq k,$$

where S is the square with vertices in $(\pm 1,0)$ and $(0,\pm 1)$. Moreover, we have

$$\Omega = \bigcup_{n=0}^{+\infty} S_n.$$

Proof. We refer for example to [S.] for a proof.

∎

Applying the construction of Lemma 3.1 we find $r_n > 0$, $(x_n, y_n) \in \Omega$ such that the sets $S_n = r_n S + (x_n, y_n) \subset \Omega$ have the stated properties. Given a function $u_0 \in W_0^{1,\infty}(\mathring{S})$, we let

$$\begin{cases} u_n \ : \ S_n \ \to \mathbb{R}, \ u_n(x,y) := r_n u_0(\frac{1}{r_n}(x - x_n, y - y_n)), \\ \\ u \ : \ \Omega \ \to \mathbb{R}, \ u(x,y) := \sum_{n=1}^{\infty} (\chi_{\mathring{S}_n} u_n)(x,y) \end{cases} \tag{3.10}$$

where $\chi_{\mathring{S}_n}$ denotes the characteristic function of the set $\mathring{S}_n$. Then we claim:

LEMMA 3.2 *The function u defined in (3.7) is in the space $W_0^{1,\infty}(\Omega)$, and we have the following formula*

$$\nabla u(x,y) = \sum_{n=1}^{\infty} (\chi_{\mathring{S}_n} \nabla u_n)(x,y) = \sum_{n=1}^{\infty} \chi_{\mathring{S}_n} \nabla u_0(\frac{1}{r_n}(x - x_n, y - y_n)) \ \textit{a.e. on } \Omega.$$

REMARK 3.2 *If we know $|\partial_y u_0| = 1$ a.e. on $\mathring{S}$, then we deduce from the disjointness of the family $\{\mathring{S}_n\}$ that also $|u_y| = 1$ is true a.e. on Ω.*

Proof of Lemma 3.2: On account of $(x_n, y_n) \in \Omega$, $S_n \subset \Omega$, the sequence $(r_n)_n$ stays bounded, thus

$$||u||_{L^\infty(\Omega)} \leq \sup_{n \in \mathbb{N}} r_n \ ||u_0||_{L^\infty(S)} < \infty.$$

In order to prove weak differentiability of the function u, we fix $\psi \in C_0^\infty(\Omega)$ and get from Lebesgue's theorem on dominated convergence

$$\int_\Omega u(x,y) \nabla \psi(x,y) dxdy = \sum_{n=1}^{\infty} \int_{\mathring{S}_n} u_n(x,y) \nabla \psi(x,y) dxdy.$$

Observing that $u_n = 0$ on ∂S_n, we can write

$$\int_{\mathring{S}_n} u_n(x,y) \nabla \psi(x,y) dxdy = - \int_{\mathring{S}_n} \nabla u_n(x,y) \psi(x,y) dxdy$$

and by the same reasoning as above (note: $||\nabla u_n||_{L^\infty(S_n)} = ||\nabla u_0||_{L^\infty(S)}$ and therefore $||\sum_{n=1}^{M} \chi_{\overset{\circ}{S}_n} \nabla u_n||_{L^\infty(\Omega)} = ||\nabla u_0||_{L^\infty(S)}$ for all $M \geq 1$)

$$-\sum_{n=1}^{\infty} \int_{\overset{\circ}{S}_n} \nabla u_n(x,y)\psi(x,y)dxdy = -\int_{\Omega} (\sum_{n=1}^{\infty} \chi_{\overset{\circ}{S}_n} \nabla u_n(x,y))\psi(x,y)dxdy,$$

which proves that

$$\sum_{n=1}^{\infty} \chi_{\overset{\circ}{S}_n} \nabla u_n \in L^\infty(\Omega, \mathbb{R}^2)$$

is the weak derivative of u. Again by dominated convergence it is obvious that

$$\sum_{n=1}^{M} \chi_{\overset{\circ}{S}_n} u_n \to u, \ \sum_{n=1}^{M} \chi_{\overset{\circ}{S}_n} \nabla u_n \to \nabla u$$

as M goes to infinity in $L^p(\Omega)$ for any finite p. Since the compact sets S_n are included in Ω, we have

$$\sum_{n=1}^{M} \chi_{\overset{\circ}{S}_n} u_n \in W_0^{1,p}(\Omega),$$

thus $u \in W_0^{1,p}(\Omega),\ p < \infty$. Lipschitz boundary of Ω guarantees that

$$W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) \ : \ B(v) = 0\},$$

where $B : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is the trace operator. Recalling that for functions $v \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, $B(v)$ is the pointwise trace, we finally deduce $u \in W_0^{1,\infty}(\Omega)$.

■

The proof of Theorem 1.2 can now be carried out as follows. Given $N \in \mathbb{N}$, we constructed in the proof of Corollary 3.1 a function $u_0 \in W_0^{1,\infty}(\overset{\circ}{S})$ such that $|\partial_y u_0| = 1$ on S and

$$\int_S \varphi(\nabla u_0(x,y))dxdy = \frac{1}{4N}[\varphi(1,-1) + \varphi(-1,-1) + \varphi(1,1) + \varphi(-1,1)].$$

Let us consider the function u defined in (3.7) for this particular choice of u_0. Lemma 3.2 implies $u \in W_0^{1,\infty}(\Omega)$, and from the remark after Lemma 3.2 we deduce $|u_y| = 1$ a.e. on Ω, thus $u \in \mathcal{W}(\Omega)$. We further have:

$$\int_\Omega \varphi(\nabla u(x,y))dxdy = \sum_{n=1}^{\infty} \int_{\overset{\circ}{S}_n} \varphi(\nabla u_0(\frac{1}{r_n}(x - x_n, y - y_n)))dxdy$$

$$= \sum_{n=1}^{\infty} r_n^2 \int_{\overset{\circ}{S}} \varphi(\nabla u_0(x,y))dxdy$$

so that

$$\int_\Omega \varphi(\nabla u(x,y))dxdy = \frac{1}{4N}[\varphi(1,-1)+\varphi(-1,-1)+\varphi(1,1)+\varphi(-1,1)] \sum_{n=1}^{\infty} r_n^2.$$

Finally we observe

$$\mathcal{L}^2(\Omega) = \sum_{n=1}^{\infty} \mathcal{L}^2(r_n S + (x_n,y_n)) = 2\sum_{n=1}^{\infty} r_n^2,$$

hence

$$\int_\Omega \varphi(\nabla u(x,y))dxdy = \frac{1}{2N}\mathcal{L}^2(\Omega)[\varphi(1,-1)+\varphi(-1,-1)+\varphi(1,1)+\varphi(-1,1)],$$

and since N was arbitrary, we have shown that $I^\infty = 0$. Moreover, it should be obvious how to obtain from the above construction a minimizing sequence in the class $\mathcal{W}(\Omega) \cap W_0^{1,\infty}(\Omega)$. This finishes the proof of Theorem 1.2.

∎

4 Remarks.

In addition to (1.1) let us assume that the integrand φ satisfies

$$\varphi(p,\pm 1) = 0 \Longrightarrow p = 0. \tag{4.1}$$

Under this condition we like to investigate if the infimum $I^\infty = 0$ is attained by some function $u \in \mathcal{W}(\Omega)$. This heavily depends on the shape of the boundary portion. For example, if $\Gamma_0 \subset \mathbb{R} \times \{b\}$ for some number $b \in \mathbb{R}$, then clearly $u(x,y) = y - b$ vanishes on Γ_0, $\partial_y u \equiv 1$ and $\nabla u(x,y) = (0,1)$, hence $\varphi(\nabla u(x,y)) = 0$ by (1.1). In order to exclude such a behaviour we let Σ denote the union of all rays starting from points $(x_0,y_0) \in \Gamma_0$ into Ω with direction $(1,0)$, and require

$$\Omega_0 := \Omega \cap \Sigma \text{ is open and nonempty.} \tag{4.2}$$

Of course, (4.2) does not hold in case $\Gamma_0 \subset \mathbb{R} \times \{b\}$.

THEOREM 4.1 *Let (1.1), (4.1) and (4.2) hold. Then we have*

$$\int_\Omega \varphi(\nabla u(x,y))dxdy > 0$$

for any $u \in \mathcal{W}(\Omega)$.

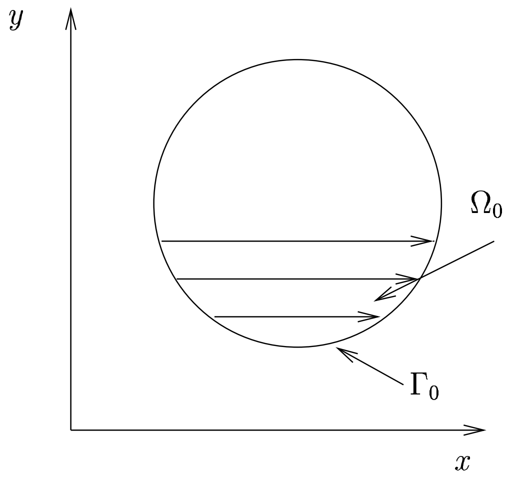

Figure3: $\Omega =$ a disc

Proof. If we assume that

$$\int_{\Omega} \varphi(\nabla u(x,y))dxdy = 0$$

for some $u \in \mathcal{W}(\Omega)$, then we get from (4.1)

$$u_x = 0 \text{ on } \Omega.$$

This implies the vanishing of u on any ray of the type defined before, hence, by (4.2), $u = 0$ on Ω_0 contradicting $u_y = \pm 1$ a.e.

■

Next we like to describe minimizing sequences in terms of Young measures (see [P.] for details about the notion Young measure)

THEOREM 4.2 *Let Ω denote a bounded Lipschitz domain in $\mathbb{R}^2$ and assume that the boundary portion Γ_0 is chosen in such a way that $\Omega_0 = \Omega$ (see (4.2)). Suppose that the integrand $\varphi : \mathbb{R}^2 \to [0,\infty)$ is a continuous function such that*

$$\varphi(p,q) = 0 \text{ if and only if } (p,q) = (0, \pm 1).$$

Let $(u_n)_n$ denote a minimizing sequence of problem (1.2) such that

$$||u_n||_{L^\infty(\Omega)},\ ||\nabla u_n||_{L^\infty(\Omega)} \leq C$$

for a finite constant C independent of n. Then

$$u_n \to 0 \text{ uniformly on } \Omega.$$

Moreover, the sequence of gradients $(\nabla u_n)_n$ *defines a unique homogeneous Young measure given by*

$$\nu_{(x,y)} = \frac{1}{2}\delta_{(0,-1)} + \frac{1}{2}\delta_{(0,1)} \text{ for a.a. } (x,y) \in \Omega,$$

where $\delta_{(0,\pm 1)}$ *are the Dirac measures at* $(0, \pm 1)$.

Proof. One proceeds as in [E.F.], we refer also to [C.] for a proof related to multiple-wells problems.

Corollary 4.1 *Let* Ω *denote a bounded Lipschitz domain in* $\mathbb{R}^2$. *Suppose that the integrand* $\varphi : \mathbb{R}^2 \to [0, \infty)$ *is a continuous function such that*

$$\varphi(p,q) = 0 \text{ if and only if } (p,q) = (0, \pm 1).$$

Let $(u_n)_n$ *denote a minimizing sequence of problem* (1.2) *such that*

$$||u_n||_{L^\infty(\Omega)},\ ||\nabla u_n||_{L^\infty(\Omega)} \leq C.$$

Suppose further that (4.2) holds. Then

$$u_n \to 0 \text{ uniformly on } \Omega_0.$$

Moreover, the sequence of gradients $(\nabla u_n)_n$ *defines a Young measure given by*

$$\nu_{(x,y)} = \alpha(x)\delta_{(0,-1)} + (1 - \alpha(x))\delta_{(0,1)} \text{ for a.a. } (x,y) \in \Omega,$$

where $\alpha : \Omega \to [0,1]$ *is a measurable function such that*

$$\alpha(x) = \frac{1}{2} \text{ for a.e. in } \Omega_0$$

Proof. The restriction of (u_n) to Ω_0 is a minimizing sequence of

$$I^\infty(\Omega_0) := \inf_{u \in \mathcal{W}(\Omega_0)} \int_{\Omega_0} \varphi(\nabla u(x,y))dxdy = 0.$$

where $\mathcal{W}(\Omega_0)$ is defined with respect to the boundary portion $\Gamma_0 \cap \partial\Omega_0$. Since $(\Omega_0)_0 = \Omega_0$ with an obvious definition of $(\Omega_0)_0$, one can apply Theorem 4.2 to get Corollary 4.1.

REMARK 4.1 *Note that* $\Omega_0 = \Omega$ *holds for the particular case* $\Gamma_0 = \partial\Omega$. *Now if* $\Omega_0 \neq \Omega$ *then the considered minimizing sequences do not necessarily converge to zero uniformly on the whole domain* Ω *and the related Young measure is in general not unique (see [E.F.] Remark 6 for an example).*

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