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Abstract

We discuss the weak form of the Ramberg/Osgood equations on two-dimensional domains and prove continuity of the stress and the strain tensor.

1 Introduction

In her recent thesis [Kn] Knees gives a very careful analysis of the Ramberg/Osgood equations first proposed by Ramberg and Osgood in [OR] as constitutive relations describing aluminium alloys. More general, these equations correspond to physically nonlinear elastic materials whose constitutive laws are of power-law type. For further details concerning the physical background we refer to Chapter 1 of [Kn]. The thesis is mainly devoted to the study of the local and also the global regularity of weak solutions of the Ramberg/Osgood equations (to be constructed in suitable function spaces) as well as to an analysis of the global regularity properties for a class of boundary transmission problems associated to Ramberg/Osgood materials.

In our note we use the local regularity results as a starting point to prove that the stress tensor and the strain tensor are Hölder continuous in the interior provided that the case of two dimensions is considered. To be precise, let us state our assumptions: suppose that $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain and fix some number $q > 2$. With $p := q/(q-1)$ (i.e. $p < 2$) we define the spaces (compare [GS] and also [FuS])

$$\begin{aligned} L^{q,2}(\Omega) &:= \{\sigma : \Omega \to \mathbb{S}^2 : \sigma^D \in L^q(\Omega),\, \operatorname{tr}\sigma \in L^2(\Omega)\}, \\ U^{p,2}(\Omega) &:= \{u : \Omega \to \mathbb{R}^2 : u \in L^p(\Omega),\, \varepsilon^D(u) \in L^p(\Omega),\, \operatorname{div} u \in L^2(\Omega)\}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{S}^2 &:= \text{space of symmetric } 2\times 2\text{-matrices}, \\ \sigma^D &:= \sigma - \frac{1}{2}(\operatorname{tr}\sigma)\mathbf{1} \quad \text{for } \sigma \in \mathbb{S}^2, \\ \varepsilon(u) &:= \frac{1}{2}(\nabla u + \nabla u^T) \quad \text{for } u\colon \Omega \to \mathbb{R}^2. \end{aligned}$$

The above spaces are normed in a standard way, and $U_0^{p,2}(\Omega)$ is the closure of $C_0^\infty(\Omega, \mathbb{R}^2)$ in $U^{p,2}(\Omega)$ w.r.t. the corresponding norm. A pair $(\sigma, u) \in L^{q,2}(\Omega) \times U^{p,2}(\Omega)$ is called a weak solution of the Ramberg/Osgood equations if

$$\int_\Omega [A\sigma + \alpha|\sigma^D|^{q-2}\sigma^D] : \tau \, dx = \int_\Omega \varepsilon(u) : \tau \, dx \tag{1.1}$$

and

$$\int_\Omega \sigma : \varepsilon(v)\, dx = \int_\Omega f \cdot v \, dx \tag{1.2}$$

holds for every choice of $(\tau, v) \in L^{q,2}(\Omega) \times U_0^{p,2}(\Omega)$. Here ":" and "·" denote the scalar products in $\mathbb{S}^2$ and $\mathbb{R}^2$, respectively. In equation (1.1) α is a positive constant, and

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$A : \mathbb{S}^2 \to \mathbb{S}^2$ is a symmetric linear operator such that $A\eta : \eta \geq \lambda|\eta|^2$ for any $\eta \in \mathbb{S}^2$ with a constant $\lambda > 0$. Finally, the "volume forces" $f\colon\, \Omega \to \mathbb{R}^2$ are supposed to be of class $L^q(\Omega)$. The "well-posedness" of the Ramberg/Osgood equations in their weak formulation (1.1), (1.2) with solutions (σ, u) in $L^{q,2}(\Omega) \times U^{p,2}(\Omega)$ is discussed in Chapter 1.3 of [Kn]. Now we can state our result:

THEOREM 1.1. *In addition to the hypotheses formulated above we assume that f is locally bounded. Then, if $\sigma \in L^{q,2}(\Omega)$ and $u \in U^{p,2}(\Omega)$ are solutions of (1.1) and (1.2), σ and $\varepsilon(u)$ are continuous functions in Ω. More precisely, the stress tensor σ and the strain tensor $\varepsilon(u)$ satisfy a local Hölder condition on Ω, i.e. for each subdomain $\Omega' \Subset \Omega$ there exists $\nu = \nu(\Omega') \in (0,1)$ s.t. σ, $\varepsilon(u) \in C^{0,\nu}(\overline{\Omega'})$.*

We remark that the proof Theorem 1.1 benefits from the local higher weak differentiability and local higher integrability results summarized in Theorem 2.3 of [Kn] which we combine with a lemma of Gehring type recently established in [BFZ] . This lemma will imply the continuity of σ and $\varepsilon(u)$, and this argument is in some sense an extension of the technique presented by Frehse and Seregin in [FrS] for obtaining regularity results in the setting of plastic materials with logarithmic hardening.

2 Proof of Theorem 1.1

Let the assumptions of Theorem 1.1 hold and consider a pair (σ, u) of solutions to (1.1) and (1.2). From [Kn], Theorem 2.3 and Lemma 2.17, it follows that

$$\sigma \in L^q(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega), \;\; |\sigma^D|^{\frac{q-2}{2}} \nabla\sigma^D \in L^2_{\mathrm{loc}}(\Omega), \;\; \sigma \in L^t_{\mathrm{loc}}(\Omega) \text{ for all } t < \infty \tag{2.1}$$

and

$$u \in W^{2,r}_{\mathrm{loc}}(\Omega) \text{ for any } r < 2, \;\; \operatorname{div} u \in L^q(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega), \tag{2.2}$$

where $W^{k,s}_{\mathrm{loc}}(\Omega)$ denotes the local variant of the standard Sobolev spaces, see, e.g. [Ad]. We fix a coordinate direction e_i, $i = 1, 2$, a number $h \neq 0$ and let $\Delta_h \rho(x) := \frac{1}{h}(\rho(x+he_i) - \rho(x))$ denote the difference quotient of a function ρ. Finally, we let $\varphi \in C^\infty_0(\Omega)$. If we abbreviate

$$W : \mathbb{S}^2 \to \mathbb{R}, \;\; W(\eta) := \frac{1}{2}(A\eta) : \eta + \frac{\alpha}{q}|\eta^D|^q,$$

then (1.1) is equivalent to $\varepsilon(u) = DW(\sigma)$ a.e. on Ω, hence $\Delta_h DW(\sigma) = \Delta_h \varepsilon(u)$ and in consequence

$$\int_\Omega \Delta_h DW(\sigma) : \Delta_h \sigma \varphi^2 \, dx = \int_\Omega \Delta_h \varepsilon(u) : \Delta_h \sigma \varphi^2 \, dx. \tag{2.3}$$

It is easy to show that

$$\Delta_h DW(\sigma) : \Delta_h \sigma \geq 0 \quad \text{a.e. on } \Omega. \tag{2.4}$$

Since $\sigma \in W^{1,2}_{\mathrm{loc}}(\Omega)$ (see (2.1)) and since we consider the case of two dimensions, it directly follows that $|\sigma^D|^{q-2}\sigma^D$ is in $W^{1,t}_{\mathrm{loc}}(\Omega)$ for any $t < 2$. Standard properties of difference quotients imply

$$\Delta_h DW(\sigma) \to \partial_i DW(\sigma) \quad \text{in } L^t_{\mathrm{loc}}(\Omega),$$

and (after passing to a subsequence) $\Delta_h DW(\sigma) \to \partial_i DW(\sigma)$ a.e. on Ω as $h \to 0$, in particular we get a.e. on Ω

$$\Delta_h DW(\sigma) : \Delta_h \sigma \to \partial_i DW(\sigma) : \partial_i \sigma \geq 0. \tag{2.5}$$

The properties (2.4) and (2.5) allow us to apply Fatou's lemma on the l.h.s. of (2.3), and we deduce

$$\int_\Omega \varphi^2 \partial_i DW(\sigma) : \partial_i \sigma \, dx \leq \liminf_{h\to 0} \int_\Omega \varepsilon(\Delta_h u) : \Delta_h \sigma \varphi^2 \, dx. \tag{2.6}$$

In order to discuss the r.h.s. of (2.6), we make use of the equation (1.2): for $P \in \mathbb{R}^{2\times 2}$ let $v := \Delta_{-h}(\varphi^2 \Delta_h \tilde{u})$, $\tilde{u} := u(x) - Px$. Clearly $\operatorname{div} v \in L^2(\Omega)$, which follows from $\Omega \subset \mathbb{R}^2$ and Sobolev's embedding theorem, hence $v \in U_0^{p,2}(\Omega)$ and v is admissible in (1.2) with the result

$$\int_\Omega \Delta_h \sigma : \varepsilon(\varphi^2 \Delta_h \tilde{u}) \, dx = -\int_\Omega \Delta_{-h}(\varphi^2 \Delta_h \tilde{u}) \cdot f \, dx,$$

or equivalently

$$\int_\Omega \varepsilon(\Delta_h u) : \Delta_h \sigma \varphi^2 \, dx = -\int_\Omega \Delta_h \sigma : (\nabla \varphi^2 \odot \Delta_h \tilde{u}) \, dx - \int_\Omega f \cdot \Delta_{-h}(\varphi^2 \Delta_h \tilde{u}) \, dx, \tag{2.7}$$

$\odot$ denoting the symmetric part of the tensor product $a \otimes b$ of vectors a, $b \in \mathbb{R}^2$. Using again (2.1) and (2.2), it is immediate that

$$\begin{aligned} \int_\Omega \Delta_h \sigma : (\nabla \varphi^2 \odot \Delta_h \tilde{u}) \, dx \quad &\to \quad \int_\Omega \partial_i \sigma : (\nabla \varphi^2 \odot \partial_i \tilde{u}) \, dx, \\ \int_\Omega f \cdot \Delta_{-h}(\varphi^2 \Delta_h \tilde{u}) \, dx \quad &\to \quad \int_\Omega f \cdot \partial_i(\varphi^2 \partial_i \tilde{u}) \, dx \end{aligned}$$

as $h \to 0$. Returning to (2.6), using (2.7) together with the convergences stated above, we have shown that

$$\int_\Omega \varphi^2 \partial_i DW(\sigma) : \partial_i \sigma \, dx \leq -\int_\Omega \partial_i \sigma : (\nabla \varphi^2 \odot \partial_i \tilde{u}) \, dx - \int_\Omega f \cdot \partial_i(\varphi^2 \partial_i \tilde{u}) \, dx. \tag{2.8}$$

From now on we agree to use the summation convention, i.e. we take the sum w.r.t. indices repeated twice. We like to emphasize that the integrals on the r.h.s. of (2.8) are well-defined (and finite), thus the function

$$H := \Big[D^2 W(\sigma)(\partial_i \sigma, \partial_i \sigma) \Big]^{\frac{1}{2}} = \Big[\partial_i DW(\sigma) : \partial_i \sigma \Big]^{\frac{1}{2}}$$

is in the space $L^2_{\text{loc}}(\Omega)$. Moreover, from the definition of the potential W it follows (see [Kn], (A.18))

$$\begin{aligned} D^2 W(\eta)(\theta, \tau) \quad = \quad & A\theta : \tau + \alpha |\eta^D|^{q-2} \theta^D : \tau^D \\ & + \alpha(q-2)|\eta^D|^{q-4}(\eta^D : \theta^D)(\eta^D : \tau^D), \end{aligned}$$

η, τ, $\theta \in \mathbb{S}^2$, and therefore it holds with suitable positive constants ω, $\overline{\omega}$

$$\omega\big(|\nabla \sigma|^2 + |\sigma^D|^{q-2} |\nabla \sigma^D|^2\big) \leq H^2 \leq \overline{\omega}\big(|\nabla \sigma|^2 + |\sigma^D|^{q-2} |\nabla \sigma^D|^2\big). \tag{2.9}$$

To proceed further consider a disc B_{2R} with compact closure in Ω and choose $\varphi \in C_0^\infty(B_{2R})$, $0 \le \varphi \le 1$, $\varphi = 1$ on B_R, $|\nabla\varphi| \le c/R$. From (2.8) we get

$$\int_{B_R} H^2\,dx \le c\left[\frac{1}{R}\int_{B_{2R}} |\nabla u - P||\nabla\sigma|\,dx + \frac{1}{R}\int_{B_{2R}} |\nabla u - P|\,dx + \int_{B_{2R}} |\Delta u|\,dx\right] \quad (2.10)$$

with c denoting a local constant depending on local L^∞-bounds for f. Let $P := \fint_{B_{2R}} \nabla u\,dx$. From Poincaré's inequality it follows that

$$\frac{1}{R}\int_{B_{2R}} |\nabla u - P|\,dx \le c\int_{B_{2R}} |\nabla^2 u|\,dx,$$

where $\nabla^2 u$ is the tensor of all second weak partial derivatives of u. Inserting this estimate into (2.10), we find

$$\int_{B_R} H^2\,dx \le c\left[\frac{1}{R}\int_{B_{2R}} |\nabla u - P||\nabla\sigma|\,dx + \int_{B_{2R}} |\nabla^2 u|\,dx\right]. \quad (2.11)$$

We discuss the first integral on the r.h.s. of (2.11): let $\gamma = 4/3$ and use Hölder's inequality as well as the Sobolev-Poincaré inequality to see

$$\begin{aligned}\frac{1}{R}\int_{B_{2R}} |\nabla u - P||\nabla\sigma|\,dx &\le \frac{1}{R}\left[\int_{B_{2R}} |\nabla u - P|^4\,dx\right]^{\frac{1}{4}}\left[\int_{B_{2R}} |\nabla\sigma|^\gamma\,dx\right]^{\frac{1}{\gamma}}\\ &\le \frac{c}{R}\left[\int_{B_{2R}} |\nabla^2 u|^\gamma\,dx\right]^{\frac{1}{\gamma}}\left[\int_{B_{2R}} |\nabla\sigma|^\gamma\,dx\right]^{\frac{1}{\gamma}}. \quad (2.12)\end{aligned}$$

We have the estimates

$$|\nabla^2 u| \le c|\nabla\varepsilon(u)|,$$

and for $l = 1, 2$ (using (1.1 and (2.9))

$$\begin{aligned}|\partial_l\varepsilon(u)| &= |A\partial_l\sigma + \partial_l(\alpha|\sigma^D|^{q-2}\sigma^D)|\\ &\le c\big[|\nabla\sigma| + |\sigma^D|^{q-2}|\nabla\sigma^D|\big]\\ &\le c(1+|\sigma^D|^2)^{\frac{q-2}{4}}\big[|\nabla\sigma| + |\sigma^D|^{\frac{q-2}{2}}|\nabla\sigma^D|\big]\\ &\le chH,\end{aligned}$$

where $h := (1+|\sigma^D|^2)^{(q-2)/4}$. Clearly $|\nabla\sigma| \le chH$, so that (2.12) implies

$$\frac{1}{R}\int_{B_{2R}} |\nabla u - P||\nabla\sigma|\,dx \le \frac{c}{R}\left[\int_{B_{2R}} (Hh)^\gamma\,dx\right]^{\frac{2}{\gamma}}. \quad (2.13)$$

Inserting the estimate (2.13) into (2.11), we find

$$\fint_{B_R} H^2\,dx \le c\left(\left[\fint_{B_{2R}} (Hh)^\gamma\,dx\right]^{2/\gamma} + \fint_{B_{2R}} |\nabla^2 u|\,dx\right). \quad (2.14)$$

Now we want to apply Lemma A.1 from the Appendix by choosing $d = 2/\gamma = 3/2$, $\overline{f} = H^\gamma$, $\overline{g} = h^\gamma$ and $\overline{h} = |\nabla^2 u|^{1/d}$. With these choices (2.14) turns into the inequality (A.1). Since $\nabla^2 u \in L^r_{\mathrm{loc}}(\Omega)$ for any $r < 2$, the assumption (A.2) clearly is satisfied for all $\beta > 0$. It remains to check if $\exp(\beta \overline{g}^d) = \exp(\beta h^2) \in L^1_{\mathrm{loc}}(\Omega)$ is true. To this purpose we recall that by (2.1) $|\sigma^D|^{\frac{q-2}{2}} |\nabla \sigma^D|$ is in $L^2_{\mathrm{loc}}(\Omega)$, hence the function $\phi := \left(1 + |\sigma^D|^2\right)^{q/4}$ is of class $W^{1,2}_{\mathrm{loc}}(\Omega)$, and by Trudinger's inequality (see Theorem 7.17 of [GT]) there exists $\beta_0 > 0$ depending on the $W^{1,2}(B_\rho)$-norm of ϕ such that

$$\int_{B_\rho} \exp(\beta_0 \phi^2)\, dx \le c(\rho) < \infty. \tag{2.15}$$

From (2.15) it follows that for any $\chi \in (0,1)$ and all $\beta > 0$ we have

$$\int_{B_\rho} \exp(\beta \phi^{2-\chi})\, dx \le c(\rho, \chi, \beta) < \infty. \tag{2.16}$$

But $h^2 = \left(1 + |\sigma^D|^2\right)^{\frac{q-2}{2}} = \phi^{2-2/q}$, hence $\exp(\beta \overline{g}^d) \in L^1_{\mathrm{loc}}(\Omega)$ by (2.16), and Lemma A.1 shows that

$$H^2 \log^{c_0 \beta}(e + H) \in L^1_{\mathrm{loc}}(\Omega)$$

for any $\beta > 0$ (note that of course the quantity H^γ in $\log(\dots)$ can be replaced by H), and in conclusion we get that

$$|\nabla \sigma|^2 \log^{c_0 \beta}(e + |\nabla \sigma|) \in L^1_{loc}(\Omega)$$

again for all $\beta > 0$. Now we may quote Example 5.3 of Kauhanen, Koskela and Malý [KKM] to get the continuity of the stress tensor σ. The continuity of the strain tensor then is a consequence of the equation $\varepsilon(u) = A\sigma + \alpha |\sigma^D|^{q-2} \sigma^D$. Alternatively, we may use Lemma A.2 with the result that (choose E as a disc of radius r and apply a scaled version of (A3)):

$$\int_{B_r} H^2\, dx \le K(s) |\log r|^{-s}$$

holds for any $s > 0$ with a constant $K(s)$ depending on s. Thus $\int_{B_r} |\nabla \sigma|^2\, dx \le K(s) |\log r|^{-s}$, and a version of the Dirichlet-growth theorem due to Frehse (see [Fr], p.287) gives $\sigma \in C^0(\Omega)$.

In a last step we are going to prove the Hölder continuity of σ and $\varepsilon(u)$. To this purpose let $B_{2R} \subset \Omega$ and observe that due to the local boundedness of σ the function H locally bounds $|\nabla \sigma|$ from above and from below (compare (2.9)). Therefore (2.10) gives

$$\begin{aligned}\int_{B_R} |\nabla \sigma|^2\, dx \quad &\le \quad c \Bigg[\frac{1}{R} \int_{T_R} |\nabla \sigma| |\nabla u - P|\, dx + \frac{1}{R} \int_{T_R} |\nabla u - P|\, dx \\ &\qquad + \int_{B_{2R}} |\Delta u|\, dx \Bigg],\end{aligned} \tag{2.17}$$

where the integrals over $T_R := B_{2R} - B_R$ result from terms involving the gradient of the cut-off function φ. In (2.17) P is still in our disposal. Hence, choosing $P = \fint_{T_R} \nabla u\, dx$, we find that

$$\frac{1}{R} \int_{T_R} |\nabla u - P|\, dx + \int_{B_{2R}} |\Delta u|\, dx \le c \int_{B_{2R}} |\nabla^2 u|\, dx \le c R^\mu$$

for any $\mu \in (0,1)$ with local constant c depending on μ. For the last estimate we used (2.2) combined with Hölder's inequality. We further have (compare (2.12))

$$\begin{aligned}\frac{1}{R}\int_{T_{2R}} |\nabla u - P||\nabla\sigma|\,dx &\leq \frac{c}{R}\left[\int_{T_{2R}} |\nabla^2 u|^\gamma\,dx\right]^{\frac{1}{\gamma}}\left[\int_{T_{2R}} |\nabla\sigma|^\gamma\,dx\right]^{\frac{1}{\gamma}}\\ &\leq \frac{c}{R}\left[\int_{T_{2R}} |\nabla\varepsilon(u)|^\gamma\,dx\right]^{\frac{1}{\gamma}}\left[\int_{T_{2R}} |\nabla\sigma|^\gamma\,dx\right]^{\frac{1}{\gamma}}\end{aligned}$$

and (by the local boundedness of σ) $|\nabla\varepsilon(u)| \leq c|\nabla\sigma|$, so that

$$\begin{aligned}\frac{1}{R}\int_{T_{2R}} |\nabla u - P||\nabla\sigma|\,dx &\leq \frac{c}{R}\left[\int_{T_{2R}} |\nabla\sigma|^\gamma\,dx\right]^{\frac{2}{\gamma}}\\ &\leq c\int_{T_{2R}} |\nabla\sigma|^2\,dx\end{aligned}$$

by Hölder's inequality. Putting together our estimates and returning to (2.17), it is shown that

$$\int_{B_R} |\nabla\sigma|^2\,dx \leq c\left[\int_{T_{2R}} |\nabla\sigma|^2\,dx + R^\mu\right]. \tag{2.18}$$

Here c stays bounded independent of R and the center x_0 of B_{2R}, provided we consider discs $B_{2R} \subset \Omega'$ for a subregion $\Omega' \Subset \Omega$. Now we use the hole-filling trick of Widman [Wi], i.e. we add $c\int_{B_R} |\nabla\sigma|^2\,dx$ on both sides of (2.18) to get

$$\rho(R) := \int_{B_R} |\nabla\sigma|^2\,dx \leq \Theta\rho(2R) + cR^\mu, \tag{2.19}$$

where $\Theta := c/(c+1) < 1$. Note that (2.19) holds for all radii R such that $R_0 := \operatorname{dist}(x_0, \partial\Omega') \geq 2R$. By induction it follows from (2.19)

$$\rho(2^{-k}R) \leq \Theta^k\rho(R) + cR^\mu\sum_{l=0}^{k-1}\Theta^l\,2^{-(k-l)\mu}$$

for all $k \in \mathbb{N}$ and $R \leq R_0/2$. W.l.o.g. we may assume that $\Theta 2^\mu > 1$. Then

$$\begin{aligned}c\sum_{l=0}^{k-1}\Theta^l 2^{-(k-l)\mu} &= c2^{-k\mu}\sum_{l=0}^{k-1}(\Theta 2^\mu)^l = c2^{-k\mu}\frac{1-\Theta^k 2^{\mu k}}{1-\Theta 2^\mu}\\ &= \frac{c}{\Theta 2^\mu - 1}(\Theta^k - 2^{-k\mu}) \leq \frac{c}{\Theta 2^\mu - 1} =: K,\end{aligned}$$

and we arrive at

$$\rho(2^{-k}R) \leq \Theta^k\rho(R) + KR^\mu. \tag{2.20}$$

By definition K is a local constant depending also on the exponent μ. Finally we let $0 < r < R \leq R_0/2$ and choose $k \in \mathbb{N}$ such that

$$R2^{-k} \leq r \leq R2^{-k+1}.$$

Then (2.20) implies

$$\rho(r) \le \frac{1}{\Theta}\Big(\frac{r}{R}\Big)^{-\frac{\log\Theta}{\log 2}}\rho(R) + KR^{\mu},$$

and if we choose $\mu < \overline{\mu} := -\frac{\log\Theta}{\log 2}$, then [Gi], Lemma 2.1, p.86, shows that $\rho(r)$ grows at most like $r^{\overline{\mu}}$, hence $\sigma \in C^{0,\overline{\mu}/2}(\Omega')$. Recalling $|\nabla\varepsilon(u)| \le c|\nabla\sigma|$ (at least locally) it also follows that $\int_{B_r} |\nabla\varepsilon(u)|^2\,dx \sim r^{\overline{\mu}}$, which completes the proof. □

Appendix. A lemma on the higher integrability of functions

The following result has been established in [BFZ], Lemma 1.2.

LEMMA A.1. *Let $d > 1$, $\beta > 0$ be given numbers. Consider functions $\overline{f}$, $\overline{g}$, $\overline{h}$ from a domain $G \subset \mathbb{R}^n$, $n \ge 2$, being non-negative and satisfying*

$$\overline{f} \in L^d_{\rm loc}(G),\ \ \exp(\beta\overline{g}^d) \in L^1_{\rm loc}(G),\ \ \overline{h} \in L^d_{\rm loc}(G).$$

Suppose further that there is a constant $C > 0$ such that

$$\Bigg[\ \ -\!\!\!\!\!\!\int_{B_R} \overline{f}^d\,dx\Bigg]^{\frac{1}{d}} \le C \ -\!\!\!\!\!\!\int_{B_{2R}} \overline{f}\,\overline{g}\,dx + C\Bigg[\ \ -\!\!\!\!\!\!\int_{B_{2R}} \overline{h}^d\,dx\Bigg]^{\frac{1}{d}} \tag{A.1}$$

holds for all balls $B_{2R} = B_{2R}(x_0) \Subset G$. Then there exists a real number $c_0 = c_0(n,d,C)$ as follows: if

$$\overline{h}^d \log^{c_0\beta}(e + \overline{h}) \in L^1_{\rm loc}(G), \tag{A.2}$$

then the same is true for $\overline{f}$.

It follows from Lemma A.1 (see Corollary 1.3 in [BFZ])

LEMMA A.2. *Suppose that $\overline{f}$, $\overline{g}$, $\overline{h}$ are the same as in Lemma A.1, and that (A.1) is true for all balls $B_{2R} = B_{2R}(x_0) \Subset B_1(0) \subset \mathbb{R}^n$. Suppose also that $\overline{h}^d \log^{c_0\beta}(e + \overline{h}) \in L^1_{\rm loc}(B_1(0))$, where c_0 is as in Lemma A.1. Then*

$$\int_E \overline{f}^d\,dx \le c\log^{-c_0\beta}\Big(e + \frac{1}{\mathcal{L}^n(E)}\Big) \tag{A.3}$$

for all measurable sets $E \subset B_{1/2}(0)$, where the constant c depends only on n, d, C, β, $\overline{f}$, $\overline{g}$ and $\overline{h}$ but not on the set E, and $\mathcal{L}^n(E)$ denotes the n-dimensional Lebesgue measure of the set E.

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