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Abstract

Unstable minimal surfaces are the unstable stationary points of the Dirichlet-Integral. In order to obtain unstable solutions, the method of the gradient flow together with the minimax-principle is generally used. The application of this method for minimal surfaces in the Euclidean spacce was presented in [St1]. We extend this theory for obtaining unstable minimal surfaces in Riemannian manifolds. In particular, we handle minimal surfaces of annulus type, i.e. we prescribe two Jordan curves of class C^3 in a Riemannian manifold and prove the existence of unstable minimal surfaces of annulus type bounded by these curves.

1 Introduction

For given curve $\Gamma_l \subset N, l = 1, ..., m$ and $\Gamma := \Gamma_1 \cup \cdots \cup \Gamma_m$, where (N, h) is a Riemannian manifold with metric $(h_{\alpha\beta})$ of dimension $n \geq 2$, the generalized Plateau Problem, denoted by $\mathcal{P}(\Gamma)$, asks for the minimal surfaces bounding Γ, possessing a following parametrization, defined on $\Sigma \subset \mathbb{R}^2$ with $\partial\Sigma = \Gamma$:

(1) $\tau_h(X) = 0$,

(2) $|X_u|_h^2 - |X_v|_h^2 = \langle X_u, X_v \rangle_h = 0$,

(3) $X|_{\partial\Sigma}$ is weakly monotone onto Γ,

where $\tau_h := \Delta X^\alpha - \Gamma^\alpha_{\beta\gamma} \nabla X^\beta X^\gamma$ is the harmonic equation in (N, h) as the Euler-Lagrange equation of the energy functional.

A regular minimal surface is called unstable if its surface is not a minimum among the neighboring surfaces with the same boundary.

In 1983 ([St1], see also [St2] [St3]), M. Struwe gave an approach to unstable minimal surfaces of disc or annulus type for a given boundary in $\mathbb{R}^n$, extending the Ljusternik-Schnirelmann Theory on convex sets in Banach Spaces. For higher topological structure in $\mathbb{R}^n$, it was studied in [JS].

Recently in [Ho], the existence of unstable minimal surfaces of higher topological structure with one boundary in a nonpositively curved Riemannian manifold was studied by applying the method in [St2]. In particular, in the first part of this paper, the Jacobi field extension operator as the derivative of the harmonic extension was studied.

In this paper, we study unstable minimal surfaces of annulus type in manifolds. The Euclidean case was studied earlier in [St3], and we want to generalize this result to manifolds satisfying some appropriate conditions, namely we will consider two boundary curves Γ_1, Γ_2 in a Riemannian manifold (N, h) such that one of the following holds.

(C1) There exists $p \in N$ with $\Gamma_1, \Gamma_2 \subset B(p,r)$, where $B(p,r)$ lies within the normal range of all of its points. Here we assume $r < \pi/(2\sqrt{\kappa})$, where κ is an upper bound of the sectional curvature of (N,h).

(C2) N is compact with nonpositive sectional curvature.

These conditions are related to the existence and the uniqueness of the harmonic extension for a given boundary parametrization.

First, we construct suitable spaces of functions, the sets of boundary parametrizations, where we have to distinguish the cases of (C1) and (C2). We also introduce a convex set which in fact serves as a tangent space for the given boundary parametrization. And we consider the following functional:

$$\mathcal{E}(x) := \frac{1}{2} \int |d\mathcal{F}(x)|_h^2,$$

where $\mathcal{F}(x)$ denotes the harmonic extension of annulus type or of two discs.

We next discuss the differentiability of $\mathcal{E}$, mainly, the situation of varing topology (from an annulus to two discs). And then, defining critical points of $\mathcal{E}$, we will see the equivalence between the harmonic extensions (in N) of critical points of $\mathcal{E}$ and minimal surfaces in N.

In section 4, we prove the Palais-Smale condition of $\mathcal{E}$. In particular, we will investigate carefully the behavior of boundary mappings which are fixed at only one point. In order to deform level sets of $\mathcal{E}$, we also discuss the construction of a suitable vector field and the corresponding flow.

The property that the energy of some annulus-type harmonic extensions is greater than the energy of two disc-type harmonic extensions with uniform positive constant, is necessary for our aim. In Euclidean spaces, this holds uniformly on any bounded set of boundary parametrizations. In Lemma 4.3, we will generate this result to the case of a Riemannian manifold, however with more restriction than in Euclidean spaces. This somewhat weaker result should be enough to prove our claim.

We can then follow the arguments in the critical point theory as in [St1] and in the main theorem we conclude, if there exists a minimal surface (of annulus type) whose energy is a strict relative minimum in $\mathcal{S}(\Gamma_1, \Gamma_2)$ (suitably defined for each case (C1) and (C2)), the existence of an unstable minimal surfaces of annulus type can be ensured under certain assumptions which are related to the solutions of $\mathcal{P}(\Gamma_i)$.
As corollaries we apply this main result to the three-dimensional sphere S^3 resp. the three-dimensional hyperbolic H^3, where the curvature is 1 resp. -1.

2 Preliminaries

2.1 Some definitions

Let (N, h) be a connected, oriented, complete Riemannian manifold of dimension $n \geq 2$ and embedded isometrically and properly into some $\mathbb{R}^k$ as a closed submanifold by η. And $d\omega$ resp. d_0 denotes the area element in $\Omega \subset \mathbb{R}^2$ resp. in $\partial\Omega$.

For $B := \{w \in \mathbb{R}^2 \mid |w| < 1\}$,

$$H^{1,2} \cap C^0(B, N) := \{f \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | f(B) \subset N\},$$

with norm, $\|f\|_{1,2;0} := \|\nabla f\|_{L^2} + \|X\|_{C^0}$. And

$$T_f H^{1,2} \cap C^0(B, N) \cong \{V \in H^{1,2} \cap C^0(B, \mathbb{R}^k) | V(\cdot) \in T_{f(\cdot)}N\} =: H^{1,2} \cap C^0(B, f^*TN)$$

with norm,

$$\|V\| := \Big(\int_B |\nabla^f V|^2 d\omega\Big)^{\frac{1}{2}} + \|V\|_\infty \cong \Big(\int_B |dV|^2 d\omega\Big)^{\frac{1}{2}} + \|V\|_\infty, \tag{1}$$

where dV means the ordinary gradient in $\mathbb{R}^k$.

Let Γ be a Jordan curve in N which is diffeomorphic to $S^1 := \partial B$, and observe that N can be equipped with another metric $\tilde{h}$ such that Γ is a geodesic in $(N, \tilde{h})$. Note that $H^{1,2} \cap C^0\big((B, \partial B), (N, \Gamma)_{\tilde{h}}\big)$ and $H^{1,2} \cap C^0\big((B, \partial B), (N, \Gamma)_h\big)$ coincide as sets. Using the exponential map in $(N, \tilde{h})$, we let

$$H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) \;\; := \;\; \{u \in H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k) | u(\partial B) = \Gamma\}$$

with norm, $\|u\|_{\frac{1}{2},2;0} := \|\nabla \mathcal{H}(u)\|_{L^2} + \|u\|_{C^0}$, here $\mathcal{H}(u)$ is the harmonic extension in $\mathbb{R}^k$, and

$$\begin{aligned} T_u H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma) \;\; &:= \;\; \{\xi \in H^{\frac{1}{2},2} \cap C^0(\partial B, u^*TN) | \xi(z) \in T_{u(z)}\Gamma, \text{ for all } z \in \partial B\} \\ &= \;\; H^{\frac{1}{2},2} \cap C^0(\partial B, u^*T\Gamma). \end{aligned}$$

Finally, the energy of $f \in H^{1,2}(\Omega, N)$ is denoted by

$$E(f) := \frac{1}{2} \int_\Omega |df|_h^2 dw.$$

2.2 The setting

Let Γ_1, Γ_2 be two Jordan curves of class C^3 in N with diffeomprphisms $\gamma^i : \partial B \to \Gamma_i, i = 1, 2$, and $\mathrm{dist}(\Gamma_1, \Gamma_2) > 0$. Moreover, for $\rho \in (0, 1)$, $A_\rho = \{w \in B \mid \rho < |w| < 1\}$ with boundary $C_1 := \partial B$ and $C_\rho := \partial B_\rho =: C_2$ (ρ, fixed).
And let

$$\mathcal{X}^i_{\text{mon}} := \{x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i) \mid \text{weakly monotone onto } \Gamma_i\}.$$

I) We first consider the following condition for $(N, h)(\supset \Gamma_1, \Gamma_2)$.

(C1) There exists $p \in N$ with $\Gamma_1, \Gamma_2 \subset B(p, r)$, where $B(p, r)$ lies within the normal range of all of its points. Here we assume $r < \pi/(2\sqrt{\kappa})$, where κ is an upper bound of the sectional curvature of (N, h).

In this paper, $B(p, r)$ denotes a geodesic ball of $p \in N$ with the properties in (C1).

We can easily observe the following property (see [Ki]).

Remark 2.1. *If $\Gamma_1, \Gamma_2 \subset N$ satisfy (C1), for each $x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$ and $\rho \in (0, 1)$ there exists $g_\rho \in H^{1,2} \cap C^0(\overline{A_\rho}, B(p, r))$ and $g^i \in H^{1,2} \cap C^0(\overline{B}, B(p, r))$ with $g_\rho|_{C_1} = x^1$, $g_\rho|_{C_\rho}(\cdot) = x^2(\frac{\cdot}{\rho})$ and $g^i|_{\partial B} = x^i$, $i = 1, 2$.*

From the results in [HKW], [JK] and the above Remark, we have a unique harmonic map of annulus and of disc type in $B(p, r) \subset N$ for a given boundary mapping in the class of $H^{\frac{1}{2},2} \cap C^0$. Now we define,

$$M^i := \{x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i) \mid x^i \text{ is weakly monotone, orientation preserving}\}.$$

Then M^i is complete, since the C^0-norm preserves the monotonicity.

Moreover, we define,

$$\begin{aligned} \mathcal{S}(\Gamma_1, \Gamma_2) &= \{X \in H^{1,2} \cap C^0(\overline{A_\rho}, B(p, r)) \mid 0 < \rho < 1,\ X|_{C_i} \text{is weakly monotone}\}, \\ \mathcal{S}(\Gamma_i) &= \{X \in H^{1,2} \cap C^0(\overline{B}, B(p, r)) | X|_{\partial B} \text{ is weakly monotone}\}. \end{aligned}$$

II) We now investigate another alternative condition for (N, h).

(C2) N is compact with nonpositive sectional curvature.

A compact Riemannian manifold is homogeneously regular and the condition of nonpositive sectional curvature for N implies $\pi_2(N) = 0$.

In order to define M^i, we first consider for $\rho \in (0, 1)$,

$$\tilde{G}_\rho := \{f \in H^{1,2} \cap C^0(\overline{A_\rho}, N) \mid f|_{C_i} \text{ is continuous and weakly monotone onto } \Gamma_i\}.$$

We may take a continuous homotopy class, denoted by $\tilde{F}_\rho \subset \tilde{G}_\rho$, so that every two elements f, g in $\tilde{F}_\rho$ are continuous homotopic (not necessarily relative), denoted by $f \sim g$. We further demand some relation $\tilde{F}_\rho \sim \tilde{F}_\sigma$ for any $\rho, \sigma \in (0,1)$, that is, for some $\tilde{f} \in \tilde{F}_\sigma$, $f \in \tilde{F}_\rho$ and some diffeomorphism $\tau_\sigma^\rho : [\sigma, 1] \to [\rho, 1]$, it holds that $\tilde{f}(r,\theta) = f(\tau_\sigma^\rho(r), \theta)$.
Clearly, letting $\tilde{F}_\rho$ fixed as above, for any $\sigma \in (0,1)$, we can find $\tilde{F}_\sigma$ with $\tilde{F}_\rho \sim \tilde{F}_\sigma$.

We now consider all the possible $H^{1,2} \cap C^0$-extensions of disc type in N, as follows:

$$\mathcal{S}(\Gamma_i) := \{X \in H^{1,2} \cap C^0(\overline{B}, N) | X|_{\partial B} \text{ is weakly monotone onto } \Gamma_i\}.$$

And we assume that $\mathcal{S}(\Gamma_i)$ is not empty for each $i = 1, 2$.

Lemma 2.1. *(i) For $X^1 \in \mathcal{S}(\Gamma_1)$ and $X^2 \in \mathcal{S}(\Gamma_2)$, there exists $f_\rho \in H^{1,2} \cap C^0(A_\rho, N)$ such that $f_\rho|_{C_1}(\cdot) = X^1|_{\partial B}(\cdot)$ and $f_\rho|_{C_\rho}(\cdot) = X^2|_{\partial B}(\frac{\cdot}{\rho})$, for $\rho \in (0,1)$.*

(ii) Moreover, there exists $\rho_0 \in (0,1)$ and a uniform positive constant C such that for some $f_\rho \in H^{1,2} \cap C^0(A_\rho, N)$, with $f_\rho|_{C_\rho}(\cdot) = X^2|_{\partial B}(\frac{\cdot}{\rho})$

$$E(f_\rho) \le C, \text{ for all } \rho \le \rho_0. \tag{2}$$

Proof. *(i)* For given $\varepsilon > 0$,take $\sigma_i > 0$ with $\operatorname{osc}_{B_{\sigma_i}} X^i < \varepsilon$.
Choose $\rho > 0$ with $\frac{\rho}{\sigma_2} < \sigma_1$, and let $\mathcal{H} : B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}} \to \mathbb{R}^k$ harmonic with $X^1|_{\partial B_{\sigma_1}} - X^1(0)$ on ∂B_{σ_1} and $X^2|_{\partial B_{\sigma_2}} - X^2(0)$ on $\partial B_{\frac{\rho}{\sigma_2}}$, then $\|\mathcal{H}\|_{C^0} < \varepsilon$.
Now let $g \in H^{1,2} \cap C^0(B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}}, N)$ with $X^1(0)$ on ∂B_{σ_1} and $X^2(0)$ on $\partial B_{\frac{\rho}{\sigma_2}}$. Such a g exists, since N is a (path-)connected Riemannian manifold.

Let $T(s, \theta) := (\frac{1}{s}\rho, \theta)$ in polar coordinates. Using the arguments and notations in the proof of Lemma 2.2, we can define f_ρ with all the desired properties as follows:

$$f_\rho := \begin{cases} X^1|_{B \backslash B_{\sigma_1}} & , \text{ on } B \backslash B_{\sigma_1}, \\ r \circ (g + \mathcal{H}) & , \text{ on } B_{\sigma_1} \backslash B_{\frac{\rho}{\sigma_2}}, \\ X^2(T^{-1}(\cdot)) & , \text{ on } B_{\frac{\rho}{\sigma_2}} \backslash B_\rho. \end{cases} \tag{3}$$

(ii) It follows from the above construction, since $\frac{\rho}{\sigma_2} < \sigma_1$, $\rho \le \rho_0$ for some $\rho_0 > 0$. $\square$

By assumption ($\mathcal{S}(\Gamma_i) \neq \emptyset$), for given $\Gamma_i \in N$ we have an annulus type extension like the above (3), and we take homotopy classes as above which include such an extension. For this setting we use the notations as above without 'tilde'.

Now letting

$$\mathcal{S}(\Gamma_1, \Gamma_2) \ := \ \{f \in F_\rho \,|\, 0 < \rho < 1\}, \tag{4}$$

define

$$\begin{aligned}
M^1 &:= \{f|_{C_1}(\cdot) \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_1) \,|\, \text{orientation preserving}, \;\; f \in \mathcal{S}(\Gamma_1, \Gamma_2)\}, \\
M^2 &:= \{f|_{C_\rho}(\cdot\rho) \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_2) \,|\, \text{orientation preserving}, \;\; f \in \mathcal{S}(\Gamma_1, \Gamma_2)\}.
\end{aligned}$$

For $x^i \in \mathcal{X}^i_{\text{mon}}$, $\mathcal{H}_\rho(x^1, x^2)$ denotes the unique $\mathbb{R}^k$-harmonic extension on A_ρ with $x^1(\cdot)$ on C_1 and $x^2(\frac{\cdot}{\rho})$ on C_ρ. And $\mathcal{H}(\cdot)$ means the $\mathbb{R}^k$-harmonic extension of disc type.

Lemma 2.2. *(i) For each $x^i_0 \in M^i$, $i = 1, 2$, there exists $\varepsilon(x^i_0) > 0$ such that*

$$\text{if } x^i \in \mathcal{X}^i_{mon} \text{ with } \|x^i - x^i_0\|_{\frac{1}{2},2;0} < \varepsilon, \text{ then } x^i \in M^i.$$

(ii) M^i is complete with respect to $\|\cdot\|_{\frac{1}{2},2;0}$.

Proof. *(i)* Let $f_\rho \in \tilde{F}_\rho$ with $f_\rho|_{C_1} = x^1_0$ and $f_\rho|_{C_\rho}(\cdot) = y^2(\frac{\cdot}{\rho})$ for some $y^2 \in M^2$. Considering submanifold coordinate neighbourhoods for $N(\overset{\eta}{\hookrightarrow} \mathbb{R}^k)$, we may take a finite covering of $f_\rho((\overline{A_\rho}))$, and by projection we obtain a smooth map $r : \mathcal{N}_\delta(f_\rho(\overline{A_\rho})) \to N$ with $r|_{\mathcal{N}_\delta(f_\rho(\overline{A_\rho}))\cap N} = Id$ for some $\delta > 0$, where $\mathcal{N}_\delta(\cdot)$ is δ-neighbourhood in $\mathbb{R}^k$.

Then, letting $\|x^i - x^i_0\|_{\frac{1}{2},2;0} < \varepsilon < \delta$, by Lemma 4.2 from [St3],

$$\begin{aligned}
&\int_{A_\rho} |d(r(f_\rho + \mathcal{H}_\rho(x^1 - x^1_0, 0)))|^2 d\omega \\
&\quad \le C(\|f_\rho\|_{C^0}, \varepsilon, N)\Big(\int_{A_\rho} |df_\rho|^2 d\omega + \int_B |d\mathcal{H}(x^1 - x^1_0)|^2 d\omega\Big) \le C(\|f_\rho\|_{1,2;0}, \varepsilon, N).
\end{aligned}$$

Now, consider $H(t, \cdot) := (1-t)\mathcal{H}_\rho(x^1 - x^1_0, 0) : [0,1] \times A_\rho \to \mathbb{R}^k$ with $\|H\|_{C^0} < \varepsilon$ and $G : [0,1] \times A_\rho \to N$ with $G(t, \cdot) = f_\rho(\cdot)$ for all $t \in [0,1]$.
Then $r(G+H) : [0,1] \times A_\rho \to N$ is a homotopy between f_ρ and $r(f_\rho + \mathcal{H}_\rho(x^1 - x^1_0, 0))$.
Hence $r(f_\rho + \mathcal{H}_\rho(x^1 - x^1_0, 0))(\sim f_\rho) \in \tilde{F}_\rho$, and $x^1 \in M^1$.
Similarly, we can prove that $x^2 \in M^2$ if $\|x^2 - x^2_0\|_{\frac{1}{2},2;0} < \varepsilon'$ for some small $\varepsilon' > 0$.

(ii) A cauchy sequence $\{x^i_n\} \subset M^i$ converges to $x^i \in H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$, and for some n, $\|x^i_n - x^i\|_{C^0} < \varepsilon$. Considering $\mathcal{H}_\rho(x^1 - x^1_n, 0)$ and $g_\rho \in F_\rho$ with boundary x^1_n on C_1 and 0 on the other boundary, we can find a homotopy in N between g_ρ and $r(g_\rho + \mathcal{H}_\rho(x^1 - x^1_0, 0))$ as in *(i)*. We may also apply this argument for x^2.
Note that x^i is weakly monotone, and hence $x^i \in M^i$. □

From the proof of Lemma 2.2, we easily observe the following: The set of x^i's which possess annulus type extensions with uniform energy with respect to $\rho \le \rho_0$ is an open and subset of $\mathcal{X}^i_{\text{mon}}$. Thus, this is a non empty connceted component of $\mathcal{X}^i_{\text{mon}}$ and must be the same as M^i, since M^i is a connected subset of $\mathcal{X}^i_{\text{mon}}$. Hence we obtain the following property.

Remark 2.2. *For each $x^i \in M^i, i = 1,2$, there exist $f_\rho \in \mathcal{S}(\Gamma_1, \Gamma_2)$ and $C > 0$ with $E(f_\rho) \leq C$ for all $\rho \leq \rho_0$ for some $\rho_0 \in (0,1)$. Clearly, this result also holds for $x^i \in M^i$ in the case of (C1).*

For the disc-type extensions for $x^i \in M^i$ the following Lemmata will be used.

Lemma 2.3. *Let (N,h) be a homogeneously regular manifold and u an absolutely continuous map on $\partial B_r(x_0)$ into $N \ni x_0$ with $\int_0^{2\pi} |u'(\theta)|_h^2 d\theta \leq \frac{C'}{\pi}$. Then there exists $f \in H^{1,2}(B_r(x_0), N) \cap C^0(\overline{B_r(x_0)}, N)$ with $f|_{\partial B_r(x_0)} = u$ and $E_{B_r(x_0)}(f) \leq \frac{C''}{C'} \int_0^{2\pi} |u'(\theta)|_h^2 d\theta$, where C'', C' are the constants from the homogeneously regularity.*

Proof. See [Mo] Lemma 9.4.8 b). □

Lemma 2.4 (From the Courant-Lebesgue Lemma). *Let $f_\rho \in H^{1,2}(A_\rho, N)$, $0 < \rho < 1$. For each $\delta \in (\rho, 1)$ there exits $\tau \in (\delta, \sqrt{\delta})$ with $\int_0^{2\pi} \left| \frac{\partial f_\rho(\tau,\theta)}{\partial \theta} \right|_h^2 d\theta \leq \frac{4E(f_\rho)}{\ln \frac{1}{\delta}}$.* □

For $x^i \in M^i$, from Remark 2.2 and the choice of $\mathcal{S}(\Gamma_1, \Gamma_2)$, we can find $f_\rho \in H^{1,2}(A_\rho, N)$ with boundary x^i such that $E(f_\rho) \leq C$ for all $\rho \leq \rho_0$. Then from Lemma 2.4 and Lemma 2.3, we have $g_\tau \in H^{1,2}(B_\tau, N)$ with boundary $f_\rho|_{\partial B_\tau}$ for some ρ. Together with g_τ and $f_\rho|_{B \setminus B_\tau}$, we obtain a map $X \in H^{1,2}(B,N)$ with boundary x^1. Similarly, we have $\tilde{X} \in H^{1,2}(B,N)$ with boundary x^2.

Moreover, the harmonic extension of disc type for each $x^i \in M^i$ in N is unique, independently of the choice of a homotopy class $\mathcal{S}(\Gamma_1, \Gamma_2)$, because of the following well-known fact.

Lemma 2.5. $\pi_2(N) = 0 \Leftrightarrow$ *Any $h_0, h_1 \in C^0(B,N)$ with $h_0|_{\partial B} = h_1|_{\partial B}$ are homotopic.*

On the other hand, using the construction (3) and by the above Lemma we can easily check that the traces of elements in $\mathcal{S}(\Gamma_i)$ are included in M^i. From [ES], [Le], [Hm], we have the following results.

Remark 2.3. *(i) For $x^i \in M^i$, there exists a unique harminic extension of disc type and of annulus type defined on A_ρ for each $\rho \in (0,1)$.*

(ii) The elements of M^i are actually the traces of $f \in \mathcal{S}(\Gamma_i)$.

III) Now let (N,h) and $\Gamma_i, i = 1,2$ satisfy (C1) or (C2).

Observing that $\partial B \cong \mathbb{R}/2\pi$, for a given oriented $y^i \in \mathcal{X}^i_{\text{mon}}$, there exists a weakly monotone map $w^i \in C^0(\mathbb{R}, \mathbb{R})$ with $w^i(\theta + 2\pi) = w^i(\theta) + 2\pi$ such that $y^i(\theta) = \gamma^i(\cos(w^i(\theta)), \sin(w^i(\theta))) =: \gamma^i \circ w^i(\theta)$. And $w^i = \tilde{w}^i + Id$ for some $\tilde{w}^i \in C^0(\partial B, \mathbb{R})$.

Denoting the Dirichlet -Integral by D and the $\mathbb{R}^k$-harmonic extension by $\mathcal{H}$, let

$$W^i_{\mathbb{R}^k} := \{w^i \in C^0(\mathbb{R}, \mathbb{R}) \mid \text{weakly monotone}, w^i(\theta + 2\pi) = w^i(\theta) + 2\pi; D(\mathcal{H}(\gamma^i \circ w^i)) < \infty\}.$$

Clearly, $W^i_{\mathbb{R}^k}$ is convex. For further details, we refer to [St1].

For $x^i \in M^i$, considering $w - w^i$ as a tangent vector along $\tilde{w}^i$, let

$$\mathcal{T}_{x^i} = \{d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w}^i) \mid w \in W^i_{\mathbb{R}^k} \text{ and } \gamma^i \circ w^i = x^i\}.$$

And $\mathcal{T}_{x^i}$ is convex in $T_{x^i}H^{\frac{1}{2},2} \cap C^0(\partial B; \Gamma_i)$, since $W^i_{\mathbb{R}^k}$ is convex. Letting $\widetilde{\exp}$ be the exponential map with respect to the metric $\tilde{h}$, we note that $\widetilde{\exp}_{x^i}\xi = \gamma^i(w)$, for $\xi = d\gamma^i((w - w^i)\frac{d}{d\theta} \circ \tilde{w}^i) \in \mathcal{T}_{x^i}$, since γ^i is geodesic in $(N, \tilde{h})$.

In case of (C1), clearly $\widetilde{\exp}_{x^i}\xi \in M^i$ for $\xi \in \mathcal{T}_{x^i}$.
For the case (C2), let us recall the proof of Lemma 2.2. Since N is compact, there exists $l_i > 0$, depending on γ^i, such that for any $x^i \in M^i$, $\widetilde{\exp}_{x^i}\xi \in M^i$, if $\|\xi\|_{\mathcal{T}_{x^i}} < l_i$.

Definition Now we define the following setting for both (C1) and (C2).

(i) With the product topology let $\mathcal{M} := M^1 \times M^2 \times (0, 1)$ and $x := (x^1, x^2, \rho) \in \mathcal{M}$ with a convex set $\mathcal{T}_x\mathcal{M} = \mathcal{T}_{x^1} \times \mathcal{T}_{x^2} \times \mathbb{R}$.

Letting $\mathcal{F}(x) = \mathcal{F}(x^1, x^2, \rho) = \mathcal{F}_\rho(x^1, x^2) : A_\rho \to N$ be the unique harmonic extension with boundary x^1 on C^1 and $x^2(\frac{\cdot}{\rho})$ on C^2, define $\mathcal{E} : \mathcal{M} \longrightarrow \mathbb{R}$ with

$$x \longmapsto E(\mathcal{F}(x)) := \frac{1}{2}\int_{A_\rho} |d\mathcal{F}_\rho(x^1, x^2)|^2_h d\omega.$$

(ii) Moreover, $\partial\mathcal{M} := M^1 \times M^2 \times \{0\} \ni x := (x^1, x^2, 0)$, with $\mathcal{T}_x\partial\mathcal{M} := \mathcal{T}_{x^1} \times \mathcal{T}_{x^2}$. And $\overline{\mathcal{M}} := \mathcal{M} \cup \partial\mathcal{M}$.

Letting $\mathcal{F}^i(x^i) : A_\rho \to N$ be the unique harmonic extension with boundary x^i, for $x = (x^1, x^2, 0) \in \partial\mathcal{M}$, let $\mathcal{E}(x) := E(\mathcal{F}^1(x^1)) + E(\mathcal{F}^2(x^2))$.

2.3 Harmonic extension operators

Let $M = A_\rho$ or $M = B$. A weak Jacobi field $\mathbf{J}$ with boundary ξ along a harmonic function f is a weak solution of

$$\int_M \langle \nabla\mathbf{J}, \nabla X\rangle + \langle tr\, R(\mathbf{J}, df)df, X\rangle d\omega = 0,$$

for all X $\in H^{1,2}(M, f^*TN)$ with $X|_{\partial M} = \xi$. And this is a natural candidate of derivative of harmonic operators $\mathcal{F}_\rho$ and $\mathcal{F}^i$.

We have the following property of the weak Jacobi fields from the arguments in [Ho].

Lemma 2.6. *The above weak Jacobi field with boundary $\eta \in T_{x^i}H^{\frac{1}{2},2} \cap C^0$ along a harmonic $\mathcal{F}$ with boundary x^i is well defined in the class $H^{1,2}$ and continuous until the boundary with*

$$\|\mathbf{J}_{\mathcal{F}}\|_0 \leq \|\mathbf{J}_{\mathcal{F}}|_{\partial M}\|_0, \quad \|\mathbf{J}_{\mathcal{F}}\|_{1,2;0} \leq C(N, \|f\|_{1,2:0})\|\mathbf{J}_{\mathcal{F}}|_{\partial M}\|_{\frac{1}{2},2;0},$$

Now we can talk about the differentiabilty of the harmonic extension operators.

Lemma 2.7. *The operators $\mathcal{F}_\rho, \mathcal{F}^i$ are partially differentiable in x^1, x^2 with respect to variations in $T_{x^1}H^{\frac{1}{2},2} \cap C^0$ resp. $T_{x^2}H^{\frac{1}{2},2} \cap C^0$, and the derivatives are the Jacobi field operators which are also continuous with respect to x^1, x^2.*

Proof. It can be proved with a similar argument to the proof of Lemma 3.1 (B), (C). We can also use the proof in [Ho]. □

3 The variational problem

3.1 Differentiability of $\mathcal{E}$ on $\overline{\mathcal{M}}$

Lemma 3.1. *We have,*

(A) $\mathcal{E}$ is continuously partially differentiable in x^1, x^2 with respect to variations in $\mathcal{T}_{x^1}$, $\mathcal{T}_{x^2}$ and the derivatives are continuous in $M^1 \times M^2$,

(B) $\mathcal{E}$ is continuous with respect to $\rho \in [0,1)$, uniformly on $\mathcal{N}_\varepsilon(x_0^i)$ for some $\varepsilon > 0$ which is independent of $x_0^i \in M^i$, $i = 1,2$,

(C) and the partial derivatives in x^1, x^2 are also continuous with respect to $\rho \in [0,1)$, uniformly on $\mathcal{N}_\varepsilon(x_0^i)$ for some $\varepsilon > 0$, independent of $x_0^i \in M^i$, $i = 1,2$,

(D) $\mathcal{E}$ is differentiable with respect to $\rho \in (0,1)$.

Proof. Here and in the sequel, the continuity will be understood in the sense of subsequence.

(A) The Dirichlet-Integral functional is in C^∞, so by Lemma 2.7 $\mathcal{E}$ is continuously partially differentiable with continuous partial derivatives on $M^1 \times M^2$.

Computation of derivatives:

Let $x = (x^1, x^2, \rho) \in \mathcal{M}$, $\xi^1 \in \mathcal{T}_{x^1}$. By Lemma 2.2, $\widetilde{\exp}_{x^1}(t\xi^1) \in M^1$, $0 \leq t \leq t_0$ for some small $t_0 > 0$. Thus,

$$\begin{aligned} \langle \delta_{x^1}\mathcal{E}, \xi^1 \rangle &:= \frac{d}{dt}\Big|_{t=0} \mathcal{E}(\widetilde{\exp}_{x^1}(t\xi^1), x^2, \rho) \\ &= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla D_{x^1}\mathcal{F}_\rho(x^1,x^2)(\xi^1)\rangle_h d\omega \\ &= \int_{A_\rho} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)\rangle_h d\omega \qquad \text{(by Lemma 2.7)}, \end{aligned} \tag{5}$$

since by computation we obtain, with $\mathcal{F}_\rho(t) := \mathcal{F}_\rho\left(\widetilde{\exp}_{x^1}(t\xi^1), x^2\right)$

$$\nabla_{\frac{d}{dt}}\left(\mathcal{F}^\alpha_{\rho,i}(t)dx^i \otimes \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}_\rho(t)\right) = \nabla \frac{d}{dt}\mathcal{F}_\rho\left(\widetilde{\exp}_{x^1}(t\xi^1), x^2\right) \left(= \nabla\left(D_{x^1}\mathcal{F}_\rho(x^1,x^2)(\xi^1)\right), t=0\right).$$

And for $\xi^2 \in \mathcal{T}_{x^2}$ by Lemma 2.7, $\langle \delta_{x^2}\mathcal{E}, \xi^2\rangle = \int_{A_\rho}\langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(0, \xi^2(\frac{\cdot}{\rho}))\rangle_h d\omega$.
Similarly, for $x = (x^1, x^2, 0) \in \partial\mathcal{M}$, $\langle \delta_{x^i}\mathcal{E}, \xi^i\rangle = \int_B \langle d\mathcal{F}^i(x^i), \nabla \mathbf{J}_{\mathcal{F}^i}(\xi^i)\rangle_h d\omega, i = 1, 2$.

(B) The continuity of $\mathcal{E}$ as $\rho \to \rho_0$ is now to prove. We discuss only the case, $\rho_0 = 0$, i.e.:

$$(6) \qquad \int_{A_\rho} \left|d\mathcal{F}_\rho(x^1,x^2)\right|_h^2 d\omega \longrightarrow \int_B \left|d\mathcal{F}^1(x^1)\right|_h^2 d\omega + \int_B \left|d\mathcal{F}^2(x^2)\right|_h^2 d\omega, \quad \rho \to 0$$

uniformly on $\mathcal{N}_\varepsilon(x_0^i)$ for some $\varepsilon > 0$ which is independent of $x_0^i \in M^i$.

We will prove the above assertion in several steps. The proof for the case $\rho_0 \in (0,1)$ is similar and somewhat easier.

Let $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2)$ and $\mathcal{F}^i := \mathcal{F}^i(x^i), i = 1, 2$.
$\mathcal{F}_\rho \in H^{1,2}(A_\rho, N)$, so by Lemma 2.4 for each δ with $0 < \rho < \delta < 1$, there exists $\nu \in (\delta, \sqrt{\delta})$ such that

$$(7) \qquad \int_0^{2\pi}\left|\frac{\partial \mathcal{F}_\rho(\nu,\theta)}{\partial\theta}\right|_h d\theta \le \sqrt{2\pi}\left(\int_0^{2\pi}\left|\frac{\partial \mathcal{F}_\rho(\nu,\theta)}{\partial\theta}\right|_h^2 d\theta\right)^{\frac{1}{2}} \le \frac{C}{\sqrt{|\ln\delta|}},$$

where C is independent of $\rho \le \rho_0$ for some $\rho_0 \in (0,1)$ from Remark 2.2.

We now construct two mappings from $\mathcal{F}_\rho$ by letting

$$(8) \qquad \begin{array}{llll} f_\nu : A_\nu \longrightarrow N & \text{with} & f_\nu(re^{i\theta}) := \mathcal{F}_\rho(re^{i\theta}), & re^{i\theta} \in A_\nu, \\ g_{\nu'} : A_{\nu'} \longrightarrow N & \text{with} & g_{\nu'}(re^{i\theta}) := \mathcal{F}_\rho(T(re^{i\theta})), & re^{i\theta} \in A_{\nu'}, \end{array}$$

where $\nu' := \frac{\rho}{\nu}$, $\nu \in (\delta, \sqrt{\delta})$ and $\delta \in (\rho, 1)$ satisfying the property (7) with $\nu', \nu \to 0$ as $\rho \to 0$ (e.g. $\delta := \sqrt{\rho}$). And $T(re^{i\theta}) = (\frac{\rho}{r}e^{i\theta})$, from $A_{\nu'}$ onto $B_\nu \backslash B_\rho$.
Then, f_ν and $g_{\nu'}$ are harmonic map into N with $f_\nu|_{\partial B} = x^1$, $g_{\nu'}|_{\partial B} = x^2$ and $\mathrm{osc}_{\partial B_\nu} f_\nu \to 0$, $\mathrm{osc}_{\partial B_{\nu'}} g_{\nu'} \to 0$ as $\rho \to 0$. Moreover, since T is conformal, by the conformal invariance of the Dirichlet-Integral, $E(\mathcal{F}_\rho) = E(\mathcal{F}_\rho|_{A_\nu}) + E(\mathcal{F}_\rho|_{B_\nu\backslash B_\rho}) = E(f_\nu) + E(g_{\nu'})$.

B-I) The convergence of $\{f_\nu\}$, $\{g_{\nu'}\}$ to $\mathcal{F}^i$.

We first investigate the modulus of continuity of harmonic maps $\{h_\nu\}$, defined on A_ν into N, which converge uniformly (C^0-norm) on ∂B with $E(h_\nu) \le L$ for some $L > 0$, independent of $\nu \le \nu_0$ for some $\nu_0 \in (0,1)$. We will discuss only the case (C2), because

the argument in the case of (C2) can clearly be applied to the case (C1);
Let $G_R := \overline{B_R(z)} \subset A_\nu$ for $\nu \leq \tilde{\nu_0}$. If $z \in \partial B$, consider $G_R := \overline{B_R(z)} \cap \overline{A_\nu}$.
Given $\varepsilon > 0$, by the Courant-Lebesgue Lemma, there exists $\delta > 0$, independent of $\nu \leq \nu_0$, such that Length of $h_\nu|_{\partial G_\delta} \leq \min\{\frac{\varepsilon}{4}, \frac{i(N)}{4}\}$.
By assumption, $i(N) > 0$. Then $h_\nu|_{\partial G_\delta} \subset B(q,s)$ for some $q \in N, s \leq \min\{\frac{\varepsilon}{2}, \frac{i(N)}{2}\}$.
We observe, h_ν is continuous on ∂G_δ, and there exists an $H^{1,2}$-extension of disc type X, whose image is in $B(q,s)$ with $X|_{\partial B_\delta} = h_\nu|_{\partial B_\delta}$ from the same argument as in the proof of Remark 2.1. Thus, by the result in [HKW], there exists a harmonic extension h' with $h'(G_\delta) \subset B(q,s) \subset B(q, \frac{\varepsilon}{2})$. From Lemma 2.5, h' is homotopic to h on G_δ, and from the energy minimizing property of harmonic maps, $h_\nu|_{G_\delta} = h'$. Hence, the functions h_ν with $\nu \leq \nu_0$ have the same modulus of continuity.
Moreover, if these mappings are with the same boundary image, the mappings are C^0-uniform bounded on each relative compact domain.

Now we apply the above result to our maps $\{\mathcal{F}_\rho, \rho \leq \rho_0\}$ in $\mathbb{R}^k$. Then, for some $\rho_0 \in (0,1)$, the functions f_ν resp. $g_{\nu'}$ have the same modulus of continuity for all $\rho \in (0, \rho_0)$, and some subsequences denoted again by f_ν $g_{\nu'}$ are locally uniform convergent.

Recall that our mappings are continuous, so by localizing in domain and image, harmonic functions as the solutions of Dirichlet Problems may be regarded as weak solutions f of the following elliptic systems in local coordinate chart of N:

$$\nabla_i \nabla_i f^\alpha = -\Gamma^\alpha_{\beta\gamma} \nabla_i f^\beta \nabla_i f^\gamma =: G^\alpha(\cdot, f(\cdot), \nabla f(\cdot)). \tag{9}$$

Letting $\nu_0 := \nu(\rho_0)$, $\nu_0' := \nu'(\rho_0)$, we can assume the same coordinate charts for the image of $\{f_\nu\}_{\nu \leq \nu_0}$ and $\{g_{\nu'}\}_{\nu' \leq \nu_0'}$, hence the same weak solution system for (9).
Moreover, since $h_{\alpha\beta}$ and $\Gamma^\alpha_{\beta\gamma}$ of N are smooth, all the structural constants of the weak systems (see [Jo] section 8.5) are independent of $\rho \leq \rho_0$.

Now consider $K^\sigma_\sigma = \{\sigma \leq |z| \leq 1 - \sigma\}$, $\sigma \in (0,1)$. From the regularity theory by [LU] and [Jo](see [Jo] section 8.5) and by the covering argument, there exists $C \in \mathbb{R}$ such that $\left\| f_\nu|_{K^\sigma_\sigma} \right\|_{H^{4,2}} \leq C$ for all $\nu \in (0, \nu_0)$. Hence by the Sobolev's embedding Theorem, for some sequence $\{\rho_i\} \subset (0,1)$, $\lim_{\rho_i \to 0} f_{\nu(\rho_i)}|_{K^\sigma_\sigma} = f'$ in $C^2(K^\sigma_\sigma, \mathbb{R}^n)$, with $\tau_h(f') = 0$ in K^σ_σ.

Now letting $\sigma := \frac{1}{n}$, choose sequence $\{f_{\nu(\rho_{n,i})}\}$ as above such that $\{\rho_{n+1,i}\}$ is a subsequence of $\{\rho_{n,i}\}$. Then by diagonalizing we have a subsequence $\{f_{\nu(\rho_{n,n})}\}, n \geq n_0$ which converges to f' locally with C^2-norm, so f' is harmonic on $B \backslash (\partial B \cup \{0\})$.

On the other hand $f_\nu|_{\partial B} = x^1$ for all ν, and f_ν converge uniformly to f' in a compact neighborhood of ∂B. Thus, f' is continuous on $\overline{B} \backslash \{0\}$ with $f'|_{\partial B} = x^1$.
We also observe that from the construction, $\operatorname{osc}_{\partial B_r} f' \to 0$ as $r \to 0$.

Moreover, for each compact $K \subset B \backslash \{0\}$, $\int_K |df'|^2 d\omega = \lim_{\rho_i \to 0} \int_K |df_{\nu(\rho_i)}|^2 \leq L$ with L, independent of K. Thus, $f' \in H^{1,2}(B \backslash \{0\}, N)$, and f' can be extended on B as a

weakly harmonic map from Lemma [Jo] Lemma 8.4.5.(see also [SkU], [Grü]).
Thus, f' can be considerd as a weakly harmonic and $f' \in C^0(\overline{B}, N) \cap C^2(B, N)$ with $f'|_{\partial B} = x^1$, and from the uniqueness property we obtain, $f' = \mathcal{F}^1(x^1)$.
We have the similar result for $g_{\nu'}$.

B-II) The convergence of energy.

Consider $\eta \circ f$, denoted again by $f := (f^a)_{a=1,\cdots,k} \in H^{1,2}(M, \mathbb{R}^k)$. Since $\eta : N \to \mathbb{R}^k$ is isometric, for $f := (f^\alpha)_{\alpha=1,\cdots,n} \in H^{1,2}(M, N)$, $\int_M |d(f^\alpha)|_h^2 d\omega = \int_M |d(f^a)|_{\mathbb{R}^k}^2 d\omega$.

Note also that for a harmonic $f \in H^{1,2}(M, N)$ with $M \subset \mathbb{R}^2$, bounded,

$$\int_M (\langle df, d\psi\rangle - \langle II \circ f(df, df), \psi\rangle) dM = 0, \tag{10}$$

for any $\psi \in H_0^{1,2} \cap C^0(M, \mathbb{R}^k)$, where II is the second fundamental form from η.

Letting $K_\sigma = \{\sigma \le |z| \le 1\}, \sigma > 0$, for $\nu \in (0, \sigma)$ we consider $\mathbb{R}^k$-harmonic maps H_ν and $\widetilde{H}_\nu$ on K_σ with $H_\nu|_{\partial K_\sigma} = f_\nu|_{\partial K_\sigma}$ and $\widetilde{H}_\nu|_{\partial K_\sigma} = \mathcal{F}^1|_{\partial K_\sigma}$.
Also let $H : B \to \mathbb{R}^k$ be the harmonic map with $H|_{\partial B} = H_\nu|_{\partial B} = \widetilde{H}_\nu|_{\partial B} = x^1$, then both of $\{H_\nu\}, \{\widetilde{H}_\nu\}$ have the same modulus of continuity until ∂B, and $\|H_\nu - H\|_{C^0; K_\sigma} \to 0, \quad \|\widetilde{H}_\nu - H\|_{C^0; K_\sigma} \to 0 \quad \text{as } \nu \to 0$.
Let $X_\nu := (f_\nu - \mathcal{F}^1) + (H_\nu - \widetilde{H}_\nu) \in H_0^{1,2} \cap C^0(K_\sigma, \mathbb{R}^k)$, and then we have that

$$\|X_\nu\|_{(C^0; K_\sigma)} \le \|f_\nu - \mathcal{F}^1\|_{C^0; K_\sigma} + \|H_\nu - H\|_{C^0; K_\sigma} + \|H - \widetilde{H}_\nu\|_{C^0; K_\sigma} \to 0 \quad \text{as } \nu \to 0.$$

Now we compute

$$\begin{aligned} &\int_{K_\sigma} \langle d(f_\nu - \mathcal{F}^1), d(f_\nu - \mathcal{F}^1)\rangle d\omega \\ &\qquad = \underbrace{\int_{K_\sigma} \langle d(f_\nu - \mathcal{F}^1), dX_\nu\rangle d\omega}_{:=I} - \underbrace{\int_{K_\sigma} \langle d(f_\nu - \mathcal{F}^1), d(H_\nu - \widetilde{H}_\nu)\rangle d\omega}_{:=II}. \end{aligned}$$

Then, from (10), as $\nu \to 0$,

$$\begin{aligned} |I| &\le \left| \int_{K_\sigma} \langle II \circ f_\nu(df_\nu, df_\nu), X_\nu\rangle d\omega \right| + \left| \int_{K_\sigma} \langle II \circ (d\mathcal{F}^1, d\mathcal{F}^1), X_\nu\rangle d\omega \right| \\ &= C\|X_\nu\|_{(C^0; K_\sigma)} \to 0. \end{aligned} \tag{11}$$

Moreover, since $H_\nu - \widetilde{H}_\nu$ is harmonic in $\mathbb{R}^k$,

$$|II| \le \int_{\partial K_\sigma} \left| \partial_r (H_\nu - \widetilde{H}_\nu) \right| d\omega \|f_\nu - \mathcal{F}^1\|_{C^0; K_\sigma} \to 0 \quad \text{as } \nu \to 0. \tag{12}$$

Thus, $\int_{K_\sigma} |d(f_\nu - \mathcal{F}^1)|^2 d\omega \to 0$, and $\int_{K_\sigma} |df_\nu|^2 \, d\omega \to \int_{K_\sigma} |d\mathcal{F}^1|^2 \, d\omega$, for any K_σ. Since $\int_{B_\sigma} |d\mathcal{F}^1|^2 d\omega \to 0$ as $\sigma \to 0$, we obtain, $\int_{A_\nu} |df_\nu|^2 \, d\omega \to \int_B |d\mathcal{F}^1|^2 \, d\omega$ as $\nu \to 0$. Similarly, it holds that $\int_{A_{\nu'}} |dg'_\nu|^2 \, d\omega \to \int_B |d\mathcal{F}^2|^2 \, d\omega \quad$ as $\nu' \to 0$.

Now for the uniform convergence on $\mathcal{N}_\varepsilon(x_0^i)$, we recall the proof of Lemma 2.2 and replace $f(\overline{A_\rho})$ by $\overline{B(p,r)}$(for (C1)) and N (compact in (C2)). Then, $\|\mathcal{F}_\rho(x^1,x^2)\|_{H^{1,2}} \leq C$, uniformly on $\mathcal{N}_\varepsilon(x_0^i)$, where the constant C is dependent on x_0^i, but ε is independent of x_0^i. And the convergence in (11), (12) is uniformly on $\mathcal{N}_\varepsilon(x_0^i)$. The proof of *(B)* is completed.

(C) We must show that for $x^i \in M^i$ and $\xi^i \in \mathcal{T}_{x^i}$,

$$\langle \delta_{x^i} \mathcal{E}_\rho, \xi^i \rangle \quad \longrightarrow \quad \langle \delta_{x^i} \mathcal{E}, \xi^i \rangle \quad \text{as } \rho \to 0, \text{ uniformly on } \mathcal{N}_\varepsilon(x_0^i) \subset M^i, i = 1, 2.$$

It suffices to show the above assertion for $i = 1$. We know that

$$\begin{aligned} \langle \delta_{x^1} \mathcal{E}_\rho, \xi^1 \rangle &= \int_{A_{\nu(\rho)}} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0) \rangle_h d\omega + \int_{B_{\nu(\rho)} \setminus B_\rho} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0) \rangle_h d\omega \\ &= \int_{A_{\nu(\rho)}} \langle d\mathcal{F}_\rho(x^1,x^2), \nabla \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0) \rangle_h d\omega + \int_{A_{\nu'}} \langle dg_{\nu'}, \nabla \mathbf{J}_{g_{\nu'}}(0, \zeta_{\nu'}) \rangle d\omega, \end{aligned}$$

where $g_{\nu'}(\cdot) = \mathcal{F}_\rho \circ T(\cdot)$ and $\zeta_{\nu'}(\nu' e^{i\theta}) = \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)(\nu e^{i\theta})$ with $\nu' := \frac{\rho}{\nu(\rho)}$.
We observe that $\mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0) \circ T$ is a Jacobi-Field along $g_{\nu'}$ by the conformal property of T.

The proof is divided into several steps.

C-I) The convergence of Jacobi fields.

First, letting $V_\nu := \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)|_{A_\nu} = v_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ f_\nu$, we show the existence of ν_0 with

$$\text{(13)} \qquad \|DV_\nu\|_2^2 := \int_{A_\nu} h_{\alpha\beta} \circ f_\nu v_{\nu,i}^\alpha v_{\nu,i}^\beta d\omega \leq C \text{ for all } \nu \in (0, \nu_0), \;\; \nu_0 \in (0,1).$$

By computation, $\|DV_\nu\|_2^2 \leq CE(V_\nu) + C(N, \|V_\nu\|_0, \|f_\nu\|_0, E(f_\nu))$, and by Lemma 2.6, $\|V_\nu\|_{C^0} \leq \|\xi_\nu^1\|_{C^0}$, so we need to show only that

$$\text{(14)} \qquad E(V_\nu) := \int_{A_\nu} \left| \nabla^{f_\nu} V_\nu \right|^2 d\omega \leq C, \quad \nu \in (0, \nu_0).$$

Let $X_\nu := x_\nu^\alpha \frac{\partial}{\partial y^\alpha} \circ f_\nu \in H^{1,2}(A_\nu, f_\nu^* TN)$, where $x_\nu^\alpha(z) := v_{2\nu_0}^\alpha(\tau_{\nu_0}^{2\nu_0}(z))$, $\nu_0 \leq |z| \leq 1$ (see section 2.2 for the definition of $\tau_{\nu_0}^{2\nu_0}$) and $x_\nu^\alpha(z) := 0$, $\nu \leq |z| \leq \nu_0$. Clearly, $\|DX_\nu\|_2^2 \leq C(\nu_0, N) \|DV_{2\nu_0}\|_2^2$ for all $\nu \leq \nu_0$.

By the minimality property of Jacobi-Field and from the Young's inequality,

$$
\begin{aligned}
&\int_{A_\nu}(\left|\nabla^{f_\nu}(V_\nu)\right|^2-\langle trR(df_\nu,V_\nu)df_\nu,V_\nu\rangle)d\omega \le \int_{A_\nu}(\left|\nabla^{f_\nu}(X_\nu)\right|^2-\langle trR(df_\nu,X_\nu)df_\nu,X_\nu\rangle)d\omega \\
\le\;& \int_{A_\nu} h_{\alpha\beta}\circ f_\nu x^\alpha_{\nu,i}x^\beta_{\nu,i}d\omega+\varepsilon\int_{A_\nu}|x^\alpha_{\nu,i}\frac{\partial}{\partial y^\alpha}\circ f_\nu|^2_h d\omega+\varepsilon^{-1}\int_{A_\nu}|x^\gamma_\nu f^\delta_{,i}\Gamma^\beta_{\gamma\delta}\circ f_\nu\frac{\partial}{\partial y^\beta}\circ f_\nu|^2_h d\omega \\
&+\int_{A_\nu} h_{\alpha\beta}\circ f_\nu x^\gamma_\nu x^\lambda_\nu f^\delta_{,i}f^\mu_{,i}\Gamma^\alpha_{\gamma\delta}\circ f_\nu\Gamma^\beta_{\lambda\mu}\circ f_\nu d\omega-\int_{A_\nu}\langle trR(df_\nu,X_\nu)df_\nu,X_\nu\rangle d\omega \\
\le\;& C(N,\varepsilon,\|f_\nu\|_0,E(f_\nu),\|V_{2\nu_0}\|_0,\|DV_{2\nu_0}\|_2^2).
\end{aligned}
$$

And $E(V_\nu)\le C,\ \nu\in(0,\nu_0)$, since $\int_{A_\nu}\langle trR(df_\nu,V_\nu)df_\nu,V_\nu\rangle)d\omega\le C(N,\|f_\nu\|_0,E(f_\nu),\|\xi^1\|_0)$.

We have proved (13), and this means that $\{(v^\alpha_\nu)|\nu\le\nu_0\}_{\alpha=1,\cdots,n}$ has the same modulus of continuity from the similar argument as in B-II) with Lemma 2.6.

With the same charts as in *(B)*, $(v^\alpha_{\nu(\rho)})\in\mathbb{R}^n,\ \nu\le\nu_0$ are weak solutions of the above system with a uniform bounded energy and the same modulus of continuity on $K_\sigma=\{\sigma\le|z|\le1\}$ with $\sigma>0$ for small ρ by Lemma 2.6.

From the similar argument as in *(B)*, this sequence converges to the Jacobi field along $\mathcal{F}^1|_{B\backslash\{0\}}$. Thus, letting $w^\alpha\frac{\partial}{\partial y^\beta}\circ\mathcal{F}^1:=\mathbf{J}_{\mathcal{F}^1}(\xi^1)$, for any compact $K\subset B\backslash\{0\}$, as $\nu(\text{or }\ \rho)\to0$

$$\|(v^\alpha_\nu(z))-(w^\alpha(z))\|_{C^0;K_\sigma}\to0,\ \ \|(v^\alpha_\nu(z))-(w^\alpha(z))\|_{C^2;K}\to0.$$

C-II) The convergence of derivatives.

Considering K_σ as above, we denote $f_\nu|_{K_\sigma}$ by f_ν and $\mathcal{F}^1|_{K_\sigma}$ by $\mathcal{F}^1$.

Note that $\exp_{\mathcal{F}^1}:\mathcal{U}(0)\to H^{1,2}\cap C^0(K_\sigma,N)$ is a diffeomorphism on some neighborhood $\mathcal{U}(0)\in H^{1,2}\cap C^0(K_\sigma,(\mathcal{F}^1)^*TN)$, since $d(\exp_{\mathcal{F}^1})_0=Id$.
Moreover, $\|f_\nu-\mathcal{F}^1|_{K_\sigma}\|_{H^{1,2}\cap C^0}\to0$ as $\nu\to0$, so there exists $\xi_\nu\in H^{1,2}\cap C^0(K_\sigma,(\mathcal{F}^1)^*TN)$ for small $\nu>0$ with $\exp_{\mathcal{F}^1}\xi_\nu=f_\nu$.

The mapping $\xi\mapsto d\exp_{\mathcal{F}^1,\xi}$ depends smoothly on $\xi_\nu\in T_{\mathcal{F}^1}H^{1,2}\cap C^0(K_\sigma,N)$, so $d\exp_{\mathcal{F}^1,\xi_\nu}\to Id$ in $H^{1,2}\cap C^0(K_\sigma)$, since $\xi_\nu\to0$ in $H^{1,2}\cap C^0(K_\sigma,(\mathcal{F}^1)^*TN)$ as $\nu\to0$, and for $W_\nu:=w^\alpha_\nu\frac{\partial}{\partial y^\alpha}\circ\mathcal{F}^1:=d\exp^{-1}_{\mathcal{F}^1,\xi_\nu}(V_\nu)$, we have, $\|w^\alpha_\nu(z)-w^\alpha(z)\|_{C^0;K_\sigma}\to0$ by (I).
Moreover, $d\mathcal{F}^1\to df_\nu$ in L^2, thus $\int_{K_\sigma}|d\exp_{\mathcal{F}^1,\xi_\nu}(d\mathcal{F}^1)-df_\nu|^2d\omega\to0$.

We next observe, for $\nabla^{\mathcal{F}^1}W_\nu=(w^\alpha_{\nu,i}+w^\gamma_\nu(\mathcal{F}^1)^\beta_{,i}\Gamma^\alpha_{\beta\gamma}(\mathcal{F}^1))dz^i\otimes\frac{\partial}{\partial y^\alpha}\circ\mathcal{F}^1$,

$$\text{(15)}\qquad \int_{K_\sigma}|d\exp_{\mathcal{F}^1,\xi_\nu}(\nabla^{\mathcal{F}^1}W_\nu)-\nabla^{f_\nu}V_\nu|^2d\omega\to0\ \text{ as }\ \nu\to0,$$

since $\|\mathcal{F}^1-f_\nu\|_{1,2;0}\to0$, $d\exp_{\mathcal{F}^1,\xi_\nu}\to Id$ in C^0, $\partial_i(d\exp_{\mathcal{F}^1,\xi_\nu})\to\partial_i(Id)=0$ in L^2.

Thus, for $X_\nu, Y_\nu \in H^{1,2}\cap C^0(K_\sigma, T^*M\otimes f_\nu^*TN)$ with $\int_{K_\sigma}|X_\nu|^2 d\omega \to 0,\ \int_{K_\sigma}|Y_\nu|^2 d\omega \to 0$, it holds that

$$d\exp_{\mathcal{F}^1,\xi_\nu}(d\mathcal{F}^1) = df_\nu + X_\nu,\ \ d\exp_{\mathcal{F}^1,\xi_\nu}(\nabla^{\mathcal{F}^1}W_\nu) = \nabla^{f_\nu}V_\nu + Y_\nu.$$

And from the Gauss Lemma, $\langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1}W_\nu\rangle_h = \langle df_\nu + X_\nu, \nabla^{f_\nu}V_\nu + Y_\nu\rangle_h$. Thus, by Hölder inequality and from (14),

$$\begin{aligned}
&\int_{K_\sigma}\Big(\langle df_\nu, \nabla^{df_\nu}V_\nu\rangle_h - \langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1}\mathbf{J}_{\mathcal{F}^1}(\xi^1)\rangle_h\Big)d\omega \\
&\quad= \int_{K_\sigma}\Big(\langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1}W_\nu\rangle_h - \langle d\mathcal{F}^1, \nabla^{\mathcal{F}^1}\mathbf{J}_{\mathcal{F}^1}(\xi^1)\rangle_h\Big)d\omega + o(1) \\
&\quad\le E(d\mathcal{F}^1)\|\nabla^{\mathcal{F}^1}W_\nu - \nabla^{\mathcal{F}^1}\mathbf{J}_{\mathcal{F}^1}(\xi^1)\|_{L^2;K_\sigma} + o(1). \qquad (16)
\end{aligned}$$

For the estimate of the last term, letting $W := \mathbf{J}_{\mathcal{F}^1}(\xi^1)$, consider $A_\nu := a_\nu^\alpha\frac{\partial}{\partial y^\alpha}\circ\mathcal{F}^1,\ \ A := a^\alpha\frac{\partial}{\partial y^\alpha}\circ\mathcal{F}^1$ such that $d\eta\big(a_\nu^\alpha\frac{\partial}{\partial y^\alpha}\circ\mathcal{F}^1\big),\ \ d\eta\big(a^\alpha\frac{\partial}{\partial y^\alpha}\circ\mathcal{F}^1\big) \in H^{1,2}\cap C^0(K_\sigma, \mathbb{R}^k)$ are harmonic in $\mathbb{R}^k$ with $A_\nu|_{\partial K_\sigma} = W_\nu|_{\partial K_\sigma}$ and $A|_{\partial K_\sigma} = W|_{\partial K_\sigma}$. Clearly, $\|d\eta(A_\nu - A)\|_{1,2;0}\to 0$.

Now, consider a test vector field $Z_\nu := W_\nu - W - A_\nu + A \in H_0^{1,2}\cap C^0(K_\sigma, (\mathcal{F}^1)^*TN)$, and then, observing that W resp. V_ν is a Jacobi-Field along $\mathcal{F}^1|_{K_\sigma}$ resp. $f_\nu|_{K_\sigma}$,

$$\begin{aligned}
(17)\ &\int_{K_\sigma}\langle\nabla^{\mathcal{F}^1}(W_\nu - W), \nabla^{\mathcal{F}^1}Z_\nu\rangle_h d\omega \\
&= \int_{K_\sigma}\{\langle\nabla^{\mathcal{F}^1}W_\nu, \nabla^{\mathcal{F}^1}Z_\nu\rangle_h - \langle trR\circ\mathcal{F}^1(W, d\mathcal{F}^1)d\mathcal{F}^1, Z_\nu\rangle_h \\
&\qquad -\langle\nabla^{f_\nu}V_\nu, \nabla^{f_\nu}(L_\nu(Z_\nu))\rangle_h + \langle trR\circ f_\nu(V_\nu, df_\nu)df_\nu, (L_\nu(Z_\nu))\rangle_h\}d\omega \\
&= \int_{K_\sigma}\{\langle\nabla^{\mathcal{F}^1}W_\nu, \nabla^{\mathcal{F}^1}Z_\nu\rangle_h - \langle trR\circ\mathcal{F}^1(W, d\mathcal{F}^1)d\mathcal{F}^1, Z_\nu\rangle_h \\
&\qquad -\langle\nabla^{\mathcal{F}^1}L_\nu^{-1}(V_\nu), \nabla^{\mathcal{F}^1}Z_\nu\rangle_h + \langle trR\circ f_\nu(V_\nu, df_\nu)df_\nu, (L_\nu(Z_\nu))\rangle_h\}d\omega + o(1)
\end{aligned}$$

with $L_\nu := d\exp_{\mathcal{F}^1,\xi_\nu}$. And this converges to 0 as $\nu\to 0$, since $L_\nu^{-1}(V_\nu) = W_\nu$, $\|Z_\nu\|_{C^0;K_\sigma}\to 0$ and $\|\mathcal{F}^1\|_{1,2;0}, \|W\|_{C^0}, \|f_\nu\|_{1,2;0}, \|V_\nu\|_{C^0} < C$ for all $\nu\in(0,\nu_0)$.

Moreover, in (17), since $\|d\eta(A_\nu - A)\|_{C^0}\to 0$, $\int_{K_\sigma}|\nabla^{\mathcal{F}^1}(A_\nu - A)|_h^2 d\omega\to 0$, note (1). Thus, (16) converges to 0 for each $\sigma\in(0,1)$. And by letting $\sigma\to 0$, we have

$$\int_{A_{\nu(\rho)}}\langle d\mathcal{F}_\rho(x^1,x^2), \nabla\mathbf{J}_{\mathcal{F}_\rho}(\xi^1,0)\rangle_h d\omega \to \int_B\langle d\mathcal{F}^1(x^1), \nabla\mathbf{J}_{\mathcal{F}^1}(\xi^1)\rangle_h d\omega, \quad \rho\to 0,$$

note that $\int_{B_\sigma}\langle d\mathcal{F}^1(x^1), \nabla\mathbf{J}_{\mathcal{F}^1}(\xi^1)\rangle_h d\omega\to 0$ as $\sigma\to 0$.

In a similar way, $\int_{A_{\nu'(\rho)}}\langle dg_{\nu'}, \nabla\mathbf{J}_{g_{\nu'}}(0,\zeta_{\nu'})\rangle d\omega \to \int_B\langle d\mathcal{F}^2(x^2), \nabla\mathbf{J}_{\mathcal{F}^2}(0)\rangle_h d\omega = 0.$

The uniform convergence on $\mathcal{N}_\varepsilon(x_0^i)$, the same as in *(B)*, is clear.

In this manner, we may also show that $\delta_{x^1}\mathcal{E}_\rho, \delta_{x^2}\mathcal{E}_\rho$ are continuous with respect to $\rho \in (0,1)$ and uniformly on $\mathcal{N}_\varepsilon(x_0^i)$. And we have proved *(C)*.

(D) By the same argumentation as in [St3] we have the following differential form.

$$\frac{\partial}{\partial t}|_{t=\rho}\mathcal{E}(x^1,x^2,t) = \int_0^{2\pi}\int_\rho^1 \left[|\partial_r \mathcal{F}_\rho|^2 - \frac{1}{r^2}|\partial_\theta \mathcal{F}_\rho|^2\right]\frac{1}{1-\rho}drd\theta \tag{18}$$

This brings to an end our proofs for Lemma 3.1. □

3.2 Critical points of $\mathcal{E}$

For given Jordan curves $\Gamma_1, \Gamma_2, \Gamma$ in (N,h) with $\mathrm{dist}(\Gamma_1,\Gamma_2) > 0$, we consider the Plateau Problem $\mathcal{P}(\Gamma_1,\Gamma_2)$ and Problem $\mathcal{P}(\Gamma)$.

Now we define for $x = (x^1, x^2, \rho) \in \overline{\mathcal{M}}$, $i = 1,2$,

$$\begin{aligned} g_i(x) &:= \sup_{\substack{\xi^i \in \mathcal{T}_{x^i} \\ \|\xi^i\| < l_i}} (-\langle \delta_{x^i}\mathcal{E}, \xi^i\rangle), \\ g_3(x) &:= \begin{cases} |\rho\cdot\partial_\rho \mathcal{E}| & , \quad \rho > 0 \\ 0 & , \quad \rho = 0, \end{cases} \\ g(x) &:= \Sigma_{j=1}^3 g_j(x). \end{aligned} \tag{19}$$

Adding to the the definition of l_i in section 2.2, we can clearly require that $l_i \leq \{1, i_{\tilde{h}}(\Gamma_i)\}$. And note that $g_j \geq 0, j = 1,2,3$, because, $g_i(x) < 0$, $i = 1,2$ means $\langle \delta_{x^i}\mathcal{E}, \xi^i\rangle \geq \sigma > 0$ for all $\xi^i \in \mathcal{T}_{x^i}$ with $\|\xi^i\| < l_i$, and since $\mathcal{T}_{x^i}$ is convex, $\langle \delta_{x^i}\mathcal{E}, t\xi^i\rangle = t\sigma \geq \sigma$, for all $t \in [0,1]$, a contradiction. Clearly, $g_3(x) \geq 0$. Now we can define the critical points of $\mathcal{E}$.

Definition $x \in \overline{\mathcal{M}}$ is a critical point of $\mathcal{E}$ if $g(x) = 0$, i.e. $g_j = 0, j = 1,2,3$.

Lemma 3.2. *g_j is continuous, and specially $g_j(x^1,x^2,\rho) \to g_j(x^1,x^2,\rho_0)$ as $\rho \to \rho_0$, uniformly on $\mathcal{N}_\varepsilon(x^i)$, for some small $\varepsilon > 0$, $j = 1,2,3$, $i = 1,2$.*

Proof. The uniform convergence of g_i with respect to $\rho \to \rho_0 \in [0,1)$ on $\mathcal{N}_\varepsilon(x^i)$ follows immediately from the uniform convergence of $\delta_{x^i}\mathcal{E}$ (see Lemma(3.1),*(C)*).

Let $\{x_n\} = \{(x_n^1, x_n^2, \rho_n)\} \subset \overline{\mathcal{M}}$ which converges strongly to $x = (x^1,x^2,\rho)$. From the above, $g_i(x_n^1,x_n^2,\rho_n) \to g_i(x_n^1,x_n^2,\rho)$, uniformly on $\{n \geq n_0\}$.

Now, let $\tilde{x}_n := (x_n^1, x_n^2, \rho)$ and $\widetilde{\exp}_{x_n^i}\xi_n^i = x^i$. Observe that $d\widetilde{\exp}_{x_n^i,\xi_n^i} \to Id$ in $H^{\frac{1}{2},2}\cap C^0$, hence for some t_0, independent of $n \geq n_0$, $\|t_0 d\widetilde{\exp}_{x_n^i,\xi_n^i}(\eta_n^i)\|_{\mathcal{T}_{x^i}} < l_i$ if $\|\eta_n^i\|_{\mathcal{T}_{x_n^i}} < l_i$, note that $\mathcal{T}_{x^i}$ is convex with 0.

Then by Lemma(3.1)*(A)*, for given $\delta > 0$ there exist $t_0(\delta)$ and $n_0(\delta)$ as above such that for each $\|\eta_n^i\|_{\mathcal{T}_{x_n^i}} < l_i$ with $n \geq n_0(\delta)$,

$$\begin{aligned} -\langle \delta_{x^i}\mathcal{E}(\tilde{x}_n), \eta_n^i \rangle &\leq -\langle \delta_{x^i}\mathcal{E}(x), d\widetilde{\exp}_{x_n^i,\xi_n^i}(\eta_n^i) \rangle + \delta \\ &\leq -\langle \delta_{x^i}\mathcal{E}(x), t_0 d\widetilde{\exp}_{x_n^i,\xi_n^i}(\eta_n^i) \rangle + 2\delta \leq g_i(x) + 2\delta. \end{aligned}$$

This implies, $g_i(\tilde{x}_n) \leq g_i(x) + 2\delta$. On the other hand, $g_i(x) \leq g_i(\tilde{x}_n) + 2\delta$, so $g_i(x_n^1, x_n^2, \rho) \to g_i(x^1, x^2, \rho)$ as $n \to \infty$.

Together with the above uniform convergence on $\mathcal{N}_\varepsilon(x^i)$ as $\rho_n \to \rho$, we conclude the cotinuity of g_i, $i = 1, 2$. The continuity and uniform continuity of g_3 is clear from the form of $\frac{\partial}{\partial \rho}\mathcal{E}$. □

Proposition 3.1. $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times [0, 1)$ *is a critical point of* $\mathcal{E}$ *if and only if* $\mathcal{F}_\rho(x^1, x^2)$ *(for* $\rho \in (0, 1)$*) resp.* $\mathcal{F}^i(x^i)$ *is a solution of* $\mathcal{P}(\Gamma_1, \Gamma_2)$ *resp.* $\mathcal{P}(\Gamma_i), i = 1, 2$.

Proof. (I) Let $x = (x^1, x^2, \rho) \in M^1 \times M^2 \times [0, 1)$ be a critical point of $\mathcal{E}$. From the result in [HKW], $\mathcal{F}$ is continuous until the boundary.
We must show that $\mathcal{F}_\rho(x^1, x^2)$(for $\rho > 0$), and $\mathcal{F}^i(x^i)$ is conformal. We will show this only for $\mathcal{F}_\rho(x^1, x^2)$, because the proof in the case of $\mathcal{F}^i(x^i)$ is similar to the case of $\mathcal{F}_\rho(x^1, x^2)$ and easier.

For $x \in \mathcal{M}$, a critical poinf of $\mathcal{E}$, we have that $\mathcal{F}_\rho(x^1, x^2)$ belongs to the class $H^{2,2}(A_\rho, \mathbb{R}^k)$, which was proved in [Ki].
We can then compute, for $\xi^1 \in \mathcal{T}_{x^1}$, letting $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2)$ and $\mathbf{J}_\rho := \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0)$,

$$\begin{aligned} \langle \delta_{x^1}\mathcal{E}, \xi^1 \rangle &= \int_{A_\rho} \langle d\mathcal{F}_\rho, \nabla_{\frac{d}{dt}} d\mathcal{F}_\rho(\widetilde{\exp}_{x^1} t\xi^1, x^2)\big|_{t=0} \rangle_h d\omega = \int_{A_\rho} \langle \frac{\partial}{\partial z^i}\mathcal{F}_\rho, \nabla_{\frac{\partial}{\partial z^i}} \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0) \rangle_h d\omega \\ &= \int_{A_\rho} \operatorname{div}(\langle \frac{\partial}{\partial z^1}\mathcal{F}_\rho, \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0) \rangle_h, \langle \frac{\partial}{\partial z^2}\mathcal{F}_\rho, \mathbf{J}_{\mathcal{F}_\rho}(\xi^1, 0) \rangle_h) d\omega \ \ (\text{since } \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^i}\mathcal{F}_\rho = 0) \\ (20) \quad &= \int_{\partial B} \langle \frac{\partial}{\partial z^1}\mathcal{F}_\rho \vec{n}, \xi^1 \rangle_h d\omega. \end{aligned}$$

Now, we can use the calculation in [St1] and obtain the conformal property of $\mathcal{F}_\rho$.

(II) Let $\mathcal{F} := \mathcal{F}_\rho(x)$(resp. $\mathcal{F}^i(x^i)$) be a minimal surface of annulus (resp.disc) type. By [HH], $\mathcal{F} \in C^1(\overline{A_\rho}, N)$ (resp.$C^1(\overline{B}, N)$). Thus, from the conformal property $\frac{d\mathcal{F}}{d\vec{n}} \cdot \frac{d}{d\theta}x^i \equiv 0$, and the computation (20) says that $g_1(x) = 0, g_2(x) = 0$, by (18) also $g_3(x) = 0$. □

4 Unstable minimal surfaces

4.1 The Palais-Smale condition

By the conformal invarianc of E, the Palais-Smale Condition((PS) condition) cannot be satisfied for some function sequence(cf. [St1] Lemma I.4.1). Hence, we need the normalization as in [St3]: Let $P_k^i \in \Gamma_i$ fixed, $k = 1, 2, 3$, $i = 1, 2$, and

$$M^{i*} = \{x^i \in M^i : x^i(\cos\frac{2\pi k}{3}, \sin\frac{2\pi k}{3}) = P_k^i \in \Gamma_i, \;\; k = 0,\, 1,\, 2\}.$$

Now define

$$\begin{aligned} \mathcal{M}^* &= \{x = (x^1, x^2, \rho) \in \mathcal{M} : x^1(1,0) = P_1^1 \in \Gamma_1\} \\ \partial\mathcal{M}^* &= \{x = (x^1, x^2, 0) \in \partial\mathcal{M} : x^i \in M^{i*}\}, \end{aligned}$$

with $\mathcal{T}_x\mathcal{M}^* = \{\xi \in \mathcal{T}_x\mathcal{M} \mid \xi^1(1,0) = 0\}$ for $x \in \mathcal{M}^*$ and $\mathcal{T}_x\partial\mathcal{M}^* = \mathcal{T}_{x^1} \times \mathcal{T}_{x^2}$ for $x \in \partial\mathcal{M}^*$. To avoid complication in the sequel, we give some explanation for the above setting;

(i) We will consider an element $x^i \in M^{i*}$ as a class which consists of $y^i \in M^i$ with $T|_{\partial B}(y^i) = x^i$ for some conformal transformation T of disc onto itself. In other words, we classify M^i in such a way that each class posesses only one element from $x^i \in M^{i*}$, if necessary, denoted by $[x^i] \in M^{i*}$, with $\|[x^i]\| = \|x^i\|, i = 1, 2$. For $[x] \in \partial\mathcal{M}^*$ with $x^i \in M^{i*}$, we define $g([x]) := g(x)$. And for $\xi \in \mathcal{T}^0_{x^i} \subset \mathcal{T}_{x^i}$, we may calculate: $\widetilde{\exp}_{[x^i]}\xi := [\widetilde{\exp}_{x^i}\xi] := [\tilde{x}^i] \in M^{i*}$, where $\widetilde{\exp}_{x^i}\xi \in M^i$, so $T(\widetilde{\exp}_{x^i}\xi) = \tilde{x}^i \in M^{i*}$, since T is a conformal map of B. We denote this correspondence by $\widetilde{\exp}_{x^i}\xi = \tilde{x}^i \in M^{1*}$, which is clearly continuous.

(ii) We consider the following topology for this setting: A neighborhood $\mathcal{U}_\varepsilon(x_0)$ of $x_0 = (x_0^1, x_0^2, 0) \in \partial\mathcal{M}^*$ consists of all $x = (x^1, x^2, \rho) \in \overline{\mathcal{M}^*}$ such that $\rho < \varepsilon$ and for each $i = 1, 2$, $\inf_{\{\text{all } \sigma\}} \|\mathcal{F}^i(x^i) \circ \sigma - \mathcal{F}^i(x^i)\|_{1,2} < \varepsilon$, where σ is a conformal diffeomorphism of B. A sequence $\{x_n = (x_n^1, x_n^2, \rho_n)\} \subset \overline{\mathcal{M}}^*$ converges strongly to $x = (x^1, x^2, 0) \in \partial\mathcal{M}^*$, if for any $\varepsilon > 0$ all but finitely many of x_n lie in $\mathcal{U}_\varepsilon(x)$.

Then, we can easily check the following:

Remark 4.1. *(i) For $x \in \mathcal{M}^*$, the value of $g_i(x)(i = 1, 2)$ in (19) does not change, even if we use $\mathcal{T}_x\mathcal{M}^*$ instead of $\mathcal{T}_x\mathcal{M}$.*

(ii) With the above topology $g_j, j = 1, 2, 3$, are continuous and uniformly continuous as $\rho \to \rho_0 \in [0, 1)$ on some ε-neighborhood of (x^1, x^2).

(iii) For $\xi = (\xi^1, \xi^2, 0) \in \mathcal{T}_x\mathcal{M}^*$ resp. $\mathcal{T}_x\partial\mathcal{M}^*$, with $\|\xi^i\|_{\frac{1}{2},2;0} \le l_i$, $\widetilde{\exp}_x\xi \in \mathcal{M}^*$ resp $\partial\mathcal{M}^*$.

Proposition 4.1 (Palais-Smale condition). *Suppose, $\{x_n\}$ is a sequence in $\overline{\mathcal{M}^*}$ such that $\mathcal{E}(x_n) \to \beta$, $g(x_n) \to 0$, as $n \to \infty$. Then there exists a subsequence of $\{x_n\}$ which converges strongly to a critical point of $\mathcal{E}$ in $\overline{\mathcal{M}^*}$.*

Proof. We prove this for the case that $\{x_n\} \subset \mathcal{M}^*$ with $0 < \rho_n < 1$, $\mathcal{E}(x_n) \to \beta$, $g_j(x_n) \to 0$. In the case that $\{x_n\} \subset \partial\mathcal{M}^*$, the proof is similar. We may suppose that $\rho_n \to \rho$.

Note that the above ρ cannot be 1, i.e. $0 \le \rho < 1$, because for any $x = (x^1, x^2, \rho) \in \mathcal{M}$, $\frac{\rho}{1-\rho} \le c\mathcal{E}(x)$, since $0 < \mathrm{dist}(\Gamma_1, \Gamma_2)$. For further details, see [St3] Lemma 4.10.

Clearly $\int_{A_\rho} |d\eta \circ \mathcal{F}_\rho(x^1, x^2)|^2 d\omega \ge \int_{A_\rho} |d\mathcal{H}_\rho(x^1, x^2)|^2 d\omega \ge C(\rho)\Sigma_i \int_B |d\mathcal{H}(x^i)|^2 d\omega$, so by [St1] Proposition II.2.2, for subsequence $\{w_n^i\}$ with $\gamma^i(w_n^i) = x_n^i$, it holds, for some integers $j^i(n)$, $\|w_n^i - 2\pi j^i(n) - w^i\|_{C^0} \to 0$ or $x_n^i = \gamma^i \circ w_n^i \to \text{const.} = a_i \in \Gamma_i$ in $L^1(\partial B)$.

We have to distinguish several cases.

(case 1) Let $\rho \in (0,1)$ and $\|w_n^i - 2\pi j_n^i - w^i\|_{C^0} \to 0$, i.e. $\|x_n^i - x^i\|_{C^0} \to 0, x^i \in M^i$, $i = 1, 2$.

First, $\gamma^i(w_n^i(\theta)) - \gamma^i(w^i(\theta)) = \underbrace{d\gamma^i(w_n^i(\theta))(w_n^i(\theta) - w^i(\theta))}_{=:I_n^i} - \underbrace{\int_{w^i(\theta)}^{w_n^i(\theta)} \int_{w'}^{w^i(\theta)} d^2\gamma^i(w'')dw''dw'}_{=:II_n^i}.$

And $\int_{A_\rho} |d\mathcal{H}_\rho(II_n^1, II_n^2)|^2 d\omega \le C(\rho)(\|\mathcal{H}(II_n^1)\|_{1,2;\infty}^2 + \|\mathcal{H}(II_n^2)\|_{1,2;\infty}^2) \to 0$, as $n \to \infty$, since $\|II_n^i\|_{\frac{1}{2},2;\infty} \le C\|w_n^i - w^i\|_\infty(|w_n^i|_{\frac{1}{2}} + |w^i|_{\frac{1}{2}})$ by [St2] (3.9).
Let $\mathcal{H}_n := \mathcal{H}_\rho(x_n^1, x_n^2)$, $\mathcal{H} := \mathcal{H}_\rho(x^1, x^2)$, $\mathcal{F}_n := \eta \circ \mathcal{F}_\rho(x_n^1, x_n^2)$, $\mathcal{F} := \eta \circ \mathcal{F}_\rho(x^1, x^2)$.
Since $\mathcal{H}_n - \mathcal{H}$ is harmonic in $\mathbb{R}^k$ and $\int_{A_\rho} \langle d\mathcal{H}, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega = o(1)$ as $n \to \infty$,

$$\int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega = \int_{A_\rho} \langle d\mathcal{F}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + o(1) = \int_{A_\rho} \langle d\mathcal{F}_n, d(\mathcal{H}_\rho(I_n^1, I_n^2))\rangle d\omega + o(1).$$

Now we consider $\xi_n^i := -I_n^i \in \mathcal{T}_{x_n^i}$, and letting $\mathbf{J}_n^1 := \mathbf{J}_{\mathcal{F}_\rho}(\xi_n^1, 0)$, $\mathbf{J}_n^2 := \mathbf{J}_{\mathcal{F}_\rho}(0, \xi_n^2)$,

$$\begin{aligned}
\int_{A_\rho} \langle d\mathcal{F}_n, d\mathcal{H}_\rho(I_n^1, I_n^2)\rangle d\omega &= \int_{A_\rho} \langle d\mathcal{F}_n, d\mathcal{H}_\rho(I_n^1, 0)\rangle d\omega + \langle d\mathcal{F}_n, d\mathcal{H}_\rho(0, I_n^2)\rangle d\omega \\
= \quad & \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + \int_{A_\rho} \langle II \circ \mathcal{F}_n(d\mathcal{F}_n, d\mathcal{F}_n), \mathcal{H}_\rho(I_n^1, 0) + \mathbf{J}_n^1\rangle d\omega \\
& + \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^2\rangle d\omega + \int_{A_\rho} \langle II \circ \mathcal{F}_n(d\mathcal{F}_n, d\mathcal{F}_n), \mathcal{H}_\rho(0, I_n^2) + \mathbf{J}_n^2\rangle d\omega \\
\le \quad & g_i(x_n^1, x_n^2, \rho)\|\xi_n^i\|_{\frac{1}{2},2;\infty} + C(\|\mathcal{F}_n\|_{1,2;0})\|\xi_n^i\|_\infty \\
\le \quad & Cg_i(x_n)\|\xi_n^i\|_{\frac{1}{2},2;\infty} + C(\|\mathcal{F}_n\|_{1,2;0})\|x_n^i - x^i\|_\infty,
\end{aligned}$$

where C is independent of $n \ge n_0$, for some n_0, since observing that $\|x_n^i - x^i\|_{C^0} \to 0$, we obtain the convergence of $g_i(x_n^1, x_n^2, \rho_{n'}))$ as $\rho_{n'} \to \rho$, uniformly on $\{x_n^i | n \ge n_0\}$, by the arguments in Remark 2.1, Remark 2.2 and Lemma 3.2.

Also note that $\|\xi_n^i\|$ are uniformly bounded, since from [St1]

$$\|d\gamma^i(w_n^i)(w_n^i - w^i)\|_{\frac{1}{2}} \le \|d\gamma^i(w_n^i)\|_{\frac{1}{2}}\|w_n^i - w^i\|_\infty + \|d\gamma^i(w_n^i)\|_\infty \|w_n^i - w^i\|_{\frac{1}{2}}.$$

Therefore $\int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega \to 0$, and $x_n^i \to x^i$ strongly in $H^{\frac{1}{2},2} \cap C^0(\partial B, \mathbb{R}^k)$.

(case 2) Let $\rho \in (0,1)$, $\|x_n^1 - x^1\|_{C^0} \to 0$, $x_n^2 = \gamma_2 \circ w_n^2 \to \text{const.} = a_2 \in \Gamma_2$ in $L^1(\partial B, \mathbb{R}^k)$.

I) First, we claim that $\mathcal{F} := \mathcal{F}_\rho(\gamma^1 \circ w^1, a^2)$ is conformal.

I-a) By assumption, $\|I_n^1\|_{C^0} = \|d\gamma^1(w_n)(w_n^1 - w^1)\|_{C^0} \to 0$.

Let $\mathcal{H}_n := \mathcal{H}_\rho(x_n^1, x_n^2)$, $\mathcal{H} := \mathcal{H}_\rho(x^1, a^2)$, $\mathcal{F}_n := \mathcal{F}_\rho(x_n^1, x_n^2)$, $\mathcal{F} := \mathcal{F}_\rho(x^1, a^2)$.
Then, for any fixed $\sigma \in (\rho, 1)$, as $n \to \infty$, letting $\mathbf{J}_\rho(\xi_n^1, 0)|_{\partial B_\sigma} =: l_n$ with $\mathbf{J}_\rho = d\eta(\mathbf{J}_\rho(\xi_n^1, 0))$,

$$\begin{aligned}
&\int_{A_\sigma} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega \\
&= \int_{A_\sigma} \langle d\mathcal{H}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + o(1) = \int_{A_\sigma} \langle d\mathcal{F}_n, d(\mathcal{H}_n - \mathcal{H})\rangle d\omega + o(1) \\
&= \int_{A_\sigma} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + \int_{A_\sigma} \langle d\mathcal{F}_n, d\mathcal{H}_\sigma(I_n^1, l_n) + d\mathbf{J}_n^1\rangle d\omega + o(1) \\
&\quad \text{(since } \|\mathcal{H}_n - \mathcal{H}\|_{\partial B_\sigma} \to 0 \text{ applying the arguments below I-b))} \\
&= \int_{A_\sigma} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + \int_{A_\sigma} \langle II \circ \mathcal{F}_n(d\mathcal{F}_n, d\mathcal{F}_n), \mathcal{H}_\sigma(I_n^1, l_n) + \mathbf{J}_n^1\rangle d\omega + o(1) \\
&= \int_{A_\sigma} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + o(1).
\end{aligned}$$

This holds for each $\sigma \in (\rho, 1)$, thus

$$\begin{aligned}
\int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega &= \lim_{\sigma \to \rho} \int_{A_\sigma} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega \le \int_{A_\rho} -\langle d\mathcal{F}_n, d\mathbf{J}_n^1\rangle d\omega + o(1) \\
&\le C g_1(x_n^1, x_n^2, \rho_n)\|\xi_n^1\|_{\frac{1}{2},2;0} + o(1), \qquad \text{for large } n \ge n_0
\end{aligned}$$

which converges to 0, by Lemma 3.2 and the uniform boundedness of $\|\xi_n^1\|_{\frac{1}{2},2;0}$.
Moreover, $\int_B |d\mathcal{H}(x_n^1 - x^1)|^2 d\omega \le \int_{A_\rho} |d(\mathcal{H}_n - \mathcal{H})|^2 d\omega + o(1)$, where $o(1) \to 0$ as $\rho \to 0$ uniformly for $n \ge n_0$, so from the above, $x_n^1 \longrightarrow x^1$ in $H^{\frac{1}{2},2} \cap C^0$.

I-b) Now letting $x_n^2 = \gamma_2 \circ w_n^2$ and $a_2 = \gamma^2 \circ w^2$, let us see the behavior of $\mathcal{F}_{\rho_n}(x_n^1, x_n^2)|_{\partial B_{\rho_n}}$.
First, there must exist $\theta_0 \in [0, 2\pi]$ such that $\left|\lim_{\theta \to \theta_0+} w^2(\theta) - \lim_{\theta \to \theta_0-} w^2(\theta)\right| = 2\pi$.
And by the Courant-Lebesgue Lemma, for given $\varepsilon > 0$ there exists $r_n \in (\delta, \sqrt{\delta})$ for some small $\delta := \delta(\varepsilon) > 0$ such that with $B_{r_n} := B_{r_n}(\theta_0)$,

$$\operatorname{osc}_{A_{\rho_n} \cap \partial B_{r_n}} \mathcal{F}_{\rho_n}(x_n^1, x_n^2) \le C\, \frac{\mathcal{E}(x_n^1, x_n^2, \rho_n)}{\ln(\delta^{-1})} \le \frac{C}{\ln(\delta^{-1})} < \varepsilon. \tag{21}$$

And letting $Y_n^2 := \Gamma_2 \backslash \mathcal{F}_{\rho_n}(B_{r_n}) \cup \mathcal{F}_{\rho_n}(\partial B_{r_n})$, we obtain, $\mathrm{dist}(Y_n^2, a_2) \to 0$ as $n \to \infty$.

Next, applying the argument in the proof of Lemma 3.1 *(B)* to $\{\mathcal{F}_{\rho_n}|_{A_{\rho_n} \backslash B_{r_n}}\} =: \tilde{\mathcal{F}}_n$, $\tilde{\mathcal{F}}_n|_K$ converges to $\mathcal{F}|_K$ in C^2. for compact $K \subset\subset (A_{\rho_n} \backslash B_{r_n})$ for large n.

I-c) Now, we investigate the behavior of Jacobi fields.

For large $n \geq n_0$, $\widetilde{\exp}_{x^1} \xi_n^1 = x_n^1$ for some $\xi_n^1 \in \mathcal{T}_{x^1}$, with $\|d\widetilde{\exp}_{x^1, \xi_n^1} \phi^1\| < l_1$, $\|\phi^1\| < l_1$. Since $d\widetilde{\exp}_{x^1, \xi_n^1} \to Id$ in $H^{\frac{1}{2},2} \cap C^0$, for $(v_n^\alpha \frac{\partial}{\partial y^\alpha} \circ \mathcal{F}_{\rho_n}) := \mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1, \xi_n^1} \phi^1, 0)$,

$$\int_{A_{\rho n}} h_{\alpha\beta} \circ \mathcal{F}_{\rho_n} v_{n,i}^\alpha v_{n,i}^\beta d\omega \leq C, \text{ independent of } n \geq n_0.$$

From the Courant-Lebesgue Lemma and $v_n^\alpha|_{\partial B_{\rho_n}} \equiv 0$, for some $\tilde{r_n} \in (\sqrt{\delta}, \sqrt{\sqrt{\delta}})$,

$$\int_{\partial(B_{\tilde{r_n}} \cap A_{\rho_n})} h_{\alpha\beta} \circ \mathcal{F}_{\rho_n} \partial_\theta v_n^\alpha \partial_\theta v_n^\beta d\theta \leq \frac{C}{|\ln \delta|} \text{ and } \|(v_n^\alpha)\|_{C^0(B_{\tilde{r_n}(\theta_0)} \cap A_{\rho_n})} \leq \frac{C}{|\ln \delta|}.$$

Hence, from Lemma 2.6, $E(\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1, \xi_n^1} \phi^1, 0)|_{B_{\tilde{r_n}}})$ is less than $\frac{C}{|\ln \delta|}$, small enough on B_{r_n}, since $r_n \leq \tilde{r_n}$. Now we choose the above δ so small that $\frac{C}{|\ln \delta|} \leq \varepsilon := \frac{1}{n}$.

I-d) Letting $\mathcal{F}_{\rho_n} := \mathcal{F}_{\rho_n}(x_n^1, x_n^2)$, by Hölder inequality

$$\begin{aligned}
0 &= \lim_{n \to \infty} g^1(x_n^1, x_n^2, \rho_n) \\
&\geq \lim_{n \to \infty} \Big(- \int_{A_{\rho_n} \backslash B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1, \xi_n^1} \phi^1, 0) \rangle d\omega - \int_{B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1, \xi_n^1} \phi^1, 0) \rangle d\omega \Big) \\
&= \lim_{n \to \infty} \Big(- \int_{A_{\rho_n} \backslash B_{r_n}} \langle d\mathcal{F}_{\rho_n}, d\mathbf{J}_{\mathcal{F}_{\rho_n}}(d\widetilde{\exp}_{x^1, \xi_n^1} \phi^1, 0) \rangle d\omega - o(1) \Big) \\
&= - \int_{A_\rho} \langle d\mathcal{F}, d\mathbf{J}_{\mathcal{F}}(\phi^1, 0) \rangle d\omega.
\end{aligned}$$

Then, the computation in [Ki] (Theorem A.1.) delivers, $\mathcal{F} := \mathcal{F}_\rho(x^1, a_2) \in H^{2,2}(A_\rho, N)$. Now, as in Proposition 3.1, we have, $\langle \frac{d\mathcal{F}}{d\vec{n}}, \frac{\partial \mathcal{F}}{\partial \theta} \rangle_h|_{\partial B} \equiv 0$, and clearly $\langle \frac{d\mathcal{F}}{d\vec{n}}, \frac{\partial \mathcal{F}}{\partial \theta} \rangle_h|_{\partial B_\rho} \equiv 0$.

As a consequence, for $z = re^{i\theta}$ $\Phi(z) = r^2 \Big| \frac{\partial}{\partial r} \mathcal{F} \Big|_h^2 - \big| \frac{\partial}{\partial \theta} \mathcal{F} \big|_h^2 - 2ir \Big\langle \frac{\partial}{\partial r} \mathcal{F}, \frac{\partial}{\partial \theta} \mathcal{F} \Big\rangle_h$ is real constant.

Now, from the form of $\frac{\partial}{\partial \rho} \mathcal{E}$ in Lemma 3.1, the above holomorphic function must be 0, if we show that $\frac{\partial}{\partial \rho} \mathcal{E}(x^1, a_2, \rho) = 0$, and for this we can here handle the same computation as in [St3] for our case. And the the conformal property of $\mathcal{F} := \mathcal{F}_\rho(x^1, a_2)$ is proved.

II) We now have a harmonic, conformal map $\mathcal{F} := \mathcal{F}_\rho(x^1, a_2) \in H^{1,2} \cap C^0(\overline{A_\rho}, N)$, and we will see that $\mathcal{F}$ must be a constant map. We use the idea as in [Jo] Theorem 8.2.3.

Consider the complex plane with positive imaginary part, $\mathbb{C}^+ = \{\theta + ir | r > 0\}$ and let

$$\mathcal{F}((r+\rho)e^{i\theta}) =: \widetilde{X}(\theta, r), \quad \text{well defined on } \mathbb{R} \times [0, 1-\rho]$$

with $\widetilde{X}(\theta, 0) = \mathcal{F}(\rho e^{i\theta}) \equiv a_2$ and $\frac{\partial^m \widetilde{X}}{\partial \theta^m}|_{\{r=0\}} \equiv 0$ for each m. Choosing an appropriate local coordinate chart in a neighborhood of a_2, we may assume that $\widetilde{X}(\theta, 0) = 0$.
Since $\mathcal{F}$ is conformal and harmonic, $\mathcal{F}|_{A_\rho \cup \partial B_\rho} \in C^\infty$(from [HKW]), and by simple computation, $\frac{\partial^m}{\partial \theta^m}\widetilde{X} \equiv \frac{\partial^m}{\partial r^m}\widetilde{X} \equiv 0$ on $\{r = 0\}$, $m \in \mathbb{N}$.

For some ρ_0, $\Omega := \{\theta + ir | \theta \in \mathbb{R},\ r \in [0, 1-\rho_0)\}$, $\Omega^- := \{\theta + ir | \theta \in \mathbb{R},\ -r \in [0, 1-\rho_0)\}$. Expanding $\widetilde{X}$ to $\Omega \cup \Omega^- =: \widetilde{\Omega}$ by reflection, $\widetilde{X} \in C^\infty(\widetilde{\Omega}, N)$. Then, from the harmonicity of $\mathcal{F}$, it holds that $|\widetilde{X}_{z\bar{z}}| \leq C|\widetilde{X}_z|$, where $\partial_z := \frac{1}{2}(\partial_\theta - i\partial_r)$, $\partial_{\bar{z}} := \frac{1}{2}(\partial_\theta + i\partial_r)$.

Furthermore, for all $m \in \mathbb{N}$, $\frac{\partial^m}{\partial \theta^m}\widetilde{X}(0) = \frac{\partial^m}{\partial r^m}\widetilde{X}(0) = 0$ and $\lim_{z=(\theta,r)\to 0} \widetilde{X}(z)|z|^{-m} = 0$. Hence $\widetilde{X}$ is constant in $\widetilde{\Omega}$ from the Hartman-Wintmer Lemma (see [Jo]). Repeating this finitely many times, we get $\mathcal{F} \equiv a_2$ on $\overline{A_\rho}$. But this cannot occur, because we have assumed that $\operatorname{dist}(\Gamma_1, \Gamma_2) > 0$. Therefore we may exclude this case.

(case3) Suppose that $x^i = \gamma_i \circ x_n^i \to$const.$=: a_i \in \Gamma_i$ in $L^1(\partial B, \mathbb{R}^k), i = 1, 2$.
Then $\Phi(\mathcal{F})$ is real constant, $\mathcal{F} := \mathcal{F}(x^1, x^2)$.

Similarly to the second case, supposing $\frac{d}{d\rho}E(\mathcal{F}) \neq 0$, we have for some fixed $t, \delta > 0$ and large $n \geq n_0$, $\left| \int_0^{2\pi} \int_{\rho+\delta}^{1-t} \left[\left|\frac{\partial}{\partial r}\mathcal{F}_{\rho_n}\right|_h^2 - \frac{1}{r^2}\left|\frac{\partial}{\partial r}\mathcal{F}_{\rho_n}\right|_h^2 \right] \frac{1}{1-\rho-\delta} dr d\theta \right| = C > 0$. Letting

$$\widetilde{\mathcal{F}}_n^\sigma := \begin{cases} \mathcal{F}_{\rho_n} & , \text{ on } A_{1-t}, \\ \mathcal{F}_{\rho_n} \circ \tau_{(1-t)\sigma}^{\rho+\delta} & , \text{ on } A_\sigma \backslash A_{1-t}, \\ \mathcal{F}_{\rho_n}(\frac{\rho+\delta}{\sigma} r, \theta) & , \text{ on } A_{\frac{\sigma \rho_n}{\rho+\sigma}} \backslash A_\sigma, \end{cases}$$

we obtain, $2\frac{d}{d\sigma}E(\widetilde{\mathcal{F}}_n^\sigma)|_{\sigma=\rho+\delta} = \int_0^{2\pi} \int_{\rho+\delta}^{1-t} \left[|\partial_r \mathcal{F}_{\rho_n}|^2 - \frac{1}{r^2}|\partial_\theta \mathcal{F}_{\rho_n}|^2 \right] \frac{1-t}{1-t-\rho-\delta} dr d\theta$.
Since $\widetilde{\mathcal{F}}_n^{\rho+\delta} = \mathcal{F}_{\rho_n}$, it follows, $\rho_n |g_3(x_n)| = |\rho_n \frac{d}{d\sigma}E(\mathcal{F}_{\rho_n})|_{\sigma=\rho_n}| = |(\rho+\delta)\frac{d}{d\sigma}E(\widetilde{\mathcal{F}}_n^{\rho+\delta})| \geq C > 0$, contradicting the assumption, $g_3(x_n) \to 0$. Thus, $\mathcal{F}_\rho(a_1, a_2)$ is also conformal. From the same argument as in **(case2)**, we can also exclude this case.

(case4) Suppose that $\rho = 0$.

It holds $\mathcal{F}^i(x_n^i) \circ \tau_n^i = \mathcal{F}^i(\tilde{x}_n^i)$, $\tilde{x}_n^i \in M^{i*}$, where τ_n^i are conformal. We may let $\tilde{x}_n^i \to x^i \in M^{i*}$ in C^0. From the topology of $\overline{\mathcal{M}^*}$ we have thus $g(\tilde{x}_n^1, \tilde{x}_n^2, 0) \to 0$ as $n \to \infty$, and $\{x_n\}$ converges to $(x^1, x^2, 0) \in \partial\mathcal{M}^*$ with $g(x^1, x^2, 0) = 0$. $\square$

4.2 Unstable minimal surfaces of annulus type

We need some Lemmata as in [St3] for our case.

Lemma 4.1. *For any $\delta > 0$, there exists a uniformly bounded, continuous vectorfield $e_\delta : M^1 \times M^2 \times [0,1) \to \mathcal{T}_{M^1} \times \mathcal{T}_{M^2} \times \mathbb{R}$, with locally Lipschitz continuity on $\mathcal{M}$ and $\partial\mathcal{M}$(separably) with the following properties,*

(i) for $\beta \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for any $x \in \mathcal{M}(\rho) := \{x = (x^1, x^2, \rho) \in \mathcal{M}\}$ with $\mathcal{E}(x) \leq \beta, 0 < \rho < \varepsilon$ it holds that $y_\delta(x) = (\widetilde{\exp}_{x^1} e^1_\delta(x^1), \widetilde{\exp}_{x^2} e^2_\delta(x^2), \rho + e^3_\delta(\rho)) \in \mathcal{M}(\rho)$, namely $e^3_\delta(\rho) = 0$ (e_δ is parallel to $\partial\mathcal{M}$ near $\partial\mathcal{M}$),

(ii) for any such $\beta, \mathcal{E}, x$ as above and any pair $T = (\tau^1, \tau^2)$ of conformal transformations of B, $y_\delta(x \circ T) = y_\delta(x) \circ T$, where $x \circ T$ satisfies $\mathcal{F}^i\left((x \circ T)^i\right) = \mathcal{F}^i(x^i) \circ T, \quad i = 1, 2,$

(iii) for any $x \in \overline{\mathcal{M}}$, $\langle d\mathcal{E}(x), e_\delta(x)\rangle_{\mathcal{T}_{x^1} \times \mathcal{T}_{x^2} \times \mathbb{R}} \leq \delta - g(x)$,

(iv) $y_\delta(x) \in \mathcal{M}^*$ resp. $\partial\mathcal{M}^*$, for all $x \in \mathcal{M}^*$ resp. $\partial\mathcal{M}^*$.

Proof. Because of Remark 4.1, one can easily prove this using the idea in the proof of the corresponding Lemma in [St3]. □

Lemma 4.2. *For a given a vector field $f : \overline{\mathcal{M}} \to \mathcal{T}_{M^1} \times \mathcal{T}_{M^2} \times \mathbb{R}$ which is locally Lipschitz continuous with the properties in Lemma 4.1, there exists a unique flow $\Phi : [0, \infty) \times \overline{\mathcal{M}^*} \to \overline{\mathcal{M}^*}$ satisfying*

$$\Phi(0, x) = x, \;\; \frac{\partial}{\partial t}\Phi(t, x) = f\left(\Phi(t, x)\right), \quad x \in \overline{\mathcal{M}^*}.$$

Proof. We use the Euler's method. Let's first define $\Phi^{(m)} : [0, \infty) \times \overline{\mathcal{M}^*} \to \overline{\mathcal{M}^*}, \;\; m \geq m_0$:

$$\begin{aligned} \Phi^{(m)}(0, x) &:= x \\ \Phi^{(m)}(t, x) &:= \widetilde{\exp}_{\Phi^{(m)}(\frac{[mt]}{m}, x)}\Big(\frac{mt - [mt]}{m} f\big(\Phi^{(m)}(\frac{[mt]}{m}, x)\big)\Big), \;\; t > 0 \end{aligned} \tag{22}$$

where $[\tau]$ denotes the largest integer which is smaller than $\tau \in \mathbb{R}$. This is well defined from the convexity of $\mathcal{T}_{x^i}$, $x^i \in M^i, i = 1, 2$ and from Lemma 4.1 *(iv)*.
Recalling a map $w^i \in C^0(\mathbb{R}, \mathbb{R})$ with $x^i = \gamma^i \circ w^i, \;\; x^i \in M^i$, consider

$$W^i := \{w^i \in C^0(\mathbb{R}, \mathbb{R}) : \gamma^i \circ w^i = x^i \text{ for some } x^i \in M^i\}, \quad W := W^1 \times W^2 \times [0, \infty).$$

Letting $\gamma(w) := (\gamma^1 \circ w^1, \gamma^2 \circ w^2, \rho)$ for $(w^1, w^2, \rho) =: w \in W$, $\gamma := (\gamma^1, \gamma^2, Id)$ and $\widetilde{f} := (\widetilde{f}^1, \widetilde{f}^2, f^3)$ with $\widetilde{f}^i(w^i) := (d\gamma^i)^{-1}(f^i(x^i)) \in C^0(\mathbb{R}/2\pi, \mathbb{R})$, there exists $\widetilde{\Phi}^{(m)}(t, w) \in W$ with $\Phi^{(m)}(t, x) = \gamma(\widetilde{\Phi}^{(m)}(t, w))$, so we can rewrite (22),

$$\widetilde{\Phi}^{(m)}(t, w) \;\; = \;\; \widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w) + \frac{mt - [mt]}{m} \widetilde{f}\big(\widetilde{\Phi}^{(m)}(\frac{[mt]}{m}, w)\big) + 2\pi l, \;\; l \in Z.$$

And for $t \in (\frac{k}{m}, \frac{k+1}{m}], k \in \mathbb{Z}$, $\widetilde{\Phi}^{(m)}(t,w) = \widetilde{\Phi}^{(m)}(0,w) + \int_0^t \widetilde{f}(\widetilde{\Phi}^{(m)}(\frac{[ms]}{m}, w))ds$.

Now we can compute further as in Euclidean case ; For any $T > 0$, $G > 0$, there exists $C(T,G)$ with $\|\Phi^{(m)}(\cdot, w)\|_{L^\infty([0,T]\times W_K, \overline{\mathcal{M}})} \le C(T,G)$ for $w \in W$ with $\|w\|_W \le G$.

Let L_1 resp. L_2 be the Lipschitz constants of f in $\{x \in \mathcal{M} \,|\, \|x\| \le C(T,G)\}$ resp. $x \in \{\partial\mathcal{M} \,|\, \|x\| \le C(T,G)\}$ and $L := \max\{C(\gamma^i)L_1, C(\gamma^i)L_2\}$.
Then for $\frac{m}{n} < 1$,

$\|\widetilde{\Phi}^{(m)}(t,w) - \widetilde{\Phi}^{(n)}(t,w)\| \le tL\frac{2}{m}C(f) + tL\|\widetilde{\Phi}^{(m)}(\cdot,w) - \widetilde{\Phi}^{(n)}(\cdot,w)\|_{L^\infty([0,t],W)}$.

Hence, for $m, n \ge m_0$, we have that

$\|\widetilde{\Phi}^{(m)}(\cdot,w) - \widetilde{\Phi}^{(n)}(\cdot,w)\|_{L^\infty([0,t],W)} \le tL(\frac{2}{m} + \frac{2}{n})C(f) + tL\|\widetilde{\Phi}^{(m)}(\cdot,w) - \widetilde{\Phi}^{(n)}(\cdot,w)\|_{L^\infty([0,t],W)}$.

Choosing $t \le \min\{T, \frac{1}{2L}\}$, $\{\widetilde{\Phi}^{(m)}\}$ converges to some function $\widetilde{\Phi}$, uniformly on $[0,t] \times \{w \in W : \|w\| \le G\}$ as $m \to \infty$. And we may conclude, $\frac{\partial}{\partial t}\widetilde{\Phi}(t,w) = \widetilde{f}\big(\widetilde{\Phi}(t,w)\big)$.

Letting $\Phi(t,w) := \gamma \circ \widetilde{\Phi}(t,w) \in \overline{\mathcal{M}}^*$, from the uniformly boundednes of f we get the flow Φ for each $x \in \overline{\mathcal{M}}$ with $\frac{\partial}{\partial t}\Phi(t,w) = d\gamma\Big(\widetilde{f}\big(\widetilde{\Phi}^{(m)}(t,w)\big)\Big) = f\big(\Phi(t,w)\big)$. Similarly, we prove that $\Phi(t,w)$ depends continuously on the initial datum, and it can be continued for $t > 0$. $\square$

In the following Lemma we have a somewhat weaker result than the corresponding Lemma 4.15 in [St3]. But this result is enough for our aim.

Lemma 4.3. *Let $\mathcal{F}^i(x_0^i)$ be a solution of $\mathcal{P}(\Gamma_i)$ for some $x_0^i \in M^i$, $i = 1,2$. And suppose that $d := dist(\mathcal{F}^1(x_0^1), \mathcal{F}^2(x_0^2)) > 0$. Then there exists $\varepsilon > 0$, $\rho_0 \in (0,1)$ and $C > 0$, dependent on $\mathcal{E}(x_0^1, x_0^2, 0)$ such that for $x^i \in M^i$ with $\|x^i - x_0^i\|_{\frac{1}{2},2;0} =: s(x^i) < \varepsilon$,*

$$\mathcal{E}(x^1, x^2, \rho) \ge \mathcal{E}(x^1, x^2, 0) + \frac{Cd^2}{|\ln\rho|}, \quad \text{for all } \rho \in (0, \rho_0).$$

Proof.(I) Let $\mathcal{F}_\rho := \mathcal{F}_\rho(x^1, x^2)$ $\mathcal{F}^i := \mathcal{F}^i(x^i), i = 1,2$. We choose σ_1 and δ such that $\sqrt{\rho} < \delta < \sigma_1 < \sqrt{\sqrt{\rho}}$. Letting $T(re^{i\theta}) := \rho\frac{1}{re^{i\theta}}$ and $\sigma_2 := \frac{\rho}{\delta}$, we take $f_{\sigma_1} := \mathcal{F}_\rho|_{A_{\sigma_1}}$ and $g_{\sigma_2} := \mathcal{F}_\rho|_{B_\delta \setminus B_\rho}(T^{-1})$, and then

$$E(\mathcal{F}_\rho) = E(f_{\sigma_1}) + E(\mathcal{F}_\rho|_{B_{\sigma_1}\setminus B_\delta}) + E(g_{\sigma_2}). \tag{23}$$

For the estimate of $E(f_{\sigma_1})$ we take $a_1 \in N$ with $\min_{a\in N} E(\mathcal{F}_{\sigma_1}(x^1, a)) = \mathcal{F}_{\sigma_1}(x^1, a_1) =: \mathcal{F}_{\sigma_1}^1$.

We next define, $\widetilde{\mathcal{F}_{\sigma_1}^1} : B \to N$ as follows: Let $\widetilde{\mathcal{F}_{\sigma_1}^1}|_{B\setminus B_{\frac{1}{2}}} := \mathcal{F}^1_{B\setminus B_{\frac{1}{2}}}$, $\widetilde{\mathcal{F}_{\sigma_1}^1}|_{B_{\frac{1}{2}}\setminus B_{\sigma_1}}$ be harmonic in N with boundary $\mathcal{F}^1|_{\partial B_{\frac{1}{2}}}$ on $\partial B_{\frac{1}{2}}$ and $\mathcal{F}^1(0)$ on ∂B_{σ_1}, and $\widetilde{\mathcal{F}_{\sigma_1}^1}|_{B_{\sigma_1}} \equiv \mathcal{F}^1(0)$.

We now estimate, $2E(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1) = \underbrace{\int_{B_{\frac{1}{2}}\setminus B_{\sigma_1}} |\nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1)|^2 d\omega}_{=:a} + \underbrace{\int_{B_{\sigma_1}} |\nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1)|^2 d\omega}_{=:b}.$

It is easy to see that $b \le C|\sigma_1|^2$, since $\mathcal{F}^1$ is regular on $B_{\frac{1}{2}}$.
We observe, $\widetilde{\mathcal{F}^1_{\sigma_1}}|_{B_{\frac{1}{2}}\setminus B_{\sigma_1}} \in H^{2,2}$, since $\widetilde{\mathcal{F}^1_{\sigma_1}}|_{\partial B_{\frac{1}{2}}}$ is regular and contant on ∂B_{σ_1}. Thus,

$$\begin{aligned} a &= \int_{\partial B_{\frac{1}{2}}} \langle \nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1)\vec{n}, \widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1\rangle d_0 + \int_{\partial B_{\sigma_1}} \langle \nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1)\vec{n}, \widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1\rangle d_0 \\ &\le C\|\mathcal{F}^1(0) - \mathcal{F}^1|_{\partial B_{\sigma_1}}\|_0 \sigma_1 \le C|\sigma_1|^2, \quad \text{with } C = C(E(\mathcal{F}^1(x^1))). \end{aligned}$$

Letting $\mathcal{F}^1_{\sigma_1}|_{B_{\sigma_1}} \equiv a_1$ it hols, $E(\mathcal{F}^1) \le E(\mathcal{F}^1_{\sigma_1}) \le E(\widetilde{\mathcal{F}^1_{\sigma_1}})$. Now from Remark 4.2,

$$\begin{aligned} E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1) &\le E(\mathcal{F}^1_{\sigma_1}) - E(\mathcal{F}^1) + o_s(1) \\ &\le E(\widetilde{\mathcal{F}^1_{\sigma_1}}) - E(\mathcal{F}^1) + o_s(1) \le E(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1) + o_s(1) \le C|\sigma_1|^2 + o_s(1), \end{aligned} \tag{24}$$

where $o_s(1) \to 0$ as $\|x^1 - x^1_0\|_{\frac{1}{2},2;0} =: s(x^1) \to 0$.
Since $E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)|_{B_{\sigma_1}} \le C|\sigma_1|^2 + o_s(1)$, we have that $E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)|_{A_{\sigma_1}} \le C|\sigma_1|^2 + o_s(1)$.

For $X^1 := f_{\sigma_1} - \mathcal{F}^1_{\sigma_1}$, $\big|\int_{A_{\sigma_1}} \nabla(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)\nabla X^1 d\omega\big| \le C\sigma_1\big(\int_{A_{\sigma_1}} |\nabla(f_{\sigma_1} - \mathcal{F}^1_{\sigma_1})|^2 d\omega\big)^{\frac{1}{2}} \le C\sigma_1$.

On the other hand,

$$\begin{aligned} |a_1 - \mathcal{F}^1(0)|^2 = \left|\int_{\sigma_1}^1 \partial_r(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1_{\sigma_1}) dr\right|^2 &\le (1-\sigma_1)\int_{\sigma_1}^1 \left|\nabla(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1_{\sigma_1})\right|^2 dr \\ &\le \frac{1-\sigma_1}{\sigma_1}(E(\widetilde{\mathcal{F}^1_{\sigma_1}} - \mathcal{F}^1) + E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)) \le C\sigma_1 + o_s(1). \end{aligned} \tag{25}$$

From the above we have that

$$\begin{aligned} \Big|\int_{A_{\sigma_1}} \langle \nabla\mathcal{F}^1_{\sigma_1}, \nabla X^1\rangle d\omega\Big| &\le \Big|\int_{A_{\sigma_1}} \langle \nabla\mathcal{F}^1, \nabla X^1\rangle d\omega\Big| + C\sigma_1 \\ &\le \|\nabla\mathcal{F}^1|_{\partial B_{\sigma_1}}\|\|(-a_1 + \mathcal{F}_\rho|_{\partial B_{\sigma_1}})\|\sigma_1 + C\sigma_1 \le C\sigma_1. \end{aligned}$$

Now we can compute, with $C \in \mathbb{R}$ depending on $E(\mathcal{F}^1)$

$$E(f_{\sigma_1}) = E(\mathcal{F}^1_{\sigma_1}) + \int_{A_{\sigma_1}} \langle \nabla\mathcal{F}^1_{\sigma_1}, \nabla X^1\rangle d\omega + E(X^1) \ge E(\mathcal{F}^1) - C\sigma_1. \tag{26}$$

Similarly we have $E(g_{\sigma_2}) \ge E(\mathcal{F}^2) - C\sigma_2$, C depends on $E(\mathcal{F}^2)$.

(II) Here we will estimate $E(\mathcal{F}_\rho|_{B_{\sigma_1}\setminus B_\delta})$.

From (25), $|a_1 - a_2| \geq ||\mathcal{F}^1(0) - \mathcal{F}^2(0)| - |a_1 - \mathcal{F}^1(0) + \mathcal{F}^2(0) - a_2|| \geq d - o_\rho(1) - o_s(1)$.
Let $\mathcal{H}_b^a(f, g)$ be the harmonic map on $B_a \backslash B_b$ in $\mathbb{R}^k$ with boundary f on ∂B_a and g on ∂B_b.
Writing $\sigma_1 =: \sigma$, $\frac{\delta}{\sigma^1} =: \tau$, $\mathcal{F}_\rho|_{\partial B_{\sigma_1}} =: p$, $\mathcal{F}_\rho|_{\partial B_\delta} =: q$,

$$\begin{aligned}
&\Big|\int \langle \nabla \mathcal{H}_\delta^\sigma(a_1, a_2), \nabla \mathcal{H}_\delta^\sigma(-a_1 + p, -a_2 + q)\rangle d\omega\Big| \\
= \;& \Big|\int \langle \nabla \mathcal{H}_\tau^1(0, -a_1 + a_2), \nabla \mathcal{H}_\tau^1(-a_1 + p(\cdot\sigma), -a_2 + q(\cdot\sigma)\rangle d\omega\Big| \\
\leq \;& \frac{2\pi}{|\ln \tau|}| - a_1 + a_2|\big(| - a_1 + p(\cdot\sigma)| + | - a_2 + q(\cdot\sigma)|\big) \leq C\frac{(o_\rho(1) + o_s(1))}{|\ln \rho|}.
\end{aligned}$$

And

$$E(\mathcal{H}_\delta^\sigma(a_1, a_2)) \;\geq\; E(\mathcal{H}_\rho^1(0, -a_1 + a_2)) = E\big((-a_1 + a_2)\frac{\ln r}{\ln \rho}\big) \geq \frac{\pi d^2}{|\ln \rho|} - C\frac{(o_\rho(1) + o_s(1))}{|\ln \rho|}.$$

Thus, with C depending only on $E(\mathcal{F}^i)$

$$\begin{aligned}
E(\mathcal{F}_\rho|_{B_\sigma \backslash B_\delta}) \;&\geq\; E(\mathcal{H}_\delta^\sigma(p, q)) = E\big(\mathcal{H}_\delta^\sigma(a_1, a_2) + \mathcal{H}_\delta^\sigma(-a_1 + p, -a_2 + q)\big) \\
&\geq\; \frac{\pi d^2}{|\ln \rho|} - C\frac{o_\rho(1) + o_s(1)}{|\ln \rho|}. \qquad (27)
\end{aligned}$$

(III) From (23), (26), (27) and the choice of σ_i,

$$\begin{aligned}
&\mathcal{E}(x^1, x^2, \rho) \geq \mathcal{E}(x^1, x^2, 0) - C\sigma_i + \frac{\pi d^2}{|\ln \rho|} - C\frac{(o_{\rho(1)} + o_{s(1)})}{|\ln \rho|} \\
&\quad \geq \;\mathcal{E}(x^1, x^2, 0) - C(\sqrt{\rho} + \sqrt{\sqrt{\rho}}) + \frac{\pi d^2}{|\ln \rho|} - C\frac{(o_\rho(1) + o_{s(1)})}{|\ln \rho|} \geq \mathcal{E}(x^1, x^2, 0) + C\frac{d^2}{|\ln \rho|},
\end{aligned}$$

for $\rho \leq \rho_0$, for some small $\rho_0 \in (0, 1)$ and small $s(x^i)$. □

Remark 4.2. *With the same notations as in Lemma 4.3, it holds that*

$$E(\mathcal{F}_{\sigma_1}^1 - \mathcal{F}^1) \leq E(\mathcal{F}_{\sigma_1}^1) - E(\mathcal{F}^1) + o_s(1).$$

Proof. Let $G^1 := \mathcal{F}^1(x_0^1)$, $G_{\sigma_1}^1 := \mathcal{F}_{\sigma_1}(x_0^1, a^1) = \min_{a \in N} E(\mathcal{F}_{\sigma_1}(x_0^1, a))$.
Observe, $\|\mathcal{F}^1(x_0^1) - \mathcal{F}^1\|_{1,2;0} \to 0$, $\|\mathcal{F}_{\sigma_1}^1 - G_{\sigma_1}^1\|_0 \to 0$ as $\|x^1 - x_0^1\|_{\frac{1}{2},2;0} =: s \to 0$. By the Hölder inequality, $\int_B \langle II \circ \mathcal{F}^1(d\mathcal{F}^1, d\mathcal{F}^1), \mathcal{F}_{\sigma_1}^1 - \mathcal{F}^1\rangle d\omega - \int_B \langle II \circ G^1(dG^1, dG^1), G_{\sigma_1}^1 - G^1\rangle d\omega \; = o_s(1)$.

Since $G^1 \in H^{2,2}$, it holds, $0 = \int_B \langle \nabla G^1, \nabla(G^1_{\sigma_1} - G^1)\rangle d\omega = \int_B \langle II \circ G^1(dG^1, dG^1), G^1_{\sigma_1} - G^1\rangle$, and $\nabla(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)\rangle d\omega = o_s(1)$. Hence

$$\begin{aligned}
2E(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1) &\leq \int_B \langle \nabla \mathcal{F}^1_{\sigma_1}, \nabla(\mathcal{F}^1_{\sigma_1} - \mathcal{F}^1)\rangle d\omega + o_s(1) \\
&\leq \int_B |\nabla \mathcal{F}^1_{\sigma_1}|^2 d\omega - \int_B \langle \nabla \mathcal{F}^1, \nabla \mathcal{F}^1_{\sigma_1} - \nabla \mathcal{F}^1\rangle d\omega - \int_B |\nabla \mathcal{F}^1|^2 d\omega + o_s(1) \\
&\leq \int_B |\nabla \mathcal{F}^1_{\sigma_1}|^2 d\omega - \int_B |\nabla \mathcal{F}^1|^2 d\omega + o_s(1).
\end{aligned}$$

□

Now we can say the following results: Let $\Gamma_1, \Gamma_2 \subset (N, h)$ satisfy (C1) or (C2).

Theorem 4.1. *Let*

$$\begin{aligned}
d &= \inf\{E(X)\,|X \in \mathcal{S}(\Gamma_1, \Gamma_2)\} \\
d^* &= \inf\{E(X^1) + E(X^2)\,|X^i \in \mathcal{S}(\Gamma_i), i = 1, 2\}.
\end{aligned}$$

If $d < d^*$, there exists a minimal surface of annulus type bounded by Γ_1 and Γ_2.

Proof. The (P.S.) condition (Lemma 4.1) and Proposition 3.1 delivers the result, following the arguments with a minimal sequence. For details we refer to [St1]. □

Theorem 4.2. *For $\mathcal{F}^1$, resp. $\mathcal{F}^2$, an absolute minimizer of E in $\mathcal{S}(\Gamma_1)$, resp. $\mathcal{S}(\Gamma_2)$, suppose that dist$(\mathcal{F}^1, \mathcal{F}^2) > 0$ and suppose furthermore there exists a strict relative minimizer of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$. Then there exists a solution of $\mathcal{P}(\Gamma_1, \Gamma_2)$ which is not a relative minimizer of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$, i.e. an unstable annulus type minimal surface or there exists a pair of solutions to $\mathcal{P}(\Gamma_1)$, $\mathcal{P}(\Gamma_2)$ one of which does not yield an absolute minimizer of E(in $\mathcal{S}(\Gamma_1)$ or $\mathcal{S}(\Gamma_2)$).*

Proof. We can write that $\mathcal{F}^i := \mathcal{F}^i(x^i)$, for some $x^i \in M^{i*}$, $i = 1, 2$, moreover, for some $y \in \mathcal{M}^*$, $\mathcal{F}(y)$ is the strict relative minimum of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$. Clearly, y is also a strict relative minimizer of $\mathcal{E}$ in $\mathcal{M}^*$. Letting $x = (x^1, x^2, 0)$, consider

$$P = \{p \in C^0([0,1], \overline{\mathcal{M}}) | p(0) = x, p(1) = y\},$$

and

$$\beta := \inf_{p \in P} \max_{t \in [0,1]} \mathcal{E}(p(t)).$$

From the (P.S.) condition, we first observe, if $\beta > \max\{\mathcal{E}(x), \mathcal{E}(y)\}$, β is then a critical value which possesses a non relative minimum critical point, and we have actually that $\beta > \mathcal{E}(y)$, since y is a strict relative minimizer. For details we refer to [St1] chapter II and [Ki].

Now supposing that any solution of $\mathcal{P}(\Gamma_i)$ is an absolute minimum of E in $\mathcal{S}(\Gamma_i)$ we have a solution of $\mathcal{P}(\Gamma_1, \Gamma_2)$, which is not a relative minimum of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$ from the E-minimality of harmonic extensions.

It remains to show that $\beta := \inf_{p \in P} \max_{t \in [0,1]} \mathcal{E}(p(t)) > \mathcal{E}(x)$.

We only need to consider $q = (q^1, q^2, \rho) \in p([0,1])$ for some $p \in P$ such that $\mathcal{E}(q^1, q^2, 0) \leq C$ for some constant C, dependent on N.

Let ε, ρ_0 be as in Lemma 4.3, and consider the set of q with $\|q^i - \tilde{x}^i\| \geq \varepsilon$ for any absolute minimizer $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, 0)$ of $\mathcal{E}$ in $\partial\mathcal{M}$. Then there exists $\delta_1 > 0$, dependent on ε, such that $\mathcal{E}(q^1, q^2, 0) \geq \mathcal{E}(x) + \delta_1$ for all but finitely many q, otherwise, we have a minimizing sequence which converges to some absolute minimizer $\tilde{x}$ by the (P.S.) condition (Lemma 4.1) and Proposition 3.1, contradicting the choice of q.

Moreover, from the uniform convergence of $\mathcal{E}$ as $\rho \to 0$ on a bounded set of q^i (see the proof of Lemma 3.1), we can choose δ_2, ρ_1 with $\delta_1 - \delta_2 > 0$, such that for all $\rho \in (0, \rho_1)$, $|\mathcal{E}(q^1, q^2, \rho) - \mathcal{E}(q^1, q^2, 0)| \leq \delta_2$.

Now let $\bar{\rho} := \min\{\rho_0, \rho_1\}$. If $\|q^i - \tilde{x}^i\| < \varepsilon$ for some $\tilde{x}$ as above, then $\mathcal{E}(q^1, q^2, \bar{\rho}) \geq \mathcal{E}(x) + \delta_3$ for some $\delta_3 > 0$, from Lemma 4.3. Otherwise we have, $\mathcal{E}(q^1, q^2, \bar{\rho}) \geq \mathcal{E}(q^1, q^2, 0) - \delta_2 \geq \mathcal{E}(x) + \delta_1 - \delta_2$, from the above choices. This compeletes the proof. $\square$

Now we apply the main result to the case of the three-dimensional sphere S^3 and the three-dimensional hyperbolic space H^3. We consider the case of condition (C1).

Corollary 4.1. *Let $\Gamma_1, \Gamma_2 \subset B(p, \pi/2)$ for some $p \in S^3$, in other words Γ_1, Γ_2 are in a (three-dimensional) hemisphere. We have then the same results as in the main theorem under the same condition.*

If there exists exactly one solution of $\mathcal{P}(\Gamma_i)$, $i = 1, 2$, the main theorem says, the existence of a minimal surface of annulus type whose energy is a strict relative minimum of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$ ensures the existence of an unstable minimal surface of annulus type.

From [LJ], the solution of $\mathcal{P}(\Gamma_i)$ is unique in the 3-dimensional hyperbolic space H^3, if the total curvature of Γ_i is less than 4π. Noting that $i(p) = \infty$ for all $p \in H^3$ we have the following result.

Corollary 4.2. *Let Γ_1, Γ_2 possess total curvature $\leq 4\pi$ in H^3 and dist$(\mathcal{F}^1, \mathcal{F}^2) > 0$. If there exists a strict relative minimizer of E in $\mathcal{S}(\Gamma_1, \Gamma_2)$, then there is an unstable minimal surface of annulus type in H^3.*

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