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On the Lipschitz property of the free boundary for parabolic obstacle problem

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Abstract

In this note w consider a parabolic obstacle problem with zero constraint. Without any additional assumptions on a free boundary we prove that a free boundary at interior points of a domain, lying near the fixed boundary, is a graph of a $C^{1+\alpha}$-function. Bibliography: 8 titles.

In this paper we study the regularity properties of a free boundary in a neighborhood of the fixed boundary of a domain for a parabolic obstacle problem with zero constraint.
For parabolic equations the simplest obstacle problem can be formulated as follows: let $\mathbb{D}$ be a domain in $\mathbb{R}^n$, $\mathbb{Q} = \mathbb{D} \times]0, T[$,

$$\mathbb{K} = \{w \in H^1(\mathbb{Q}) : \quad w \geqslant 0 \quad \text{a.e. in } \mathbb{Q}, \quad w = \phi \quad \text{on } \partial' \mathbb{Q}\},$$

where ϕ be a nonnegative function defined on the parabolic boundary $\partial'\mathbb{Q}$ of the cylinder $\mathbb{Q}$. It is required to find a function $u \in \mathbb{K}$ such that

$$\int_{\mathbb{D}} \partial_t u(w-u)dx + \int_{\mathbb{D}} DuD(w-u)dx + \int_{\mathbb{D}} (w-u)dx \geqslant 0$$

a.e. in $t \in]0, T[$, and for all $w \in \mathbb{K}$.
It is known that a solution u of the problem, formulated above, satisfies (in the sense of distributions) the equation

$$\Delta u - \partial_t u = \chi_\Omega \quad \text{in } \mathbb{Q},$$

where $\Omega = \{(x,t) \in \mathbb{Q} : u(x,t) > 0\}$.
It cannot be ruled out that the free boundary $\partial\Omega$ touches $\partial'\mathbb{Q}$ at points where $\phi = 0$, that is, there can exist the points of contact.
Even the regularity of the free boundary on a distance from the fixed boundary for this problem has been investigated earlier only in the special case of the Stefan problem ([C1]), where boundary conditions guarantee the additional information $\partial_t u \geqslant 0$. The nonnegativity of the time-derivative of solutions has been used in [C1] to prove the fact that $\partial_t u$ is continuous at the points of the free boundary.
The latter (i.e., the continuity of $\partial_t u$) is quite essential for investigation the regularity properties of the free boundary. For instance, I. Athanasopoulos and S. Salsa have proved the following:

Theorem. ([AtSa]) *Let* $v(x,t) \geqslant 0$ *in* $D_R := B_R(x^0)\times]t^0 - R^2, t^0 + R^2[$, *with* $(x^0,t^0) \in \partial\{v > 0\}$, *let* v *be a solution of the equation* $\Delta v - \partial_t v = 1$ *in* $D_R \cap \{u > 0\}$, *and let* $\partial_t v \in C(D_{R-\varepsilon})$ *for any* $\varepsilon > 0$.
Suppose also that in some spatial direction, say e_1, v *is monotone, (i.e.,* $D_{e_1} v \geqslant 0$) *and* $\partial\{v > 0\}$ *is* x_1*-graph of a Lipschitz function* f. *Then* f *is a* $C^{1+\alpha}$*-function for some* $0 < \alpha < 1$.

It should be especially noted that the above Theorem relates to $C^{1+\alpha}$-regularity of $\partial\{v > 0\}$ only at interior points of D_R. Unfortunately, at the contact points where the free boundary meets the fixed boundary is impossible to establish $C^{1+\alpha}$-regularity. There exists the counterexample showing that the free boundary $\partial\{v > 0\}$ does not touch the fixed boundary in t-direction.
The main result of the present paper is the proof of the Lipschitz continuity of the free boundary in a neighborhood of the fixed boundary. In particular, from here it follows that $\partial\Omega \cap \mathbb{Q}$ is a graph of a $C^{1+\alpha}$-function near the fixed boundary $\partial'\mathbb{Q}$.
Our arguments are based on the blow-up technique, the various monotonicity formulas and the results of [ASU2], concerning the global solutions of the parabolic obstacle problem with zero constraint (i.e., solutions in the entire half-space $\{(x,t) \in \mathbb{R}^{n+1} : x_1 > 0\}$). It should be noted that our arguments do not require any additional assumptions on the free boundary.
Except of the monotonicity formula due to L. A. Caffarelli and C. Kenig (see [C2], [CK] as well as Lemma 2.1 [ASU2]), we use also the Weiss functional introduced by G. S. Weiss during the studies of some kind the free boundary problems in the whole space $\mathbb{R}^{n+1}$. Changing Weiss's notation we denote this functional by W. For a solution u defined in $\mathbb{R}^{n+1}$ and $r > 0$ it is defined as follows:

$$W(r, x^*, t^*, u) := \\ = \frac{1}{r^4} \int_{t^*-4r^2}^{t^*-r^2} \int_{\mathbb{R}^n} \left(|Du|^2 + 2u + \frac{u^2}{t - t^*} \right) G(x - x^*, t^* - t)\, dx dt,$$

where (x^*, t^*) is a free boundary point, whereas

$$G(x,t) = \frac{\exp\left(-|x|^2/4t\right)}{(4\pi t)^{n/2}} \quad \text{for} \quad t > 0 \qquad \text{and} \qquad G(x,t) = 0 \quad \text{for} \quad t \leqslant 0.$$

In the paper [W] it was showed that the functional W is a nondecreasing with respect to r and that the equality $\frac{\partial W}{\partial r} = 0\ \forall r > 0$ is equivalent to the parabolic homogenuity of degree 2 for the function u.
In addition, for our purposes it was essential to take into consideration a local version of the Weiss functional. In particular, it permits us to make

a conclusion about homogenuity of blow-ups limits. We observe that for interior counterpart of our problem such a local version of W was introdused in [CPS]. In the present paper we introduce a modified local version of W which takes into account the existence of the homogenious Dirichlet condition on the fixed boundary.

This paper is organized as follows. Section 1 is devoted to a local version of the Weiss monotonicity formula. In Section 2 we prove that $\partial_t u$ is continuous at the points of the free boundary lying in a neighborhood of the fixed boundary. Finally, in Section 3 we analyze the free boundary near the fixed boundary in the context of regularity theory.

Notations and definitions.

Throughout the paper we will use the following notations:

$z = (x,t)$ are points in $\mathbb{R}^{n+1}$, where $x = (x_1, x') = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $n \geqslant 2$, and $t \in \mathbb{R}^1$; $\mathbb{R}^{n+1}_b = \{(x,t) \in \mathbb{R}^{n+1} : x_1 > b\}$, where $b \in \mathbb{R}$;

$\mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1}_0$;

$\Pi_b = \{(x,t) \in \mathbb{R}^{n+1} : x_1 = b\}$;

$\Pi = \Pi_0$;

$e_1, \dots, e_n$ is the standard basis in $\mathbb{R}^n$;

e_0 is the standard basis in $\mathbb{R}^1_t$;

χ_Ω denotes the characteristic function of the set Ω ($\Omega \subset \mathbb{R}^{n+1}$);

$v_+ = \max\{v, 0\}$;

$B_r(x^0)$ denotes the open ball in $\mathbb{R}^n$ with center x^0 and radius r;

$B^+_r(x^0) = B_r(x^0) \cap \mathbb{R}^{n+1}_+$;

$B_r = B_r(0)$;

$S_r(x^0) = \{x \in \mathbb{R}^n : |x - x^0| = r\}$;

$S_r = S_r(0)$;

$Q_r(z^0) = Q_r(x^0, t^0) = B_r(x^0) \times]t^0 - r^2, t^0[$ is the open cylinder in $\mathbb{R}^{n+1}$;

$Q^+_r(z^0) = Q^+_r(x^0, t^0) = Q_r(x^0, t^0) \cap \mathbb{R}^{n+1}_+$;

$Q_r = Q_r(0,0)$;

$Q^+_r = Q^+_r(0,0)$.

If $Q = \mathbb{R}^{n+1}_b \cap Q_r(x^0, t^0)$ then $\widehat{Q} = \{(x, t^0) : x_1 > b, |x - x^0| < r\}$ is the top of Q and $\partial' Q$ is it's parabolic boundary, i.e., topological boundary minus the top of the cylinder.

D_i denotes the differential operator with respect to x_i; $\partial_t = \frac{\partial}{\partial t}$;

$D = (D_1, D') = (D_1, D_2, \dots, D_n)$ denotes the spatial gradient;

$D^2 = D(D)$ denotes the Hessian;

D_ν denotes the operator of differentiation along the direction $\nu \in \mathbb{R}^n$, i.e., $|\nu| = 1$ and $D_\nu u = \sum_{i=1}^n \nu_i D_i u$;

$H = \Delta - \partial_t$ is the heat operator.

We adopt the convention that the index τ runs from 2 to n. We also adopt the convention regarding summation with respect to repeated indices.
For a non-negative $C^1_x \cap C^0_t$-function u defined in $\mathbb{R}^{n+1}_b \cup \Pi_b$ we introduce the sets
$\Lambda(u) = \{(x,t) \in \mathbb{R}^{n+1}_b \cup \Pi_b : u(x,t) = |Du(x,t)| = 0\}$;
$\Omega(u) = \{(x,t) \in \mathbb{R}^{n+1}_b : u(x,t) > 0\} = \mathbb{R}^{n+1}_b \setminus \Lambda(u)$;
$\Gamma(u) = \partial\Omega(u) \cap \Lambda(u)$ is the free boundary;
$\Gamma(u) \cap \Pi_b$ is the set of contact points.
We use letters M, N, C (with or without indices) to denote various constants. To indicate that, say, N depends on some parameters, we list them in the parentheses: $N(\dots)$.

Let M be a constant, $M \geqslant 1$. We denote by $P^+_r(M,b)$, the **class of "local non-negative solutions"** to the problem, i.e., we say a continuous function u (not identically zero) belongs to the class $P^+_r(M,b)$ if u satisfies:

(a) $H[u] = \chi_\Omega$ in $Q_r \cap \mathbb{R}^{n+1}_b$, for some open set $\Omega = \Omega(u) \subset Q_r \cap \mathbb{R}^{n+1}_b$, and $u = |Du| = 0$ in $\{Q_r \cap \mathbb{R}^{n+1}_b\} \setminus \Omega(u)$,

(b) $u \geqslant 0$ in $Q_r \cap \mathbb{R}^{n+1}_b$, $\quad u = 0$ on $\Pi_b \cap Q_r$,

(c) $\operatorname*{ess\,sup}_{Q_r \cap \mathbb{R}^{n+1}_b} \{|D^2u| + |\partial_t u|\} = M$

and the first equation in (a) is understood in the sence of distributions. For simplicity of notation we will write $P^+_r(M)$ instead of $P^+_r(M,0)$.
We denote by $P^+_\infty(M,b)$, the **class of "global non-negative solutions"** to the problem in the entire "half space" $\mathbb{R}^{n+1}_b$ with quadratic growth in x and linear growth in t, i.e., solutions in $\mathbb{R}^{n+1}_b$ satisfying

$$\operatorname*{ess\,sup}_{\mathbb{R}^{n+1}_b} \big\{|D^2u| + |\partial_t u|\big\} \leqslant M. \tag{0.1}$$

More precisely, we say a continuous function (not identically zero) belongs to the class $P^+_\infty(M,b)$ if u satisfies:

(a') $H[u] = \chi_\Omega$ in $\mathbb{R}^{n+1}_b$, for some open set $\Omega = \Omega(u)$, and $u = |Du| = 0$ in $\mathbb{R}^{n+1}_b \setminus \Omega(u)$,

(b') $u \geqslant 0$ in $\mathbb{R}^{n+1}_b$, $\quad u = 0$ on Π_b,

(c') u satisfies inequality (0.1),

and equation in (a') is understood in the sence of distributions.
We also define the class $P^+_\infty(M,-\infty)$ corresponding formally to $b=-\infty$. In this case the whole space $\mathbb{R}^{n+1}$ is considered instead of $\mathbb{R}^{n+1}_b$, $\Pi_b=\emptyset$ and we omit the condition $u|_{\Pi_b}=0$. For any $b\in[-\infty,0]$ and a global solution $u\in P^+_\infty(M,b)$ from results of [ASU2] it follows that

$$-1\leqslant\partial_t u\leqslant 0. \tag{0.2}$$

Let $a>0$, $b\leqslant 0$ be some constants, let $u\in P^+_{2a}(M,b)$, and let $z^0=(x^0,t^0)\in\Gamma(u)$. For $r>0$ we consider the scaling

$$u_r(x,t)=\frac{u(rx+x^0,r^2t+t^0)}{r^2}. \tag{0.3}$$

By the standard compactness methods we may let r tend to zero and obtain (for a subsequence) a global solution $u_0\in P^+_\infty(M,-\infty)$. This process is referred to as blowing up, and the global solution u_0 thus obtained is called a **blow-up** of the function u at the point z^0.

§1. The Monotonicity Formula

Let $z^*=(x^*,t^*)$ be an arbitrary point in $\mathbb{R}^{n+1}$, let a and r be positive constants, and let v be a continuous function in $\mathcal{Q}_{a,r}(z^*):=B_a(x^*)\times]t^*-4r^2,t^*-r^2[$, satisfying $|Dv|\in L_2(\mathcal{Q}_{a,r}(z^*))$.
We define the local Weiss functional (cf. [W]) as

$$\begin{aligned}&W_a(r,x^*,t^*,v):=\\&\qquad=\frac{1}{r^4}\int_{t^*-4r^2}^{t^*-r^2}\int_{B_a(x^*)}\left(|Dv|^2+2v+\frac{v^2}{t-t^*}\right)G(x-x^*,t^*-t)\,dxdt,\end{aligned}$$

where

$$G(x,t)=\frac{\exp\left(-|x|^2/4t\right)}{(4\pi t)^{n/2}}\quad\text{for}\quad t>0\qquad\text{and}\qquad G(x,t)=0\quad\text{for}\quad t\leqslant 0.$$

Lemma 1.1. *Let v and z^* be as above.*
Then the equality

$$W_a(\lambda r,x^*,t^*,v)=W_{a/r}(\lambda,0,0,v_r) \tag{1.1}$$

holds for any $\lambda<a/r$ and $v_r(x,t)=r^{-2}\cdot v(rx+x^*,r^2t+t^*)$.

We omit the simple proof.

Lemma 1.2. *Let $a > 0$, $b \geqslant 0$ be given constants, let $u \in P_{2a}^{+}(M, -b)$, and let*

$$z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_a \cap \mathbb{R}^{n+1}_{-b}.$$

Suppose that we extend the function u as zero across the plane Π_{-b} to the set $Q_a(z^0) \cap \{x_1 < -b\}$ and preserve the notation u for the extended function. Then

$$\begin{aligned} \frac{dW_a(r, x^0, t^0, u)}{dr} &= \frac{1}{r}\int_{-4}^{-1} \int\limits_{B_{a/r}} \frac{|\partial' u_r|^2}{-t} G(x, -t)\, dx dt + 2J_a(r) \\ &+ \frac{x_1^0 + b}{r^2} \int_{-4}^{-1} \int\limits_{B_{a/r} \cap \{x_1 = \frac{-x_1^0 - b}{r}\}} |D_1 u_r|^2 G(x, -t)\, dx dt, \end{aligned} \tag{1.2}$$

where u_r is defined by formula (0.3),

$$\partial' u_r(x, t) := x \cdot Du_r(x, t) + 2t\partial_t u_r(x, t) - 2u_r(x, t), \tag{1.3}$$

$$\begin{aligned} J_a(r) := &\int_{-4}^{-1} \int\limits_{S_{a/r}} \frac{\partial' u_r}{r} (\vec{\gamma} \cdot Du_r) G(x, -t)\, dx dt \\ -&\frac{a}{2r^2} \int_{-4}^{-1} \int\limits_{S_{a/r}} \left(|Du_r|^2 + 2u_r + \frac{(u_r)^2}{t} \right) G(x, -t)\, dx dt, \end{aligned}$$

and $\vec{\gamma}$ is the unit vector of the outward normal to $S_{a/r}$.

Proof. Using (1.1) and taking into account the evident relation

$$\frac{d}{dr}(D_i u_r) = D_i \left(\frac{du_r}{dr} \right)$$

we obtain

$$\frac{d}{dr} W_a(r, x^0, t^0, u) = \frac{d}{dr} W_{a/r}(1, 0, 0, u_r) = I_1 + I_2, \tag{1.4}$$

where

$$I_1 = 2\int_{-4}^{-1} \int\limits_{B_{a/r}\cap\{x_1 > \frac{-x_1^0-b}{r}\}} \left[Du_r \cdot D\left(\frac{du_r}{dr}\right) + \frac{du_r}{dr} + \frac{u_r}{t}\frac{du_r}{dr}\right] G(x,-t)dxdt,$$

$$I_2 = -\frac{a}{r^2}\int_{-4}^{-1} \int\limits_{S_{a/r}\cap\{x_1 > \frac{-x_1^0-b}{r}\}} \left(|Du_r|^2 + 2u_r + \frac{(u_r)^2}{t}\right) G(x,-t)dxdt$$
$$- \frac{x_1^0+b}{r^2}\int_{-4}^{-1} \int\limits_{B_{a/r}\cap\{x_1 = \frac{-x_1^0-b}{r}\}} \left(|Du_r|^2 + 2u_r + \frac{(u_r)^2}{t}\right) G(x,-t)dxdt.$$

Then, integrating by parts the term $2Du_r \cdot D\left(\frac{d}{dr}u_r\right) G(x,-t)$ in I_1 and using the identity

$$D_iG(x,-t) = \frac{x_i}{2t}G(x,-t)$$

we obtain

$$I_1 = 2\int_{-4}^{-1} \int\limits_{B_{a/r}\cap\{x_1 > \frac{-x_1^0-b}{r}\}} \frac{du_r}{dr}\left[-\Delta u_r - \frac{x_i}{2t}D_iu_r + 1 + \frac{u_r}{t}\right] G(x,-t)dxdt$$
$$+ 2\int_{-4}^{-1} \int\limits_{S_{a/r}\cap\{x_1 > \frac{-x_1^0-b}{r}\}} \frac{du_r}{dr}\left(\vec{\gamma}\cdot Du_r\right) G(x,-t)dxdt$$
$$- 2\int_{-4}^{-1} \int\limits_{B_{a/r}\cap\{x_1 = \frac{-x_1^0-b}{r}\}} \frac{du_r}{dr}D_1u_rG(x,-t)dxdt. \tag{1.5}$$

Observe that according to the assumption $u \in P_{2a}^+(M,-b)$ we have for $(x,t) \in \mathcal{E} := \left\{B_{a/r} \cap \{x_1 = \frac{-x_1^0-b}{r}\}\right\} \times]-4,-1[$ the equalities

$$u_r = D'u_r = 0, \tag{1.6}$$

and, consequently,

$$\left(|Du_r|^2 + 2u_r + \frac{(u_r)^2}{t}\right)\Bigg|_{\mathcal{E}} = |D_1u_r|^2. \tag{1.7}$$

Moreover, taking into account the identity

$$\frac{du_r}{dr} = \frac{\partial' u_r}{r}, \tag{1.8}$$

and using (1.6) and (1.3), we get

$$\left(-\frac{du_r}{dr}D_1u_r\right)\bigg|_{\mathcal{E}} = -\frac{1}{r}\left(\frac{-x_1^0-b}{r}D_1u_r + 2t\partial_t u_r\right)\cdot D_1u_r. \tag{1.9}$$

Observe that $\partial_t u_r = 0$ on $\mathcal{E}\backslash\Gamma(u_r)$. Moreover, $\partial_t u_r$ is bounded while $D_1u_r = 0$ at the points of $\mathcal{E}\cap\Gamma(u_r)$. Therefore, $(2t\partial_t u_r D_1 u_r)\,|_{\mathcal{E}} = 0$ and relation (1.9) takes the form

$$\left(-\frac{du_r}{dr}D_1u_r\right)\bigg|_{\mathcal{E}} = \frac{x_1^0+b}{r^2}|D_1u_r|^2. \tag{1.10}$$

Substituting (1.7) and (1.10) into (1.5), and using (1.8) we obtain the following representation:

$$\begin{aligned} I_1+I_2 = {} & 2\int_{-4}^{-1}\int\limits_{B_{a/r}\cap\{x_1>\frac{-x_1^0-b}{r}\}} \frac{\partial' u_r}{r}\left[1-H[u]-\frac{\partial' u_r}{2t}\right]G(x,-t)dxdt \\ & + 2J_a(r) + \frac{x_1^0+b}{r^2}\int_{-4}^{-1}\int\limits_{B_{a/r}\cap\{x_1=\frac{-x_1^0-b}{r}\}} |D_1u_r|^2G(x,-t)dxdt. \end{aligned} \tag{1.11}$$

From the assumption $u\in P_{2a}^+(M,-b)$ it follows that $\Gamma(u_r)$ has zero Lebesgue measure and

$$H[u_r] = \chi_{\{u_r>0\}} \quad \text{in} \quad \mathcal{Q} := \left\{B_{a/r}\cap\{x_1>(-x_1^0-b)/r\}\right\}\times]-4,-1[.$$

Therefore, we have for $(x,t)\in\mathcal{Q}$ the equality

$$\frac{\partial' u_r}{r}\left[1-H[u]-\frac{\partial' u_r}{2t}\right] = -\frac{|\partial' u_r|^2}{2rt} \tag{1.12}$$

The proof is completed by combining (1.4), (1.11) and (1.12). □

Remark. There exists a universal constant $C_0 = C_0(n,M)$ such that

$$|J_a(r)| \leqslant \frac{C_0}{r^{n+3}}\left(1+\frac{a}{r^2}\right)\exp\left(-a^2/4r^2\right).$$

Moreover,

$$\lim_{a\to+\infty}|J_a(r)| = 0 \quad \forall r>0, \tag{1.13}$$

$$\lim_{r\to0^+}|J_a(r)| = 0 \quad \forall a>0. \tag{1.14}$$

Corollary 1.3. *Let $a > 0$, $b \geqslant 0$ be given constants, and let $u \in P_{2a}^{+}(M, -b)$. Then for any $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_a \cap \mathbb{R}_{-b}^{n+1}$ the function $W_a(r, x^0, t^0, u)$ has a limit as $r \to 0^+$, i.e.,*

$$\lim_{r \to 0^+} W_a(r, x^0, t^0, u) = \omega(x^0, t^0, u). \tag{1.15}$$

We will call the value $\omega(x^0, t^0, u)$ the "transition energy" of function u at the point (x^0, t^0). From (1.1) it follows that

$$\omega(x^0, t^0, u) = \int_{-4}^{-1} \int_{\mathbb{R}^n} \left(|Du_0|^2 + 2u_0 + \frac{(u_0)^2}{t} \right) G(x, -t) dx dt, \tag{1.16}$$

where u_0 is a blow-up of the solution u at the point (x^0, t^0).

§2. Regularity Properties of Solutions

Lemma 2.1. *Let $u \in P_2^{+}(M)$, let $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_1^{+}$, and let u_0 be a blow-up of the solution u at the point z^0.*
Then u_0 is a homogenious function of degree 2 on the set $\mathbb{R}^{n+1} \cap \{t \leqslant 0\}$, i.e.,

$$u_0(\varkappa x, \varkappa^2 t) = \varkappa^2 u_0(x, t) \qquad \forall \varkappa > 0, \quad \forall (x, t) \in \mathbb{R}^{n+1} \cap \{t \leqslant 0\}.$$

Remark. We observe that the statement of Lemma 2.1 concerns only the blow-ups of u at some <u>fixed</u> point $z^0 \in \Gamma(u)$.

Proof. We consider a sequence $r_k \to 0^+$ as $k \to \infty$ and define u_k as

$$u_k(x, t) = \frac{u(r_k x + x^0, r_k^2 t + t^0)}{r_k^2}.$$

By the definition $u_0(x, t) = \lim_{k \to \infty} u_k(x, t)$.
From (1.15) and (1.1) it follows that for arbitrary $\lambda > \mu > 0$ we have

$$\begin{aligned} 0 \underset{k \to \infty}{\longleftarrow} & W_1(\lambda r_k, x^0, t^0, u) - W_1(\mu r_k, x^0, t^0, u) \\ = W_{1/r_k}(\lambda, 0, 0, u_k) - W_{1/r_k}(\mu, 0, 0, u_k) &= \int_{\mu}^{\lambda} \frac{dW_{1/r_k}(\theta, 0, 0, u_k)}{d\theta} d\theta. \end{aligned} \tag{2.1}$$

On the other hand, according to (1.2) we have

$$\frac{dW_{1/r_k}(\theta,0,0,u_k)}{d\theta}=\frac{1}{\theta}\int_{-4}^{-1}\int\limits_{B^+_{1/r_k\theta}}\frac{|\partial'(u_k)_\theta|^2}{-t}G(x,-t)dxdt+2J_{1/r_k}(\theta)$$

$$=\frac{1}{\theta^5}\int_{-4\theta^2}^{-\theta^2}\int\limits_{B^+_{1/r_k}}\frac{|\partial'(u_k)|^2}{-t}G(x,-t)dxdt+2J_{1/r_k}(\theta). \tag{2.2}$$

Now, combining (2.1), (2.2), taking into account estimate (1.13) and letting $k\to\infty$ we get the identity

$$\partial' u_0=x\cdot Du_0+2t\partial_t u_0-2u_0\equiv 0 \qquad \forall t\in[-\sqrt{\lambda},-\sqrt{\mu}]$$

Therefore, u_0 is homogenious of degree 2 for all t from the interval $[-\sqrt{\lambda},-\sqrt{\mu}]$. It remains only to recall that λ and μ are arbitrary positive constants satisfying the inequality $\lambda>\mu$. This finishes the proof. $\square$

Lemma 2.2. *Let* $u_0\in P^+_\infty(M,-\infty)$, *let* $(0,0)\in\Gamma(u_0)$, *and let*

$$W_\infty(r,0,0,u_0):=\frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}\left(|Du_0|^2+2u_0+\frac{(u_0)^2}{t}\right)G(x,-t)dxdt, \quad r>0.$$

Then the equality

$$\frac{dW_\infty(r,0,0,u_0)}{dr}=0 \qquad \forall r>0$$

implies the homogenuity of degree 2 for the function u_0, *i.e.,*

$$u_0(\varkappa x,\varkappa^2 t)=\varkappa^2 u_0(x,t) \qquad \forall\varkappa>0,\quad \forall(x,t)\in\mathbb{R}^{n+1}\cap\{t<0\}.$$

Proof. The proof of this statement is given in [W]. $\square$

Lemma 2.3. *Let* $u\in P^+_2(M)$.
For any $\varepsilon>0$ *there exists* $\rho^*=\rho^*(\varepsilon)>0$ *such that if* $z^0=(x^0,t^0)\in\Gamma(u)\cap Q_1$ *and* $0\leqslant x^0_1<\rho^*$ *then for* $\rho\in[x^0_1,\rho^*]$ *we have*

$$\sup_{Q^+_\rho(z^0)}\left|u(x,t)-\frac{\left((x_1-x^0_1)_+\right)^2}{2}\right|\leqslant\varepsilon\rho^2, \tag{2.3}$$

$$\sup_{Q^+_\rho(z^0)}\left|D_1u(x,t)-\left(x_1-x^0_1\right)_+\right|\leqslant\varepsilon\rho, \tag{2.4}$$

$$\sup_{Q^+_\rho(z^0)}\left|D_\tau u(x,t)\right|\leqslant\varepsilon\rho, \qquad \tau=2,\dots,n. \tag{2.5}$$

Proof. We begin with considering the first inequality. Suppose, towards a contradiction, that (2.3) fails. Then there exists a number $\varepsilon_0 > 0$ and sequences $u^j \in P_2^+(M)$, $\rho_j \searrow 0$, and $z^j = (x^j, t^j) \in \Gamma(u^j) \cap Q_1$ such that $\rho_j > x_1^j \geqslant 0$ and

$$\sup_{Q^+_{\rho_j}(z^j)} \left| u_j(x,t) - \frac{\left((x_1 - x_1^j)_+\right)^2}{2} \right| > \varepsilon_0 \rho_j^2. \tag{2.6}$$

Next, we define v_j as

$$v_j(x,t) = \frac{u_j(\rho_j x + x^j, \rho_j^2 t + t^j)}{\rho_j^2}$$

for $(x,t) \in Q_{1/\rho_j} \cap \mathbb{R}^{n+1}_{-b_j}$, where $b_j = \frac{x_1^j}{\rho_j}$, $b_j \in [0,1]$. Observe that for each function v_j we have $(0,0) \in \Gamma(v_j)$ and $v_j|_{x_1=-b_j} = 0$. Moreover, for a subsequence and in appropriate space, v_j converges to a global solution $v_0 \in P^+_\infty(M, -b)$, where $b = \lim\limits_{j\to\infty} b_j$, $b \in [0,1]$.
Taking into account the relations $(0,0) \in \Gamma(v_0)$ and $v_0|_{x_1=-b} = 0$, we get from Theorem II [ASU2] that $v_0 \equiv \frac{((x_1)_+)^2}{2}$. Therefore, for all sufficiently large j we have the inequality

$$\sup_{Q_1 \cap \{x_1 > -b_j\}} \left| v_j(x,t) - \frac{\left((x_1)_+\right)^2}{2} \right| \leqslant \frac{\varepsilon_0}{2}. \tag{2.7}$$

On the other hand, the inequality (2.6) implies

$$\sup_{Q_1 \cap \{x_1 > -b_j\}} \left| v_j(x,t) - \frac{\left((x_1)_+\right)^2}{2} \right| = \sup_{Q_1 \cap \{x_1 > -b_j\}} \left| \frac{u_j(\rho_j x + x^j, \rho_j^2 t + t^j)}{\rho_j^2} - \frac{\left((x_1)_+\right)^2}{2} \right|$$
$$= \sup_{Q^+_{\rho_j}(z^j)} \left| \frac{u_j(y,\tau)}{\rho_j^2} - \frac{\left((y_1 - x_1^j)_+\right)^2}{2\rho_j^2} \right| > \varepsilon_0.$$

The latter inequality gives the contradiction with (2.7) and completes the proof of (2.3).
It remains only to observe that the estimates (2.4) and (2.5) are proved in the same way as (2.3). $\square$

Lemma 2.4. *Let $u \in P_2^+(M)$, let $\varepsilon_0 \in]0, \frac{1}{16(2n+1)}[$, and let $N_1 > 0$ and N_τ (with $\tau = 2, \dots, n$) be some constants.*

Then for arbitrary point $z^0=(x^0,t^0)\in Q_1^+$ and $\rho<1$ the inequality

$$\rho\left(\sum_{j=1}^n N_jD_ju\right)-u\geqslant -\varepsilon_0\rho^2 \quad in \quad Q^+_{\rho/2}(z^0)$$

implies

$$\rho\left(\sum_{j=1}^n N_jD_ju\right)-u\geqslant 0 \quad in \quad Q^+_{\rho/4}(z^0).$$

Proof. Suppose the conclusion of the lemma fails. Then there is a function $u\in P_2^+(M)$ and some points $z^0\in Q_1^+$ and $z^*=(x^*,t^*)\in Q^+_{\rho/4}(z^0)$ such that

$$\rho\left(\sum_{j=1}^n N_jD_ju(x^*,t^*)\right)-u(x^*,t^*)<0. \tag{2.8}$$

Let

$$w(x,t)=\rho\left(\sum_{j=1}^n N_jD_ju(x,t)\right)-u(x,t)+\frac{1}{2n+1}\left(|x-x^*|^2-(t-t^*)\right).$$

Then w is caloric in $Q_{\rho/4}(z^*)\cap\Omega(u)$, and, by (2.8), $w(x^*,t^*)<0$. Observe also that $u\geqslant 0$ implies $D_1u\geqslant 0$ on Π, and, consequently, $w\geqslant 0$ on the set $\partial\Omega(u)\cap Q_{\rho/4}(z^*)$. Hence by the maximum principle the negative infimum of w is attained on $\partial' Q^+_{\rho/4}(z^*)\cap\Omega(u)$. We thus obtain

$$-\frac{\rho^2}{16(2n+1)}\geqslant \inf_{\partial' Q^+_{\rho/4}(z^*)\cap\Omega(u)}\left\{\rho\left(\sum_{j=1}^n N_jD_ju\right)-u\right\}\geqslant -\varepsilon_0\rho^2,$$

which is a contradiction. This proves the lemma. □

Lemma 2.5. *Let $u\in P_2^+(M)$.*
There exists $\rho_0>0$ such that if $z^0=(x^0,t^0)\in\Gamma(u)\cap Q_{1/2}$ and $x_1^0\leqslant\frac{\rho_0}{8}$ then for any $\rho\in[x_1^0,\rho_0]$ we have

$$\rho\cdot D_1u-u\geqslant 0 \quad in \quad Q^+_{\rho/4}(z^0).$$

Proof. We fix ε_0 from Lemma 2.4 and set $\varepsilon=\varepsilon_0/2$. Now, successive application of Lemmas 2.3 and 2.4 finishes the proof. □

Lemma 2.6. *Let u_0 be non-negative, continuous and homogenious function degree 2 in* $\mathbb{R}^{n+1} \cap \{t \leqslant 0\}$, *satisfying the inequality* (0.1). *Suppose also that*

$$H[u_0] = \chi_{\{u_0>0\}} \quad \text{in} \quad \mathbb{R}^{n+1} \cap \{t \leqslant 0\},$$
$$(0,0) \in \Gamma(u),$$
$$D_1 u_0 \geqslant 0 \quad \text{in} \quad \mathbb{R}^{n+1} \cap \{t \leqslant 0\}.$$

Then, either for some direction $e \in \mathbb{R}^n$ such that $e \cdot e_1 \geqslant 0$ we have

$$u_0(x,t) = \frac{\left((x \cdot e)_+\right)^2}{2}, \tag{2.9}$$

or, in some rotated system of x-coordinates,

$$u_0(x,t) = \sum_{i=1}^{n} \frac{a_i}{2} x_i^2 - ct, \tag{2.10}$$

where $\sum_{i=1}^{n} a_i = 1 - c, \quad a_i \geqslant 0,\ c \geqslant 0.$

Proof. It is evident that only two situations may arise: interior of $\Lambda(u_0) = \emptyset$ and interior of $\Lambda(u_0) \neq \emptyset$.
For case <u>interior of $\Lambda(u_0) = \emptyset$</u> we observe that the function v_0 defined as

$$v_0(x,t) = u_0(x,t) - \frac{x_1^2}{2},$$

is caloric in $\mathbb{R}^{n+1} \cap \{t \leqslant 0\}$, and it has quadratic growth with respect to x and linear growth with respect to t. So, by Liouville's theorem (see Lemma 2.1 [ASU1]) we get that v_0, and, consequently u_0, is a polinomial of degree two, i.e., there exist constants $a_i \geqslant 0$, $c \geqslant 0$ such that the exact representation (2.10) takes place.

For case <u>interior of $\Lambda(u_0) \neq \emptyset$</u> we need a more detailed analysis. In view of homogenuity of u_0 the fact, that the set of interior points of $\Lambda(u_0)$ is not empty, implies the existence of interior point of $\Lambda(u_0)$ for all values of $t \leqslant 0$. Next, arguing in the same way as in **Step 2** of Theorem II [ASU2] we can prove that the function u_0 is the one-space dimensional, i.e., $u_0 = u_0(y,t)$ where $y = (x \cdot e)$ for some $e \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $t \leqslant 0$. This result follows by dimensional reduction based on the version of the monotonicity formula due to L. A. Caffarelli and C. Kenig.
Therefore, due to homogenuity of u_0, the relation

$$u_0(0,t) = -mt > 0, \qquad 0 \leqslant m \leqslant 1, \tag{2.11}$$

holds for all $t < 0$.
If $m = 0$ then Theorem I [ASU2] immediately gives (2.9). We would like to show that $m = 0$ is the only possibility.
Let $m > 0$. From the assumption $D_1 u_0 \geqslant 0$ in $\mathbb{R} \cap \{t \leqslant 0\}$ it follows that $D_e u_0$ does not change sign. For the definitness we suppose that $D_e u_0 \geqslant 0$ (otherwise we just change e by $-e$). Now, combining (2.11) with the inequality $D_e u_0 \geqslant 0$ we obtain $u_0(y,t) > 0$ on the set $\mathcal{D} := \{(y,t) : y \in \mathbb{R}_+,\ t < 0\}$. Therefore, $H[u_0] = 1$ and $H[\partial_t u_0] = 0$ on $\mathcal{D}$.
Now we define the function v as

$$v(y,t) = \begin{cases} \partial_t u_0(y,t) + m, & \text{if} \quad y \geqslant 0, t \leqslant 0 \\ -\partial_t u_0(-y,t) + m, & \text{if} \quad y < 0, t \leqslant 0. \end{cases}$$

Obviously, the function v is bounded and caloric in $\mathbb{R} \cap \{t < 0\}$. Therefore, by Liouville's theorem we get $v \equiv 0$ in $\mathbb{R} \cap \{t < 0\}$, and elementary integrations imply the exact representation for u_0 on the set $\{y \geqslant 0, t \leqslant 0\}$

$$u_0(y,t) = -mt + \frac{1-m}{2} y^2. \tag{2.12}$$

Observe that from (2.12) we immediately get

$$D_e u_0 = 0 \quad \text{on} \quad \Pi \cap \{t \leqslant 0\}. \tag{2.13}$$

On the other hand, from (0.2) it follows that

$$D_e D_e u_0 \geqslant 0 \quad \text{in} \quad \mathbb{R} \cap \{t \leqslant 0\}. \tag{2.14}$$

Combination (2.13) and (2.14) gives $D_e u_0 \equiv 0$ on the set $\{y < 0, t \leqslant 0\}$. The latter means that on the set $\{y < 0, t \leqslant 0\}$ we have exact representation

$$u_0(y,t) = -mt. \tag{2.15}$$

Representation (2.15) together with the equation $H[u_0] = 1$ gives $m = 1$. However, the case $m = 1$ is impossible since for $u_0 = -t$ the set of interior points of $\Lambda(u_0)$ is empty. $\square$

Corollary 2.7. *Let $u \in P_2^+(M)$, let $z^0 = (x^0, t^0) \in \Gamma(u)$, and let u_0 be a blow-up of the solution u at the point z^0.*
Then, either for some direction $e \in \mathbb{R}^n$ such that $e \cdot e_1 \geqslant 0$ we have

$$u_0(x,t) = \frac{(x \cdot e)_+^2}{2} \quad \textit{and} \quad \omega(x^0, t^0, u) = W_\infty(1,0,0,u_0) = \frac{15}{4}, \tag{2.16}$$

or, in some rotated system of x-coordinates,

$$u_0(x,t) = \sum_{i=1}^{n} \frac{a_i}{2} x_i^2 - ct \quad \textit{and} \quad \omega(x^0, t^0, u) = W_\infty(1,0,0,u_0) = \frac{15}{2}. \tag{2.17}$$

Here a_i and c are the constants from Lemma 2.6, *while $\omega(x^0,t^0,u)$ is the "transition energy" at the point z^0 defined by* (1.15).

Proof. By Lemmas 2.1 and 2.6 the first equalities in (2.16) and (2.17) are already obtained.
Further, in view of (1.16), for the case $u_0(x,t) = (x \cdot e)_+^2/2$ we have

$$\omega(x^0,t^0,u) = \int_{-4}^{-1} dt \int_{\mathbb{R}^{n-1}} dy_2 \dots dy_n \int_0^\infty \left(2y_1^2 + \frac{y_1^4}{4t}\right) G(y,-t) dy_1.$$

Analogously, in view of (1.16), for the case $u_0(x,t) = \sum_{i=1}^n \frac{a_i}{2} x_i^2 - ct$ we have

$$\begin{aligned}\omega(x^0,t^0,u) = \int_{-4}^{-1} dt \int_{\mathbb{R}^n} \Bigg(&\sum_{i=1}^n a_i^2 x_i^2 + \sum_{i=1}^n a_i x_i^2 - 2ct \\ &+ c^2 t - c \sum_{i=1}^n a_i x_i^2 + (4t)^{-1} \sum_{i,j=1}^n a_i a_j x_i^2 x_j^2 \Bigg) G(x,-t) dx.\end{aligned}$$

Now the second equalities in (2.16) and (2.17) follow immediately from the direct calculations of the integrals introduced above. □

Lemma 2.8. *There exists $\frac{1}{2} > \delta > 0$ such that if $u \in P_2^+(M)$, and*

$$z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_1 \cap \{x_1 \leqslant \delta\}$$

then

$$\omega(x^0,t^0,u) = \frac{15}{4}. \tag{2.18}$$

Here $\omega(x^0,t^0,u)$ is the "transition energy" at the point z^0 defined by (1.15).

Proof. We set $R := x_1^0$ and consider the scaling

$$u_R(x,t) = \frac{u(Rx + x^0, R^2 t + t^0)}{R^2}.$$

If R is small enough then the function u_R is close to a global solution $u_0 \in P_\infty^+(M,-1)$. Moreover, from (1.1), Lemma 2.3 and (2.16) it follows that

$$W_1(R, x^0, t^0, u) = W_{1/R}(1,0,0,u_R) \leqslant \frac{15}{4} + \varepsilon(R),$$

where $\varepsilon(R)$ tends to zero as R goes to 0^+.
Therefore, by (1.2) and (1.14) we have for $r < R < \delta$

$$W_1(r, x^0, t^0, u) \leqslant W_1(R, x^0, t^0, u) + \frac{N_1(n, M)}{R^{n+5}} \exp\left(-\frac{1}{4R^2}\right) \leqslant 5, \qquad (2.19)$$

provided δ is small enough.
Now, the equality (2.18) follows from (2.19) and Corollary 2.7. □

Remark. Let $u \in P_2^+(M)$, let δ be the constant from Lemma 2.8, and let $z = (x, t) \in \Gamma(u) \cap Q_1 \cap \{x_1 \leqslant \delta\}$. Then the convergence $W_1(\rho, x, t, u)$ to $\frac{15}{4}$ as $\rho \searrow 0^+$ is uniform with respect to $z = (x, t)$. This fact follows from Dini's theorem, since for $\rho > 0$ the functions $W_1(\rho, x, t, u)$ are continuous with respect to (x, t), the limit function is constant, and convergence, up to exponentially small terms, is monotonuous.
In particular, if $\rho_k \searrow 0^+$ as $k \to \infty$, and $z^k = (x^k, t^k) \longrightarrow z^0 = (x^0, t^0)$, where $z^k \in \Gamma(u) \cap Q_1 \cap \{x_1 \leqslant \delta\}$, then

$$\lim_{k \to \infty} W_1(\rho_k, x^k, t^k, u) = \frac{15}{4}. \qquad (2.20)$$

Theorem 2.9. *Let $u \in P_2^+(M)$.*
Then $\partial_t u$ is continuous on the set $Q_{1/2} \cap \{0 \leqslant x_1 < \delta\}$, where δ is the contstant from Lemma 2.8.

Proof. Consider a point $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_{1/2} \cap \{0 \leqslant x_1 < \delta\}$. We claim that

$$\lim_{\Omega(u) \ni z \to z^0} \partial_t u(z) = 0. \qquad (2.21)$$

As we know from [ASU2] the corresponding upper limit in (2.21) is non-positive. Therefore, we only need to prove that the lower limit in (2.21) is non-negative.
Suppose that

$$\lim_{\Omega(u) \ni z \to z^0} \inf \partial_t u(z) =: -m, \quad m \geqslant 0. \qquad (2.22)$$

Then there exists a sequence $z^k = (x^k, t^k) \in \Omega(u)$ such that $z^k \to z^0$ as $k \to \infty$ and

$$\lim_{k \to \infty} \partial_t u(z^k) = -m.$$

We denote by $K_r(z^k)$, $r > 0$ the cylinder $K_r(z^k) = B_r(x^k) \times]t^k - r^2, t^k + r^2[$, and for each point z^k we define the corresponding distance to the free boundary as follows:

$$r_k = \sup\left\{r > 0 : K_r(z^k) \subset \Omega(u)\right\}.$$

It is clear that $r_k \to 0$ as $k \to \infty$.
Also we define u_k as

$$u_k(x,t) = \frac{u(r_k x + x^k, r_k^2 t + t^k)}{r_k^2},$$

and observe that $K_1(0,0) \subset \Omega(u_k)$ and $\partial_t u_k(0,0) = \partial_t u(x^k, t^k)$ tends to $-m$ as $k \to \infty$.
Then u_k converges (for a subsequence) to a global solution $u_0 \in P_\infty^+(M, \lim\limits_{k\to\infty} \frac{-x_1^k}{r_k})$, satisfying the following properties:

$$H[u_0] = 1 \quad \text{in} \quad K_1(0,0),$$
$$\partial_t u_0(0,0) = -m,$$
$$\partial_t u_0(x,t) \geqslant -m \quad \text{for all } (x,t) \in K_1(0,0),$$

where the latter follows from the convergence of $\partial_t u_k$ to $\partial_t u_0$ at each point of $K_1(0,0)$ and (2.22).
Thus the function $\partial_t u_0$ is caloric in $K_1(0,0)$ and takes a local minimum at the origin. Therefore, one can conclude by the maximum principle that

$$\partial_t u_0(x,t) \equiv -m \quad \text{in} \quad Q_1. \tag{2.23}$$

We now proceed to get a contradiction with (2.23) if $m \neq 0$. It is easy to see that only two situations may arise:
(a) $R_k = \mathrm{O}(r_k)$ as $k \to \infty$, where $R_k := x_1^k$;
(b) $r_k = \mathrm{o}(R_k)$ as $k \to \infty$.
For $m > 0$ in the case **(a)** the contradiction with (2.23) follows immediately by the results of [ASU2].

For case **(b)** we need a more detailed analysis. Observe that in the case **(b)** the limit function u_0 is a global solution defined in the whole space $\mathbb{R}^{n+1}$.
By the definition of r_k, for each k there exists a point $(y^k, \tau^k) \in \Gamma(u_k) \cap \partial K_1(0,0)$ and a corresponding point $(r_k y^k + x^k, r_k^2 \tau^k + t^k) \in \Gamma(u) \cap \partial K_{r_k}(z^k)$ such that

$$(r_k y^k + x^k, r_k^2 \tau^k + t^k) \to (x^0, t^0) \quad \text{as} \quad k \to \infty. \tag{2.24}$$

We denote by (y^0, τ^0) the limit of (y^k, τ^k) as $k \to \infty$. It is obvious that $(y^0, \tau^0) \in \Gamma(u_0) \cap \partial K_1(0,0)$. Further, using (1.1),(2.24),(2.20) and (1.15) we obtain for any $\rho > 0$

$$\begin{aligned} W_1(\rho, y^0, \tau^0, u_0) &= \lim_{k\to\infty} W_1(\rho, y^k, \tau^k, u_k) \\ &= \lim_{k\to\infty} W_{r_k}(\rho r_k, r_k y^k + x^k, r_k^2 \tau^k + t^k, u) \\ &= \omega(x^0, t^0, u) = \frac{15}{4}. \end{aligned} \tag{2.25}$$

From Lemma 2.2 it follows that a limit u_0 is a homogenious function of degree 2 for $t \leqslant \tau^0$. More precisely, for any $x \in \mathbb{R}^n$, $t \leqslant \tau^0$ and $\lambda > 0$ we have

$$u_0(y^0 + \lambda(x - y^0), \tau^0 + \lambda^2(t - \tau^0)) = \lambda^2 u_0(x, t).$$

In addition, from Lemma 2.5 it follows that for any $x \in \mathbb{R}^n$ and $t \leqslant \tau^0$ we have

$$D_1 u_0 \geqslant 0 \quad \text{in} \quad \mathbb{R}^{n+1} \cap \{t \leqslant \tau^0\}.$$

Now, using Lemma 2.6, Corollary 2.7 and (2.25) we obtain for $t \leqslant \tau^0$ the representation

$$u_0(x, t) = \frac{((x - y^0) \cdot e)_+^2}{2}, \tag{2.26}$$

where e is a direction in $\mathbb{R}^n$ satisfying $\cos(e \cdot e_1) \geqslant 0$.
The solution u_0 can be uniquely continued for $t > \tau^0$ by the same expression.
It remains only to note that representation (2.26) contradicts (2.23) if $m \neq 0$.
The proof is completed. $\square$

§3. Regularity Properties of a Free Boundary

Lemma 3.1. *Let $u \in P_2^+(M)$, let δ be the constant from Lemma 2.8, and let $0 < \varepsilon_1 < \frac{1}{16(2n+1)}$, $N_1 > 0$, N_0, and N_τ (with $\tau = 2, \dots, n$) be some constants. Then for arbitrary $z^0 = (x^0, t^0) \in Q_{1/2}^+ \cap \{0 \leqslant x_1^0 \leqslant \delta/2\}$ and $\rho < \delta/2$ the inequality*

$$\rho \left(\sum_{i=1}^{n} N_i D_i u \right) + \rho^2 N_0 \partial_t u - u \geqslant -\varepsilon_1 \rho^2 \quad \text{in} \quad Q_{\rho/2}(z^0)$$

implies

$$\rho \left(\sum_{i=1}^{n} N_i D_i u \right) + \rho^2 N_0 \partial_t u - u \geqslant 0 \quad \text{in} \quad Q_{\rho/4}(z^0).$$

Proof. Suppose the conclusion of the lemma fails.
Then there is a function $u \in P_2^+(M)$ and some points $z^0 \in Q_{1/2} \cap \{0 \leqslant x_1^0 \leqslant \delta/2\}$ and $z^* = (x^*, t^*) \in Q_{\rho/4}(z^0)$ such that

$$\rho \left(\sum_{i=1}^{n} N_i D_i u(x^*, t^*) \right) + \rho^2 N_0 \partial_t u(x^*, t^*) - u(x^*, t^*) < 0. \tag{3.1}$$

Let

$$w(x,t) = \rho \left(\sum_{i=1}^{n} N_i D_i u(x,t) \right) + \rho^2 N_0 \partial_t u(x,t)$$
$$- u(x,t) + \frac{1}{2n+1} \left(|x - x^*|^2 - (t - t^*) \right).$$

Then w is caloric in $Q_{\rho/4}(z^*) \cap \Omega(u)$, and, by (3.1), $w(x^*, t^*) < 0$.
We observe also that Theorem 2.9 and the condition $u \geqslant 0$ imply the inequalities

$$w \geqslant 0 \quad \text{on} \quad \Pi, \qquad\qquad w \geqslant 0 \quad \text{on} \quad \Gamma(u) \cap Q_{\rho/4}(z^*).$$

Hence by the maximum principle the negative infimum of w is attained on $\partial' Q_{\rho/4}(z^*) \cap \Omega(u)$. We thus obtain

$$-\frac{\rho^2}{16(2n+1)} \geqslant \inf_{\partial' Q_{\rho/4}(z^*) \cap \Omega(u)} \left\{ \rho \left(\sum_{i=1}^{n} N_i D_i u \right) + \rho^2 N_0 \partial_t u - u \right\} \geqslant -\varepsilon_1 \rho^2,$$

which is a contradiction. This proves the lemma. □

Lemma 3.2. *Let $u \in P_2^+(M)$, and let $(0,0) \in \Gamma(u)$.*
There exist constants $\rho > 0$ and $\vartheta \in]0, \frac{\pi}{2}[$, and the whole cone of directions

$$\mathcal{K}_\vartheta := \{ e = \sum_{i=0}^{n} \alpha_i e_i : \ \alpha_1 \geqslant \cos\vartheta, \ \sum_{i=0}^{n} \alpha_i^2 = 1 \}$$

such that for any point $z = (x,t) \in Q_\rho^+$ and any direction $e \in \mathcal{K}_\vartheta$ we have

$$\frac{\partial u}{\partial e}(z) \geqslant 0.$$

Proof. We fix ε_1 from Lemma 3.1, and choose $\varepsilon \leqslant \varepsilon_1/n$ and $\rho \leqslant \min\{\rho^*, \rho_0, \delta/2\}$ where $\rho^* = \rho^*(\varepsilon_1)$, ρ_0 and δ are the constants from Lemmas 2.3, 2.5 and 2.8, respectively.
Applying Lemmas 2.3 and 2.5 we obtain

$$\rho D_1 u - u \geqslant 0 \qquad \text{in} \quad Q_{\rho/4}^+, \tag{3.2}$$
$$\sup_{Q_\rho^+} |D_\tau u| \leqslant \varepsilon \rho, \qquad \tau = 2, \dots, n. \tag{3.3}$$

Moreover, from Theorem 2.9 it follows that

$$\sup_{Q_\rho^+} \left|\partial_t u\right| \leqslant \varepsilon, \tag{3.4}$$

if ρ is small enough.
Further, we choose ρ so small that the inequalities (3.2), (3.3), and (3.4) hold. Combining (3.2), (3.3), and (3.4), we get

$$\rho \left(D_1 u + \sum_{\tau=2}^{n} N_\tau D_\tau u \right) + \rho^2 N_0 \partial_t u - u \geqslant -\varepsilon_1 \rho^2 \quad \text{in} \quad Q_{\rho/8}^+,$$

with constants N_τ and N_0 satisfying $|N_\tau| \leqslant 1$ and $|N_0| \leqslant 1$, respectively.
Now, applying Lemma 3.1 we obtain the statement of the lemma with

$$|\alpha_0| \leqslant \rho \left(1 + n + \rho^2\right)^{-1/2}, \qquad \alpha_1 \geqslant \left(1 + n + \rho^2\right)^{-1/2},$$
$$|\alpha_\tau| \leqslant \left(1 + n + \rho^2\right)^{-1/2}, \quad \tau = 2, \ldots, n.$$

□

Theorem 3.3. *Let $u \in P_2^+(M)$, and let $(0,0)$ be a contact point. There exists $r > 0$ such that $\partial\Omega(u) \cap Q_r^+$ can be represented in the form*

$$x_1 = f(x_2, \ldots, x_n, t) \tag{3.5}$$

with Lipschitz continuous function f.

Proof. By Lemma 2.5 we know that u is monotone in direction e_1. This implies that $\partial\Omega(u)$ can be represented by formula (3.5).
In addition, by Lemma 3.2 for each point $z = (x,t) \in \Omega(u) \cap Q_r^+$ there exists a whole cone of directions $\mathcal{K}_\vartheta$ in which the function u is nondecreasing. Hence f is Lipschitz continuous. □

Corollary 3.4. *There exists a universal constant $r_0 = r_0(n, M)$ such that if $u \in P_2^+(M)$ and $(0,0) \in \Gamma(u)$ then $\partial\Omega(u) \cap Q_{r_0}^+$ is the graph of a $C^{1+\alpha}$-function for some $0 < \alpha < 1$.*

Proof. The proof of this statement is now an easy consequence of Theorem 3.3, Lemma 3.2 and the result of [AtSa]. □

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