Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint

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Preprint No. 75

Saarbrücken 2002

Universität des Saarlandes

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submitted: December 12, 2002

Preprint No. 75

Saarbrücken 2002

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A commutant lifting theorem on analytic polyhedra

Calin Ambrozie[1] and Jörg Eschmeier

In 1967 Sarason observed that commutant lifting results for contractions on Hilbert spaces can be used to solve interpolation problems for bounded analytic functions on the unit disc. This idea was considerably extended in the work of Sz.-Nagy and Foiaş on Hilbert space contractions ([20],[26]). Each contraction $T \in B(H)$ of type $C_{\cdot 0}$ is, up to unitary equivalence, the compression of the multiplication operator with the independent variable $M_z \in B(H^2(\mathbb{D}, E))$ on a suitable vector-valued Hardy space to a closed linear subspace $M \subset H^2(\mathbb{D}, E)$ which is invariant for the adjoint $M_z^* \in B(H^2(\mathbb{D}, E))$. A functional version of the commutant lifting theorem, which is suitable for applications to interpolation problems, says that each contraction X in the commutant of the compression $T = P_M M_z|M$ can be realized as the compression $X = P_M M_f|M$ of a multiplication operator M_f given by a bounded analytic function $f \in H^\infty(\mathbb{D}, B(E))$ with supremum norm $\|f\|_\infty$ bounded by one. Since the commutant of $M_z \in B(H^2(\mathbb{D}, E))$ consists precisely of all multiplication operators M_f with $f \in H^\infty(\mathbb{D}, B(E))$, the above result can be seen as the $C_{\cdot 0}$-case of the abstract commutant lifting theorem for contractions.

If one replaces the Hardy space on the unit disc $\mathbb{D}$ in $\mathbb{C}$ by the Hardy space on the open Euclidean unit ball $\mathbb{B}$ in $\mathbb{C}^n$ and uses the commuting n-tuple $M_z = (M_{z_1}, \ldots, M_{z_n}) \in B(H^2(\mathbb{B}, E))^n$ consisting of the multiplication operators with the coordinate functions instead of the operator M_z on $H^2(\mathbb{D}, E)$, then the multivariable analogue of the above functional form of the commutant lifting theorem becomes wrong, even in the scalar-valued case $E = \mathbb{C}$. An easy way to show this, is to use the well-known observation that the ball version of the classical Nevanlinna-Pick theorem for functions in $H^\infty(\mathbb{B})$ is wrong (cf. [22]).

By von Neumann's inequality, the supremum norm of a bounded analytic function f in $H^\infty(\mathbb{D}, B(E))$ can be described as

$$\|f\|_\infty = \sup\{\|f(T)\|;\ T \in B(H) \text{ with } \|T\| < 1\}.$$

Here H is a fixed separable, infinite-dimensional Hilbert space, and the operator $f(T)$ is formed with the help of an appropriate vector-valued analytic functional calculus.

[1]The first-named author expresses his thanks to the Department of Mathematics of the University of Saarbrücken for its kind hospitality during the first stage of the preparation of the present work.

If one substitutes $\mathbb{B}$ for $\mathbb{D}$ and replaces single contractions by the class of all n-contractions as introduced by Arveson in [9], then the space of all analytic functions $f \in \mathcal{O}(\mathbb{B}, B(E))$, for which the norm defined by the supremum on the right in the above formula is finite, forms a contractively embedded Banach subalgebra of $H^\infty(\mathbb{B}, B(E))$. For this class of analytic functions, interpolation results of Nevanlinna-Pick or Carathéodory-Fejér type have been proved, and closely related versions of the above functional form of the commutant lifting theorem can be proved (see [7], [13]], [18], [23]). Based on Agler's characterization of Schur class functions on the open unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$ ([1]), corresponding interpolation and commutant lifting results have been proved over the unit polydisc in $\mathbb{C}^n$ ([2], [12]).

In the present paper we replace the ball and unit polydisc by a bounded analytic polyhedron of the form

$$D = \{z \in W;\ \|d(z)\| < 1\} \subset \mathbb{C}^n,$$

where $d : W \to \mathbb{C}^{p,q}$ is a matrix-valued analytic function on an open neighbourhood W of $\overline{D}$ in $\mathbb{C}^n$. Instead of contractions we use the class $\mathcal{C}$ of all commuting n-tuples $T \in B(H)^n$ on a fixed separable, infinite-dimensional Hilbert space H such that $\sigma(T) \subset W$ and $\|d(T)\| < 1$. Since in this situation the Taylor spectrum $\sigma(T)$ of T is contained in D, the formula

$$\|f\|_{\mathcal{S}} = \sup\{\|f(T)\|;\ T \in \mathcal{C}\}$$

can be used to define a contractively embedded Banach subalgebra $\mathcal{S}(d, B(E))$ of $H^\infty(D, B(E))$, called the Schur space on D. We replace the Hardy space by suitable Hilbert spaces $\mathcal{H}$ of analytic functions on D defined by a reproducing kernel function C. Our main result (Theorem 3.7) shows in particular that, for each M_z^*-invariant finite-dimensional subspace $M \subset \mathcal{H} \otimes E$, the operators X in the commutant of the compression $T = P_M M_z|M \in B(M)^n$, which possess a lifting to a multiplication operator M_f given by a Schur class function $f \in \mathcal{S}(d, B(E))$ with Schur norm $\|f\|_{\mathcal{S}} \leq 1$, can be characterized by a positivity condition which is formulated in terms of the reproducing kernel C and the boundary function d of the domain D.

The result obtained in this way is of a quite general nature. It applies in particular to the ball and the polydisc and, more generally, to all symmetric domains, since in these cases natural examples of reproducing kernel Hilbert spaces are known to exist. As an application we derive interpolation results for functions $f \in \mathcal{S}(d, B(E))$ of Nevanlinna-Pick and Carathéodory-Fejér type.

The paper is a continuation of joint work of the first named author with D. Timotin on von Neumann type inequalities and intertwining lifting results over suitable domains in $\mathbb{C}^n$ ([4],[5]). We make essential use of a recent result of Ball and Bolotnikov [11] which gives different characterizations of Schur class functions on polynomial polyhedra.

§0 Preliminaries

Let H and K be complex Hilbert spaces. We write $B(H,K)$ for the set of all bounded linear operators from H to K and $H \otimes K$ for the Hilbertian tensor product of H and K. For an open set U in $\mathbb{C}^n$ and a given Banach space X, we denote by $\mathcal{O}(U,X)$ the Fréchet space of all analytic X-valued functions on U. If $T \in B(H)^n$ is a commuting tuple of bounded linear operators on H, then $\sigma(T)$ is defined as the Taylor spectrum of T, and we write $\Phi : \mathcal{O}(\sigma(T)) \to B(H)$, $f \mapsto f(T)$, for Taylor's analytic functional calculus of T. For the definition and basic properties of these notions from multivariable operator theory, we refer the reader to [17] or [27].

For given Hilbert spaces E and $E_.$ and each open neighbourhood U of $\sigma(T)$, there is a unique continuous linear map

$$\Phi_{E,E_.} : \mathcal{O}(U, B(E,E_.)) \cong \mathcal{O}(U)\tilde{\otimes} B(E,E_.) \to B(H \otimes E, H \otimes E_.)$$

with the property that $\Phi_{E,E_.}(f \otimes A) = f(T) \otimes A$ for $f \in \mathcal{O}(U)$ and $A \in B(E,E_.)$. For simplicity, we write again $f(T)$ instead of $\Phi_{E,E_.}(f)$. If $S \subset \mathbb{C}^n$ is an arbitrary subset, then we define $\tilde{S} = \{z \in \mathbb{C}^n;\ \overline{z} \in S\}$. For $f \in \mathcal{O}(U, B(E,E_.))$, the function

$$\tilde{f} : \tilde{U} \to B(E_., E), \quad \tilde{f}(z) = f(\overline{z})^*$$

is analytic again. An elementary exercise, using the corresponding scalar-valued result and the density of the linear span of all elementary tensors, shows that the identity $\tilde{f}(T^*) = f(T)^*$ holds for all functions $f \in \mathcal{O}(U, B(E,E_.))$.

If p, q are positive integers, then we identify the space $B(H^q, H^p)$ of all bounded linear operators from H^q to H^p with the space $B(H)^{p,q}$ of all $(p \times q)$-operator matrices with entries in $B(H)$. For $p = q$, we regard the map

$$\operatorname{tr} : B(H)^{p,p} \to B(H), \quad (A_{ij}) \mapsto \sum_{j=1}^{p} A_{jj}$$

as a generalization of the ordinary trace for scalar matrices.

Let Λ be an arbitrary set. An operator-valued function $K : \Lambda \times \Lambda \to B(H)$ is called positive definite if $\sum_{i,j=1}^{s} \langle K(\lambda_i, \lambda_j) c_i, c_j \rangle \geq 0$ whenever s is a positive integer, $\lambda_1, \ldots, \lambda_s \in \Lambda$ and $c_1, \ldots, c_s \in H$. By a well-known theorem of Kolmogorov and Aronszajn a function $K : \Lambda \times \Lambda \to B(H)$ is positive definite if and only if there is a Hilbert space G and a function $k : \Lambda \to B(H,G)$ such that $K(\lambda, \mu) = k(\mu)^* k(\lambda)$ for $\lambda, \mu \in \Lambda$.

§1 **Functional Hilbert spaces**

Let $D \subset \mathbb{C}^n$ be an open set with $0 \in D$. Throughout this note we shall denote by $\mathcal{H}$ a fixed functional Hilbert space consisting of analytic functions on D. More precisely, $\mathcal{H}$ is a Hilbert space consisting of complex-valued analytic functions on D such that all point evaluations

$$\delta_w : \mathcal{H} \to \mathbb{C}, \quad f \mapsto f(w) \qquad (w \in D)$$

are continuous. For each point $w \in D$, there is a unique function $C_w \in \mathcal{H}$ such that

$$\langle f, C_w \rangle = f(w) \qquad (f \in \mathcal{H}).$$

The map $D \to \mathcal{H}$, $w \mapsto C_w$, is weakly anti-analytic, and hence it is anti-analytic as a map with values in $\mathcal{H}$. Consequently, on the open set $\Delta = \{(z, \overline{w});\ z, w \in D\} \subset \mathbb{C}^{2n}$, we can define an analytic function $C : \Delta \to \mathbb{C}$ by

$$C(z, w) = C_{\overline{w}}(z).$$

In addition, we shall suppose that the domain D and the functional Hilbert space $\mathcal{H}$ satisfy the following conditions:

(i) $\mathcal{H}$ contains the constant functions and $\|1\| = 1$,

(ii) the coordinate functions are multipliers of $\mathcal{H}$, and the tuple $Z = (Z_1, \ldots, Z_n) \in B(\mathcal{H})^n$ consisting of the multiplication operators $Z_j : \mathcal{H} \to \mathcal{H}$, $f \mapsto z_j f$, has the property that $\sigma(Z) = \overline{D}$,

(iii) $\overline{D}$ is polynomially convex and the polynomials form a dense subset of $\mathcal{H}$,

(iv) the function C has no zeros, $C_0 \equiv 1$ and C^{-1} extends to a holomorphic function defined on an open neighbourhood of $\overline{\Delta}$.

For a given Hilbert space E, we shall identify the Hilbertian tensor product $\mathcal{H} \otimes E$ with a linear subspace of the space $\mathcal{O}(D, E)$ of all E-valued analytic functions on D via the injective linear map

$$j : \mathcal{H} \otimes E \to \mathcal{O}(D, E), \quad (jh)(z) = (\delta_z \otimes 1_E)(h).$$

Let $E, E_.$ be fixed Hilbert spaces, and let $f \in \mathcal{O}\big(U, B(E, E_.)\big)$ be an analytic function defined on an open set $U \supset \overline{D}$. Then the multiplication operator

$$T_f : \mathcal{H} \otimes E \to \mathcal{H} \otimes E_., \quad h \mapsto fh$$

is well defined and continuous linear. To verify this, it suffices to show that

$$\big(f(Z)h\big)(z) = f(z)h(z) \qquad (f \in \mathcal{H} \otimes E,\ z \in D),$$

where $f(Z)$ is formed with the operator-valued analytic functional calculus explained in the preliminaries. Since both sides of the above equation depend in a continuous bilinear way on f and h, it suffices to consider the particular case where $f \in \mathcal{O}(U)$ and $h \in \mathcal{H}$.

To settle this case, one can use the observation that $Z_j^* C_z = \overline{z}_j C_z$ $(1 \le j \le n)$ to deduce that

$$\big(f(Z)h\big)(z) = \big\langle h, f(Z)^* C_z \big\rangle = \big\langle h, \tilde{f}(Z^*) C_z \big\rangle = \big\langle h, \tilde{f}(\overline{z}) C_z \big\rangle = f(z)h(z).$$

By choosing E and $E_.$ as finite-dimensional Hilbert spaces, we obtain that each matrix-valued analytic map $d \in \mathcal{O}(\overline{D}, \mathbb{C}^{p,q})$ induces a continuous linear operator $T_d = d(Z) : \mathcal{H}^q \to \mathcal{H}^p$.

A closed subspace $M \subset \mathcal{H} \otimes E$ will be called $*$-invariant if $(Z_j \otimes 1_E)^* M \subset M$ for $j = 1, \ldots, n$.

Lemma 1.1 Suppose that $M \subset \mathcal{H} \otimes E$ is a $*$-invariant closed subspace.

(a) The compression $T = P_M(Z \otimes 1_E)|M \in B(M)^n$ is a commuting tuple with $\sigma(T) \subset \overline{D}$ such that

$$f(Z \otimes 1_E)^* M \subset M \quad \text{and} \quad f(T) = P_M f(Z \otimes 1_E)|M$$

holds for each function $f \in \mathcal{O}(\overline{D})$.

(b) For $h \in \mathcal{H} \otimes E$, $z \in D$ and $x \in E$, we have the identity

$$\langle h, C_z \otimes x \rangle = \langle h(z), x \rangle.$$

(c) Each function $f \in \mathcal{O}(\overline{D}, B(E, E_.))$ induces a well-defined multiplication operator $T_f : \mathcal{H} \otimes E \to \mathcal{H} \otimes E_.$, $h \mapsto fh$. For $z \in D$ and $x \in E_.$, we have

$$T_f^*(C_z \otimes x) = C_z \otimes f(z)^* x.$$

Proof (a) Since the adjoint $T^* = Z^* \otimes 1_E|M$ is a commuting tuple, the same is true for T. The approximate point spectrum of T^* satisfies

$$\sigma_\pi(T^*) \subset \sigma_\pi(Z^* \otimes 1_E) \subset \sigma(Z^* \otimes 1_E) = \sigma(Z^*) = \{\overline{z};\ z \in \overline{D}\}.$$

Since the Taylor spectrum is always contained in the polynomially convex hull of the approximate point spectrum ([25]), we conclude that $\sigma(T) \subset \overline{D}$. Let $f \in \mathcal{O}(\overline{D})$. Since $\sigma(Z^* \otimes 1_E) \cup \sigma(T^*) \subset \{\overline{z};\ z \in \overline{D}\}$, the space M is invariant for $\tilde{f}(Z^* \otimes 1_E) = f(Z \otimes 1_E)^*$. Hence it follows that (cf. Lemma 2.5.8 in [17])

$$f(T)^* = \tilde{f}(T^*) = \tilde{f}(Z^* \otimes 1_E)|M = f(Z \otimes 1_E)^*|M.$$

(b) The observation that both sides of the claimed identity are continuous linear in h reduces the assertion to the case of elementary tensors $h = h_0 \otimes y$, $h_0 \in \mathcal{H}$ and $y \in E$, where it is obvious.

(c) The first part of (c) has been proved before. The remaining part follows from (b) and the observation that

$$\big\langle g, T_f^*(C_z \otimes x) \big\rangle = \big\langle f(z) g(z), x \big\rangle = \big\langle g, C_z \otimes f(z)^* x \big\rangle$$

for $g \in \mathcal{H} \otimes E$, $z \in D$ and $x \in E_.$. $\square$

For a commuting tuple $T \in B(H)^n$ on a Hilbert space H, we denote by

$$M_T = (L_T, R_{T^*}) \in B(B(H))^{2n}$$

the commuting $(2n)$-tuple consisting of the tuple $L_T = (L_{T_1}, \ldots, L_{T_n})$ of left multiplication operators $L_{T_j} : B(H) \to B(H)$, $X \mapsto T_j X$, and of the tuple $R_{T^*} = (R_{T_1^*}, \ldots, R_{T_n^*})$ of right multiplication operators $R_{T_j^*} : B(H) \to B(H)$, $X \mapsto XT_j^*$. It is well known (see [16]) that $\sigma(M_T) = \sigma(T) \times \sigma(T^*)$. For a given analytic function $f \in \mathcal{O}(\sigma(T) \times \sigma(T^*))$, we use the notation

$$f(T, T^*) = f(M_T)(1_H) \in B(H).$$

Suppose that $T \in B(H)^n$ and $S \in B(K)^n$ are commuting tuples of bounded operators on Hilbert spaces H and K and that $X \in B(H, K)$ intertwines T and S componentwise in the sense that

$$XT_i = S_i X \qquad (1 \leq i \leq n).$$

Then the operator $\Delta_X : B(H) \to B(K)$, $A \mapsto XAX^*$, intertwines the multiplication tuples $M_T \in B(B(H))^{2n}$ and $M_S \in B(B(K))^{2n}$ componentwise. If f is an analytic complex-valued function on an open neighbourhood of $(\sigma(T) \times \sigma(T^*)) \cup (\sigma(S) \times \sigma(S^*))$, then

$$\Delta_X f(M_T) = f(M_S)\Delta_X \qquad \text{(Lemma 2.5.8 in [17])}.$$

Lemma 1.2 With the notation from above, we obtain the identity

$$\frac{1}{C}\big(Z \otimes 1_E, (Z \otimes 1_E)^*\big) = P_E,$$

where $P_E \in B(\mathcal{H} \otimes E)$ denotes the orthogonal projection onto the subspace of all constant functions.

Proof Our hypothesis that $C_0 \equiv 1$ implies in particular that

$$P_E f = f(0) \qquad (f \in \mathcal{H} \otimes E).$$

Let $U \supset \sigma(Z)$ and $V \supset \sigma(Z^*)$ be open neighbourhoods. To prove the assertion it suffices to show that, for any function $f \in \mathcal{O}(U \times V)$, the identity

$$\big\langle f(Z \otimes 1_E, Z^* \otimes 1_E) C_\lambda \otimes x, C_\mu \otimes y \big\rangle = f(\mu, \overline{\lambda}) C(\mu, \overline{\lambda}) \langle x, y \rangle$$

holds for all $\lambda, \mu \in D$ and $x, y \in E$. Since both sides of this equation are continuous linear in f, we may suppose that $f = g \otimes h$ with $g \in \mathcal{O}(U)$ and $h \in \mathcal{O}(V)$. In this case, we obtain that

$$\begin{aligned} f(Z \otimes 1_E,\, Z^* \otimes 1_E) &= g(L_{Z \otimes 1_E})\, h(R_{Z^* \otimes 1_E})\, (1_{\mathcal{H} \otimes E}) \\ &= L_{g(Z \otimes 1_E)}\, R_{h(Z^* \otimes 1_E)}\, (1_{h \otimes E}) = g(Z \otimes 1_E)\, h(Z^* \otimes 1_E) \\ &= \big(g(Z)\, h(Z^*)\big) \otimes 1_E. \end{aligned}$$

Thus the proof is completed by the observation that

$$\big\langle h(Z^*)C_\lambda,\ g(Z)^*C_\mu\big\rangle = h(\overline{\lambda})\ g(\mu)\ \langle C_\lambda, C_\mu\rangle = (g\otimes h)\ (\mu,\overline{\lambda})\ C(\mu,\overline{\lambda}).$$

$\square$

Let $M \subset \mathcal{H}\otimes E$ and $M_. \subset \mathcal{H}\otimes E_.$ be $*$-invariant closed subspaces. Then

$$T = P_M(Z\otimes 1_E)|M \in B(M)^n, \quad T_. = P_{M_.}(Z\otimes 1_{E_.})|M_. \in B(M_.)^n$$

are commuting n-tuples with $\sigma(T)\cup\sigma(T_.) \subset \overline{D}$. For every function $f \in \mathcal{O}(\overline{\Delta})$ and every operator $X \in B(\mathcal{H}\otimes E)$, the identity

$$f(M_T)(P_M X|M) = P_M\big[f(M_{Z\otimes 1_E})(X)\big]|M$$

holds. Indeed, it suffices to check this identity for functions f of the form

$$f = g\otimes h, \qquad g\in\mathcal{O}(U),\ h\in\mathcal{O}(V),$$

where $U \supset \sigma(Z)$ and $V \supset \sigma(Z^*)$ are fixed open neighbourhoods. In this case the assertion easily follows as an application of Lemma 1.1(a).

Proposition 1.3 For every operator $X \in B(M, M_.)$ which intertwines the commuting tuples $T \in B(M)^n$ and $T_. \in B(M_.)^n$ componentwise, we have the identity

$$\big\langle \frac{1}{C}(M_{T_.})(XX^*)h, k\big\rangle = \langle P_E X^*h, X^*k\rangle$$

for all $h, k \in M_.$.

Proof Using Lemma 1.2 we obtain the identity

$$\frac{1}{C}(M_{T_.})(XX^*) = \Delta_X\big(\frac{1}{C}(T,T^*)\big) = \Delta_X(P_M P_E|M),$$

which is equivalent to the assertion. $\square$

§2 Fractional transforms and the Schur class

In this section we make the additional assumption that the open set $D\subset\mathbb{C}^n$ is a generalized analytic polyhedron in the sense that there are an open neighbourhood W of $\overline{D}$ and an analytic function $d = (d_{jk}) : W \to B(\mathbb{C}^q, \mathbb{C}^p) \cong \mathbb{C}^{p,q}$ such that $d(0) = 0$ and

$$D = \{z\in W;\ \|d(z)\| < 1\}.$$

Let us fix a separable infinite-dimensional Hilbert space H. For a given commuting tuple $X \in B(H)^n$ with $\sigma(X)\subset W$, the operator-valued analytic functional calculus of X applied to the function d gives rise to the matrix-operator

$$d(X) = (d_{jk}(X)) \in B(H^q, H^p).$$

Using the spectral mapping theorem for Taylor's analytic functional calculus, one can prove the following result.

Lemma 2.1 Let $X \in B(H)^n$ be a commuting tuple with $\sigma(X) \subset W$. Then

$$\sup_{z \in \sigma(X)} \|d(z)\| \; \leq \|d(X)\|.$$

$\square$

This result was proved in [5] for the case that the coefficients d_{jk} of d are polynomial functions. Since the same proof, without any changes, applies to our more general situation, we omit the details. Note that, if $\|d(X)\| < 1$ in the setting of Lemma 2.1, then $\sigma(X) \subset D$. Hence, for any operator-valued analytic function $f \in \mathcal{O}(D, B(E, E_\cdot))$ with given Hilbert spaces E and $E_\cdot$, we can form the operator

$$f(X) \in B(H \otimes E,\, H \otimes E_\cdot).$$

Define $\mathcal{C}$ as the set of all commuting tuples $X \in B(H)^n$ with $\sigma(X) \subset W$ and $\|d(X)\| < 1$. For $f \in \mathcal{O}(D, B(E, E_\cdot))$, we call

$$\|f\|_{\mathcal{S}} = \sup\{\|f(X)\|;\ X \in \mathcal{C}\}$$

the Schur norm of f. Since, for each point $z \in D$, the n-tuple $z1 = (z_1 1_H, \ldots, z_n 1_H)$ belongs to $\mathcal{C}$ and since $f(z1) = 1_H \otimes f(z)$ for each function $f \in \mathcal{O}(D, B(E, E_\cdot))$, it follows that $\|f\|_{\mathcal{S}} \geq \|f\|_{\infty, D}$ for each such function f.

The linear space

$$\mathcal{S}(d, B(E, E_\cdot)) = \{f \in \mathcal{O}(D, B(E, E_\cdot));\ \|f\|_{\mathcal{S}} < \infty\}$$

becomes a Banach space if equipped with the Schur norm $\|\cdot\|_{\mathcal{S}}$.

Its unit ball

$$\mathcal{S} = \mathcal{S}_d(E, E_\cdot) = \{f \in \mathcal{O}(D, B(E, E_\cdot));\ \|f\|_{\mathcal{S}} \leq 1\}$$

is called the $B(E, E_\cdot)$-valued Schur class on D (with respect to d). In the scalar case $E = E_\cdot = \mathbb{C}$, we simply write $\mathcal{S}_d$ instead of $\mathcal{S}_d(\mathbb{C}, \mathbb{C})$.

Let L and $L_\cdot$ be Hilbert spaces, and let the matrix operator

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(L \oplus E,\, L_\cdot \oplus E_\cdot)$$

be a contraction. We shall use the well-known and elementary fact that, for each bounded operator $X \in B(L_\cdot, L)$ with $\|X\| < 1$, the operator

$$d + c(1_L - Xa)^{-1} Xb \in B(E, E_\cdot)$$

is a contraction again. Fix an arbitrary Hilbert space K. With

$$L = K^p \cong \mathbb{C}^p \otimes K, \quad L_. = K^q \cong \mathbb{C}^q \otimes K,$$

and with X replaced by $d_K(z) = d(z) \otimes 1_K$ $(z \in D)$, we see that the analytic function $f_U : D \to B(E, E_.)$ defined by

$$f_U(z) = d + c(1_{K^p} - d_K(z)a)^{-1} d_K(z) b$$

is sup-norm bounded by one. To keep the notation simpler, we shall usually write $d(z)$ instead of $d_K(z)$ again.

For the case that d is a polynomial function, the following result is contained in [5].

Proposition 2.2 Let K be a Hilbert space. Suppose that the matrix operator

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(K^p \oplus E,\ K^q \oplus E_.)$$

is a contraction. Then $f_U \in \mathcal{S}_d(E, E_.)$.

Proof For completeness sake, we indicate the elementary proof. It suffices to observe that, for any given tuple $X \in \mathcal{C}$, the operator

$$f_U(X) = 1_H \otimes d + (1_H \otimes c)\big(1_{H\otimes K^p} - d_K(X)(1_H \otimes a)\big)^{-1} d_K(X)(1_H \otimes b)$$

is a contraction again. The reason is that the matrix-operator obtained from U by replacing its coefficients by the tensor products with the identity operator on H is a contraction again. □

Our next aim is to derive some useful characterizations of Schur class functions, which in particular show that the converse of Proposition 2.2 holds. For the case that the coefficients of d are polynomial functions, this result has been announced in [11]. Since our setting is slightly more general, we indicate a sketch of the main ideas.

Lemma 2.3 Let $A \in B(H^r) = B(H)^{r,r}$ be a positive operator. Then

$$\|A\| \le r\, \|\mathrm{tr} A\|.$$

Proof Using the Cauchy-Schwarz inequality for the positive definite map

$$\{1, \dots, r\}^2 \to B(H), \quad (i,j) \mapsto A_{ji}$$

one obtains the estimate

$$\|A_{ji}\| \le (\|A_{ii}\| \|A_{jj}\|)^{1/2} \le \max_k \|A_{kk}\|$$

for $i, j = 1, \dots, r$. On the other hand, for any fixed index $k \in \{1, \dots, r\}$, we have

$$\|A_{kk}\| \le \sup_{\|x\|=1} \max_i \langle A_{ii} x, x\rangle \le \|\mathrm{tr} A\|.$$

To complete the proof it suffices to recall that $\|A\| \leq r \max_{j,i} \|A_{ji}\|$. $\square$

Suppose that $\delta : S \to B(\mathbb{C}^r, \mathbb{C}^s)$ is a map on an arbitrary set S such that $\|\delta(z)\| < 1$ for $z \in S$.

Lemma 2.4 Suppose that $\Gamma : S \times S \to B(H^r)$ is positive definite such that

$$F : S \times S \to B(H), \quad F(z,w) = \mathrm{tr}\big((1 - \delta(z)\delta(w)^*)\Gamma(z,w)\big)$$

has sup-norm bounded by one. Then the estimate

$$\|\Gamma(z,w)\| \leq r\big((1 - \|\delta(z)\|^2)(1 - \|\delta(w)\|^2)\big)^{-(1/2)}$$

holds for all $z, w \in S$.

Proof By Lemma 2.3 we obtain that

$$\begin{aligned}
\|\Gamma(z,z)\| &\leq \|(1 - \delta(z)\delta(z)^*)^{-1/2}\|^2 \|(1 - \delta(z)\delta(z)^*)^{1/2}\Gamma(z,z)(1 - \delta(z)\delta(z)^*)^{1/2}\| \\
&\leq r\|(1 - \delta(z)\delta(z)^*)^{-1}\| \|F(z,z)\| \\
&\leq r(1 - \|\delta(z)\|^2)^{-1}
\end{aligned}$$

for all $z \in S$. To complete the proof it suffices to apply the Cauchy-Schwarz inequality to the positive definite map Γ. $\square$

After these preparations we can prove our version of the theorem from Ball and Bolotnikov [11] referred to above.

Theorem 2.5 Let $S \subset D$ and $f : S \to B(E, E_{\cdot})$ be given. Then the following are equivalent:

(i) f extends to a map in $\mathcal{S}$;

(ii) there is a positive definite map $\Gamma : S \times S \to B(E^q)$ such that

$$1 - f(w)^* f(z) = \mathrm{tr}\big((1 - d^t(z)d^t(w)^*)\Gamma(z,w)\big) \quad (z, w \in S);$$

(iii) there is a Hilbert space $\mathcal{G}$ and a map $G : S \to B(E, \mathcal{G}^q)$ such that

$$1 - f(w)^* f(z) = G(w)^*(1 - d(w)^* d(z))G(z) \quad (z, w \in S);$$

(iv) there is a Hilbert space K and a unitary operator

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(K^p \oplus E, K^q \oplus E_{\cdot})$$

such that $f(z) = d + cd(z)(1_{K^q} - ad(z))^{-1}b$ for $z \in S$;

(ii)' there is a positive definite map $\Gamma : S \times S \to B(E^p)$ such that

$$1 - f(w)f(z)^* = \mathrm{tr}\big((1 - d^t(z)^* d^t(w))\Gamma(z,w)\big) \quad (z, w \in S);$$

(iii)' there is a Hilbert space $\mathcal{G}$ and a map $G : S \to B(\mathcal{G}^p, E)$ such that

$$1 - f(w)f(z)^* = G(w)(1 - d(w)d(z)^*)G(z)^* \quad (z, w \in S);$$

(iv)' there is a Hilbert space K and a unitary operator U as in condition (iv) such that $f = f_U$ on S.

Proof (i) $\Rightarrow$ (ii). Let $f \in \mathcal{S}$ be a Schur class function, and let $S \subset D$ be finite. Set $\delta(z) = d^t(z)$. Suppose that $\dim(E) < \infty$. It suffices to show that in this case there is a positive definite map Γ as in condition (ii). Then the general case can be deduced by choosing, for each finite set $M \subset D$ and each finite-dimensional subspace $F \subset E$, a positive definite map $\Gamma_{M,F} : M \times M \to B(F^q)$ such that

$$P_F(1 - f(w)^* f(z))|F = \mathrm{tr}\Big((1 - \delta(z)\delta(w)^*)\Gamma_{M,F}(z,w)\Big)$$

holds for $z, w \in M$. The trivial extensions $\Delta_{M,F} : D \times D \to B(E^q)$ defined by setting $\Delta_{M,F} = 0$ on $(D \times D) \setminus (M \times M)$ and

$$\Delta_{M,F}(z,w) = \Gamma_{M,F}(z,w)P_{F^q} \quad (z, w \in M)$$

form a net in the compact Hausdorff space

$$\prod_{(z,w) \in D \times D} \big(\{X \in B(E^q);\ \|X\| \le r(z,w)\}, \tau_{\mathrm{WOT}}\big),$$

where $r(z,w)$ are suitable real numbers given by Lemma 2.4 and τ_{WOT} refers to the weak operator topology. It is elementary to check that the limit Γ of any convergent subnet of $(\Delta_{M,F})$ will give a representation of $1 - f(w)^* f(z)$ on $D \times D$ as in condition (ii).

Thus suppose that $S \subset D$ is finite and that $\dim(E) < \infty$. The proof of this implication follows by a standard separation argument due to Agler [1]. The set $V = B(E)^{S \times \tilde{S}}$ of all functions $S \times \tilde{S} \to B(E)$ is a finite-dimensional normed space with respect to the sup-norm. It suffices to show that the function

$$F : S \times \tilde{S} \to B(E), \quad F(z,w) = 1 - f(\overline{w})^* f(z)$$

belongs to the subset $C \subset V$ consisting of all functions $g \in V$ for which there is a positive definite map $\Gamma : S \times S \to B(E^q)$ such that

$$g(z, \overline{w}) = \mathrm{tr}\big((1 - \delta(z)\delta(w)^*)\Gamma(z,w)\big) \quad (z, w \in S).$$

By Lemma 2.4 the set C is a closed proper cone in V. Assume that $F \notin C$. Since F and the elements of C are self-adjoint with respect to the involution on V defined by

$$f^*(z,w) = f(\overline{w}, \overline{z})^*,$$

a well-known separation theorem (Theorem 2.7 in [21]) allows us to choose a linear functional $L : V \to \mathbb{C}$ such that $L > 0$ on $C \setminus \{0\}$ and $L(F) < 0$. Let $H_0 = B(E, \mathbb{C})^S$ be the vector space of all functions $g : S \to B(E, \mathbb{C})$. For $g, h \in H_0$, define a function $g \times h$ in V by

$$g \times h(z, w) = h(\overline{w})^* g(z).$$

All functions of the form $g \times g$, $g \in H_0$, belong to C. Indeed, by Schur's theorem the map $\Gamma : S \times S \to B(E)$ defined by

$$\Gamma(z, w) = \sum_{m=0}^{\infty} \left(\langle (d_{k1}(z))_k, (d_{k1}(w))_k \rangle_{\mathbb{C}^p}\right)^m g(w)^* g(z)$$

is positive definite and satisfies

$$g \times g(z, \overline{w}) = \operatorname{tr}\big((1 - \delta(z)\delta(w)^*)\pi_1^* \Gamma(z, w)\pi_1\big) \quad (z, w \in S),$$

where $\pi_1 : E^q \to E$ is the projection onto the first coordinate. Thus H_0 is a finite-dimensional Hilbert space relative to the inner product

$$\langle g, h \rangle = L(g \times h).$$

The tuple $X \in L(H_0)^n$ defined by $(X_j f)(z) = z_j f(z)$ for $z \in S$ and $j = 1, \ldots, n$ is commutative with $\sigma(X) = S$. Since for each non-zero element $h \in H_0^q$ the relation

$$\begin{aligned}\|h\|^2 - \|d(X)h\|^2 &= L\left(\sum_{i,j=1}^{q} \Big(\delta_{ij} - \sum_{k=1}^{p} d_{kj}(z)(d_{ki})^\sim(w)\Big) h_i(\overline{w})^* h_j(z)\right) \\ &= L\Big(\operatorname{tr}[(1 - \delta(z)\delta(\overline{w})^*)\Gamma_h(z, \overline{w})]\Big) > 0\end{aligned}$$

holds with $\Gamma_h(z, w) = (h_i(w)^* h_j(z))_{1 \le i,j \le q}$, we conclude that $\|d(X)\| < 1$.

Since by hypothesis $f \in \mathcal{S}$, it follows that $\|f(X)\| \le 1$. For fixed vectors $g, h \in H_0$ and $x, y \in E$, the identity

$$\langle u(X)g \otimes x, v(X)h \otimes y \rangle_{H_0 \otimes E_.} = L\Big(\langle u(z)x, v(\overline{w})y \rangle_{E_.}\ g \times h(z, w)\Big)$$

holds for each pair of functions $u, v \in \mathcal{O}(D, B(E, E_.))$. It suffices to check this assertion for elementary tensors $u = u_0 \otimes A$, $v = v_0 \otimes B$, which is straightforward.

Fix an orthonormal basis $(e_j)_{j=1}^r$ of E and define, for $j = 1, \ldots, r$,

$$f_j : S \to B(E, \mathbb{C}), \quad f_j(z)x = \langle x, e_j \rangle.$$

Then $f_i \times f_j \equiv \langle \cdot, e_i \rangle e_j$ on $S \times \tilde{S}$. The identity

$$F(z, w) = \sum_{i,j=1}^{r} \langle F(z, w)e_i, e_j \rangle \ \langle \cdot, e_i \rangle e_j,$$

valid for all $(z,w) \in S \times \tilde{S}$, implies that

$$\begin{aligned} L(F) &= L\Big(\sum_{j=1}^{r}(f_j \times f_j)(z,w) - \sum_{i,j=1}^{r}\langle f(z)e_i, f(\overline{w})e_j\rangle f_i \times f_j(z,w)\Big) \\ &= \Big\|\sum_{j=1}^{r} f_j \otimes e_j\Big\|^2_{H_0\otimes E} - \Big\|f(X)\big(\sum_{j=1}^{r} f_j \otimes e_j\big)\Big\|^2_{H_0\otimes E.} \geq 0. \end{aligned}$$

This contradiction completes the proof of the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Suppose that Γ is given as in condition (ii). Then there is a Hilbert space $\mathcal{G}$ and a map $\tilde{G} : S \to B(E^q, \mathcal{G})$ such that $\Gamma(z,w) = \tilde{G}(w)^*\tilde{G}(z)$ for $z, w \in S$. Write $\tilde{G}(z) = (G_1(z), \ldots, G_q(z))$ and define

$$G : S \to B(E, \mathcal{G}^q), \quad G(z) = (G_1(z), \ldots, G_q(z))^t.$$

Then, with $\delta(z) = d^t(z)$, we obtain that

$$\begin{aligned} &\operatorname{tr}\big((1 - \delta(z)\delta(w)^*)\Gamma(z,w)\big) \\ &= \sum_{j=1}^{q} G_j(w)^*G_j(z) - \sum_{i,j=1}^{q} G_i(w)^*\Big(\sum_{k=1}^{p}\overline{d_{ki}(w)}d_{kj}(z)\Big)G_j(z) \\ &= G(w)^*(1 - d(w)^*d(z))G(z) \end{aligned}$$

for $z, w \in S$.

(iii) $\Rightarrow$ (iv). Let G be a map as in condition (iii). Then, for $z, w \in S$ and $x, y \in E$, we obtain the identity

$$\langle d(z)G(z)x, d(w)G(w)y\rangle_{\mathcal{G}^p} + \langle x, y\rangle_E = \langle G(z)x, G(w)y\rangle_{\mathcal{G}^q} + \langle f(z)x, f(w)y\rangle_{E.}.$$

Hence there is a Hilbert space $K \supset \mathcal{G}$ and a unitary operator U of the form described in condition (iv) such that

$$U\begin{pmatrix} d(z)G(z)x \\ x \end{pmatrix} = \begin{pmatrix} G(z)x \\ f(z)x \end{pmatrix}$$

for all $z \in S$ and $x \in E$. By solving the corresponding system of linear equations

$$\begin{aligned} ad(z)G(z) + b &= G(z), \\ cd(z)G(z) + d &= f(z), \end{aligned}$$

we obtain that, for all $z \in S$,

$$f(z) = d + cd(z)(1_{K^q} - ad(z))^{-1}b.$$

(iv) $\Rightarrow$ (iv)'. Suppose that U represents f as in condition (iv). Using the identity

$$(1_{K^q} - ad(z))^{-1} = 1_{K^q} \; + \; a(1_{K^p} - d(z)a)^{-1}d(z)$$

we see that

$$
\begin{aligned}
f(z) &= d + cd(z)(1_{K^q} - ad(z))^{-1}b \\
&= d + cd(z)b + cd(z)a(1_{K^p} - d(z)a)^{-1}d(z)b \\
&= d + c(1_{K^p} - d(z)a)^{-1}d(z)b.
\end{aligned}
$$

In Proposition 2.2 we proved that (iv)' implies (i). Completely analogous to the above arguments one can prove the implications (i) $\Rightarrow$ (ii)' $\Rightarrow$ (iii)' $\Rightarrow$ (iv)'. Thus the proof of Theorem 2.5 is complete. □

§3 **A commutant lifting theorem**

Let $\mathcal{H}$ be a functional Hilbert space consisting of analytic functions defined on an open set D in $\mathbb{C}^n$ such that $\mathcal{H}$ and D satisfy the conditions described at the beginning of §1. As in §2 we suppose that there is an open neighbourhood W of $\overline{D}$ and an analytic function $d : W \to B(\mathbb{C}^q, \mathbb{C}^p)$ such that $d(0) = 0$ and

$$D = \{z \in W;\ \|d(z)\| < 1\}.$$

For $0 < r < 1$, the set $W_r = \{z \in W;\ \|d(z)\| < 1/r\}$ is an open neighbourhood of $\overline{D}$. Recall that the operator-valued analytic functional calculus of the multiplication tuple $Z = (Z_1, \ldots, Z_n) \in B(\mathcal{H})^n$ gives rise to the matrix operator $d(Z) \in B(\mathcal{H}^q, \mathcal{H}^p)$. Throughout this section we make the additional assumption that

$$\|d(Z)\| \leq 1.$$

For given Hilbert spaces E and $E_.$, we consider the multiplier space

$$M(E, E_.) = \{f \in \mathcal{O}(D, B(E, E_.));\ f\,\mathcal{H} \otimes E \subset \mathcal{H} \otimes E_.\}.$$

By the closed graph theorem each function $f \in M(E, E_.)$ induces a continuous linear multiplication operator $T_f : \mathcal{H} \otimes E \to \mathcal{H} \otimes E_.$, $g \mapsto fg$. It is well known ([14],[15]) that a function $f \in \mathcal{O}(D, B(E, E_.))$ belongs to the closed unit ball of $M(E, E_.)$ with respect to the multiplier norm $\|f\| = \|T_f\|$ if and only if the induced map

$$K_f : D \times D \to B(E_.), \quad (z, w) \mapsto C(w, \overline{z})(1_{E_.} - f(w)f(z)^*)$$

is positive definite. In particular, the closed unit ball of $M(E, E_.)$ contains together with each pointwise convergent sequence of multipliers its limit.

Lemma 3.1 Let E and $E_.$ be Hilbert spaces. Then $\mathcal{S}(d, B(E, E_.)) \subset M(E, E_.)$ and

$$\|T_f\| \leq \|f\|_{\mathcal{S}}$$

for all $f \in \mathcal{S}(d, B(E, E_.))$.

Proof Let $f \in \mathcal{S}_d(E, E_.)$ be a Schur class function. By Theorem 2.5 there is a unitary matrix operator $U \in B(K^p \oplus E,\ K^q \oplus E_.)$ such that $f = f_U$. As usual we denote the coefficients of U by a, b, c, d (cf. Proposition 2.2). On D, the function f is the pointwise limit of the functions $f_r \in \mathcal{O}(W_r, B(E, E_.))$ defined by

$$f_r(z) = d + c\big(1_{K^p} - r d_K(z) a\big)^{-1} r d_K(z) b$$

as r tends to one from below. The representation

$$f_r(Z) = (1_{\mathcal{H}} \otimes d) + (1_{\mathcal{H}} \otimes c)\Big(1_{\mathcal{H}\otimes K^p} - r d_K(Z)(1_{\mathcal{H}} \otimes a)\Big)^{-1} r d_K(Z)(1_{\mathcal{H}} \otimes b)$$

shows that the restrictions $f_r|D$ are contractive multipliers. By the remarks preceding the lemma, this observation completes the proof. $\square$

A second important consequence of the hypothesis that $\|d(Z)\| \leq 1$ is the following result, which should be compared with Lemma 3.2 in [12].

Lemma 3.2 Let K and $E_.$ be Hilbert spaces. Suppose that $a \in B(K^p, K^q)$ and $c \in B(K^p, E_.)$ are bounded operators with $a^*a + c^*c \leq 1_{K^p}$. Then, for $x \in K^p$, the function $\Omega x : D \to E_.$ defined by

$$(\Omega x)(z) = c\big(1_{K^p} - d_K(z) a\big)^{-1} x$$

belongs to $\mathcal{H} \otimes E_.$. The map $\Omega : K^p \to \mathcal{H} \otimes E_.$, $x \mapsto \Omega x$, is a linear contraction with the property that

$$\Omega^*(C_z \otimes y) = \big(1_{K^p} - a^* d_K(z)^*\big)^{-1} c^* y \quad (z \in D,\, y \in E_.).$$

Proof Define $\alpha = 1_{\mathcal{H}} \otimes a \in B(\mathcal{H} \otimes K^p, \mathcal{H} \otimes K^q)$. Let us fix an arbitrary real number r with $0 < r < 1$. Then the operator

$$\delta = \delta_r = r d(Z) \otimes 1_K \in B(\mathcal{H}^q \otimes K, \mathcal{H}^p \otimes K) \cong B(\mathcal{H} \otimes K^q, \mathcal{H} \otimes K^p)$$

satisfies $\|\delta_r\| \leq r$. The functions

$$\omega = \omega_r : W_r \to B(K^p, E_.), \quad z \mapsto c(1 - r d_K(z) a)^{-1},$$

$$\varphi = \varphi_r : W_r \to B(K^p), \quad z \mapsto (1 + r d_K(z) a)(1 - r d_K(z) a)^{-1}$$

induce multipliers $T_\omega \in M(K^p, E_.)$ and $T_\varphi \in M(K^p)$ such that

$$\begin{aligned}
\tfrac{T_\varphi + T_\varphi^*}{2} &= \tfrac{1}{2}\big[(1+\delta\alpha)(1-\delta\alpha)^{-1} + (1-\alpha^*\delta^*)^{-1}(1+\alpha^*\delta^*)\big] \\
&= (1-\alpha^*\delta^*)^{-1}(1-\alpha^*\delta^*\delta^*\alpha)(1-\delta\alpha)^{-1} \\
&\geq (1-\alpha^*\delta^*)^{-1}(1-r^2\alpha^*\alpha)(1-\delta\alpha)^{-1} \\
&\geq r^2(1-\alpha^*\delta^*)^{-1}(1_{\mathcal{H}} \otimes c^*)(1_{\mathcal{H}} \otimes c)(1-\delta\alpha)^{-1} \\
&= r^2 T_\omega^* T_\omega.
\end{aligned}$$

In particular, it follows that

$$r^2 \left\| T_{\omega_r}(C_0 \otimes x) \right\|^2 \leq (1/2) \left\langle (T_{\varphi_r} + T_{\varphi_r}^*) C_0 \otimes x,\, C_0 \otimes x \right\rangle = \|x\|^2$$

for $0 < r < 1$ and $x \in K^p$. Since, for $x \in K^p$, $y \in E_\cdot$ and $z \in D$, we have

$$\left\langle T_{\omega_r}(C_0 \otimes x), C_z \otimes y \right\rangle = \left\langle x, \omega_r(z)^* y \right\rangle = \left\langle \omega_r(z)x, y \right\rangle,$$

it follows that the limit $\lim_{r \uparrow 1} \left\langle T_{\omega_r}(C_0 \otimes x), y \right\rangle$ exists for $x \in K^p$ and y in the linear span of all vectors $C_z \otimes y$, $z \in D$ and $y \in E_\cdot$. Because of

$$\overline{\lim_{r \uparrow 1}} \left\| T_{\omega_r}(C_0 \otimes x) \right\| \leq \|x\|$$

the above limit exists for all $x \in K^p$ and $y \in \mathcal{H} \otimes E_\cdot$ and can be represented by a linear contraction $\Omega' : K^p \to \mathcal{H} \otimes E_\cdot$ in the sense that

$$\langle \Omega' x, y \rangle = \lim_{r \uparrow 1} \left\langle T_{\omega_r}(C_0 \otimes x), y \right\rangle \quad (x \in K^p, y \in \mathcal{H} \otimes E_\cdot).$$

In view of the identity

$$\left\langle (\Omega' x)(z), y \right\rangle = \lim_{r \uparrow 1} \left\langle \omega_r(z)x, y \right\rangle = \left\langle (\Omega x)(z), y \right\rangle$$

it is clear that $\Omega = \Omega' \in B(K^p, \mathcal{H} \otimes E_\cdot)$ is a well-defined contraction. The proof is completed by the observation that

$$\left\langle \Omega^*(C_z \otimes y), x \right\rangle = \left\langle y, (\Omega x)(z) \right\rangle = \left\langle (1 - a^* d_K(z)^*)^{-1} c^* y, x \right\rangle$$

holds for all elements $y \in E_\cdot$, $x \in K^p$ and $z \in D$. □

Let E and $E_\cdot$ be complex Hilbert spaces. Suppose that $M \subset \mathcal{H} \otimes E$ and $M_\cdot \subset \mathcal{H} \otimes E_\cdot$ are $*$-invariant closed subspaces. Denote by

$$T = P_M(Z \otimes 1_E)|M \in B(M)^n, \quad T_\cdot = P_{M_\cdot}(Z \otimes 1_{E_\cdot})|M_\cdot \in B(M_\cdot)^n$$

the compressions of $Z \otimes 1_E$ and $Z \otimes 1_{E_\cdot}$ to M and $M_\cdot$, respectively. Our aim is to find positivity conditions that characterize those operators $X : M \to M_\cdot$ which intertwine T and $T_\cdot$ and which possess a lifting to a multiplier $T_f : \mathcal{H} \otimes E \to \mathcal{H} \otimes E_\cdot$ with a Schur class symbol f.

For this purpose, we fix an orthonormal basis $(e_k)_{k \geq 0}$ of $\mathcal{H}$ with $\mathbb{C}[z] = \mathrm{LH}\{e_k;\ k \geq 0\}$ and define $A_k = e_k(T_\cdot) \in B(M_\cdot)$ for $k \geq 0$. In addition to our previous hypotheses we make the assumption that the set

$$M_0 = \{x \in M_\cdot;\ \sum_{k \geq 0} \|A_k^* x\|^2 < \infty\}$$

is dense in $M_\cdot$. Note that $M_0 \subset M_\cdot$ is a linear subspace which is invariant for each operator in the commutant of $T_\cdot^*$. For any given operator $B \in B(M_\cdot)$, the map

$$\tilde{B} : M_0 \times M_0 \to \mathbb{C}, \quad \tilde{B}(x, y) = \sum_{k \geq 0} \langle A_k B A_k^* x, y \rangle$$

is a well-defined sesquilinear form. More generally, let $p \geq 1$ be a fixed natural number. Then, in exactly the same way, the p-fold direct sums $A_k^{(p)} \in B(M_\cdot^p)$ can be used to associate with each operator $B \in B(M_\cdot^p)$ a sesquilinear form

$$\tilde{B} : M_0^p \times M_0^p \to \mathbb{C}, \quad \tilde{B}(x,y) = \sum_{k \geq 0} \langle A_k^{(p)} B A_k^{*(p)} x, y \rangle.$$

For $1 \leq i \leq p$, we denote by

$$\iota_i : M_\cdot \to M_\cdot^p, \quad x \mapsto (\delta_{ij} x)_{j=1}^p,$$

$$\pi_i : M_\cdot^p \to M_\cdot, \quad (x_j)_{j=1}^p \mapsto x_i$$

the canonical inclusions and projections.

Lemma 3.3 Let $B = (B_{ij}) \in B(M_\cdot^p)$ be a given operator. Then, for $\rho, \sigma = 1, \ldots, q$, $\mu, \nu = 1, \ldots, p$ and all vectors $x, y \in M_0$, we obtain the identities

(a) $\tilde{B}(\iota_\nu x, \iota_\mu y) = (B_{\mu\nu})^\sim(x,y)$,

(b) $\tilde{B}\big(d^t(T_\cdot)^*(\iota_\rho x),\, d^t(T_\cdot)^*(\iota_\sigma y)\big) = \sum_{i,j=1}^p (B_{ij})^\sim\big(d_{j\rho}(T_\cdot)^* x, d_{i\sigma}(T_\cdot)^* y\big)$.

Proof (a) By definition we have

$$\begin{aligned}\tilde{B}(\iota_\nu x,\, \iota_\mu y) &= \sum_{k=0}^\infty \langle A_k^{(p)} B A_k^{*(p)}(\iota_\nu x), (\iota_\mu y)\rangle \\ &= \sum_{k=0}^\infty \langle A_k(\pi_\mu B \iota_\nu) A_k^* x, y\rangle = (B_{\mu\nu})^\sim(x,y).\end{aligned}$$

(b) Let $C = (C_{ij}) \in B(M_\cdot)^{p,q}$ be a matrix such that all entries of C belong to the commutant of $T_\cdot^* \in B(M_\cdot)^n$. Then as an application of part (a) we obtain that

$$\tilde{B}\big(C(\iota_\rho x),\, C(\iota_\sigma y)\big) = \sum_{i,j=1}^p \tilde{B}\big(\iota_j(C_{j\rho} x),\, \iota_i(C_{i\sigma} y)\big) = \sum_{i,j=1}^p (B_{ij})^\sim(C_{j\rho} x,\, C_{i\sigma} y).$$

By choosing $C = d^t(T_\cdot)^*$ we obtain the claimed formula. □

For $B = (B_{ij}) \in B(M_\cdot^p)$, we define a sesquilinear form $\tau B : M_0 \times M_0 \to \mathbb{C}$ by setting

$$\tau B(x,y) = \sum_{j=1}^p (B_{jj})^\sim(x,y) - \sum_{i,j=1}^p \sum_{k=1}^q (B_{ij})^\sim\big(d_{jk}(T_\cdot)^* x, d_{ik}(T_\cdot)^* y\big).$$

Lemma 3.4 Let $\Gamma = \langle \cdot, y\rangle y \in B(M_\cdot^p)$ be a positive rank-one operator given by a vector $y \in M_\cdot^p$. Then there is a linear map $S = S_y : M_0 \to \mathcal{H}^p$ such that

(i) $\tau\Gamma(x,x) = \|Sx\|^2 - \|d(Z)^* Sx\|^2$ for all $x \in M_0$,

(ii) $\langle Sx, q\rangle = \langle x, \sum\limits_{j=1}^{p} q_j(T.)y_j\rangle$ for all $x \in M_0$ and $q = (q_j) \in \mathbb{C}[z]^p$.

Proof For $x \in M_0^p$, the map $t_x : \mathbb{C}[z] \to \mathbb{C}, \quad q \mapsto \langle x, q(T.)^{(p)}y\rangle$, is antilinear such that

$$\sum_{k=0}^{\infty} |t_x(e_k)|^2 = \sum_{k=0}^{\infty} \langle A_k^{(p)} \Gamma A_k^{*(p)} x, x\rangle = \tilde{\Gamma}(x,x) < \infty.$$

Hence we obtain a well-defined linear map $t : M_0^p \to \mathcal{H}$ by setting

$$t(x) = \sum_{k=0}^{\infty} t_x(e_k)e_k.$$

By definition it follows that $\langle t(x), e_k\rangle = t_x(e_k)$ for all $k \in \mathbb{N}$. Since $\mathbb{C}[z]$ is the linear span of the polynomials e_k ($k \in \mathbb{N}$) and since each function $f \in \mathcal{O}(\overline{D})$ is the uniform limit of polynomials on an open neighbourhood of $\overline{D}$, we obtain that

$$\langle t(x), f\rangle = \langle x, f(T.)^{(p)}y\rangle$$

for $x \in M_0^p$ and $f \in \mathcal{O}(\overline{D})$. Furthermore, for $x, x' \in M_0^p$, we have

$$\langle t(x),\, t(x')\rangle = \sum_{k=0}^{\infty} t_x(e_k)\overline{t_{x'}(e_k)} = \tilde{\Gamma}(x,x').$$

The observation that, for $r, s \in \mathcal{O}(\overline{D})$ and $x \in M_0^p$, the identity

$$\big\langle t\big(r(T.)^{*(p)}x\big), s\big\rangle = \big\langle r(T.)^{*(p)}x,\, s(T.)^{(p)}y\big\rangle$$

$$= \big\langle x, (rs)(T.)^{(p)}y\big\rangle = \big\langle t(x), rs\big\rangle = \big\langle r(Z)^* t(x), s\big\rangle$$

holds, allows us to conclude that the intertwining relation

$$t \circ r(T.)^{*(p)} = r(Z)^* \circ t$$

holds on M_0^p. Write t as a row operator $t = (t_1, \ldots, t_p) \in B(M_0^p, \mathcal{H})$ and define

$$S : M_0 \to \mathcal{H}^p, \quad x \mapsto (t_j x)_{j=1}^p.$$

Using Lemma 3.3 (a) we obtain that

$$\big\langle t(\iota_j x), t(\iota_i x')\big\rangle = \tilde{\Gamma}(\iota_j x, \iota_i x') = (\Gamma_{ij})^{\sim}(x,x')$$

for $i, j = 1, \ldots, p$ and $x, x' \in M_0$, and hence that

$$(\Gamma_{ij})^{\sim}\big(d_{jk}(T.)^* x,\, d_{ik}(T.)^* x\big) = \big\langle t\big(d_{jk}(T.)^{*(p)}\iota_j, x\big),\, t\big(d_{ik}(T.)^{*(p)}\iota_i x\big)\big\rangle$$

$$= \big\langle d_{jk}(Z)^*(sx)_j,\, d_{ik}(Z)^*(Sx)_i\big\rangle$$

for all $i, j = 1, \ldots, p$, $k = 1, \ldots, q$ and $x \in M_0$. Taking the sum over all indices i, j and k, we see that

$$\sum_{i,j=1}^{p} \sum_{k=1}^{q} (\Gamma_{ij})^{\sim}\big(d_{jk}(T.)^* x,\, d_{ik}(T.)^* x\big)$$
$$= \sum_{i,j=1}^{p} \big\langle \big(d(Z)d(Z)^*\big)_{i,j} (Sx)_j,\, (Sx)_i \big\rangle$$
$$= \|d(Z)^* Sx\|^2$$

for all $x \in M_0$. Now, condition (i) follows from the observation that

$$\sum_{j=1}^{p} (\Gamma_{ij})^{\sim}(x, x) = \sum_{j=1}^{p} \big\langle (Sx)_j,\, (Sx)_j \big\rangle = \|Sx\|^2$$

for all vectors $x \in M_0$. At the same time we find that

$$\langle Sx, q\rangle = \sum_{j=1}^{p} \big\langle t(\iota_j x),\, q_j \big\rangle = \sum_{j=1}^{p} \big\langle \iota_j x,\, q_j(T.)^{(p)} y \big\rangle$$
$$= \sum_{j=1}^{p} \big\langle x, q_j(T.) y_j \big\rangle = \big\langle x, \sum_{j=1}^{p} q_j(T.) y_j \big\rangle$$

for $x \in M_0$ and $q = (q_j) \in \mathbb{C}[z]^p$. □

It follows from part (i) of Lemma 3.4 that $\tau\Gamma$ is a positive semi-definite sesquilinear form for each positive rank-one operator $\Gamma \in B(M_.^p)$. Using part (ii) of Lemma 3.4 and the density of $M_0 \subset M_.$, one finds that Γ has to be zero if $\tau\Gamma = 0$.

Lemma 3.5 Suppose that $\Gamma = \langle \cdot, y \rangle y \in B(M_.^p)$ is a positive rank-one operator. If $\tau\Gamma(x, x) = 0$ for all $x \in M_0$, then $\Gamma = 0$.

Proof Choose a linear map $S : M_0 \to \mathcal{H}^p$ as in Lemma 3.4. The hypothesis that $\tau\Gamma \equiv 0$ implies that

$$\langle Sx, Sx\rangle = \big\langle d(Z)d(Z)^* Sx, Sx \big\rangle \quad (x \in M_0).$$

Since $d(Z)d(Z)^* \in B(\mathcal{H}^p)$ is a positive contraction, it follows that $d(Z)d(Z)^* Sx = Sx$ for $x \in M_0$. Because of $d(Z)^*(C_0 \otimes \alpha) = C_0 \otimes (d(0)^* \alpha) = 0$ for $\alpha \in \mathbb{C}^p$ we find that

$$\langle Sx,\, C_0 \otimes \alpha\rangle = \langle Sx - d(Z)d(Z)^* Sx,\, C_0 \otimes \alpha\rangle = 0$$

for $x \in M_0$ and $\alpha \in \mathbb{C}^p$. Since $C_0 \equiv 1$, part (ii) of Lemma 3.4 allows us to conclude that

$$0 = \langle Sx,\, C_0 \otimes \alpha\rangle = \langle x, \sum_{j=1}^{p} \overline{\alpha}_j y_j \rangle = \sum_{j=1}^{p} \alpha_j \langle x, y_j\rangle$$

for all $x \in M_0$ and $\alpha = (\alpha_j) \in \mathbb{C}^p$. Hence $y = 0$ as was to be shown. □

Using the fact that each positive operator $\Delta \in B(M_.^p)$ can be represented as the limit of a strongly convergent, increasing series of positive rank-one operators, one can improve the above results considerably.

Proposition 3.6 Let $\Delta \in B(M^p_\cdot)$ be a positive operator. Then $\tau\Delta \geq 0$, and $\tau\Delta(x,x) = 0$ holds for all $x \in M_0$ if and only if $\Delta = 0$.

Proof Define $\gamma = \Delta^{1/2}$ and fix an orthonormal basis $(u_i)_{i \in I}$ of $M^p_\cdot$. Then the orthogonal projections

$$\pi_J = \sum_{j \in J} \langle \cdot, u_j \rangle u_j \quad (J \subset I \text{ finite})$$

form an increasing net of positive operators converging strongly to the identity operator. Fix a vector $x \in M^p_0$ and a real number $\varepsilon > 0$. Then there is a natural number $k = k(x, \varepsilon)$ such that

$$0 \leq \tilde{\Delta}(x,x) - \sum_{j=0}^{k} \langle A^{(p)}_k \Delta A^{*(p)}_k x, x \rangle < \varepsilon/2.$$

For k fixed, we can choose a finite set $J_0 \subset I$ such that

$$0 \leq \sum_{j=0}^{k} \Big(\langle \gamma A^{*(p)}_k x, \gamma\, A^{*(p)}_k x \rangle - \langle \pi_J \gamma A^{*(p)}_k x, \gamma A^{*(p)}_k x \rangle \Big) < \varepsilon/2$$

for all finite sets $J \subset I$ containing J_0. Hence, for the same sets J, we have

$$0 \leq \tilde{\Delta}(x,x) - \sum_{j=0}^{k} \langle \gamma \pi_J \gamma A^{*(p)}_k x, A^{*(p)}_k x \rangle < \varepsilon.$$

Thus we obtain that $0 \leq \tilde{\Delta}(x,x) - (\gamma\pi_J\gamma)^{\sim}(x,x) < \varepsilon$ for all finite sets $J \supset J_0$. Consequently we have shown that

$$\lim_J (\gamma\pi_J\gamma)^{\sim}(x,x) = \tilde{\Delta}(x,x)$$

for all $x \in M^p_0$.

Fix a vector $x \in M_0$. Since, for all $J \subset I$ finite, the operators $\gamma\pi_J\gamma = \sum_{j \in J} \langle \cdot, \gamma u_j \rangle \gamma u_j$ are finite sums of positive rank-one operators, Lemma 3.4 shows that the expressions

$$\tau(\gamma\pi_J\gamma)(x,x) = \sum_{j=1}^{p} (\gamma\pi_J\gamma)^{\sim}(\iota_j x, \iota_j x) - \sum_{k=1}^{q} (\gamma\pi_J\gamma)^{\sim}\big(d^t(T_\cdot)^* \iota_k x,\, d^t(T_\cdot)^* \iota_k x\big)$$

form an increasing net of non-negative real numbers. But then

$$\tau(\Delta)(x,x) = \lim_J \tau(\gamma\pi_J\gamma)(x,x) \geq 0.$$

If this limit is zero for each $x \in M_0$, then

$$\tau(\langle \cdot, \gamma u_j \rangle \gamma u_j) \equiv 0 \quad (j \in I),$$

and Lemma 3.5 implies that $\gamma u_j = 0$ for all $j \in I$. This observation completes the proof. $\square$

Let as before $T \in B(M)^n$ and $T_. \in B(M_.)^n$ be compressions of $Z \otimes 1_E$ and $Z \otimes 1_{E_.}$ to $*$-invariant closed subspaces $M \subset \mathcal{H} \otimes E$ and $M_. \subset \mathcal{H} \otimes E_.$, respectively. Under the hypothesis that the set

$$M_0 = \{x \in M_. \,;\, \sum_{k=0}^{\infty} \|e_k(T_.)^* x\|^2 < \infty\}$$

is dense in $M_.$, we characterize those operators $X \in B(M, M_.)$ intertwining T and $T_.$ that possess a lifting to a multiplier with Schur class symbol.

Theorem 3.7 Let $X \in B(M, M_.)$ be a bounded operator with $XT_i = T_{.i}X$ for $i = 1, \ldots, n$. Then the following conditions are equivalent:

(i) there exists a function $F \in \mathcal{S}$ such that $XP_M = P_{M_.}T_F$;

(ii) there exists a positive operator $\Gamma = (\Gamma_{ij}) \in B(M_.^p)$ such that

$$\frac{1}{C}(M_{T_.})(1 - XX^*) = \sum_{j=1}^{p} \Gamma_{jj} - \sum_{i,j=1}^{p} \sum_{k=1}^{q} d_{ik}(T_.)\Gamma_{ij} d_{jk}(T_.)^*;$$

(iii) there exists a Hilbert space K and a unitary operator

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(K^p \oplus E,\ K^q \oplus E_.)$$

such that if $F(z) = d + c(1_{K^p} - d_K(z)a)^{-1} d_K(z) b$, then $XP_M = P_{M_.}T_F$.

Proof The equivalence of conditions (i) and (iii) follows from Theorem 2.5.

To prove the implication (ii) $\Rightarrow$ (iii), suppose that $\Gamma \in B(M_.)^p$ is a positive operator as in condition (ii). Define $K_0 = M_.^p$ and $L = \Gamma^{1/2}$. Write $L = (L_1, \ldots, L_p)^t \in B(K_0, M_.^p)$ as a column operator. Then $\Gamma_{ij} = L_i L_j^*$ for $1 \leq i, j \leq p$ and, using Proposition 1.3, we obtain the identity

$$\|P_{E_.}h\|^2 - \|P_E X^* h\|^2 = \|(L_j^* h)_{j=1}^p\|^2 - \|\Big(\sum_{j=1}^{p} L_j^* d_{jk}(T_.)^* h\Big)_{k=1}^{q}\|^2$$

for all $h \in M_.$. Hence the map

$$\Big(\sum_{j=1}^{p} L_j^* d_{jk}(T_.)^* h\Big)_{k=1}^{q} \oplus (P_{E_.}h) \mapsto (L_j^* h)_{j=1}^p \oplus (P_E X^* h)$$

defines an isometry V from the linear subspace of $K_0^q \oplus E_.$ consisting of the vectors on the left-hand side into the space $K_0^p \oplus E$. Choose a Hilbert space $K \supset K_0$ and a unitary operator $U \in B(K^p \oplus E_., K^q \oplus E_.)$ such that U^* extends V. Let a, b, c, d be the coefficients of U (as in condition (iii) of Proposition 3.7). Defining $\Phi_j = L_j P_{K_0} \in B(K, M_.)$ for $j = 1, \ldots, p$ we obtain that

$$a^*\Big(\sum_{j=1}^{p} \Phi_j^* d_{jk}(T_.)^* h\Big)_{k=1}^{q} + c^* P_{E_.} h = (\Phi_j^* h)_{j=1}^{p}$$

$$b^*\Big(\sum_{j=1}^{p} \Phi_j^* d_{jk}(T_.)^* h\Big)_{k=1}^{q} + d^* P_{E_.} h = P_E X^* h$$

for all $h \in M_.$. According to Lemma 3.2 the unitary operator U induces a contraction $\Omega = (\Omega_1, \ldots, \Omega_p) : K^p \to \mathcal{H} \otimes E_.$ uniquely determined by

$$\Omega^*(C_z \otimes x) = (1_{K^p} - a^* d_K(z)^*)^{-1} c^* x \quad (z \in D,\ x \in E_.).$$

It follows that

$$a^* d_K(z)^* \Omega^*(C_z \otimes x) + c^* x = \Omega^*(C_z \otimes x),$$

$$b^* d_K(z)^* \Omega^*(C_z \otimes x) + d^* x = P_E T_{F_U}^*(C_z \otimes x)$$

for $x \in E_.$ and $z \in D$. Since the vectors $C_z \otimes x$ span $\mathcal{H} \otimes E_.$ topologically, the identity

$$d_K(z)^* \Omega^*(C_z \otimes x) = \Big(\sum_{j=1}^{p} \Omega_j^*(d_{jk}(Z)^* \otimes 1_{E_.})(C_z \otimes x)\Big)_{k=1}^{q}$$

implies that

$$a^*\Big(\sum_{j=1}^{p} \Omega_j^*(d_{jk}(Z)^* \otimes 1_{E_.}) h\Big)_{k=1}^{q} + c^* P_{E_.} h = (\Omega_j^* h)_{j=1}^{p}$$

$$b^*\Big(\sum_{j=1}^{p} \Omega_j^*(d_{jk}(Z)^* \otimes 1_{E_.}) h\Big)_{k=1}^{q} + d^* P_{E_.} h = P_E T_{F_U}^* h$$

for all $h \in \mathcal{H} \otimes E_.$. Define $\Phi_j' = P_{M_.} \Omega_j \in B(K, M_.)$ and $\Psi_j = \Phi_j - \Phi_j' \in B(K, M_.)$ for $j = 1, \ldots, p$. Since by Lemma 1.1 (a) the identity

$$\big(d_{jk}(Z)^* \otimes 1_{E_.}\big)|M_. = d_{jk}(T_.)^*,$$

holds, we see that

$$a^*\Big(\sum_{j=1}^{p} \Psi_j^* d_{jk}(T_.)^* h\Big)_{k=1}^{q} = (\Psi_j^* h)_{j=1}^{p}$$

for all $h \in M_.$.

Since a^* is a contraction, we find that

$$\sum_{j=1}^{p} \|\Psi_j^* h\|^2 \leq \sum_{k=1}^{q} \|\sum_{j=1}^{p} \Psi_j^* d_{jk}(T_.)^* h\|^2 \quad (h \in M_.).$$

Hence $\Delta = (\Psi_i \Psi_j^*)_{i,j} \in B(M_\cdot^p)$ is a positive operator such that

$$\sum_{j=1}^{p} \Delta_{jj} - \sum_{i,j=1}^{p} \sum_{k=1}^{q} d_{ik}(T_\cdot)\Delta_{ij} d_{jk}(T_\cdot)^* \;\leq\; 0.$$

By Proposition 3.6 the positivity of the operator Δ implies that the induced sesquilinear form $\tau\Delta = M_0 \times M_0 \to \mathbb{C}$ is positive. Using the definition of $\tau\Delta$ (see the section preceding Lemma 3.4) one immediately obtains that $\tau\Delta$ is negative. Hence $\tau\Delta(x,x) = 0$ for all $x \in M_0$, and again by Proposition 3.6, it follows that $\Delta = 0$. But then

$$\Phi_j^* = \Phi_j^{'*} = \Omega_j^* | M_\cdot \quad (1 \leq j \leq p).$$

Comparing the above representations of $P_E X^* h$ and $P_E T_{F_U}^* h$, we obtain that

$$P_E X^* h = P_E T_{F_U}^* h \quad (h \in M_\cdot).$$

Then, for any constant function $x \in E \subset \mathcal{H} \otimes E$ and any multiindex $\alpha \in \mathbb{N}^n$, we have the identity

$$\big\langle X^* h, (Z^\alpha \otimes 1_E)x \big\rangle = \langle T^{*\alpha} X^* h, x \rangle = \langle X^* T_\cdot^{*\alpha} h, x \rangle$$

$$= \langle T_{F_U}^* T_\cdot^{*\alpha} h, x \rangle = \langle Z^{*\alpha} \otimes 1_E \, T_{F_U}^* h, \alpha \rangle = \big\langle T_{F_U}^* h, (Z^\alpha \otimes 1_E)\alpha \big\rangle.$$

Since $\mathcal{H} \otimes E$ is spanned topologically by the elements of the form $z^\alpha \otimes x$ ($\alpha \in \mathbb{N}^n$, $x \in E$), we find that $X^* h = T_{F_U}^* h$ for each $h \in M_\cdot$. Thus the proof of the implication (ii) $\Rightarrow$ (iii) is complete.

We complete the proof of Theorem 3.7 by showing that (iii) $\Rightarrow$ (ii). Suppose that $F = F_U$ for a unitary matrix operator as in condition (iii). The hypothesis that $X P_M = P_{M_\cdot} T_F$ is easily seen to be equivalent to the condition that $T_F^* M_\cdot \subset M$ and $X^* = T_F^* | M_\cdot$. Exactly as in the previous part of the proof, we apply Lemma 3.2 to the first column of the matrix operator U to define a contraction $\Omega = (\Omega_1, \ldots, \Omega_p) : K^p \to \mathcal{H} \otimes E_\cdot$. Reversing the arguments given in the proof of the implication (ii) $\Rightarrow$ (iii) we obtain that

$$U^* \Big(\big(\sum_{j=1}^{p} \Omega_j^* (d_{jk}(Z)^* \otimes 1_{E_\cdot}) h \big)_{k=1}^{q} \oplus (P_{E_\cdot} h) \Big) = (\Omega^* h) \oplus P_E T_F^* h$$

for each $h \in \mathcal{H} \otimes E_\cdot$. Using Proposition 1.3 and the fact that U^* is isometric, we see that

$$\big\langle \tfrac{1}{C}(M_{T_\cdot})(1 - XX^*)h, h \big\rangle = \|P_{E_\cdot} h\|^2 - \|P_E X^* h\|^2$$

$$= \|\Omega^* h\|^2 - \|\Big(\sum_{j=1}^{p} \Omega_j^* d_{jk}(T_\cdot)^* h \Big)_{k=1}^{q}\|^2$$

for all $h \in M_\cdot$. Now it is elementary to check that condition (ii) holds with

$$\Gamma = (P_{M_\cdot} \Omega_i \Omega_j^* | M_\cdot)_{ij} \in B(M_\cdot^p).$$

Thus the proof of Theorem 3.7 is complete. $\square$

Remark 3.8 Suppose that $\sigma(T_.) \subset D$. Then $\sigma(M_{T_.}) = \sigma(T_.) \times \sigma(T_.^*) \subset \Delta$ and, since the series $C(z,w) = \sum_{k=0}^{\infty} e_k(z)\tilde{e}_k(w)$ converges uniformly on all compact subsets of Δ, we obtain the representation

$$C(T_., T_.^*) = \sum_{k=0}^{\infty} e_k(T_.)e_k(T_.)^*.$$

It follows that

$$\sum_{k=0}^{\infty} \|e_k(T_.)^* x\|^2 = \langle C(T_., T_.^*)x, x\rangle < \infty$$

for each vector $x \in M_.$, and hence Theorem 3.7 is applicable in this case.

Next we want to indicate that the convergence condition used to prove Theorem 3.7 is also automatically satisfied in the case where $M_. \subset \mathcal{H} \otimes E_.$ is a finite-dimensional $*$-invariant subspace. For this purpose, define a sequence of analytic functions $f_m \in \mathcal{O}(\overline{\Delta})$ by

$$f_m(z,w) = 1 - \sum_{k=0}^{m-1} e_k(z)\tilde{e}_k(w)C(z,w)^{-1}.$$

Exactly as in [3] (Lemma 14 and Corollary 15) one can show that, without any condition on the $*$-invariant subspace $M_. \subset \mathcal{H} \otimes E_.$, the convergence condition

$$\mathrm{SOT} - \lim_{m\to\infty} f_m(T_., T_.^*) = 0.$$

holds. Note that, for a given operator $X \in B(M_.)$, the sequence $\big(f_m(M_{T_.})(X)\big)_m$ converges to zero in the strong operator topology if and only if

$$\mathrm{SOT} - \sum_{k=0}^{\infty} e_k(T_.)\frac{1}{C}(M_{T_.})(X)e_k(T_.)^* = X.$$

If the operators $X, Y \in B(M, M_.)$ intertwine the tuples T and $T_.$ componentwise, then $f_m(M_{T_.})(XY^*) = Xf_m(T_., T_.^*)Y^*$ converges pointwise to zero as a sequence in m. In particular, it follows that, for given multipliers $f, g \in M(E, E_.)$, the sequence of operators

$$f_m(M_{T_.})(P_{M_.}T_f T_g^*|M_.) = P_{M_.} f_m(M_{Z\otimes 1_{E_.}})(T_f T_g^*)|M_.$$

converges to zero pointwise. Hence we find that the series $\sum_{k=0}^{\infty} e_k(T_.)Y_{f,g}\, e_k(T_.)^*$ converges strongly at least for every operator $Y_{f,g}$ of the form

$$Y_{f,g} = \frac{1}{C}(M_{T_.})(P_{M_.}T_f T_g^*|M_.) = P_{M_.}(T_f P_{E_.} T_g^*)|M_. \qquad \big(f, g \in M(E, E_.)\big).$$

Lemma 3.9 Suppose that $\dim(M_.) < \infty$. Then we have

(a) $\mathrm{LH}\{P_{M_.}(T_f P_{E_.} T_g^*)|M_.;\ f, g \in M(E, E_.)\} = B(M_.)$,

(b) the series $\sum\limits_{k=0}^{\infty} e_k(T_.)e_k(T_.)^*$ is norm-convergent.

Proof (a) Let us denote the linear hull on the left by L. If $f, g \in \mathcal{O}(D)$ are multipliers of $\mathcal{H}$, then it is an elementary exercise to deduce, for any pair of operators A, B in $B(E, E_.)$, the identity

$$T_{f\otimes A}P_{E_.}T^*_{g\otimes B} = (\langle \cdot, g\rangle f) \otimes (AB^*) \in B(\mathcal{H} \otimes E_.).$$

Let $y \in M_.$ be a fixed vector. It suffices to show that the rank-one operator $\langle \cdot, y\rangle y \in B(M_.)$ can be approximated by operators in L. For this in turn, it is enough to show that the corresponding rank-one operator $\langle \cdot, y\rangle y \in B(\mathcal{H} \otimes E_.)$ can be approximated in $B(\mathcal{H} \otimes E_.)$ by linear combinations of operators of the form $T_f P_{E_.} T_g^*$, where $f, g \in M(E, E_.)$. Since

$$\mathcal{H} \otimes E_. = \bigvee \{p \otimes x;\ p \in \mathbb{C}[z] \text{ and } x \in E_.\},$$

it suffices to observe that, for $p, q \in \mathbb{C}[z]$ and $x, y \in E_.$,

$$\langle \cdot, q \otimes y\rangle p \otimes x = (\langle \cdot, q\rangle p) \otimes (\langle \cdot, y\rangle x) = T_{p\otimes A}P_{E_.}T^*_{q\otimes B}$$

with suitably chosen rank-one operators $A, B \in B(E, E_.)$.

(b) Since the identity operator $1_{M_.}$ belongs to L, the remarks preceding the lemma show that the series $\sum_{k=0}^{\infty} e_k(T_.)e_k(T_.)^*$ converges strongly. □

As an application of the last lemma we see that in the case where $\dim(M_.) < \infty$, the convergence condition

$$\sum_{k=0}^{\infty} \|e_k(T_.)^* x\|^2 < \infty$$

holds for all vectors $x \in M_.$. Hence Theorem 3.7 is applicable in this case.

§4 Interpolation for Schur class functions

The above results can be used to solve interpolation problems for Schur class functions. For simplicity, we only treat the scalar-valued case, that is, in the following we make the assumption that $E = E_. = \mathbb{C}$.

Let $\mathcal{H}$ be a functional Hilbert space consisting of analytic functions on an open set

$$D = \{z \in W;\ \|d(z)\| < 1\}$$

in $\mathbb{C}^n$ such that $\mathcal{H}$ and D satisfy all hypotheses described at the beginning of Section 3. We fix a finite subset $S \subset D$ and suppose that, for each $s \in S$, a finite set $A_s \subset \mathbb{N}^n$ is given with the property that, for each $\alpha \in A_s$, we have

$$\{\gamma \in \mathbb{N}^n; \gamma \leq \alpha\} \subset A_s.$$

Here the order relation $\gamma \leq \alpha$ is defined componentwise. For each $s \in S$, let $(c_{s,\alpha})_{\alpha \in A_s}$ be a fixed family of complex numbers. Our aim in the following is to find conditions that characterize the existence of functions $f \in \mathcal{S} = \mathcal{S}_d$ in the Schur class such that

$$f^{(\alpha)}(s) = c_{s,\alpha} \quad (s \in S, \alpha \in A_s).$$

Since every Schur class function $f \in \mathcal{S}$ is a multiplier of $\mathcal{H}$ with $\|T_f\| \leq 1$, we find at least sufficient conditions for the existence of interpolating functions with multiplier norm bounded by one.
Since by the closed graph theorem the inclusion mapping $\mathcal{H} \subset \mathcal{O}(D)$ is continuous, for each point $w \in D$ and each multiindex $\alpha \in \mathbb{N}^n$, there is a unique function $C_w^\alpha \in \mathcal{H}$ with the property that

$$\langle f, C_w^\alpha \rangle = f^{(\alpha)}(w) \quad (f \in \mathcal{H}).$$

Let, as before, $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ consisting of polynomials. Then

$$C(z, w) = \sum_{k=0}^{\infty} e_k(z) \tilde{e}_k(w),$$

where the series converges uniformly on compact subsets of $\Delta = \{(z, \bar{w}); z, w \in D\}$. Hence, for $\alpha \in \mathbb{N}^n$ and $z, w \in D$, we find that

$$(\partial_w^\alpha C)(z, \bar{w}) = \sum_{k=0}^{\infty} (\partial^\alpha e_k)^\sim(\bar{w}) e_k(z) = \sum_{k=0}^{\infty} \overline{(\partial^\alpha e_k)(w)} e_k(z) = \sum_{k=0}^{\infty} \langle C_w^\alpha, e_k \rangle e_k(z) = C_w^\alpha(z).$$

Since, for $z, w \in D$, the identity

$$C_z(w) = C(w, \bar{z}) = \overline{C(z, \bar{w})} = C(z, \cdot)^\sim(w)$$

holds, it follows that

$$\overline{(\partial^\alpha C_z)(w)} = (\partial^\alpha C(z, \cdot))(\bar{w}) = C_w^\alpha(z).$$

Lemma 4.1 Let $f \in \mathcal{O}(D)$ be a multiplier of $\mathcal{H}$. Then the identity

$$T_f^* C_w^\alpha = \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} \overline{(\partial^{\alpha - \gamma} f)(w)} C_w^\gamma$$

holds for all points $w \in D$ and each multiindex $\alpha \in \mathbb{N}^n$.

Proof It suffices to observe that

$$T_f^* C_w^\alpha = \langle C_w^\alpha, T_f C_z \rangle = \overline{(f C_z)^{(\alpha)}(w)} = \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} \overline{(\partial^{\alpha - \gamma} f)(w)} C_w^\gamma(z)$$

for all $z, w \in D$ and $\alpha \in \mathbb{N}^n$. $\square$

By Lemma 4.1 the subspace

$$M = M_{.} = \mathrm{LH}\{C_s^\alpha;\ s \in S \text{ and } \alpha \in A_s\}$$

is invariant for the commuting tuple $Z^* \in B(\mathcal{H})^n$. As before, we denote by $T \in B(M)^n$ the compression of Z to M. Since the generating vectors C_s^α for M are linearly independent, there is a unique operator $X \in B(M)$ such that

$$X^* C_s^\alpha = \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} \bar{c}_{s,\alpha-\gamma} C_s^\gamma$$

for all $s \in S$ and $\alpha \in A_s$. The existence of a Schur class function $f \in \mathcal{S}$ with

$$f^{(\alpha)}(s) = c_{s,\alpha} \qquad (s \in S \text{ and } \alpha \in A_s)$$

is equivalent to the existence of a function $f \in \mathcal{S}$ with $T_f^* M \subset M$ and $T_f^*|M = X^*$. It is elementary to check that $XT_j = T_j X$ for $j = 1, \ldots, n$. Hence Theorem 3.7 can be applied to characterize the solvability of the above interpolation problem.

Our aim is to prove this interpolation result in a form which directly generalizes the classical theorem known for the case of the unit disc. For this aim, let P denote the orthogonal projection from $\mathcal{H}$ onto the constant functions. Then we obtain that

$$PC_s^\alpha = \langle C_s^\alpha, 1\rangle = \delta_{\alpha,0} 1 \quad (s \in S,\ \alpha \in A_s).$$

Using Lemma 4.1 and the results from Section 1, one easily finds that

$$\langle \frac{1}{C}(M_T)(1 - XX^*) C_s^\alpha, C_t^\beta \rangle = \delta_{(\alpha,\beta),0} - \bar{c}_{s,\alpha} c_{t,\beta}$$

for all $s, t \in S$ and $\alpha \in A_s, \beta \in A_t$. Our result is formulated in terms of a scalar matrix $G = (G_{\rho,\sigma})_{\rho,\sigma \in \Lambda}$, where Λ is the index set

$$\Lambda = \{\rho = (j, s, \alpha);\ j = 1, \ldots, p \text{ and } s \in S, \alpha \in A_s\}.$$

Theorem 4.2 (Carathéodory-Fejér problem) For $S \subset D$ finite and finite families $(c_{s,\alpha})_{\alpha \in A_s}$ $(s \in S)$ as above, the following are equivalent:

(i) there is a function $f \in \mathcal{S}$ with $f^{(\alpha)}(s) = c_{s,\alpha}$ for all $s \in S$ and $\alpha \in A_s$;

(ii) there is a positive semidefinite matrix $G = (G_{\rho,\sigma})_{\rho,\sigma \in \Lambda}$ of complex numbers such that the equations

$$\begin{aligned}\delta_{(\alpha,\beta),0} - \bar{c}_{s,\alpha} c_{t,\beta} \ &= \sum_{j=1}^{p} G_{(j,t,\beta),(j,s,\alpha)} \\ - \sum_{\substack{0 \leq \delta \leq \beta \\ 0 \leq \lambda \leq \alpha}} \binom{\beta}{\delta} \binom{\alpha}{\lambda} \sum_{i,j=1}^{p} &\left(\sum_{k=1}^{q} (\partial^{\beta-\delta} d_{ik})(t) \overline{(\partial^{\alpha-\lambda} d_{jk})(s)} \right) G_{(i,t,\delta),(j,s,\lambda)}\end{aligned}$$

hold for all $s, t \in S$ and $\alpha \in A_s$, $\beta \in A_t$.

Proof By Theorem 3.7 the validity of condition (i) is equivalent to the existence of a positive operator $\Gamma = (\Gamma_{ij}) \in B(M^p)$ such that

$$\frac{1}{C}(M_T)(1 - XX^*) = \sum_{j=1}^{p} \Gamma_{jj} - \sum_{i,j=1}^{p} \sum_{k=1}^{q} d_{ik}(T)\Gamma_{ij}d_{jk}(T)^*.$$

Suppose that Γ is such a matrix operator. Each coefficient $\Gamma_{ij} \in B(M)$ has a representation of the form

$$\Gamma_{ij}C_s^\alpha = \sum_{t\in S,\beta\in A_t} \Gamma_{t\beta}^{s\alpha}(i,j)C_t^\beta \quad (s \in S, \alpha \in A_s).$$

By applying both sides of the first of the last two equations to the vector C_s^α, and then forming the inner product with C_t^β, one obtains the identity

$$\delta_{(\alpha,\beta),0} - \overline{c}_{s,\alpha}c_{t,\beta} = \sum_{u\in S,\nu\in A_u} \Big[\sum_{j=1}^{p} \Gamma_{u\nu}^{s\alpha}(j,j)(\partial^{(\beta,\nu)}C)(t,\overline{u})$$

$$- \sum_{i,j=1}^{p} \sum_{k=1}^{q} \sum_{\substack{0\leq\delta\leq\beta \\ 0\leq\lambda\leq\alpha}} \binom{\beta}{\delta}\binom{\alpha}{\lambda}(\partial^{\beta-\delta}d_{ik})(t)\overline{(\partial^{\alpha-\lambda}d_{jk})(s)}(\partial^{(\delta,\nu)}C)(t,\overline{u})\Big]$$

for all $s, t \in S$ and $\alpha \in A_s, \beta \in A_t$. The scalar matrix $G = (G_{\rho\sigma})_{\rho\sigma\in\Lambda}$ with coefficients defined by

$$G_{(j,t,\beta),(i,s,\alpha)} = \sum_{u\in S,\nu\in A_u} \Gamma_{u\nu}^{s\alpha}(j,i)(\partial^{(\beta,\nu)}C)(t,\overline{u})$$

clearly satisfies condition (ii). The positivity of the matrix operator $\Gamma = (\Gamma_{ij}) \in B(M^p)$ is equivalent to the positive semidefiniteness of the matrix G. To check this, it suffices to observe that, for every vector $m = (m_i) \in M^p$ with $m_i = \sum_{s\in S,\alpha\in A_s} v_{(i,s,\alpha)}C_s^\alpha$ for $i = 1, \ldots, p$, the identity

$$\sum_{i,j=1}^{p} \langle\Gamma_{ji}m_i, m_j\rangle = \sum_{\rho,\sigma\in\Lambda} G_{\sigma\rho}v_\rho\overline{v}_\sigma$$

holds. Thus it is clear that condition (i) implies condition (ii).

If conversely, a positive semidefinite matrix G as in condition (ii) is given, then the last equality can be used to define a positive matrix operator $\Gamma \in B(M^p)$. By reversing the above arguments one finds that Γ satisfies the equation contained in condition (ii) of Theorem 3.7, where the operator X is defined as above.

□

In the particular case that $A_s = \{0\}$ for each $s \in S$, the preceding result yields a solution of the Nevanlinna-Pick problem for Schur class functions. The same result and various similar characterizations, even for an arbitrary subset $S \subset D$, are contained in Theorem 2.5. As an example we state the result below, for which we also cite [5].

Theorem 4.3 Let $S \subset D$ be an arbitrary subset, and let $(c_s)_{s \in S}$ be a family of complex numbers. Then there is a Schur class function $f \in \mathcal{S}$ such that $f(s) = c_s$ for all $s \in S$ if and only if there is a positive definite function $\Gamma = (\Gamma_{ij}) : S \times S \to B(\mathbb{C}^p) \cong \mathbb{C}^{p,p}$ such that

$$1 - \overline{c}_s c_t = \sum_{j=1}^{p} \Gamma_{jj}(s,t) - \sum_{i,j=1}^{p} \sum_{k=1}^{q} d_{ik}(t) \overline{d_{jk}(s)} \Gamma_{ij}(s,t)$$

holds for all $s, t \in S$.

Proof The assertion follows directly from the equivalence of conditions (i) and (ii)' in Theorem 2.5. $\square$

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