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Abstract

We consider the problem

$$I^\infty = \inf_{u\in\mathcal{W}} \int_\Omega \varphi(\nabla u(x,y))\ dxdy$$

in the class $\mathcal{W} = \{u \in W^{1,\infty}(\Omega) : u/_{\Gamma_0} = 0,\ |u_y| = 1\ a.e.\}$, where Ω is either the rectangle $(0,1)^2$ or the parallelogram $\{(x,y) \in \mathbb{R}^2 : 0 < y < 1,\ y < x < y+1\}$, and Γ_0 denotes the boundary part $\{0\} \times [0,1]$ in the first case, for the parallelogram we let $\Gamma_0 = \{(x,x) :\ 0 \leq x \leq 1\}$. The function $\varphi : \mathbb{R}^2 \to [0,\infty)$ is an elastic potential with wells in $(0,\pm 1)$. We prove that $I^\infty = 0$ by considering minimizing sequences which differ substantially for both cases.

Exemples de microstructures relatifs à la théorie de transformations martensitiques de phase

Résumé: On considère le problème

$$I^\infty = \inf_{u\in\mathcal{W}} \int_\Omega \varphi(\nabla u(x,y))\ dxdy$$

sur la classe $\mathcal{W} = \{u \in W^{1,\infty}(\Omega) : u/_{\Gamma_0} = 0,\ |u_y| = 1\ p.p.\}$, où Ω désigne à la fois le rectangle $R = (0,1) \times (0,1)$ ou le parallélogramme $P = \{(x,y) \in \mathbb{R}^2 : 0 < y < 1, y < x < y+1\}$, et $\Gamma_0 = \{0\} \times [0,1]$ si $\Omega = R$, dans le cas du parallélogramme on pose $\Gamma_0 = \{(x,x) : x \in (0,1)\}$. La fonction $\varphi : \mathbb{R}^2 \longrightarrow [0,\infty)$ est une densité d'énergie ayant deux puits de potentiel $(0,\pm 1)$. Nous montrons que $I^\infty = 0$ moyennant la construction de suites minimisantes qui diffèrent d'une manière considérable dans les deux cas.

Version Française Abrégée Soit Ω un sous-ensemble de $\mathbb{R}^2$ qui désigne à

la fois le rectangle $R = (0,1)^2$ ou le parallélogramme $P = \{(x,y) \in \mathbb{R}^2 : 0 < y < 1,\ y < x < y+1\}$. On note par Γ_0 la partie de la frontière de Ω définie par:

$$\Gamma_0 := \begin{cases} \{0\} \times [0,1], \text{ si } \Omega = R, \\ \{(x,x) : x \in [0,1]\}, \text{ si } \Omega = P. \end{cases}$$

Nous nous intéressons au problème de minimisation suivant:

$$I^\infty := \inf_{u\in\mathcal{W}} \int_\Omega \varphi(\nabla u(x,y))dxdy \tag{0.1}$$

sur la classe $\mathcal{W} := \mathcal{W}(\Omega) = \{u \in W^{1,\infty}(\Omega) : |u_y| = 1 \ p.p., \ u = 0 \text{ sur } \Gamma_0\}$. La densité d'énergie $\varphi : \mathbb{R}^2 \to [0, \infty)$ est une fonction borélienne bornée sur les ensembles bornés et ayant deux puits de potentiel $(0, \pm 1)$ i.e.

$$\varphi(0,1) = \varphi(0,-1) = 0. \tag{0.2}$$

Les résultats obtenus dans cette note sont regroupés dans le théorème suivant:

Théorème:

a) Sous les hypothèses ci-dessus on a $I^\infty = 0$.

b) On suppose que φ *est continue et vérifie* $\varphi(\xi, \eta) = 0$ *si et seulement si* $(\xi, \eta) = (0, \pm 1)$. *Alors on a:*

i) Le problème (0.1), n'atteint pas son infimum.

ii) Soit $\{u_n\} \in \mathcal{W}$ *une suite minimisante du problème (0.1)*

telle que

$$\sup_n \|u_n\|_{W^{1,\infty}(\Omega)} < \infty.$$

Alors

$$u_n \to 0 \ \textit{uniformément sur} \ \Omega$$

et

$$\nu_{(x,y)} = \frac{1}{2}\,\delta_{(0,-1)} + \frac{1}{2}\,\delta_{(0,1)} \quad \textit{p.p. dans } \Omega,$$

où $\nu_{(x,y)}$ *est la mesure de Young associée à* $\{\nabla u_n\}$.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ either denote the rectangle $R = (0,1)^2$ or the parallelogram $P = \{(x,y) : 0 < y < 1,\ y < x < y+1\}$ with boundary part

$$\Gamma_0 := \begin{cases} \{0\} \times [0,1], & \text{if } \Omega = R \\ \{(x,x) : x \in [0,1]\}, & \text{if } \Omega = P. \end{cases}$$

Then we consider the minimization problem

$$I^\infty := \inf_{u \in \mathcal{W}} \int_\Omega \varphi(\nabla u(x,y))\, dxdy \tag{1.1}$$

on the class $\mathcal{W} = \mathcal{W}(\Omega) = \{u \in W^{1,\infty}(\Omega) : |u_y| = 1 \text{ a.e. },\ u = 0 \text{ on } \Gamma_0\}$. Here $\varphi : \mathbb{R}^2 \to [0,\infty)$ denotes a Borel function which is bounded on bounded sets and such that

$$\varphi(0,1) = \varphi(0,-1) = 0. \tag{1.2}$$

For example, if we think of a model in martensitic phase transformation, φ could be the elastic energy density of the martensite with two wells in $(0,1)$ and $(0,-1)$ representing the stress free states of two variants of the martensite. Γ_0 stands for the austenite-twinned martensite interface, and the boundary condition $u = 0$ on Γ_0 refers to elastic compatibility with the austenite phase in the extreme case of complete rigidity of the austenite (see [1], [2] and [6]) . Problems of type (1.1) without the constraint $|u_y| = 1$ were discussed by Chipot and Collins (see [3] and [4]), we refer to the work of Kohn and Müller (compare [7], [8], see also Winter [10]) where this constraint is introduced and where they analyse a minimization problem including surface energy on the Sobolev class $W^{1,2}(\Omega)$. Our results can be summarized as follows.

THEOREM 1.1. *a) Under the above assumptions we have* $I^\infty = 0$.

b) In addition to (1.2) assume that φ *is continuous and that* $\varphi(\xi,\eta) = 0$ *if and only if* $(\xi,\eta) = (0,\pm 1)$. *Then the following statements are true:*

i) Problem (1.1) cannot attain its infimum.

ii) Let $\{u_n\} \in \mathcal{W}$ *denote a minimizing sequence to problem (1.1) such that* $\sup_n \|u_n\|_{W^{1,\infty}(\Omega)} < \infty$. *Then*

$$u_n \to 0 \text{ uniformly on } \Omega$$

and

$$\nu_{(x,y)} = \frac{1}{2}\,\delta_{(0,-1)} + \frac{1}{2}\,\delta_{(0,1)} \quad a.e. \ on \ \Omega,$$

where $\nu_{(x,y)}$ *is the Young measure associated to* $\{\nabla u_n\}$.

A complete proof of this theorem together with the analysis of the situation for more complicated geometries will be given in the paper [5] where also some of the assumptions concerning the energy density φ are removed. The main point is part a) of Theorem 1.1 where we construct minimizing sequences which differ substantially for the rectangle and the parallelogram, for example, in case $\Omega = P$ we find a minimizing sequence $\{u_n\}$ s.t. $|(u_n)_{yy}|(\Omega) < \infty$ holds for each n which of course can not be expected for the rectangle (see [W]).

REMARK 1.1. *a) For i) of Theorem 1.1 b) it is enough to assume in addition to (1.2) that* $\varphi(\xi,\eta) = 0$ *implies* $\xi = 0$.

b) If Ω *denotes the parallelogram* P *and if for example* φ *has wells in* $(0,1)$, $(0,-1)$, $(1,-1)$, *then* $u(x,y) = x - y$ *is a solution of (1.1). So our assumption in Theorem 1.1 b) seems to be quite natural, and for obtaining the uniqueness result of ii) in fact it cannot be weakened.*

2 Construction of minimizing sequences

We start with the case $\Omega = P$. Let N denote some large integer and define $\delta = \frac{1}{N+1}$, $a_i = i\delta$, $i = 0, \ldots, N+1$. We further let

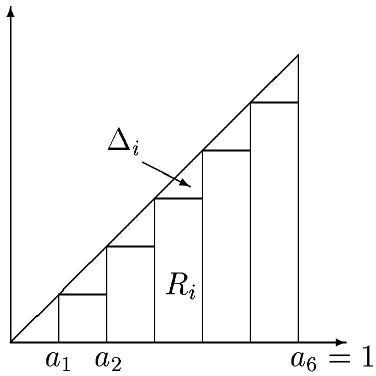
$$h(t) := \begin{cases} t, & 0 \le t \le \frac{\delta}{2} \\ \delta - t, & \frac{\delta}{2} \le t \le \delta. \end{cases}$$

and extend h periodically from $[0,\delta]$ to $[0,1]$. The triangle

$$\Delta = \{(x,y) \in \mathbb{R}^2 \ : \ 0 \le x \le 1, 0 \le y \le x\}$$

is subdivided as follows

$$\begin{aligned}
\Delta &= \bigcup_{i=1}^{N} R_i \;\cup\; \bigcup_{i=0}^{N} \Delta_i, \\
R_i &= [a_i, a_{i+1}] \times [0, a_i], \; i = 1, \dots, N, \\
\Delta_i &= \big\{(x,y) \in \mathbb{R}^2 : a_i \le x \le a_{i+1}, a_i \le y \le x\big\}, i = 0, \dots, N.
\end{aligned}$$



The function $u : \Delta \to \mathbb{R}$ defined as

$$u(x,y) = \begin{cases} h(y) \;\text{ if }\; (x,y) \in \bigcup\limits_{i=1}^{N} R_i \\ \frac{x-a_i}{\delta}\; h\Big(\frac{\delta}{x-a_i}(y-a_i) + a_i\Big) \;\text{ if }\; (x,y) \in \Delta_i \end{cases}$$

belongs to $W^{1,\infty}(\Delta)$ and satisfies $u = 0$ on Γ_0, as well as $|u_y| = 1$ a.e. and

$$u_x = 0 \;\text{ on }\; \bigcup_{i=1}^{N} R_i.$$

Finally we extend u to the whole parallelogram P by letting

$$u(x,y) := u(1,y) = h(y)$$

if $(x,y) \in P, x \ge 1$. Clearly $u \in \mathcal{W}(\Omega)$, and (1.2) implies

$$\begin{aligned}
&\int_\Omega \varphi(\nabla u(x,y)) \, dxdy = \sum_{i=0}^{N} \int_{\Delta_i} \varphi(\nabla u(x,y)) dxdy \le \\
&c_1 \cdot \sum_{i=0}^{N} \mathcal{L}^2(\Delta_i) \le c_2 \, \delta \le \tfrac{c_3}{N}
\end{aligned}$$

with positive constants c_k independent of δ. In this way we can generate a minimizing sequence $u_n \in \mathcal{W}(\Omega)$ such that

$$\int_\Omega \varphi(\nabla u_n)dxdy \le \frac{c_4}{n} \underset{n\to\infty}{\longrightarrow} 0,$$

and from the construction it also follows that $|(u_n)_{yy}|(\Omega) \le c_5\, n$.

In case $\Omega = R$ we need a modification of a construction due to Kohn and Müller (see [7]) which we summarize in the next lemma , the details are given in [5].

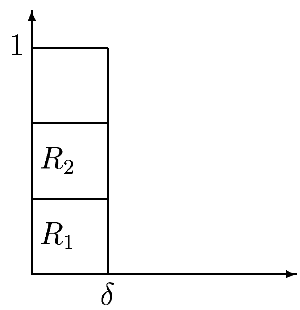
LEMMA 2.1. *There exists a function $w \in W^{1,\infty}(R)$ such that*

$$|w_y| = 1 \ a.e., \quad w = 0 \quad on \ \ \Gamma_o \ and$$

$$w(x,0) = w(x,1) = 0 \quad for\ all\ x \in [0,1].$$

Given Lemma 2.1 we want to show that $I^\infty = 0$. Let $\delta = \frac{1}{N}$, $N \in \mathbb{N}$, and define

$$R_i := [0,\delta] \times [(i-1)\delta, i\delta], i = 1, \ldots, N,$$



$u(x,y) := \delta w\big(\frac{x}{\delta}, \frac{y-(i-1)\delta}{\delta}\big)$, if $(x,y) \in R_i$, where w is taken from Lemma 2.1. The boundary behaviour of w implies that u is a well-defined Lipschitz function on $\bigcup_{i=1}^{N} R_i$. If we let $u(x,y) = u(\delta,y)$ for $(x,y) \in R$ with $x \ge \delta$, then $u \in \mathcal{W}(\Omega)$ and

$$\int_\Omega \varphi(\nabla u(x,y))dxdy = \sum_{i=1}^{N} \int_{R_i} \varphi(\nabla u(x,y))dxdy \le c_6 \cdot \delta = \frac{c_6}{N}$$

Altogether Theorem 1.1 a) is established.

3 Nonexistence of minimizers and Young measures

The following Poincaré type inequality is crucial (see [10] and [5]).

LEMMA 3.1. *Consider $u \in W^{1,\infty}(\Omega)$ such that $u = 0$ on Γ_0. Then there is a constant γ such that*

$$\int_\Omega |u(x,y)|dxdy \leq \gamma \int_\Omega |u_x(x,y)|dxdy.$$

Now, assume that $u \in \mathcal{W}(\Omega)$ is minimizing, i.e.

$$\int_\Omega \varphi(\nabla u(x,y))dxdy = 0. \tag{3.1}$$

If the wells of φ are located on $\{0\} \times \mathbb{R}$ then (3.1) implies $u_x = 0$, and from Lemma 3.1 we get $u = 0$ contradicting $|u_y| \equiv 1$ a.e.

Let $\{u_n\}$ denote a minimizing sequence as in Theorem 1.1 b)ii). Then there exists a function $u \in W^{1,\infty}(\Omega)$, $u/_{\Gamma_0} = 0$, such that

$$u_n \to u \text{ uniformly, } \nabla u_n \overset{*}{\rightharpoonup} \nabla u \text{ in } L^\infty(\Omega) \text{ weak} - *$$

at least for a subsequence. Now the bounded sequence of gradients generates a Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega}$ (see [9]), in particular

$$\int_\Omega \varphi(\nabla u_n)dxdy \to \int_\Omega \int_{\mathbb{R}^2} \varphi(\xi,\eta)d\nu_{(x,y)}(\xi,\eta)dxdy,$$

thus $\int_{\mathbb{R}^2} \varphi(\xi,\eta)d\nu_{(x,y)}(\xi,\eta) = 0$ a.e., hence $\operatorname{spt}(\nu_{(x,y)}) \subset \{(0,\pm 1)\}$ which means

$$\nu_{(x,y)} = \alpha(x,y)\delta_{(0,-1)} + (1-\alpha(x,y))\delta_{(0,1)} \tag{3.2}$$

for some measurable function α such that $0 \leq \alpha(x,y) \leq 1$. This implies (writing again (ξ,η) for the variables in $\mathbb{R}^2$)

$$\int_\Omega |(u_n)_x|dxdy \longrightarrow \int_\Omega \int_{\mathbb{R}^2} |\xi|d\ \nu_{(x,y)}(\xi,\eta) = 0,$$

lower semicontinuity gives

$$\int_\Omega |u_x| dxdy \leq \liminf_{n\to\infty} \int_\Omega |(u_n)_x| dxdy,$$

thus $u_x = 0$ and therefore $u \equiv 0$ on account of Lemma 3.1. This shows $\nabla u_n \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ weak - $*$, in particular

$$0 = \lim_{n\to\infty} \int_D (u_n)_y dxdy = \int_D \int_{\mathbb{R}^2} \eta d\nu_{(x,y)}(\xi, \eta) dxdy \tag{3.3}$$

for any Borel set $D \subset \Omega$. From (3.2) and (3.3) we deduce

$$\int_D (1 - 2\alpha(x,y)) dxdy = 0,$$

i.e. $\alpha(x,y) \equiv \frac{1}{2}$ a.e.

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