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Abstract

Given an integrand f of linear growth and assuming an ellipticity condition of the form

$$D^2f(Z)(Y,Y) \;\geq\; c\big(1+|Z|^2\big)^{-\frac{\mu}{2}}|Y|^2 \quad , \; 1<\mu\leq 3\,,$$

we consider the variational problem $J[w] = \int_\Omega f(\nabla w)\,dx \to \min$ among mappings w: $\mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ with prescribed Dirichlet boundary data. If we impose some boundedness condition, then the existence of a generalized minimizer u^* is proved such that $\int_{\Omega'} |\nabla u^*| \log^2(1+|\nabla u^*|^2)\,dx \leq c(\Omega')$ for any $\Omega' \Subset \Omega$. Here the limit case $\mu=3$ is included. Moreover, if $\mu<3$ and if $f(Z)=g(|Z|^2)$ is assumed in the vectorvalued case, then we show local $C^{1,\alpha}$-regularity and uniqueness up to a constant of generalized minimizers. These results substantially improve earlier contributions of [BF3] where only the case of exponents $1<\mu<1+2/n$ could be considered.

AMS Subject Classification: 49N60, 49N15, 49M29, 35J
Key words: linear growth, minimizers, regularity, duality, BV-functions

Contents

1 Introduction

Suppose we are given a smooth, strictly convex (in the sense of definitions) integrand $f\colon \mathbb{R}^{nN} \to \mathbb{R}$ of linear growth (compare Assumption 2.1 for details). Then we consider the variational problem

$$(\mathcal{P}) \qquad J[w] \;=\; \int_\Omega f(\nabla w)\,dx \;\to\; \min$$

among mappings $w \in u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)$ where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and u_0 is of class $W^1_1(\Omega;\mathbb{R}^N)$.
One prominent (scalar) example is the minimal surface case $f(Z) = \sqrt{1+|Z|^2}$. A variety of references is available for the study of this variational integrand. With regard to the following discussion we just want to mention the monographs [GI], [GMS2] and the apriori estimates given in [LU2] and [GMS1].
The theory of perfect plasticity provides another famous variational integrand of linear growth (the assumptions of smoothness and strict convexity however are not satisfied in this case). Here we like to refer to the studies of Seregin (see, for instance, [SE1]–[SE4]) and to the recent monograph [FS2].
In any case, on account of the lack of compactness in the non-reflexive Sobolev space $W^1_1(\Omega;\mathbb{R}^N)$, problem $(\mathcal{P})$ in general fails to have solutions. Thus one either has to study suitable relaxations (possibility i)) or we may pass to the dual variational problem (possibility ii)).

ad i): since the integrand f under consideration is of linear growth, any J-minimizing sequence $\{u_m\}$, $u_m \in u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)$, is uniformly bounded in the space $BV(\Omega;\mathbb{R}^N)$. This ensures the existence of a subsequence (not relabelled) and a function u in $BV(\Omega;\mathbb{R}^N)$ such that $u_m \to u$ in $L^1(\Omega;\mathbb{R}^N)$. So we define the set $\mathcal{M}$ of all generalized minimizers of problem $(\mathcal{P})$ via

$$\mathcal{M} \;=\; \Big\{u \in BV\big(\Omega;\mathbb{R}^N\big) : \;\; u \text{ is the } L^1\text{–limit of a } J\text{–minimizing sequence from } u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)\Big\}\,.$$

ad ii): following [ET] it is possible to write

$$J[w] \;=\; \sup_{\tau \in L^\infty(\Omega;\mathbb{R}^{nN})} l(w,\tau)\,, \qquad w \in u_0 + \overset{\circ}{W}{}^1_1\big(\Omega;\mathbb{R}^N\big)\,,$$

where the Lagrangian $l(w,\tau)$ for $(w,\tau) = (u_0+\varphi,\tau) \in \big(u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)\big) \times L^\infty(\Omega;\mathbb{R}^{nN})$ is defined through the formula

$$l(w,\tau) \;:=\; \int_\Omega \tau : \nabla w\,dx - \int_\Omega f^*(\tau)\,dx \;=\; l(u_0,\tau) + \int_\Omega \tau : \nabla\varphi\,dx\,,$$

and where f^* denotes the conjugate function of f. If we let

$$R: \quad L^\infty\big(\Omega;\mathbb{R}^{nN}\big) \to \overline{\mathbb{R}}\,,$$

$$R(\tau) \;:=\; \inf_{u\in u_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} l(u,\tau) \;=\; \begin{cases} -\infty\,, & \text{if} \quad \operatorname{div}\tau \neq 0\,, \\[2ex] l(u_0,\tau)\,, & \text{if} \quad \operatorname{div}\tau = 0\,, \end{cases}$$

then the dual problem reads as

$(\mathcal{P}^*)$ to maximize R among all functions in $L^\infty\big(\Omega;\mathbb{R}^{nN}\big)$.

Although the set $\mathcal{M}$ of generalized minimizers of problem $(\mathcal{P})$ may be very "large", the solution of the dual problem is unique. This is a well known fact from duality theory (compare [ET]), a generalization (without imposing any conditions on the conjugate function) is given in [BI]. Moreover, the dual solution σ admits are clear physical or geometrical interpretation: in the minimal surface case the dual solution corresponds to the normal of the surface, in the theory of perfect plasticity we obtain the stress tensor. Let us finally mention that (see again [ET])

$$\inf_{u\in u_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J(u) \;=\; \sup_{\tau\in L^\infty(\Omega;\mathbb{R}^{nN})} R(\tau)\,.$$

Next, some known results are briefly summarized.
i) In the minimal surface case it is possible to benefit from the geometric structure of the problem (compare Remark 2.3). A class of integrands with this structure is studied, for instance, in [GMS1] following the apriori gradient bounds given in [LU2]. It turns out that generalized minimizers have (locally) Hölder continuous derivatives, they are unique up to a constant and the dual solution σ is of class $C^{0,\alpha}_{loc}$ for any $0<\alpha<1$.
ii) In the theory of perfect plasticity only partial regularity of the stress tensor is known (compare [SE3]). Even in the twodimensional setting $n=2$ we just have some additional information on the singular set (see [SE4]). As an approximation, plastic materials with logarithmic hardening are studied, i.e. the integrand under consideration is given by

$$f(Z) \;:=\; |Z|\,\log(1+|Z|)\,.$$

This integrand is of nearly linear growth, and, as a consequence, a unique solution of Problem $(\mathcal{P})$ exists. The solution is known to be of class $C^{1,\alpha}_{loc}$ implying the stress tensor to be (locally) continuous (see [FRS], [FS1], [MS] — generalizations are given in [FM], [BFM], [BF2]).

iii) Considering the general situation of strictly convex, smooth integrands with linear growth, singularities have to be expected (see [GMS1] for examples). However, partial $C^{1,\alpha}$-regularity of one particular generalized minimizer u^* is established in [BF1]. This is done by proving u^* to be a local minimizer of a relaxed problem studied in [AG]. Finally, the duality relation $\sigma = \nabla f(\nabla u^*)$ ensures partial $C^{0,\alpha}$-regularity of the stress tensor.
iv) In contrast to the nearly linear logarithmic hardening, the idea in [BF3] is to study a regular class of variational integrals with linear growth. Here, on one hand, existence and regularity results are comparable to the minimal surface situation. On the other hand no geometric structure conditions are imposed.
As an example one may think of

$$f(Z) \; = \; \int_0^{|Z|} \int_0^s (1+t^2)^{-\frac{\mu}{2}} \, dt \, ds \, ,$$

where $\mu > 1$ is some fixed real number. If, as a substitute for the geometric structure, ellipticity is assumed to be "good enough", i.e. if $\mu < 1 + 2/n$ is assumed, then $C^{1,\alpha}_{loc}$-regularity of generalized minimizers (which again are unique up to a constant) and local Hölder continuity of the stress tensor are valid.
Let us shortly discuss the limitation $\mu < 1 + 2/n$. Given a suitable regularization u_δ, it is shown that

$$\omega_\delta \; := \; \big(1 + |\nabla u_\delta|^2\big)^{\frac{2-\mu}{4}}$$

is uniformly bounded in the class $W^1_{2,loc}(\Omega)$. This provides no information at all if the exponent is negative, i.e. if $\mu > 2$. An application of Sobolev's inequality, which needs the bound $\mu < 1 + 2/n$, proves local uniform higher integrability of the gradients.
In a similar way, the DeGiorgi type reasoning of [BF3] leads to the same limitation on the ellipticity exponent μ.

The purpose of our paper is to cover the whole scale of μ-elliptic integrands with linear growth (as introduced in [BF3]) up to $\mu = 3$. This is the limit induced by the minimal surface example (see Remark 2.3).
As an additional assumption, the boundary values u_0 are supposed to be of class $L^\infty(\Omega;\mathbb{R}^N)$ (with the approximation arguments of [BF1] w.l.o.g $u_0 \in L^\infty \cap W^1_2(\Omega;\mathbb{R}^N)$). Moreover, a maximum principle is imposed:

ASSUMPTION 1.1 *Let u_δ denote the unique minimizer of*

$$J_\delta[w] \; := \; \frac{\delta}{2} \int_\Omega |\nabla w|^2 \, dx + J[w] \, , \qquad w \in u_0 + \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^N) \, ,$$

$\delta \in (0,1)$. *Then there is a real number M, independent of δ, such that*

$$\|u_\delta\|_{L^\infty(\Omega;\mathbb{R}^N)} \;\le\; M\,\|u_0\|_{L^\infty(\Omega;\mathbb{R}^N)} \; .$$

REMARK 1.2 *Alternatively Assumption 1.1 may be replaced by*

$$\|u_\delta\|_{L^\infty_{loc}(\Omega;\mathbb{R}^N)} \;\le\; K$$

for some real number K not depending on δ. In this case no restriction on the boundary values is needed.

REMARK 1.3 *Of course there are a lot of contributions on the boundedness of solutions of variational problems. Let us mention [TA] in the scalar case, a maximum principle for $N > 1$ is given in [DLM]. Let us also remark that in the case of nonstandard growth conditions, a boundedness assumption serves as an important tool in [CH] and [ELM].*

Given some preliminary results on the regularization (see Section 3), we exploit these hypotheses in the main sections 4–6 to obtain uniform apriori gradient estimates for the sequence $\{u_\delta\}$.
In contrast to [BF3] we do not differentiate the Euler equation in Sections 4 and 5 by the way avoiding Sobolev's inequality.
As outlined in Section 4, a generalized minimizer $u^* \in W^1_{1,loc}(\Omega;\mathbb{R}^N)$ is found in the first step (in fact, integrability is slightly better, compare Theorem 2.5). Thus, following [BF1], u^* is a local minimizer of $\int \nabla f(\nabla w)\,dx$ (see Corollary 2.6).
It turns out that in the limit case $\mu = 3$ we have to stop at this point, i.e. full regularity in the minimal surface case depends on the geometric structure of the problem (again compare Remark 2.3).
However, if $\mu < 3$ and if some additional assumptions are imposed in the vectorial setting, then Section 5 proves uniform local L^p-integrability of the gradients for any $1 < p < \infty$ (see Theorem 5.1).
Once this is established, uniform local apriori gradient bounds for the sequence $\{u_\delta\}$ are shown in Theorem 6.1. Here DeGiorgi's technique is modified: since on one hand we benefit from Hölder's inequality, on the other hand we have to check carefully that the iteration works (see the definition of β).
Finally, in Section 7, the proof of the main Theorem 2.7 is completed.

2 Assumptions and main results

The boundary values u_0 are supposed to be of class $L^\infty \cap W^1_2(\Omega;\mathbb{R}^N)$. As mentioned above, the case $u_0 \in L^\infty \cap W^1_1(\Omega;\mathbb{R}^N)$ is covered with the help of the approximation arguments given in [BF1].
The class of integrands under consideration is defined by

ASSUMPTION 2.1 *There exist positive constants ν_1, ν_2, ν_3 and a real number $1 < \mu \le 3$ such that for any $Z \in \mathbb{R}^{nN}$*

i) $f \in C^2(\mathbb{R}^{nN})$;

ii) $|\nabla f(Z)| \;\le\; \nu_1$;

iii) for any $Y \in \mathbb{R}^{nN}$ we have

$$\nu_2 \big(1+|Z|^2\big)^{-\frac{\mu}{2}} |Y|^2 \;\le\; D^2 f(Z)(Y,Y) \;\le\; \nu_3 \big(1+|Z|^2\big)^{-\frac{1}{2}} |Y|^2 .$$

REMARK 2.2 *Assumption 2.1 implies the following structure conditions.*

i) There are real numbers $\nu_4 > 0$ and ν_5 such that for any $Z \in \mathbb{R}^{nN}$

$$\nabla f(Z) : Z \;\ge\; \nu_4 \big(1+|Z|^2\big)^{\frac{1}{2}} - \nu_5 ,$$

where we use the symbol $Y : Z$ to denote the standard scalar product in $\mathbb{R}^{nN}$.

ii) The integrand f is of linear growth in the sense that for real numbers $\nu_6 > 0$, ν_7, $\nu_8 > 0$, ν_9 and for any $Z \in \mathbb{R}^{nN}$

$$\nu_6 |Z| - \nu_7 \;\le\; f(Z) \;\le\; \nu_8 |Z| + \nu_9 .$$

iii) The integrand satisfies a balancing condition: there is a positive number ν_{10} such that

$$|D^2 f(Z)|\,|Z|^2 \;\le\; \nu_{10} \big(1+f(Z)\big) \quad \textit{holds for any } Z \in \mathbb{R}^{nN} .$$

Proof. ad *i*): replace f by $\bar{f}$: $\mathbb{R}^{nN} \to \mathbb{R}$,

$$\bar{f}(Z) \;:=\; f(Z) - \nabla f(0) : Z \quad \text{for all } Z \in \mathbb{R}^{nN} .$$

A partial integration gives a real number c such that we have for all $w \in u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)$

$$\bar{J}[w] \;:=\; \int_\Omega f(\nabla w)\,dx - \int_\Omega \nabla f(0) : \nabla w\,dx \;=\; J[w] + c .$$

Thus minimizing sequences and generalized minimizers of J and $\bar{J}$ respectively coincide, and w.l.o.g. $\nabla f(0) = 0$ may be assumed. This implies by Assumption 2.1 *iii*)

$$
\begin{aligned}
\nabla f(Z) : Z &= \int_0^1 \frac{d}{d\theta} \nabla f(\theta Z) : Z \, d\theta \\
&= \int_0^1 D^2 f(\theta Z)\big(Z, Z\big) \, d\theta \\
&\geq \nu_2 \int_0^1 \big(1 + \theta^2 |Z|^2\big)^{-\frac{\mu}{2}} |Z|^2 \, d\theta \\
&= \nu_2 |Z| \int_0^{|Z|} \big(1 + \rho^2\big)^{-\frac{\mu}{2}} \, d\rho \, ,
\end{aligned} \tag{2.1}
$$

i.e. $\nabla f(Z) : Z$ is at least of linear growth and *i*) follows.
ad *ii*): the upper bound is immediate by Assumption 2.1 *ii*). Proving the left-hand inequality we observe that (2.1) gives $\nabla f(Z) : Z \geq 0$ for any $Z \in \mathbb{R}^{nN}$. W.l.o.g. we additionally assume $f(0) = 0$ to write (using *i*))

$$
\begin{aligned}
f(Z) &= \int_0^1 \frac{d}{d\theta} f\big(\theta Z\big) \, d\theta \\
&\geq \int_{1/2}^1 \nabla f\big(\theta Z\big) : \theta Z \, d\theta \\
&\geq \frac{1}{2} \left[\nu_4 \left(1 + \frac{|Z|^2}{4}\right)^{\frac{1}{2}} - \nu_5 \right] ,
\end{aligned}
$$

hence *ii*) is clear as well.
ad *iii*): this assertion follows from *ii*) and the right-hand side of Assumption 2.1 *iii*). ■

A comparison of the minimal surface integrand with the above definition provides the following

REMARK 2.3 *The minimal surface example* $f(Z) = \sqrt{1 + |Z|^2}$ *satisfies Assumption 2.1 with the limit exponent* $\mu = 3$. *On the other hand, there is much better information on account of the geometric structure of this exam-*

ple, in particular we have

$$\frac{c_1}{\sqrt{1+|Z|^2}}\left[|Y|^2-\frac{(Y:Z)^2}{1+|Z|^2}\right] \;\leq\; D^2 f(Z)(Y,Y)$$
$$\leq\; \frac{c_2}{\sqrt{1+|Z|^2}}\left[|Y|^2-\frac{(Y:Z)^2}{1+|Z|^2}\right]$$

for all Z, $Y \in \mathbb{R}^n$ with some real numbers c_1, c_2.
Given an integrand satisfying this condition, Ladyzhenskaya/Ural'tseva ([LU2]) and Giaquinta/Modica/Souček ([GMS1]) then use Sobolev's inequality for functions defined on minimal hypersurfaces (compare [MI] and [BGM]) as an essential tool for proving their regularity results.

Finally, the vectorial setting $N > 1$ needs some additional

REMARK 2.4 *Suppose that $N > 1$ and that Assumptions 1.1 and 2.1 hold. This is sufficient to prove a higher integrability result for generalized minimizers — even in the limit case $\mu = 3$ (see Theorem 2.5 and Corollary 2.6).*
The main theorem in the vectorial setting however is obtained for integrands f with some additional "special structure" in the sense that

$$f(Z) \;=\; g\big(|Z|^2\big) \quad \textit{for all } Z \in \mathbb{R}^{nN} \tag{2.2}$$

with g: $[0,\infty) \to [0,\infty)$ of class $C^2(\mathbb{R})$.
Note that (2.2) is not needed to prove a maximum principle (compare [DLM]). As an immediate consequence, (2.2) gives

$$\frac{\partial^2 f}{\partial z^i_\alpha\, \partial z^j_\beta}(Z) \;=\; 4\, g''\big(|Z|^2\big)\, z^i_\alpha\, z^j_\beta + 2\, g'\big(|Z|^2\big)\, \delta^{ij}\, \delta_{\alpha\beta}\,.$$

In addition to (2.2) we impose some Hölder condition on the second derivatives: there are real numbers $\alpha \in (0,1]$, $K > 0$ such that

$$\big|D^2 f(Z) - D^2 f(\tilde{Z})\big| \;\leq\; K\, |Z-\tilde{Z}|^\alpha\,. \tag{2.3}$$

Now let us give a precise formulation of the results.

THEOREM 2.5 *If $N \geq 1$ and if Assumptions 1.1 and 2.1 are supposed to be true in the limit case $\mu = 3$, then there is a generalized minimizer $u^* \in \mathcal{M}$ such that* *i)* $\nabla^s u^* \equiv 0$, *i.e.* $\nabla u^* \equiv \nabla^a u^*$, *where the absolutely continuous part of* ∇u^* *w.r.t. the Lebesgue measure is denoted by* $\nabla^a u^*$, *the singular part by* $\nabla^s u^*$.

ii) *For any* $\Omega' \Subset \Omega$ *there is a constant* $c(\Omega')$ *satisfying*

$$\int_{\Omega'} |\nabla u^*| \log^2\left(1 + |\nabla u^*|^2\right) dx \;\leq\; c(\Omega') \;<\; \infty\,.$$

In [BF1] the behaviour of generalized solutions is studied on suitable balls, which means that no mass of ∇u^* is concentrated on the boundary of these balls. As a result the following Corollary immediately is verified.

COROLLARY 2.6 *The generalized minimizer* u^* *given in Theorem 2.5 is a local minimizer of the functional* $\int_\Omega f(\nabla w)\,dx$.

A slight improvement of the ellipticity condition yields:

THEOREM 2.7 *Suppose Assumptions 1.1 and 2.1 to be true with* $\mu < 3$. *In the case* $N > 1$ *we additionally impose (2.2) and (2.3).*

i) *Each generalized minimizer* $u \in \mathcal{M}$ *is in the space* $C^{1,\alpha}(\Omega;\mathbb{R}^N)$ *for any* $0 < \alpha < 1$.

ii) *The dual solution* σ *is of class* $C^{0,\alpha}(\Omega;\mathbb{R}^{nN})$ *for any* $0 < \alpha < 1$. *Moreover,* σ *has weak derivatives in the space* $L^2_{loc}(\Omega;\mathbb{R}^{nN})$.

iii) *For* $u,\ v \in \mathcal{M}$ *we have* $\nabla u = \nabla v$, *i.e. up to a constant uniqueness of generalized minimizers holds true.*

REMARK 2.8 *Although we concentrate on generalized minimizers, the (local) continuity of the dual solution is needed to obtain iii) from i). The fact* $\sigma \in W^1_{2,loc}(\Omega;\mathbb{R}^{nN})$ *is well known (compare [SE1], [BF1]) and just mentioned for the sake of completeness.*

To finish this section we fix some notation.

i) With a slight abuse of notation, constants are denoted by c without being relabelled.

ii) We take the sum w.r.t. repeated Greek indices $\alpha = 1,\dots,n$ and w.r.t. repeated Latin indices $i = 1,\dots,N$.

iii) We always assume that $x_0 \in \Omega$ and that $B_r(x_0) \Subset \Omega$ is satisfied for each ball under consideration.

3 Regularization

As mentioned above, Problem $(\mathcal{P})$ is approximated in the following way: consider for any $\delta \in (0,1)$ the functional

$$J_\delta[w] \;:=\; \frac{\delta}{2}\int_\Omega |\nabla w|^2\, dx + J[w]\,, \qquad w \in u_0 + \overset{\circ}{W}{}^1_2\big(\Omega;\mathbb{R}^N\big)\,,$$

and denote by u_δ the unique solution of

$$(\mathcal{P}_\delta) \qquad\qquad J_\delta[w] \;\to\; \min\,, \qquad w \in u_0 + \overset{\circ}{W}{}^1_2\big(\Omega;\mathbb{R}^N\big)\,.$$

Letting $f_\delta := \frac{\delta}{2}|\cdot|^2 + f$, observe that the minimality of u_δ implies $J_\delta(u_\delta) \le J_\delta(u_0) \le J_1(u_0)$, hence

$$\int_\Omega f_\delta(\nabla u_\delta)\, dx \;\le\; c \tag{3.1}$$

follows for some real number c. Moreover, by the definition of u_δ,

$$\int_\Omega \nabla f_\delta(\nabla u_\delta) : \nabla\varphi\, dx \;=\; 0 \quad \text{for all } \varphi \in \overset{\circ}{W}{}^1_2\big(\Omega;\mathbb{R}^N\big)\,. \tag{3.2}$$

With the notation $\sigma_\delta = \nabla f_\delta(\nabla u_\delta)$ we may assume on account of (3.1) that $\sigma_\delta \rightharpoondown: \sigma$ in $L^2(\Omega;\mathbb{R}^{nN})$ as $\delta \to 0$, and following [BF1] it is easily seen that

LEMMA 3.1 *i) The sequence $\{u_\delta\}$ is a J-minimizing sequence. Hence, the L^1-cluster points of $\{u_\delta\}$ provide generalized minimizers in the above sense.*

ii) The limit σ of the sequence $\{\sigma_\delta\}$ maximizes the dual problem $(\mathcal{P}^*)$.

Next, a preliminary lemma (nevertheless an essential tool) is proved.

LEMMA 3.2 *Suppose that Assumption 2.1 is true and that we have (2.2) in the case $N > 1$.*

i) There is a real number $c > 0$ such that for any $s \ge 0$, for any $\eta \in C_0^\infty(\Omega)$, $0 \le \eta \le 1$ and for any $\delta \in (0,1)$

$$\begin{aligned}\int_\Omega & D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\,\Gamma_\delta^s \eta^2\, dx \\ &\le\; c\int_\Omega D^2 f_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla\eta, \partial_\gamma u_\delta \otimes \nabla\eta)\,\Gamma_\delta^s\, dx\,,\end{aligned}$$

where we have abbreviated $\Gamma_\delta = 1 + |\nabla u_\delta|^2$.

ii) Letting $A(k,r) = A_\delta(k,r) = \{x \in B_r(x_0) : \Gamma_\delta > k\}$, $k > 0$, there is a real number $c > 0$, independent of δ, such that for any $\eta \in C_0^\infty(B_r(x_0))$, $0 \le \eta \le 1$ and for any $\delta \in (0,1)$

$$\begin{aligned}\int_{A(k,r)} &\Gamma_\delta^{-\frac{\mu}{2}} |\nabla \Gamma_\delta|^2 \eta^2 \, dx \\ &\le \; c \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)(e_j \otimes \nabla\eta, e_j \otimes \nabla\eta)\, (\Gamma_\delta - k)^2 \, dx \, .\end{aligned}$$

Here e_j denotes the j^{th} coordinate vector.

REMARK 3.3 *Following the proof of Lemma 3.2 we see that (2.2) is not needed for assertion i) if $s = 0$.*

Proof of Lemma 3.2. ad *i)*: Using the standard difference quotient technique it is easily seen that u_δ is of class $W^2_{2,loc}(\Omega; \mathbb{R}^N)$.
Moreover, since $|D^2 f_\delta|$ is bounded, $\nabla f_\delta(\nabla u_\delta)$ is of class $W^1_{2,loc}(\Omega; \mathbb{R}^{nN})$ with partial derivatives (almost everywhere)

$$\partial_\gamma\big(\nabla f_\delta(\nabla u_\delta)\big) \;=\; D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \cdot\big)\, , \;\; \gamma = 1, \dots, n \, .$$

Now, given $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$, we take $\partial_\gamma \varphi$, $\gamma = 1, \dots, n$, as an admissible choice in the Euler equation (3.2). A partial integration implies by the above remarks

$$\int_\Omega D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla\varphi)\, dx \;=\; 0 \quad \text{for all } \; \varphi \in C_0^\infty\big(\Omega; \mathbb{R}^N\big)\, , \tag{3.3}$$

and, using standard approximation arguments, (3.3) is seen to be true for all $\varphi \in W_2^1(\Omega; \mathbb{R}^N)$ which are compactly supported in Ω. Next we cite [LU1], Chapter 4, Theorem 5.2, in the scalar case and the Uhlenbeck/Ural'tseva estimates (see [UH], [UR], we refer to [GM], Theorem 3.1) if $N > 1$ to see that $u_\delta \in W^1_{\infty,loc}(\Omega; \mathbb{R}^N)$. As a consequence, $\varphi = \eta^2\, \partial_\gamma u_\delta\, \Gamma_\delta^s$ with η given above is admissible in (3.3) (recall the product and chain rules for Sobolev functions). Summarizing the results we arrive at

$$\begin{aligned}\int_\Omega & D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big)\, \Gamma_\delta^s \eta^2 \, dx \\ &+ s \int_\Omega D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla\Gamma_\delta\big)\, \Gamma_\delta^{s-1} \eta^2 \, dx \\ &= -2 \int_\Omega D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla\eta\big)\, \eta\, \Gamma_\delta^s \, dx \, .\end{aligned} \tag{3.4}$$

In the scalar case $N = 1$ the second integral on the left-hand side can be neglected on account of

$$D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \Gamma_\delta\big) \;=\; \frac{1}{2} D^2 f_\delta(\nabla u_\delta)\big(\nabla \Gamma_\delta, \nabla \Gamma_\delta\big) \;\geq\; 0 \text{ a.e.}$$

In the vectorial setting $N > 1$ we first consider the case $s = 0$. Then the second term on the left-hand side trivially vanishes without any additional assumption (compare Remark 3.3). If $s > 0$, then (2.2) is needed: given a weakly differentiable function ψ: $\Omega \to \mathbb{R}$, and letting $f_\delta(Z) = g_\delta(|Z|^2)$ we obtain almost everywhere

$$\begin{aligned}
& D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \psi\big) \\
&= 4\, g_\delta'' \partial_\alpha u_\delta^i\, \partial_\gamma \partial_\alpha u_\delta^i\, \partial_\beta u_\delta^j\, \partial_\gamma u_\delta^j\, \partial_\beta \psi + 2\, g_\delta'\, \partial_\gamma \partial_\alpha u_\delta^i\, \partial_\gamma u_\delta^i\, \partial_\alpha \psi \\
&= 2\, g_\delta''\, \partial_\gamma |\nabla u_\delta|^2\, \partial_\beta \psi\, \partial_\beta u_\delta^j\, \partial_\gamma u_\delta^j + g_\delta'\, \partial_\alpha |\nabla u_\delta|^2\, \partial_\alpha \psi \\
&= 2\, g_\delta''\, \partial_\gamma \Gamma_\delta\, \partial_\beta \psi\, \partial_\gamma u_\delta^j \partial_\beta u_\delta^j + g_\delta'\, \partial_\alpha \Gamma_\delta\, \partial_\alpha \psi \\
&= \frac{1}{2} \frac{\partial^2 f_\delta}{\partial z_\beta^j \partial z_\gamma^j}(\nabla u_\delta)\, \partial_\beta \psi\, \partial_\gamma \Gamma_\delta \\
&= \frac{1}{2} D^2 f_\delta(\nabla u_\delta)\big(e_j \otimes \nabla \psi, e_j \otimes \nabla \Gamma_\delta\big)\,.
\end{aligned} \tag{3.5}$$

Choosing $\psi = \Gamma_\delta$ we see that in the vectorial setting the second integral on the left-hand side of (3.4) is non-negative as well. In any case we obtain for any $\varepsilon > 0$

$$\begin{aligned}
& \int_\Omega D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big)\, \Gamma_\delta^s \eta^2\, dx \\
&\quad \leq\; c \int_\Omega \Big[D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big)\Big]^{\frac{1}{2}} \eta\, \Gamma_\delta^{\frac{s}{2}} \\
&\qquad\qquad \cdot \Big[D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma u_\delta \otimes \nabla \eta, \partial_\gamma u_\delta \otimes \nabla \eta\big)\Big]^{\frac{1}{2}} \Gamma_\delta^{\frac{s}{2}}\, dx \\
&\quad \leq\; c \Big\{ \varepsilon \int_\Omega D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big)\, \Gamma_\delta^s \eta^2\, dx \\
&\qquad\qquad + \varepsilon^{-1} \int_\Omega D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma u_\delta \otimes \nabla \eta, \partial_\gamma u_\delta \otimes \nabla \eta\big) \Gamma_\delta^s\, dx \Big\}\,.
\end{aligned}$$

If ε is sufficiently small, then we may absorb the first integal on the right-hand side and $i)$ is proved.

ad ii): This time we choose $\varphi = \eta^2 \, \partial_\gamma u_\delta \, \max\big[\Gamma_\delta - k, 0\big]$. Moreover, given a measurable function $w\colon \Omega \to \mathbb{R}$ and writing $w^+ = \max[w, 0]$, we recall (see, for instance, [GT], Lemma 7.6, p. 152) that for $w \in W^1_p(\Omega)$

$$\nabla w^+ \; = \; \begin{cases} \nabla w & \text{if} \;\; w > 0\,, \\ 0 & \text{if} \;\; w \leq 0\,. \end{cases}$$

Then the same arguments as before prove φ to be admissible in (3.3):

$$\begin{aligned} & \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big)\,(\Gamma_\delta - k)\,\eta^2\,dx \\ (3.6) \qquad & \quad + \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \Gamma_\delta\big)\,\eta^2\,dx \\ & \quad = \; -2 \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \eta\big)\,\eta\,(\Gamma_\delta - k)\,dx\,. \end{aligned}$$

Here the non-negative first integral on the left-hand side is neglected and the second integral is estimated as above:

$$\begin{aligned} & \frac{1}{2} \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(e_j \otimes \nabla \Gamma_\delta, e_j \otimes \nabla \Gamma_\delta\big)\,\eta^2\,dx \\ (3.7) \qquad & \quad \leq \; \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \Gamma_\delta\big)\,\eta^2\,dx\,. \end{aligned}$$

According to (3.5), the right-hand side of (3.6) satisfies almost everywhere

$$(3.8) \;\; D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \eta\big) \;\; = \;\; \frac{1}{2} D^2 f_\delta(\nabla u_\delta)\big(e_j \otimes \nabla \eta, e_j \otimes \nabla \Gamma_\delta\big)\,.$$

(3.6)–(3.8) imply with the Cauchy-Schwarz inequality

$$\begin{aligned} & \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(e_j \otimes \nabla \Gamma_\delta, e_j \otimes \nabla \Gamma_\delta\big)\,\eta^2\,dx \\ & \quad \leq \; c \,\Big\{ \varepsilon \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(e_j \otimes \nabla \Gamma_\delta, e_j \otimes \nabla \Gamma_\delta\big)\,\eta^2\,dx \\ & \qquad + \varepsilon^{-1} \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(e_j \otimes \nabla \eta, e_j \otimes \nabla \eta\big)\,(\Gamma_\delta - k)^2\,dx \Big\}\,, \end{aligned}$$

hence ii) is proved recalling Assumption 2.1 iii) and chosing $\varepsilon > 0$ sufficiently small. ■

4 Higher integrability in the limit case $\mu = 3$

In this section we consider the limit case $\mu = 3$ and prove local uniform integrability of $|\nabla u_\delta| \log^2(1 + |\nabla u_\delta|^2)$ by the way establishing Theorem 2.5. Here the discussion of the vectorial setting does not depend on additional conditions (compare Remark 2.4).

THEOREM 4.1 *Suppose we are given Assumptions 1.1 and 2.1 in the limit case $\mu = 3$. Then for any $\Omega' \Subset \Omega$ there is a real number $c(\Omega')$ – independent of δ – satisfying*

$$\int_{\Omega'} |\nabla u_\delta| \log^2\left(1 + |\nabla u_\delta|^2\right) dx \;\le\; c(\Omega') \;<\; \infty\,.$$

Proof. This time we have to show that $\varphi = u_\delta\, \omega_\delta^2\, \eta^2$, $\omega_\delta = \log(\Gamma_\delta)$, $\eta \in C_0^\infty(B_{2r}(x_0))$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$, is admissible in the Euler equation (3.2). Since condition (2.2) is dropped in this section, we now refer to the discussion of asymptotically regular integrands given in [CE]; a generalization is proved in [GM], Theorem 5.1. As a result, u_δ is seen to be of class $W^2_{2,loc} \cap W^1_{\infty,loc}(\Omega;\mathbb{R}^N)$ which proves φ to be admissible. Alternatively, we could replace ω_δ by a suitable truncation $\omega_{\delta,M}$ and prove Theorem 4.1 passing to the limit $M \to \infty$.

With the above choice, the Euler equation reads as

$$\begin{aligned}
&\int_{B_{2r}(x_0)} \nabla f(\nabla u_\delta) : \nabla u_\delta \omega_\delta^2 \eta^2\, dx + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \omega_\delta^2 \eta^2\, dx \\
&\quad = -\int_{B_{2r}(x_0)} \nabla f(\nabla u_\delta) : u_\delta \otimes \left[\nabla \omega_\delta^2 \eta^2 + \nabla \eta^2 \omega_\delta^2\right] dx \\
&\qquad -\delta \int_{B_{2r}(x_0)} \nabla u_\delta : u_\delta \otimes \left[\nabla \omega_\delta^2 \eta^2 + \nabla \eta^2 \omega_\delta^2\right] dx\,.
\end{aligned} \tag{4.1}$$

Remark 2.2 *i*) proves the left-hand side of (4.1) to be greater than or equal to

$$\int_{B_{2r}(x_0)} \left[\nu_4 \Gamma_\delta^{\frac{1}{2}} \omega_\delta^2 \eta^2 - \nu_5 \omega_\delta^2 \eta^2\right] dx + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \omega_\delta^2 \eta^2\, dx\,. \tag{4.2}$$

Since $|\nabla f|$ and $|u_\delta|$ are bounded, we find an upper bound for the right-hand side of (4.1) (using Young's inequality with $\varepsilon > 0$ fixed)

$$
\begin{aligned}
\text{r.h.s} \ \leq\ & c \int_{B_{2r}(x_0)} \eta^2 \Big[\varepsilon\, \Gamma_\delta^{\frac{1}{2}}\, \omega_\delta^2 + \varepsilon^{-1}\, \Gamma_\delta^{-\frac{1}{2}}\, |\nabla \omega_\delta|^2\Big]\, dx \\
& + c(r) \int_{B_{2r}(x_0)} \omega_\delta^2\, dx \\
& + c\,\delta \int_{B_{2r}(x_0)} \eta^2 \Big[\varepsilon\, |\nabla u_\delta|^2 \omega_\delta^2 + \varepsilon^{-1}\, |\nabla \omega_\delta|^2\Big]\, dx \\
& + c(r)\, \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|\, \omega_\delta^2\, dx\, .
\end{aligned}
\tag{4.3}
$$

Clearly $\int_{B_{2r}(x_0)} \omega_\delta^2\, dx$ and $\delta \int_{B_{2r}(x_0)} |\nabla u_\delta| \omega_\delta^2\, dx$ are uniformly bounded w.r.t. δ (compare (3.1)). Hence (4.1)–(4.3) imply after absorbing terms (for ε sufficiently small)

$$
\begin{aligned}
\int_{B_r(x_0)} \Gamma_\delta^{\frac{1}{2}}\, \omega_\delta^2\, dx \ \leq\ & c \Big[1 + \int_{B_{2r}(x_0)} \Gamma_\delta^{-\frac{1}{2}}\, |\nabla \omega_\delta|^2\, \eta^2\, dx \\
& \qquad + \delta \int_{B_{2r}(x_0)} |\nabla \omega_\delta|^2\, \eta^2\, dx\Big]\, .
\end{aligned}
\tag{4.4}
$$

Given (4.4) observe that a.e.

$$
|\nabla \omega_\delta|^2 \ \leq\ c\, \frac{1}{1 + |\nabla u_\delta|^2}\, |D^2 u_\delta|^2\, ,
$$

thus we may use Assumption 2.1, *iii)*, with $\mu = 3$, Lemma 3.2 (letting $s = 0$ and recalling Remark 3.3) as well as Remark 2.2 *iii)* and (3.1) to obtain the final result

$$
\begin{aligned}
\int_{B_r(x_0)} \Gamma_\delta^{\frac{1}{2}}\, \omega_\delta^2\, dx \ \leq\ & c \Big[1 + c \int_{B_{2r}(x_0)} \Big(\Gamma_\delta^{-\frac{1}{2}} + \delta\Big)\, \Gamma_\delta^{-1}\, |D^2 u_\delta|^2\, \eta^2\, dx\Big] \\
\leq\ & c \Big[1 + c \int_{B_{2r}(x_0)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big)\, \eta^2\, dx\Big] \\
\leq\ & c \Big[1 + c(r) \int_{B_{2r}(x_0)} \big|D^2 f_\delta(\nabla u_\delta)\big|\, |\nabla u_\delta|^2\, dx\Big] \ \leq\ c\, .
\end{aligned}
$$

■

5 L^p–estimates in the case $\mu < 3$

From now on we concentrate on the case $\mu < 3$. Moreover, we impose some additional structure in the vectorial setting $N > 1$ (compare Remark 2.4). Then it is possible to modify the arguments of Section 4 such that the results obtained there may be iterated. This gives uniform L^p–estimates in the sense of

THEOREM 5.1 *Suppose that $\mu < 3$, that we have Assumptions 1.1, 2.1 and that (2.2) is satisfied. Then for any $1 < p < \infty$ and for any $\Omega' \Subset \Omega$ there is a constant $c(p, \Omega')$, which does not depend on δ, such that*

$$\|\nabla u_\delta\|_{L^p(\Omega';\mathbb{R}^{nN})} \;\le\; c(p,\Omega') \;<\; \infty\,.$$

REMARK 5.2 *As an immediate consequence a generalized minimizer $u^* \in \mathcal{M}$ is found which is of class $W^1_{p,loc}(\Omega;\mathbb{R}^N)$ for any $1 < p < \infty$.*

Proof. Fix a ball $B_{r_0}(x_0) \Subset \Omega$ and assume that there is real number $\alpha_0 \ge 0$ such that (uniformly w.r.t. δ)

$$\text{(5.1)} \qquad \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{1+\alpha_0}{2}}\,dx + \delta \int_{B_{r_0}(x_0)} \Gamma_\delta^{1+\frac{\alpha_0}{2}}\,dx \;\le\; c\,.$$

Note that by (3.1) this assumption is true for $\alpha_0 = 0$. Next define $\alpha = \alpha_0 + 3 - \mu$ and choose $\varphi = u_\delta \Gamma_\delta^{\frac{\alpha}{2}} \eta^2$, $\eta \in C_0^\infty(B_{r_0}(x_0))$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{r_0/2}(x_0)$, $|\nabla \eta| \le c/r_0$. As outlined in the proof of Lemma 3.2 u_δ is of class $W^2_{2,loc} \cap W^1_{\infty,loc}(\Omega;\mathbb{R}^N)$, hence φ is admissible in (3.2) with the result

$$\begin{aligned}
&\int_{B_{r_0}(x_0)} \nabla f(\nabla u_\delta) : \nabla u_\delta \Gamma_\delta^{\frac{\alpha}{2}} \eta^2\,dx + \delta \int_{B_{r_0}(x_0)} |\nabla u_\delta|^2 \Gamma_\delta^{\frac{\alpha}{2}} \eta^2\,dx \\
\text{(5.2)} \qquad &\le\; c(\alpha) \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha-1}{2}} |D^2 u_\delta| \eta^2\,dx + c(\alpha)\,\delta \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha}{2}} |D^2 u_\delta| \eta^2\,dx \\
&\quad + c \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha}{2}} |\nabla \eta^2|\,dx + c\delta \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha+1}{2}} |\nabla \eta^2|\,dx\,.
\end{aligned}$$

Here Assumption 1.1 and the boundedness of ∇f (compare Assumption 2.1 *ii*)) are already used. Analogous to the previous section, the left-hand side of (5.2) is estimated with the help of Remark 2.2, *i*):

$$\begin{aligned}
l.h.s. \;\ge\; &\nu_4 \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{1+\alpha}{2}} \eta^2\,dx - \nu_5 \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha}{2}} \eta^2\,dx \\
&+ \delta \int_{B_{r_0}(x_0)} \Gamma_\delta^{1+\frac{\alpha}{2}} \eta^2\,dx - \delta \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha}{2}} \eta^2\,dx\,.
\end{aligned}$$

The right-hand side of (5.2) is handled via (fix $\varepsilon > 0$ and use Young's inequality)

$$
\begin{aligned}
r.h.s \ \le\ & c \int_{B_{r_0}(x_0)} \eta^2 \Big[\varepsilon \Gamma_\delta^{\frac{1+\alpha}{2}} + \varepsilon^{-1} \Gamma_\delta^{-\frac{1+\alpha}{2}} \Gamma_\delta^{\alpha-1} |D^2 u_\delta|^2\Big]\, dx \\
&+ c \int_{B_{r_0}(x_0)} \Big[\varepsilon \Gamma_\delta^{\frac{1+\alpha}{2}} \eta^2 + \varepsilon^{-1} \Gamma_\delta^{-\frac{1+\alpha}{2}} \Gamma_\delta^{\alpha} |\nabla \eta|^2\Big]\, dx \\
&+ c\delta \int_{B_{r_0}(x_0)} \eta^2 \Big[\varepsilon \Gamma_\delta^{1+\frac{\alpha}{2}} + \varepsilon^{-1} \Gamma_\delta^{-1-\frac{\alpha}{2}} \Gamma_\delta^{\alpha} |D^2 u_\delta|^2\Big]\, dx \\
&+ c\delta \int_{B_{r_0}(x_0)} \Big[\varepsilon \Gamma_\delta^{1+\frac{\alpha}{2}} \eta^2 + \varepsilon^{-1} \Gamma_\delta^{-1-\frac{\alpha}{2}} \Gamma_\delta^{1+\alpha} |\nabla \eta|^2\Big]\, dx\, .
\end{aligned}
$$

Hence, absorbing terms, (5.2) yields

$$
\begin{aligned}
& \int_{B_{r_0/2}(x_0)} \Gamma_\delta^{\frac{1+\alpha}{2}}\, dx + \delta \int_{B_{r_0/2}(x_0)} \Gamma_\delta^{1+\frac{\alpha}{2}}\, dx \\
& \le\ c \Big[\int_{B_{r_0}(x_0)} \eta^2 \Gamma_\delta^{\frac{\alpha-3}{2}} |D^2 u_\delta|^2\, dx \\
& \qquad + \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha-1}{2}} |\nabla \eta|^2\, dx + \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha}{2}} \eta^2\, dx \Big] \\
& \quad + c\delta \Big[\int_{B_{r_0}(x_0)} \eta^2 \Gamma_\delta^{\frac{\alpha}{2}-1} |D^2 u_\delta|^2\, dx \\
& \qquad + \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha}{2}} |\nabla \eta|^2\, dx + \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha}{2}} \eta^2\, dx \Big] \\
& =:\ c \sum_{i=1}^{3} I_i + c\delta \sum_{i=4}^{6} I_i\, .
\end{aligned}
\tag{5.3}
$$

Starting with I_1, we recall that by definition $\mu + \alpha - 3 = \alpha_0 \ge 0$, thus Assumption 2.1 *iii)* and Lemma 3.2 *i)* give

$$
\begin{aligned}
I_1 \ =\ & \int_{B_{r_0}(x_0)} \eta^2 \Gamma_\delta^{-\frac{\mu}{2}} |D^2 u_\delta|^2 \Gamma_\delta^{\frac{\mu+\alpha-3}{2}}\, dx \\
\le\ & c \int_{B_{r_0}(x_0)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big) \Gamma_\delta^{\frac{\alpha_0}{2}} \eta^2\, dx \\
\le\ & c(r_0) \int_{B_{r_0}(x_0)} \Big[\Gamma_\delta^{-\frac{1}{2}} + \delta\Big] \Gamma_\delta^{1+\frac{\alpha_0}{2}}\, dx \ \le\ c\, ,
\end{aligned}
$$

where the last inequality is due to Assumption (5.1). An upper bound (not depending on δ) for I_3 is found since we may assume w.l.o.g. that $\mu \ge 2$.

This clearly proves I_2 to be bounded independent of δ as well. Studying I_4 let us first assume that $\alpha \le 2$. Then, again by Lemma 3.2 i)

$$\begin{aligned}\delta I_4 &\le \int_{B_{r_0}(x_0)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big)\,\eta^2\,dx\\ &\le c(r_0)\int_{B_{r_0}(x_0)} \Big[\Gamma_\delta^{-\frac{1}{2}} + \delta\Big]\,\Gamma_\delta\,dx \;\le\; c\,.\end{aligned}$$

In the case $\alpha > 2$, Lemma 3.2 i) gives

$$\begin{aligned}\delta I_4 &\le \int_{B_{r_0}(x_0)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big)\,\Gamma_\delta^{\frac{\alpha}{2}-1}\eta^2\,dx\\ &\le c(r_0)\int_{B_{r_0}(x_0)} \Big[\Gamma_\delta^{-\frac{1}{2}} + \delta\Big]\,\Gamma_\delta^{1+\frac{\alpha}{2}-1}\,dx \;\le\; c\,,\end{aligned}$$

where we once more recall (5.1) and observe that $\frac{\alpha}{2} - 1 = (\alpha_0 + 1 - \mu)/2 \le \alpha_0/2$. This condition trivially bounds δI_5 and δI_6 independent of δ, and we have proved with (5.3): suppose that (5.1) holds for some given $r_0 > 0$ and $\alpha_0 > 0$. Then there is a constant, independent of δ such that

$$\int_{B_{r_0/2}(x_0)} \Gamma_\delta^{\frac{1+\alpha_0+3-\mu}{2}}\,dx + \delta \int_{B_{r_0/2}(x_0)} \Gamma_\delta^{1+\frac{\alpha_0+3-\mu}{2}}\,dx \;\le\; c\,. \tag{5.4}$$

We now claim that for any $n \in \mathbb{N}$ there is a constant $c(n)$, independent of δ, such that

$$\int_{B_{r_0/2^n}(x_0)} \Gamma_\delta^{\frac{1+n(3-\mu)}{2}}\,dx + \delta \int_{B_{r_0/2^n}(x_0)} \Gamma_\delta^{1+\frac{n(3-\mu)}{2}}\,dx \;\le\; c\,. \tag{5.5}$$

In fact, as mentioned above, $\alpha_0 = 0$ is an admissible choice to obtain (5.5) from 5.4 in the case $n = 1$. Next assume by induction that (5.5) is true for some $n \in \mathbb{N}$. Then we may take $\alpha_0 = n(3-\mu)$ in (5.1) and (5.4) establishes (5.5) with n replaced by $n+1$, thus the claim is proved. Obviously this implies Theorem 5.1. ∎

REMARK 5.3 *If we omit condition (2.2) in the vectorial setting, then analogous arguments prove higher integrability up to a finite number* $1 < p(\mu)$.

6 Apriori gradient bounds

In this section the DeGiorgi type arguments, outlined for example in [BF3], are modified: on one hand, given Theorem 5.1, we benefit from Hölder's inequality. This decreases on the other hand the exponent of iteration (see definition of β). Nevertheless it turns out that Lemma 6.3 still is applicable to obtain

THEOREM 6.1 *Consider a ball $B_{R_0}(x_0) \Subset \Omega$. With the assumptions of Theorem 5.1 there is a local constant $c > 0$ such that for any $\delta \in (0,1)$*

$$\|\nabla u_\delta\|_{L^\infty(B_{R_0/2},\mathbb{R}^{nN})} \;\leq\; c\,.$$

Before proving Theorem 6.1 we recall the definition

$$A(k,r) \;=\; \big\{x \in B_r(x_0) : \Gamma_\delta > k\big\}\,,\;\; B_r(x_0) \Subset \Omega\,,\;\; k > 0\,,$$

and establish the following result.

LEMMA 6.2 *Fix some $x_0 \in \Omega$ and suppose $0 < r < R < R_0$ such that $B_{R_0}(x_0) \Subset \Omega$. Then there is a real number c, independent of r, R, R_0, k and δ, satisfying*

$$\begin{aligned} &\int_{A(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}}\,dx \\ \text{(6.1)}\qquad &\leq\; \frac{c}{(R-r)^{\frac{n}{n-1}}}\Big[\int_{A(k,R)} (\Gamma_\delta - k)^2\,dx\Big]^{\frac{1}{2}\frac{n}{n-1}}\Big[\int_{A(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\,dx\Big]^{\frac{1}{2}\frac{n}{n-1}}\,. \end{aligned}$$

Proof of Lemma 6.2. Recalling the notion w^+ of Section 3, Sobolev's inequality yields with $\eta \in C_0^\infty(B_R(x_0))$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(x_0)$, $|\nabla\eta| \leq c/(R-r)$,

$$\text{(6.2)}\qquad \begin{aligned} \int_{A(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}}\,dx \;&\leq\; \int_{B_R(x_0)} \Big[\eta\,(\Gamma_\delta - k)^+\Big]^{\frac{n}{n-1}}\,dx \\ &\leq\; c\Big[\int_{B_R(x_0)} \big|\nabla\big[\eta\,(\Gamma_\delta - k)^+\big]\big|\,dx\Big]^{\frac{n}{n-1}} \\ &\leq\; c\Big[\int_{A(k,R)} \big|\nabla\big[\eta\,(\Gamma_\delta - k)\big]\big|\,dx\Big]^{\frac{n}{n-1}} \\ &\leq\; c\Big[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\Big]\,. \end{aligned}$$

Here we have

$$\begin{aligned}
I_1^{\frac{n}{n-1}} &= \left[\int_{A(k,R)} |\nabla\eta| \big(\Gamma_\delta - k\big)\, dx\right]^{\frac{n}{n-1}}\\
&\le \left[\int_{A(k,R)} |\nabla\eta|^2 \big(\Gamma_\delta - k\big)^2\, dx\right]^{\frac{1}{2}\frac{n}{n-1}} \left[\int_{A(k,R)} 1\, dx\right]^{\frac{1}{2}\frac{n}{n-1}}\\
&\le \frac{c}{(R-r)^{\frac{n}{n-1}}} \left[\int_{A(k,R)} \big(\Gamma_\delta - k\big)^2\, dx\right]^{\frac{1}{2}\frac{n}{n-1}} \left[\int_{A(k,R)} 1\, dx\right]^{\frac{1}{2}\frac{n}{n-1}},
\end{aligned}$$

thus $I_1^{\frac{n}{n-1}}$ is seen to be bounded from above by the right-hand side of 6.1. Estimating I_2, Lemma 3.2 *ii*) is needed with the result

$$\begin{aligned}
I_2^{\frac{n}{n-1}} &= \left[\int_{A(k,R)} \eta\, |\nabla\Gamma_\delta|\, dx\right]^{\frac{n}{n-1}}\\
&\le \left[\int_{A(k,R)} \eta^2\, |\nabla\Gamma_\delta|^2\, \Gamma_\delta^{-\frac{\mu}{2}}\, dx\right]^{\frac{1}{2}\frac{n}{n-1}} \left[\int_{A(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\, dx\right]^{\frac{1}{2}\frac{n}{n-1}}\\
&\le c\left[\int_{A(k,R)} D^2 f_\delta(\nabla u_\delta)\big(e_j\otimes\nabla\eta, e_j\otimes\nabla\eta\big)\big(\Gamma_\delta - k\big)^2\right]^{\frac{1}{2}\frac{n}{n-1}}\\
&\qquad\qquad \times \left[\int_{A(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\, dx\right]^{\frac{1}{2}\frac{n}{n-1}}\\
&\le \frac{c}{(R-r)^{\frac{n}{n-1}}} \left[\int_{A(k,R)} \big(\Gamma_\delta - k\big)^2\, dx\right]^{\frac{1}{2}\frac{n}{n-1}} \left[\int_{A(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\, dx\right]^{\frac{1}{2}\frac{n}{n-1}},
\end{aligned}$$

hence (6.2) proves the lemma. ■

We now come to the **Proof of Theorem 6.1.** Consider the left-hand side of (6.1): for any real number $s > 1$ Hölder's inequality implies

$$\begin{aligned}
\int_{A(k,r)} \big(\Gamma_\delta - k\big)^2\, dx &= \int_{A(k,r)} \big(\Gamma_\delta - k\big)^{\frac{n}{n-1}\frac{1}{s}} \big(\Gamma_\delta - k\big)^{2-\frac{n}{n-1}\frac{1}{s}}\, dx\\
&\le \left[\int_{A(k,r)} \big(\Gamma_\delta - k\big)^{\frac{n}{n-1}}\, dx\right]^{\frac{1}{s}}\\
&\qquad \times \left[\int_{A(k,r)} \big(\Gamma_\delta - k\big)^{\left(2-\frac{n}{n-1}\frac{1}{s}\right)\frac{s}{s-1}}\right]^{\frac{s-1}{s}}.
\end{aligned}$$

Hence, on account of Theorem 5.1 there is a real number $c_1(s,n,B_{R_0}(x_0))$, independent of δ,

$$c_1(s,n,B_{R_0}(x_0)) \;:=\; \sup_{\delta>0}\Big[\int_{B_{R_0}(x_0)} \Gamma_\delta^{\frac{1}{s-1}\left(2s-\frac{n}{n-1}\right)}\,dx\Big]^{\frac{s-1}{s}} \;<\; \infty\,,$$

such that

$$\int_{A(k,r)} \big(\Gamma_\delta - k\big)^2\,dx \;\le\; c_1(s,n,B_{R_0}(x_0))\Big[\int_{A(k,r)} \big(\Gamma_\delta - k\big)^{\frac{n}{n-1}}\,dx\Big]^{\frac{1}{s}}\,. \tag{6.3}$$

Studying the right-hand side of (6.1), we fix a second real number $t>1$ and applying Hölder's inequality once more it is seen that

$$\int_{A(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\,dx \;\le\; \big|A(k,R)\big|^{\frac{1}{t}}\Big[\int_{A(k,R)} \Gamma_\delta^{\frac{\mu}{2}\frac{t}{t-1}}\,dx\Big]^{\frac{t-1}{t}}\,.$$

Similar as above, one defines

$$c_2(t,\mu,B_{R_0}(x_0)) \;:=\; \sup_{\delta>0}\Big[\int_{B_{R_0}(x_0)} \Gamma_\delta^{\frac{\mu}{2}\frac{t}{t-1}}\,dx\Big]^{\frac{t-1}{t}} \;<\; \infty$$

to obtain

$$\int_{A(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\,dx \;\le\; c_2(t,\mu,B_{R_0}(x_0))\,\big|A(k,R)\big|^{\frac{1}{t}}\,. \tag{6.4}$$

Summarizing the results we arrive at

$$\begin{aligned}
\int_{A(k,r)} \big(\Gamma_\delta - k\big)^2\,dx \;&\overset{(6.3)}{\le}\; c\Big[\int_{A(k,r)} \big(\Gamma_\delta - k\big)^{\frac{n}{n-1}}\,dx\Big]^{\frac{1}{s}}\\
&\overset{(6.1)}{\le}\; \frac{c}{(R-r)^{\frac{n}{n-1}\frac{1}{s}}}\Big[\int_{A(k,R)} \big(\Gamma_\delta - k\big)^2\,dx\Big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}}\\
&\qquad\qquad \times\Big[\int_{A(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\,dx\Big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}}\\
&\overset{(6.4)}{\le}\; \frac{c}{(R-r)^{\frac{n}{n-1}\frac{1}{s}}}\Big[\int_{A(k,R)} \big(\Gamma_\delta - k\big)^2\,dx\Big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}}\\
&\qquad\qquad \times\big|A(k,R)\big|^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}\,.
\end{aligned} \tag{6.5}$$

As the next step, define for k and $r < R$ as above

$$\begin{aligned} \tau(k,r) &:= \int_{A(k,r)} \big(\Gamma_\delta - k\big)^2 \, dx \,, \\ a(k,r) &:= \big|A(k,r)\big| \,. \end{aligned}$$

With this notation (6.5) reads as

$$\tau(k,r) \le \frac{c}{(R-r)^{\frac{n}{n-1}\frac{1}{s}}} \big[\tau(k,R)\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}} \big[a(k,R)\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\frac{1}{t}} \,. \tag{6.6}$$

Given two real numbers $h > k > 0$, we now observe

$$a(h,R) = \int_{A(h,R)} dx \le \int_{A(h,R)} \big(\Gamma_\delta - k\big)^2 (h-k)^{-2} \, dx \,,$$

thus $h > k > 0$ implies

$$a(h,R) \le \frac{1}{(h-k)^2} \tau(k,R) \,. \tag{6.7}$$

With (6.6) and (6.7) it is proved that for $h > k > 0$

$$\begin{aligned} \tau(h,r) &\le \frac{c}{(R-r)^{\frac{n}{n-1}\frac{1}{s}}} \big[\tau(h,R)\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}} \frac{1}{(h-k)^{\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}} \big[\tau(k,R)\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\frac{1}{t}} \\ &\le \frac{c}{(R-r)^{\frac{n}{n-1}\frac{1}{s}}} \frac{1}{(h-k)^{\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}} \big[\tau(k,R)\big]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left(1+\frac{1}{t}\right)} \,. \end{aligned}$$

If s and t are chosen sufficiently close to 1 (depending on n), then it is achieved that

$$\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left[1+\frac{1}{t}\right] =: \beta > 1 \,.$$

With this choice of s and t we additionally let

$$\alpha := \frac{n}{n-1}\frac{1}{s}\frac{1}{t} > 0 \,, \qquad \gamma := \frac{n}{n-1}\frac{1}{s} > 0 \,,$$

hence the following lemma, stated for example in [ST], Lemma 5.1, p.219, may be applied.

LEMMA 6.3 *Assume that $\varphi(h,\rho)$ is a non-negative real valued function defined for $h > k_0$ and $\rho < R_0$. Suppose further that for fixed ρ the function is non-increasing in h and that is non-decreasing in ρ if h is fixed. Then*

$$\varphi(h,\rho) \;\le\; \frac{C}{(h-k)^\alpha (R-\rho)^\gamma} \big[\varphi(k,R)\big]^\beta \,, \quad h > k > k_0 \,, \quad \rho < R < R_0 \,,$$

with some positive constants C, α, $\beta > 1$, γ, implies for all $0 < \sigma < 1$

$$\varphi\big(k_0 + d, R_0 - \sigma R_0\big) \;=\; 0 \,,$$

where the quantity d is given by

$$d^\alpha \;=\; \frac{2^{(\alpha+\beta)\beta/(\beta-1)}\, C \,\big[\varphi(k_0,R_0)\big]^{\beta-1}}{\sigma^\gamma R_0^\gamma} \,.$$

This lemma yields

$$\tau(d, R_0/2) \;=\; \int_{A(d,R_0/2)} \big(\Gamma_\delta - d\big)^2 \, dx \;=\; 0 \,,$$

and, as a consequence,

$$\Gamma_\delta \;\le\; d \quad \text{on } B_{R_0/2}(x_0) \,. \tag{6.8}$$

Here the quantity d is uniformly bounded w.r.t. δ if and only if there is a constant (independent of δ) such that

$$\tau(0, R_0) \;=\; \int_{B_{R_0}(x_0)} \Gamma_\delta^2 \, dx \;\le\; c \,.$$

This fact is proved in Theorem 5.1 and the apriori estimate Theorem 6.1 follows from (6.8). ■

7 Proof of the main theorem

Once Theorem 6.1 is established, Theorem 2.7 follows exactly as outlined in [BF3]. Let us first sketch the main arguments to obtain local $C^{1,\alpha}$-regularity for weak $\{u_\delta\}$-cluster points u^*: recall Corollary 2.6 which implies the Euler equation

$$\int_\Omega \nabla f(\nabla u^*) : \nabla \varphi \, dx \;=\; 0 \quad \text{for all } \varphi \in C_0^1\big(\Omega; \mathbb{R}^N\big) \,.$$

In the scalar case $N = 1$ we argue with the standard difference quotient technique and, since u^* is Lipschitz, it follows that u^* is of class $W^2_{2,loc}(\Omega;\mathbb{R}^N)$. Then, letting $v = \partial_\gamma u^*$, one arrives at

$$\int_\Omega D^2 f(\nabla u^*)\big(\nabla v, \nabla\varphi\big)\, dx \;=\; 0 \quad \text{for all } \varphi \in C^1_0\big(\Omega;\mathbb{R}^N\big)$$

(compare (3.3)), where the coefficients $\frac{\partial^2 f}{\partial z_\alpha \partial z_\beta}(\nabla u^*)$ are uniformly elliptic on $\Omega' \Subset \Omega$. Theorem 8.22 of [GT] finally proves Hölder continuity of v.
In the vectorvalued case an auxiliary integrand $\tilde{f}$ is constructed following the lines of [MS]. As a result, Theorem 3.1 of [GM] may be applied since we also have imposed the Hölder condition (2.3). Thus $C^{1,\alpha}$-regularity is proved in the vectorial setting as well.
Next consider the dual solution σ. As it is proved in [BF1] or [BF3], the duality relation

$$\sigma \;=\; \nabla f(\nabla u^*) \quad \text{for a.a. } x \in \Omega$$

holds true for any weak cluster point u^* of the sequence $\{u_\delta\}$. Since these cluster points are known to have (locally) Hölder continuous first derivatives, $\sigma \in C^{0,\alpha}(\Omega;\mathbb{R}^{nN})$ is immediate.
Uniqueness of generalized minimizers up to a constant is shown in Section 5 of [BF3] — the idea is due to [SE4]: a suitable variation of σ is seen to be admissible on account of $\sigma \in C^{0,\alpha}(\Omega;\mathbb{R}^{nN})$. This, together with the uniqueness of σ, yields

$$\nabla u \;=\; \nabla f^*(\sigma)$$

for any generalized minimizer $u \in \mathcal{M}$, and Theorem 2.7 is proved. ■

References

[AG] Anzellotti, G., Giaquinta, M., Convex functionals and partial regularity. Arch. Rat. Mech. Anal. 102 (1988), 243–272.

[BI] Bildhauer, M., A uniqueness theorem for the dual problem associated to a variational problem with linear growth. Zap. Nauchn. Sem. St.-Petersburg Odtel. Math. Inst. Steklov (POMI) 271 (2000), 83–91.

[BF1] Bildhauer, M., Fuchs, M., Regularity for dual solutions and for weak cluster points of minimizing sequences of variational problems with linear growth. Zap. Nauchn. Sem. St.-Petersburg Odtel. Math. Inst. Steklov (POMI) 259 (1999), 46–66.

[BF2] Bildhauer, M.,Fuchs, M., Partial regularity for variational integrals with (s,μ,q)–growth. To appear in Calc. Var.

[BF3] Bildhauer, M., Fuchs, M., On a class of variational integrals with linear growth satisfying the condition of μ-ellipticity. Preprint Bonn University/SFB 256 No. 681.

[BFM] Bildhauer, M., Fuchs, M., Mingione, G., Apriori gradient bounds and local $C^{1,\alpha}$–estimates for (double) obstacle problems under nonstandard growth conditions. Preprint Bonn University/SFB 256 No. 647.

[BGM] Bombieri, E., DeGiorgi, E., Miranda, M., Una maggiorazione a priori relative alle ipersuperfici minimali non parametriche. Arch. Rat. Mech. Anal. 32 (1969), 255–267.

[CE] Chipot, M., Evans, L.C., Linearization at infinity and Lipschitz estimates for certain problems in the calculus of variations. Proc. Roy. Soc. Edinburgh 102 A (1986), 291–303.

[CH] Choe, H. J., Interior behaviour of minimizers for certain functionals with linear growth. Nonlinear Analysis, Theory, Methods & Appl. 19.10 (1992), 933-945.

[DLM] D'Ottavio, A., Leonetti, F., Musciano, C., Maximum principle for vector valued mappings minimizing variational integrals. Atti Sem. Mat. Fis. Univ. Modena (1998).

[ET] Ekeland, I., Temam, R., Convex analysis and variational problems. North–Holland, Amsterdam 1976.

[ELM] Esposito, L., Leonetti, F., Mingione, G., Regularity for minimizers of functionals with p-q growth. Nonlinear Diff. Equations Appl. 6 (1999), 133–148.

[FRS] Frehse, J. Seregin, G., Regularity of solutions to variational problems of the deformation theory of plasticity with logarithmic hardening. Proc. St. Petersburg Math. Soc. 5 (1998), 184-222. English translation: Amer. Math. Soc. Transl. II, 193 (1999), 127–152.

[FM] Fuchs, M., Mingione, G., Full $C^{1,\alpha}$–regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth. Manus. Math. 102 (2000), 227–250.

[FS1] Fuchs, M., Seregin, G., A regularity theory for variational integrals with $L\log L$–growth. Calc. Var. 6 (1998), 171–187.

[FS2] Fuchs, M., Seregin, G., Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Lecture Notes in Mathematics 1749, Springer, Berlin-Heidelberg, 2000.

[GG] Giaquinta, M., Giusti, E., On the regularity of the minima of variational integrals. Acta Math. 148 (1982), 31-45.

[GM] Giaquinta, M., Modica, G., Remarks on the regularity of the minimizers of certain degenerate functionals. Manus. Math. 57 (1986), 55–99.

[GMS1] Giaquinta, M., Modica, G., Souček, J., Functionals with linear growth in the calculus of variations. Comm. Math. Univ. Carolinae 20 (1979), 143–171.

[GMS2] Giaquinta, M., Modica, G., Souček, J., Cartesian currents in the calculus of variations I/II. Ergebnisse der Mathematik und ihrer Grenzgebiete 37/38, Springer, Berlin-Heidelberg, 1998.

[GT] Gilbarg, D., Trudinger, N.S., Elliptic partial differential equations of second order. Grundlehren der math. Wiss. 224, second ed., revised third print., Springer, Berlin–Heidelberg–New York, 1998.

[GI] Giusti, E., Minimal surfaces and functions of bounded variation. Monographs in Mathematics 80, Birkhäuser, Boston-Basel-Stuttgart, 1984.

[LU1] Ladyzhenskaya, O.A., Ural'tseva, N.N., Linear and quasilinear elliptic equations. Nauka, Moskow, 1964. English translation: Academic Press, New York 1968.

[LU2] Ladyzhenskaya, O.A., Ural'tseva, N.N., Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations. Comm. Pure Appl. Math. 23 (1970), 667–703.

[MS] Mingione, G., Siepe, F., Full $C^{1,\alpha}$ regularity for minimizers of integral functionals with $L \log L$ growth. Z. Anal. Anw. 18 (1999), 1083–1100.

[MI] Miranda, M., Disequaglianze di Sobolev sulle ipersuperfici minimali. Rend. Sem. Mat. Univ. Padova 38 (1967), 69–79.

[SE1] Seregin, G., Differentiability of extremals of variational problems in the mechanics of perfect elastoplastic media. Differentsial'nye Uravneniya, 23.11 (1987), 1981–1991. English Translation: Differential Equations, 23 (1987).

[SE2] Seregin, G.A., Differential properties of extremals of variational problems arising in plasticity theory. Differentsial'nye Uravneniya 26 (1990), 1033–1043 (in Russian). English translation: Differential Equations 26 (1990).

[SE3] Seregin, G.A., Differential properties of solutions of variational problems for functionals with linear growth. Problemy Matematicheskogo Analiza, Vypusk 11, Isazadel'stvo LGU (1990), 51–79. English translation: J, Soviet Math. 64 (1993), 1256–1277.

[SE4] Seregin, G., Twodimensional variational problems in plasticity theory. Izv. Russian Academy of Sciences 60 (1996), 175–210. English translation: Izvestiya Mathematics 60.1 (1996).

[ST] Stampacchia, G., Le problème de Dirichlet pour les équations elliptiques du second ordre á coefficients discontinus. Ann. Inst. Fourier Grenoble 15.1 (1965), 189–258.

[TA] Talenti, G., Boundedness of minimizers. Hokkaido Math. J. 19 (1990), 259-279.

[UH] Uhlenbeck, K., Regularity for a class of nonlinear elliptic systems. Acta Math. 138 (1977), 219–240.

[UR] Ural'tseva. N.N., Quasilinear degenerate elliptic systems (Russian), Leningrad Odtel. Mat. Inst. Steklov (LOMI) 7 (1968), 184–222.