Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint

The effect of a penalty term involving higher order derivatives on the distribution of phases in an elastic medium with a two–well elastic potential

Michael Bildhauer, Martin Fuchs and Victor Osmolovskii

Preprint No. 24

Saarbrücken 2000

Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

The effect of a penalty term involving higher order derivatives on the distribution of phases in an elastic medium with a two–well elastic potential

Michael Bildhauer

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken, Germany
E-Mail: bibi@math.uni-sb.de

Martin Fuchs

Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken, Germany
E-Mail: fuchs@math.uni-sb.de

Victor Osmolovskii

Mathematics & Mechanics Faculty
St. Petersburg State University
198904 St. Petersburg, Russia

submitted: December 6, 2000

Preprint No. 24

Saarbrücken 2000

Edited by
FR 6.1 – Mathematik
Im Stadtwald
D–66041 Saarbrücken
Germany

Fax:	+ 49 681 302 4443
e-mail:	preprint@math.uni-sb.de
WWW:	http://www.math.uni-sb.de/

Abstract

We consider the problem of minimizing

$$I[u,\chi,h,\sigma] = \int_\Omega \Big(\chi f_h^+\big(\varepsilon(u)\big) + (1-\chi)\, f^-\big(\varepsilon(u)\big)\Big)\, dx$$
$$+\sigma\left(\int_\Omega |\Delta u|^2\, dx\right)^{p/2},$$

$0 < p < 1$, $h \in \mathbb{R}$, $\sigma > 0$, among functions u: $\mathbb{R}^d \supset \Omega \to \mathbb{R}^d$, $u_{|\partial\Omega} = 0$, and measurable characteristic functions χ: $\Omega \to \mathbb{R}$. Here f_h^+, f^- denote quadratic potentials defined on the space of all symmetric $d \times d$ matrices, h is the minimum energy of f_h^+ and $\varepsilon(u)$ denotes the symmetric gradient of the displacement field. An equilibrium state $\hat{u}$, $\hat{\chi}$ of $I[\cdot,\cdot,h,\sigma]$ is termed one–phase if $\hat{\chi} \equiv 0$ or $\hat{\chi} \equiv 1$, two–phase otherwise. We investigate in which way the distribution of phases is affected by the choice of the parameters h and σ.

AMS Subject Classification: 74 B 05, 74 G 65, 74 N 99 .
Key words: elastic materials, phase transition, equilibrium states, regularization.

1 Introduction

We consider an elastic medium which can exist in two different phases. If the medium occupies a bounded region $\Omega \subset \mathbb{R}^d$ (assumed to be of class C^2), then the energy density of the first (second) phase is given by

$$f_h^+\big(\varepsilon(u)\big) = \big\langle A^+\big(\varepsilon(u)-\xi^+\big), \varepsilon(u)-\xi^+\big\rangle + h\,,$$
$$\Big(f^-\big(\varepsilon(u)\big) = \big\langle A^-\big(\varepsilon(u)-\xi^-\big), \varepsilon(u)-\xi^-\big\rangle\Big)$$

where $u = (u^1,\dots,u^d)$: $\Omega \to \mathbb{R}^d$ is the field of displacements, $\varepsilon(u) = \frac{1}{2}(\partial_i u^j + \partial_j u^i)_{1\le i,j\le d}$ denotes the corresponding strain tensor, and $A^\pm$: $\mathbb{S}^d \to \mathbb{S}^d$ are linear, symmetric operators defined on the space $\mathbb{S}^d$ of all symmetric $d\times d$ matrices having the meaning of the tensors of elastic moduli of the first and the second phase. Finally, $\xi^\pm \in \mathbb{S}^d$ denote the stress–free strains of the i^{th} phase, and we use the symbol $\langle \varepsilon, \varkappa\rangle := \mathrm{tr}\,(\varepsilon\, \varkappa)$ for the scalar product in $\mathbb{S}^d$. Thus, the energy density of each phase is a quadratic function of the linear strain, where the energy density of the first phase depends in addition on the parameter $h \in \mathbb{R}$. Let us state the hypotheses imposed on the data: $A^{\pm}$ are assumed to be positive, i.e. for some number $\nu > 0$ we have

$$\nu\,|\varepsilon|^2 \;\leq\; \langle A^{\pm}\varepsilon, \varepsilon\rangle \;\leq\; \nu^{-1}\,|\varepsilon|^2 \quad \text{for all } \varepsilon \in \mathbb{S}^d, \tag{1.1}$$

hence the parameter h measures the difference between the minima of f_h^+ and f^-. As a second condition concerning the tensors of elastic moduli we require that for some number $\mu \in (0, \nu)$

$$\big|\big\langle (A^+ - A^-)\,\varepsilon, \varepsilon\big\rangle\big| \;\leq\; \mu\,|\varepsilon|^2 \quad \text{for all } \varepsilon \in \mathbb{S}^d \tag{1.2}$$

is satisfied. Finally, we suppose that

$$A^+\xi^+ \;\neq\; A^-\xi^- \tag{1.3}$$

is valid. Clearly, (1.2) holds in case that $A^+ = A^-$ for which (1.3) reduces to the condition $\xi^+ \neq \xi^-$. If χ denotes the characteristic function of the set occupied by the first phase, then it is natural to take the functional

$$J[u, \chi, h] \;:=\; \int_\Omega \Big(\chi\, f_h^+\big(\varepsilon(u)\big) + (1-\chi)\, f^-\big(\varepsilon(u)\big)\Big)\, dx \tag{1.4}$$

as the total deformation energy of the medium and to define an equilibrium state of J as a minimizing pair $(\hat{u}, \hat{\chi})$ consisting of a deformation $\hat{u}$ and a measurable characteristic function $\hat{\chi}$. Following standard convention we say that the equilibrium state is one–phase if $\hat{\chi} \equiv 0$ or $\hat{\chi} \equiv 1$, two–phase otherwise. Let us consider displacement fields u vanishing on $\partial\Omega$. Then the domain of definition of the functional $J[\cdot, \cdot, h]$ is the space of all pairs (u, χ) with $u \in X := \overset{\circ}{W}{}_2^1(\Omega; \mathbb{R}^d)$ (equipped with the norm $\|u\|_X := \|\varepsilon(u)\|_{L^2(\Omega;\mathbb{S}^d)}$) and χ denoting an arbitrary measurable characteristic function $\Omega \to \mathbb{R}$. Unfortunately, the variational problem $J[\cdot, \cdot, h] \to \min$ may fail to have solutions as it is shown by an example in [MO]. One way to overcome this difficulty is to introduce the quasiconvex envelope $\tilde{f}_h$ of the integrand $f_h := \min\{f_h^+, f^-\} \leq \chi\, f_h^+ + (1-\chi)\, f^-$ (see [DA] for a definition) and to pass to the relaxed problem

$$\int_\Omega \tilde{f}_h\big(\varepsilon(u)\big)\, dx \;\to\; \min \quad \text{in } X$$

(note that by Dacorogna's formula $u \equiv 0$ is a solution; nontrivial solutions were produced in [OS2]), we refer the reader to [DA], [KO], [SE] for a more detailed outline of this approach and for further references. From the physical point of view (compare [GR]) it is also reasonable to consider a regularization of the functional J from (1.4) taking the area of the separating surface between the different phases into account, i.e. we replace J by the energy

$$J[u, \chi, h, \sigma] \;=\; J[u, \chi, h] + \sigma \int_\Omega |\nabla \chi| \,, \tag{1.5}$$

where $\sigma > 0$ denotes a parameter, and the characteristic function χ is required to be an element of the space $BV(\Omega)$ of all functions having bounded variation (see for example [GIU] for definitions). This model was investigated in [OS1], [OS2], [BFO] establishing various existence results for the functional from (1.5), in particular, in the paper [BFO] we showed how the distribution of phases depends on the choices for the parameters h and σ.
In the present note we regularize $J[u, \chi, h]$ by adding a penalty term involving higher order derivatives of the displacement field. In principal, this model was proposed by Kohn and Müller in [KM] and [MU]. To be precise, suppose that a number $0 < p < 1$ is fixed, and let for $\sigma > 0$

$$I[u, \chi, h, \sigma] \;:=\; J[u, \chi, h] + \sigma \left(\int_\Omega |\Delta u|^2 \, dx \right)^{\frac{p}{2}} \,, \tag{1.6}$$

where now $u \in H := W_2^2(\Omega; \mathbb{R}^d) \cap X$ and (as in (1.4))

$$\chi \in M \;:=\; \big\{ \text{measurable characteristic functions } \Omega \to \mathbb{R} \big\} \,.$$

With a slight abuse of notation we sometimes only assume $\chi \in L^\infty(\Omega)$, $0 \le \chi \le 1$ a.e., equilibrium states of I however are always defined w.r.t. $H \times M$. Note, that on account of $\partial\Omega \in C^2$, the quantity

$$\|u\|_H \;:=\; \|\Delta u\|_{L^2(\Omega; \mathbb{R}^d)}$$

introduces a norm on the space H being equivalent to the W_2^2–norm which is a consequence of the Calderon–Zygmund regularity results. Our main result now concerns the analysis of the effect of the parameters $h \in \mathbb{R}$ and $\sigma > 0$ on the distribution of phases, we have

THEOREM 1.1 *Let (1.1)–(1.3) hold. Then, for each $h \in \mathbb{R}$ and all $\sigma > 0$, the functional $I[\cdot, \cdot, h, \sigma]$ attains its minimum on the set $H \times M$. There are two bounded, continuous functions $h^\pm(\sigma)$, $\sigma > 0$, and a number $\sigma^* > 0$ with the following properties*

$$h^+(\sigma) \;>\; \hat{h} \quad on \;\; (0,\sigma^*)\,, \qquad h^+(\sigma) \;\equiv\; \hat{h} \quad for \;\; \sigma \geq \sigma^*\,;$$

$$h^-(\sigma) \;<\; \hat{h} \quad on \;\; (0,\sigma^*)\,, \qquad h^-(\sigma) \;\equiv\; \hat{h} \quad for \;\; \sigma \geq \sigma^*\,;$$

$$\hat{h} \;:=\; \langle A^-\xi^-,\xi^-\rangle - \langle A^+\xi^+,\xi^+\rangle\,;$$

h^+ *strictly decreases on* $(0,\sigma^*)$, h^- *is strictly increasing on* $(0,\sigma^*)$.

The graphs of $h^\pm$ *divide the half–plane of parameters* $\sigma > 0$, $h \in \mathbb{R}$, *into three open regions*

$$A \;:=\; \big\{(\sigma,h)\,:\; \sigma > 0,\, h > h^+(\sigma)\big\}\,,$$

$$B \;:=\; \big\{(\sigma,h)\,:\; 0 < \sigma < \sigma^*,\, h^-(\sigma) < h < h^+(\sigma)\big\}\,,$$

$$C \;:=\; \big\{(\sigma,h)\,:\; \sigma > 0,\, h < h^-(\sigma)\big\}\,,$$

in which we have the following distribution of phases:

i) for $(\sigma,h) \in A$ *we only have the one–phase equilibrium* $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$;

ii) for $(\sigma,h) \in C$ *only the one–phase equilibrium* $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$ *exists;*

iii) for $(\sigma,h) \in B$ *only two–phase states of equilibria exist.*

On the graphs of $h^\pm$ *we have the following distribution of equilibrium states:*

iv) for $h = h^+(\sigma)$, $0 < \sigma < \sigma^*$, *we have the one–phase equilibrium state* $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ *and at least one two–phase equilibrium;*

v) for $h = h^-(\sigma)$, $0 < \sigma < \sigma^*$, *we have the one–phase equilibrium state* $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$ *and at least one two–phase equilibrium;*

vi) for $h = \hat{h}$, $\sigma > \sigma^*$, *the equilibrium states consist of the pairs* $\hat{u} \equiv 0$, $\hat{\chi} \equiv$ any measurable characteristic function;

vii) for $h = \hat{h}$, $\sigma = \sigma^*$ *there exist the equilibrium states* $\hat{u} \equiv 0$, $\hat{\chi} \equiv$ arbitary measurable characteristic function *and at least one two–phase equilibrium state with* $\hat{u} \not\equiv 0$.

REMARK 1.2 *a) Except for the behaviour at* $h = \hat{h}$ *together with* $\sigma \geq \sigma^*$ *(see vi) and vii)) Theorem 1.1 corresponds in a qualitative sense to Theorem 2.1 in [BFO]. Of course we do not claim that the functions* $h^{\pm}$ *as well as the numbers* σ^* *are the same in both cases.*

b) The different behaviour for the choice $h = \hat{h}$, $\sigma \geq \sigma^*$ *originates from the fact that in this case the penalty term* $\sigma \left(\int_\Omega |\Delta u|^2 \, dx \right)^{p/2}$ *does not create a formation of phases.*

c) In [OS2] the reader will find further comments on the above model, moreover, the choice $p < 1$ *is explained.*

Concerning the regularity of solutions we have the following

THEOREM 1.3 *With the above notation let* $(\hat{u}, \hat{\chi}) \in H \times M$ *denote an equilibrium state of* $I[\cdot, \cdot, h, \sigma]$, $\sigma > 0$. *Then* $\hat{u}$ *is of class* $C^{2,\alpha}(\Omega; \mathbb{R}^d)$ *for any* $0 < \alpha < 1$.

REMARK 1.4 *For* $h \in \mathbb{R}$, $\sigma > 0$ *and* $u \in H$ *let (recall* $f_h = \min\{f_h^+, f^-\}$*)*

$$\tilde{I}[u, h, \sigma] \;=\; \int_\Omega f_h\big(\varepsilon(u)\big) \, dx + \sigma \, \|\Delta u\|^p_{L^2(\Omega;\mathbb{R}^d)} \, .$$

Clearly, the variational problem

$$\tilde{I} \;\to\; \min \qquad on \;\; H$$

has at least one solution $\hat{u}$ *(compare also Lemma 2.2 and Theorem 2.3 below). For* $u \in H$ *let*

$$\chi_u \;:=\; \begin{cases} 0 & if \;\; f_h^+\big(\varepsilon(u)\big) \geq f^-\big(\varepsilon(u)\big) \, , \\ 1 & \quad otherwise \, . \end{cases}$$

Then we have

$$I[u, \chi, h, \sigma] \;\geq\; \tilde{I}[u, h, \sigma] \;\geq\; \tilde{I}[\hat{u}, h, \sigma] \;=\; I[\hat{u}, \chi_{\hat{u}}, h, \sigma]$$

for any $u \in H$ *and any measurable characteristic function* χ. *Thus* $\hat{u}$ *generates a minimizing pair* $(\hat{u}, \chi_{\hat{u}})$ *of* $I[\cdot, \cdot, h, \sigma]$. *Conversely, consider an equilibrium state* $(\check{u}, \check{\chi})$ *of* $I[\cdot, \cdot, h, \sigma]$. *Observing (recall* $f_h \leq \check{\chi} \, f_h^+ + (1 - \check{\chi}) \, f^-$*)*

$$\tilde{I}[\check{u}, h, \sigma] \;\leq\; I[\check{u}, \check{\chi}, h, \sigma] \;\leq\; I[u, \chi_u, h, \sigma] \;=\; \tilde{I}[u, h, \sigma] \quad for \; all \;\; u \in H$$

we deduce $\tilde{I}[\cdot,h,\sigma]$*–minimality of* $\check{u}$*. So there is a one–to–one correspondence between the minimizing deformation fields of both functionals. But the deformation field* u *alone does not serve the complete information, for example, in case* $u \equiv 0$ *there exist various possibilities for the distribution of phases as described in Theorem 1.1.*
As an alternative to the model proposed in Theorem 1.1 we may associate to each $\tilde{I}[\cdot,h,\sigma]$*–minimizing deformation field* $\hat{u}$ *the function* $\chi_{\hat{u}}$ *and introduce the notion of one (two)–phase equilibrium states* $(\hat{u},\chi_{\hat{u}})$ *as before. Then again we get the statements of Theorem 1.1 where in part vi) and vii) the phrase "*$\hat{\chi} =$ *any measurable characteristic function" has to be replaced by the requirement* $\hat{\chi} = \chi_0$*. Obviously the number of equilibrium states* $(\hat{u},\chi_{\hat{u}})$ *generated by* $\tilde{I}[\cdot,h,\sigma]$*–minimizers* $\hat{u}$ *is in general much smaller than the number of equilibria considered in the first model: if* $\hat{\chi}$ *is a measurable characteristic function satisfying*

$$\int_\Omega f_h\big(\varepsilon(\hat{u})\big)\,dx \;=\; \int_\Omega \Big(\hat{\chi}\, f_h^+\big(\varepsilon(\hat{u})\big) + (1-\hat{\chi})\, f^-\big(\varepsilon(\hat{u})\big)\Big)\,dx\,,$$

then $(\hat{u},\hat{\chi})$ *is a minimizing pair for* $I[\cdot,\cdot,h,\sigma]$*. But since we are mainly interested in the qualitative behaviour of the distribution of phases depending on* h *and* σ*, we do not see any principal difference between both models except for the different behaviour at* $h = \hat{h}$*,* $\sigma \geq \sigma^*$*.*

REMARK 1.5 *At the end, let us briefly discuss some situations for which the non–uniqueness w.r.t. the function* χ *can be removed. Let* $(\hat{u},\hat{\chi})$ *denote an equilibrium state of* $I[u,\chi,h,\sigma]$ *with* $\hat{\chi} := \chi_{\hat{u}}$*. We introduce the sets*

$$\begin{aligned} E^{+(-)} \;&:=\; \Big[f_h^+\big(\varepsilon(\hat{u})\big) \;>(<)\; f^-\big(\varepsilon(\hat{u})\big)\Big]\,,\\ E^0 \;&:=\; \Big[f_h^+\big(\varepsilon(\hat{u})\big) \;=\; f^-\big(\varepsilon(\hat{u})\big)\Big]\,, \end{aligned}$$

and consider $\chi \in L^\infty(\Omega)$*,* $0 \leq \chi \leq 1$*. Then*

$$I[\hat{u},\hat{\chi},h,\sigma] \;=\; I[\hat{u},\chi,h,\sigma] \tag{1.7}$$

if and only if

$$\int_{E^+} (\hat{\chi}-\chi)\Big(f_h^+\big(\varepsilon(\hat{u})\big) - f^-\big(\varepsilon(\hat{u})\big)\Big)\,dx \\ + \int_{E^-} (\hat{\chi}-\chi)\Big(f_h^+\big(\varepsilon(\hat{u})\big) - f^-\big(\varepsilon(\hat{u})\big)\Big)\,dx \;=\; 0\,.$$

Since $\hat\chi = \chi_{\hat u} = \begin{cases} 0 & on \;\; E^+ \\ 1 & on \;\; E^- \end{cases}$ *, we see*

$$\chi \; = \; \hat\chi \quad on \;\; E^+ \cup E^- \,, \tag{1.8}$$

and the "non–uniqueness" can be excluded for the case that E_0 *is a set of Lebesgue measure zero. In order to find a sufficient condition for* $|E_0| = 0$ *let us assume that* $\hat u \not\equiv 0$*. Then* $\|\Delta \hat u\|_{L^2(\Omega;\mathbb{R}^d)} > 0$ *and for any* $v \in H$ *the expression* $\|\Delta \hat u + t\Delta v\|_{L^2(\Omega;\mathbb{R}^d)} > 0$ *is differentiable at* $t = 0$*. For* $\chi \in L^\infty(\Omega)$*,* $0 \le \chi \le 1$*, with (1.8) and all* $v \in H$ *we have according to (1.7)*

$$\frac{d}{dt}_{|t=0} I[\hat u + tv, \chi, h, \sigma] \; = \; 0 \,, \quad i.e.$$

$$\begin{aligned} & 2\int_\Omega \chi \Big\langle A^+\big(\varepsilon(\hat u) - \xi^+\big) - A^-\big(\varepsilon(\hat u) - \xi^-\big), \varepsilon(v)\Big\rangle \, dx \\ & +2\int_\Omega \big\langle A^-\varepsilon(v), \varepsilon(\hat u) - \xi^-\big\rangle \, dx + p\sigma \left(\int_\Omega |\Delta \hat u|^2\right)^{\frac{p}{2}-1} \int_\Omega \Delta \hat u \cdot \Delta v \, dx \\ & = \; 0 \,. \end{aligned} \tag{1.9}$$

Let $|E_0| > 0$*. Then we use (1.9) with* $\chi = 0$ *on* E_0 *and with* $\chi = \Phi$ *on* E_0*, where* $\Phi \in L^\infty(E_0)$*,* $0 \le \Phi \le 1$*. Subtracting the results we get*

$$\int_{E_0} \Phi \Big\langle A^+\big(\varepsilon(\hat u) - \xi^+\big) - A^-\big(\varepsilon(\hat u) - \xi^-\big), \varepsilon(v)\Big\rangle \, dx \; = \; 0 \,,$$

and since Φ *can be chosen arbitrarily, this turns into*

$$\Big\langle A^+\big(\varepsilon(\hat u) - \xi^+\big) - A^-\big(\varepsilon(\hat u) - \xi^-\big), \varepsilon(v)\Big\rangle \; = \; 0$$

a.e. on E_0*. Consider a Lebesgue point* $x_0 \in E_0$ *of* $\varepsilon(\hat u)$ *and let* $v(x) = \eta(x)\, x_k\, E^l$ *where* $\eta \in C_0^\infty(\Omega)$*,* $\eta \equiv 1$ *near* x_0*, and* E^l *is the* l^{th} *standard unit–vector in* $\mathbb{R}^d$*. Then* $\varepsilon(v)(x_0) = \big(\delta_{ik}\,\delta^{jl}\big)_{1\le i,j\le d}$ *and the above identity implies*

$$A^+\big(\varepsilon(\hat u) - \xi^+\big) - A^-\big(\varepsilon(\hat u) - \xi^-\big) \; = \; 0$$

on E_0*, hence*

$$(A^+ - A^-)\,\varepsilon(\hat u) \; = \; A^+\xi^+ - A^-\xi^- \,,$$

and we get a contradiction if we assume that

$$A^+\xi^+ - A^-\xi^- \notin \mathrm{Im}\,(A^+ - A^-) \tag{1.10}$$

holds. For example we have (1.10) in case $A^+ = A^-$ together with $\xi^+ \neq \xi^-$. Thus the assumption $\hat{u} \not\equiv 0$ combined with (1.10) shows $|E_0| = 0$ and we can associate to $\hat{u}$ a unique function χ such that (1.7) is valid.

Our paper is organized as follows: in Section 2 we prove some existence and lower semicontinuity results concerning the functional I from (1.6). Section 3 contains a series of lemmata which are used in Section 4 and Section 5 to prove satement i)–vii) of Theorem 1.1. In a last section we prove Theorem 1.3.

2 Some existence results

From now on we assume that all the conditions stated in Section 1 are valid.

LEMMA 2.1 *Let $h \in \mathbb{R}$, $\sigma \geq 0$ be given. Then we have for any $(u, \chi) \in H \times M$*

$$\frac{\nu}{2}\,\|u\|_X^2 + \sigma\,\|u\|_H^p \;\leq\; I[u,\chi,h,\sigma] + h\,|\Omega| + \frac{4+\nu^2}{\nu^3}\big(|\xi^+|^2 + |\xi^-|^2\big)\,.$$

Proof. Assumption (1.1) implies

$$\begin{aligned} I[u,\chi,h,\sigma] \;\geq\; & \nu \int_\Omega |\varepsilon(u)|^2\,dx - |h|\,|\Omega| + \sigma\,\|u\|_H^p \\ & -\frac{1}{\nu}\int_\Omega \big(|\xi^+|^2 + |\xi^-|^2\big)\,dx \\ & -2\int_\Omega \Big(\big|\langle A^+\varepsilon(u),\xi^+\rangle\big| + \big|\langle A^-\varepsilon(u),\xi^-\rangle\big|\Big)\,dx\,. \end{aligned}$$

The lemma is proved by combining this inequality with
$|\langle A^\pm\varepsilon,\tilde{\varepsilon}\rangle| \leq \sqrt{\langle A^\pm\varepsilon,\varepsilon\rangle}\,\sqrt{\langle A^\pm\tilde{\varepsilon},\tilde{\varepsilon}\rangle}$. ■

Next we establish a lower semicontinuity result

LEMMA 2.2 *Consider sequences* $\{u_n\}$, $\{\chi_n\}$, $\{h_n\}$ *and* $\{\sigma_n\}$, $u_n \in H$, $\chi_n \in L^\infty(\Omega)$, $0 \le \chi_n \le 1$, $h_n \in \mathbb{R}$, $\sigma_n \ge 0$ *such that* $u_n \rightharpoonup u$ *in* H, $\chi_n \rightharpoonup \chi$ *in* $L^2(\Omega)$, $h_n \to h$ *and* $\sigma_n \to \sigma$ *as* $n \to \infty$. *Then we have*

$$I[u,\chi,h,\sigma] \;\le\; \liminf_{n\to\infty} I[u_n,\chi_n,h_n,\sigma_n]\,.$$

Proof. The uniform L^∞–bound together with the weak L^2–convergence of the sequence $\{\chi_n\}$ yields

$$\chi_n \overset{n\to\infty}{\rightharpoonup} \chi \;\text{ in }\; L^s(\Omega) \;\text{ for any }\; s<\infty\,, \qquad 0\le\chi\le 1 \;\text{ a.e.}$$

The weak H–convergence of the sequence $\{u_n\}$ gives in addition

$$\varepsilon(u_n) \overset{n\to\infty}{\rightharpoonup} \varepsilon(u) \;\text{ in }\; L^r(\Omega;\mathbb{S}^d) \quad \text{for some }\; r>2\,,$$

thus

$$I[u_n,\chi_n,h_n,0] \;\to\; I[u,\chi,h,0] \quad \text{as }\; n\to\infty\,.$$

Moreover, again by weak convergence of the sequence $\{u_n\}$,

$$\|u\|_H^p \;\le\; \liminf_{n\to\infty} \|u_n\|_H^p\,,$$

i.e. we get the estimate

$$\begin{aligned} I[u,\chi,h,\sigma] &= I[u,\chi,h,0] + \sigma\,\|u\|_H^p \\ &\le \liminf_{n\to\infty} I[u_n,\chi_n,h_n,0] + \liminf_{n\to\infty}\left(\sigma_n\,\|u_n\|_H^p\right) \\ &\le \liminf_{n\to\infty}\left(I[u_n,\chi_n,h_n,0] + \sigma_n\,\|u_n\|_H^p\right) \\ &= \liminf_{n\to\infty} I[u_n,\chi_n,h_n,\sigma_n]\,. \end{aligned}$$

As a consequence we obtain the following existence theorem ■

THEOREM 2.3 *The functional* $I[\cdot,\cdot,h,\sigma]$, $h\in\mathbb{R}$, $\sigma>0$, *attains its minimum on the set* $H\times M$.

Proof. Lemma 2.1 immediately gives

$$\gamma \;:=\; \inf_{(u,\chi)\in H\times M} I[u,\chi,h,\sigma] \;>\; -\infty\,,$$

and we may consider a minimizing sequence (u_n, χ_n) s.t. (again recall Lemma 2.1)

$$u_n \;\rightharpoonup:\; \hat{u} \text{ in } H\,, \qquad \chi_n \;\rightharpoonup:\; \tilde{\chi} \text{ in } L^2(\Omega) \qquad \text{as } n \to \infty\,.$$

We do not know that $\tilde{\chi}$ is an element of M, however $0 \le \tilde{\chi} \le 1$ and, by Lemma 2.2,

$$I[\hat{u}, \tilde{\chi}, h, \sigma] \;\le\; \liminf_{n\to\infty} I[u_n, \chi_n, h, \sigma]\,. \tag{2.1}$$

Therefore, if $\hat{\chi}$ is defined via

$$\hat{\chi} \;:=\; \begin{cases} 0 & \text{on the set } \left[f_h^+\big(\varepsilon(\hat{u})\big) \ge f^-\big(\varepsilon(\hat{u})\big)\right], \\ 1 & \text{on the set } \left[f_h^+\big(\varepsilon(\hat{u})\big) < f^-\big(\varepsilon(\hat{u})\big)\right], \end{cases}$$

and if we observe (2.1) together with

$$\begin{aligned} \tilde{\chi}\, f_h^+\big(\varepsilon(\hat{u})\big) + (1-\tilde{\chi})\, f^-\big(\varepsilon(\hat{u})\big) &= \tilde{\chi}\Big(f_h^+\big(\varepsilon(\hat{u})\big) - f^-\big(\varepsilon(\hat{u})\big)\Big) + f^-\big(\varepsilon(\hat{u})\big) \\ &\ge \hat{\chi}\Big(f_h^+\big(\varepsilon(\hat{u})\big) - f^-\big(\varepsilon(\hat{u})\big)\Big) + f^-\big(\varepsilon(\hat{u})\big)\,, \end{aligned}$$

$(\hat{u}, \hat{\chi}) \in H \times M$ is seen to be an equilibrium state of I. ■

Next, consider the energies of one–phase deformations, i.e. we let

$$\begin{aligned} I^+[u, h, \sigma] \;&:=\; I[u, 1, h, \sigma] \;=\; \int_\Omega f_h^+\big(\varepsilon(u)\big)\, dx + \sigma\, \|u\|_H^p\,, \\ I^-[u, \sigma] \;&:=\; I[u, 0, h, \sigma] \;=\; \int_\Omega f^-\big(\varepsilon(u)\big)\, dx + \sigma\, \|u\|_H^p\,, \qquad u \in H\,. \end{aligned}$$

LEMMA 2.4 *On H the functionals $I^\pm$ attain their unique minima at $u^\pm \equiv 0$.*

Proof. For any $u \in H$ we have

$$\begin{aligned} I^+[u, h, \sigma] &= \int_\Omega \Big[\big\langle A^+\big(\varepsilon(u) - \xi^+\big), \varepsilon(u) - \xi^+\big\rangle + h\Big]\, dx + \sigma\, \|u\|_H^p \\ &= \int_\Omega \big\langle A^+\varepsilon(u), \varepsilon(u)\big\rangle\, dx + |\Omega|\, \langle A^+\xi^+, \xi^+\rangle + h\,|\Omega| + \sigma\, \|u\|_H^p \\ &\ge |\Omega|\, \langle A^+\xi^+, \xi^+\rangle + h\,|\Omega|\,, \end{aligned}$$

where equality holds if and only if $u \equiv 0$. An analogous inequality is true for I^- and the lemma is proved. ■

We finish this section by introducing the quantity $I_0(h) := \min\{I^+[0,h,\sigma], I^-[0,\sigma]\}$, i.e.

$$
\begin{aligned}
I_0(h) &= \begin{cases} |\Omega|\left(\langle A^+\xi^+,\xi^+\rangle + h\right), & h \le \hat{h}\,, \\ |\Omega|\,\langle A^-\xi^-,\xi^-\rangle\,, & h \ge \hat{h}\,, \end{cases} \\
\hat{h} &:= \langle A^-\xi^-,\xi^-\rangle - \langle A^+\xi^+,\xi^+\rangle\,,
\end{aligned}
$$

which measures the dependence of the energy of one–phase equilibria on the parameter h.

3 Auxiliary results

In this section we prove (under the hypotheses stated in Section 1) a series of auxiliary results which are needed in Section 4 to show Theorem 1.1. We start with two lemmata estimating the X–norm of equilibrium states.

LEMMA 3.1 *Consider an equilibrium state* $(\hat{u},\hat{\chi})$ *of* $I[\cdot,\cdot,h,\sigma]$*. Then*

$$
\sigma\,\|\hat{u}\|_H^p + (\nu-\mu)\,\|\hat{u}\|_X^2 \;\le\; 2\,|A^-\xi^- - A^+\xi^+|\,\sqrt{|\Omega|}\,\|\hat{u}\|_X \tag{3.1}
$$

holds true, in particular, there is a constant R, not depending on h, σ, such that

$$
\|\hat{u}\|_X \;=\; \|\varepsilon(\hat{u})\|_{L^2(\Omega;\mathbb{S}^d)} \;\le\; R\,. \tag{3.2}
$$

Proof. The minimizing property yields $I[\hat{u},\hat{\chi},h,\sigma] \le I[0,\hat{\chi},h,\sigma]$, i.e.

$$
\begin{aligned}
\sigma\,\|\hat{u}\|_H^p + \int_\Omega \hat{\chi}\left\langle (A^+ - A^-)\,\varepsilon(\hat{u}),\varepsilon(\hat{u})\right\rangle dx + \int_\Omega \langle A^-\varepsilon(\hat{u}),\varepsilon(\hat{u})\rangle\,dx& \\
+2\int_\Omega \hat{\chi}\,\langle \varepsilon(\hat{u}), A^-\xi^- - A^+\xi^+\rangle\,dx \;\le\; 0\,,&
\end{aligned}
$$

thus the assertions follow from (1.1)–(1.3). ■

LEMMA 3.2 *There is a real number $\delta > 0$ such that we have for any equilibrium state $(\hat{u}, \hat{\chi})$ of $I[\cdot,\cdot,h,\sigma]$, $\hat{u} \not\equiv 0$,*

$$\|\hat{u}\|_X^{1-p} \;\geq\; \delta\,\sigma\,. \tag{3.3}$$

Proof. From the Calderon–Zygmund regularity results (compare, for example [GT], Theorem 9.14 and 9.15) we deduce the existence of a positve number $\kappa = \kappa(\Omega, d)$ such that

$$\|\hat{u}\|_X \;=\; \|\varepsilon(\hat{u})\|_{L^2(\Omega;\mathbb{S}^d)} \;\leq\; \|\hat{u}\|_{W_2^2(\Omega;\mathbb{R}^d)} \;\leq\; \kappa\,\|\Delta\hat{u}\|_{L^2(\Omega;\mathbb{R}^d)} \;=\; \kappa\,\|\hat{u}\|_H\,.$$

(3.1) gives

$$\begin{aligned}\sigma\,\|\hat{u}\|_H^p \;&\leq\; 2\,|A^+\xi^+ - A^-\xi^-|\,\sqrt{|\Omega|}\,\|\hat{u}\|_X \\ &\leq\; 2\,|A^+\xi^+ - A^-\xi^-|\,\sqrt{|\Omega|}\,\|\hat{u}\|_X^{1-p}\,\kappa^p\,\|\hat{u}\|_H^p\,,\end{aligned}$$

implying Lemma 3.2 since

$$\|\hat{u}\|_X^{1-p} \;\geq\; \sigma\,\frac{1}{2\,|A^+\xi^+ - A^-\xi^-|\,\sqrt{|\Omega|}\,\kappa^p} \;=:\; \sigma\,\delta\,.$$

■

In the next lemma we investigate the relation between one–phase equilibrium states and the vanishing of the associated deformation field.

LEMMA 3.3 *Consider an equilibrium state $(\hat{u}, \hat{\chi})$ of $I[\cdot,\cdot,h,\sigma]$. Then:*

a) if $(\hat{u}, \hat{\chi})$ is one–phase, i.e. $\hat{\chi} \equiv 0$ or $\hat{\chi} \equiv 1$, then $\hat{u} \equiv 0$;

b) if $h \neq \hat{h}$ and if $\hat{u} \equiv 0$, then $(\hat{u}, \hat{\chi})$ is a one–phase equilibrium;

c) if $h = \hat{h}$ and if $\hat{u} \equiv 0$, then any $\chi \in M$ provides an equilibrium state $(0, \chi)$.

Proof. Assume that $\hat{\chi} \equiv 1$ ($\hat{\chi} \equiv 0$), thus $I[\cdot, \hat{\chi}, h, \sigma] = I^+[\cdot, h, \sigma]$ $(= I^-[\cdot, \sigma])$, hence by Lemma 2.4 $\hat{u} \equiv 0$ and a) is verified. Next observe that for any $\chi \in M$

$$\begin{aligned}I[0,\chi,h,\sigma] \;&=\; \left[\langle A^+\xi^+, \xi^+\rangle - \langle A^-\xi^-, \xi^-\rangle + h\right]\int_\Omega \chi\,dx + |\Omega|\,\langle A^-\xi^-, \xi^-\rangle \\ &=\; (h - \hat{h})\int_\Omega \chi\,dx + |\Omega|\,\langle A^-\xi^-, \xi^-\rangle\,.\end{aligned}$$

In the case $h > \hat{h}$ it is seen that

$$I[0,\chi,h,\sigma] \;\geq\; |\Omega|\,\langle A^-\xi^-,\xi^-\rangle\,,$$

and equality is true if and only if $\chi \equiv 0$. This proves part b) for $h > \hat{h}$, the case $h < \hat{h}$ is treated in the same manner. Finally $h = \hat{h}$ implies $I[0,\chi,h,\sigma] = |\Omega|\,\langle A^-\xi^-,\xi^-\rangle$ for any $\chi \in M$, thus we have c). ■

As a next step we ensure that the existence of one–phase (two–phase) equilibria depends continuously on h and σ.

LEMMA 3.4 *Given two sequences $\{h_n\}$, $\{\sigma_n\}$ assume that $h_n \to h_0$ and $\sigma_n \to \sigma_0 > 0$ as $n \to \infty$. As usual denote by $(\hat{u}_n, \hat{\chi}_n)$, $(\hat{u}_0, \hat{\chi}_0)$ equilibrium states of $I[\cdot,\cdot,h_n,\sigma_n]$ and $I[\cdot,\cdot,h_0,\sigma_0]$, respectively.*

a) If $\hat{u}_n \equiv 0$ ($\hat{u}_n \not\equiv 0$) at least for a subsequence, then there exists an equilibrium state $(\hat{u}_0, \hat{\chi}_0)$ satisfying $\hat{u}_0 \equiv 0$ ($\hat{u}_0 \not\equiv 0$).

b) If $\hat{\chi}_n \equiv 0$ ($\hat{\chi}_n \equiv 1$) for a subsequence, then $I[\cdot,\cdot,h_0,\sigma_0]$ admits an equilibrium state satisfying $\hat{u}_0 \equiv 0$, $\hat{\chi}_0 \equiv 0$ ($\hat{\chi}_0 \equiv 1$).

c) If $h_0 \neq \hat{h}$ and if $0 \not\equiv \hat{\chi}_n \not\equiv 1$, again at least for a subsequence, then there is a solution with $0 \not\equiv \hat{\chi}_0 \not\equiv 1$.

Proof. From Lemma 2.1 we deduce

$$\begin{aligned}
\frac{\nu}{2}\,\|\hat{u}_n\|_X^2 + \sigma_n\,\|\hat{u}_n\|_H^p \;&\leq\; I[\hat{u}_n,\hat{\chi}_n,h_n,\sigma_n] + h_n\,|\Omega| + \frac{4+\nu^2}{\nu^3}\big(|\xi^+|^2 + |\xi^-|^2\big)\\
&\leq\; I[0,0,h_n,\sigma_n] + h_n\,|\Omega| + \frac{4+\nu^2}{\nu^3}\big(|\xi^+|^2 + |\xi^-|^2\big)\,,
\end{aligned}$$

hence (recall that $\sigma_0 > 0$) there is a real number $c > 0$ such that $\|\hat{u}_n\|_H \leq c < +\infty$. Passing to a subsequence (not relabelled) we may assume that

$$\hat{u}_n \;\rightharpoonup:\; \hat{u}_0 \quad \text{in } H \text{ as } n \to \infty\,,$$

Sobolev's embedding theorem then gives the existence of a real number $r > 1$ such that

$$\hat{u}_n \;\to\; \hat{u}_0 \quad \text{in } W^1_{2r}\big(\Omega;\mathbb{R}^d\big) \text{ as } n \to \infty\,.$$

Moreover, we may assume (again passing to a subsequence if necessary) that

$$\hat{\chi}_n \overset{n\to\infty}{\rightharpoondown} \tilde{\chi}_0 \quad \text{in } L^2(\Omega)\,, \qquad 0 \le \tilde{\chi}_0 \le 1 \text{ a.e.}\,,$$

and applying Lemma 2.2 we see for all $(u,\chi) \in H \times M$

$$\begin{aligned} I[\hat{u}_0, \tilde{\chi}_0, h_0, \sigma_0] \;&\le\; \liminf_{n\to\infty} I[\hat{u}_n, \hat{\chi}_n, h_n, \sigma_n] \;\le\; \liminf_{n\to\infty} I[u, \chi, h_n, \sigma_n] \\ &=\; I[u, \chi, h_0, \sigma_0]\,. \end{aligned}$$

As done in the proof of Theorem 2.3 (compare also Remark 1.4 and Remark 1.5) we may replace $\tilde{\chi}_0$ by a characteristic function $\hat{\chi}_0 \in M$, which provides an admissible minimizer $(\hat{u}_0, \hat{\chi}_0)$ of $I[\cdot,\cdot,h_0,\sigma_0]$.

ad a) If $\hat{u}_n = 0$ for a subsequence, then by the above arguments we clearly may take $\hat{u}_0 \equiv 0$. If $\hat{u}_n \not\equiv 0$ for a subsequence, Lemma 3.2 gives $\|\hat{u}_n\|_X^{1-p} \ge \delta\,\sigma_n$, hence strong convergence in $W^1_{2r}(\Omega;\mathbb{R}^d)$ proves $\|\hat{u}_0\|_X^{1-p} \ge \delta\,\sigma_0$, i.e. $\hat{u}_0 \not\equiv 0$.

ad b) The case $\hat{\chi}_n \equiv 0$ for a subsequence shows (with the above notation) $\tilde{\chi}_0 \equiv 0$ and $(\hat{u}_0, 0)$ is seen to be minimizing. The first assertion of Lemma 3.3 ensures the statement $\hat{u}_0 \equiv 0$. The case $\hat{\chi}_n \equiv 1$ is covered by the same arguments.

ad c) We may assume that $h_n \neq \hat{h}$ for all n sufficiently large. Moreover, by Lemma 3.3 b) we then observe that $\hat{u}_n \not\equiv 0$, in conclusion Lemma 3.2 gives $\|\hat{u}_n\|_X^{1-p} \ge \delta\,\sigma_n$ and therefore the limit $\hat{u}_0$ does not vanish. The claim now follows from Lemma 3.3 a). ■

The volume of the phases depends in a monotonic manner on the parameter h, more precisely

LEMMA 3.5 *Denote by $(\hat{u}_i, \hat{\chi}_i)$ equilibrium states of $I[\cdot,\cdot,h_i,\sigma]$, $i = 1,2$. Then we have*

$$(h_1 - h_2)\Big(\|\hat{\chi}_1\|_{L^1(\Omega)} - \|\hat{\chi}_2\|_{L^1(\Omega)}\Big) \;\le\; 0\,.$$

Proof. The proof is an imediate consequence of

$$\begin{aligned} I[\hat{u}_1, \hat{\chi}_1, h_1, \sigma] \;&\le I[\hat{u}_2, \hat{\chi}_2, h_1, \sigma]\,, \\ I[\hat{u}_2, \hat{\chi}_2, h_2, \sigma] \;&\le I[\hat{u}_1, \hat{\chi}_1, h_2, \sigma]\,. \end{aligned}$$

■

REMARK 3.6 *If there exists an equilibrium state* $(\hat{u}_0, \hat{\chi}_0)$ *of* $I[\cdot,\cdot,h_0,\sigma]$ *satisfying* $\hat{\chi}_0 \equiv 0$ $(\hat{\chi}_0 \equiv 1)$*, then by Lemma 3.5 for* $h > h_0$ $(h < h_0)$ *any equilibrium state* $(\hat{u}, \hat{\chi})$ *of* $I[\cdot,\cdot,h,\sigma]$ *is one–phase, i.e.* $\hat{\chi} \equiv 0$ $(\hat{\chi} \equiv 1)$*.*

If we want two–phase equilibria to exist, then we have to restrict the admissible values for the parameters h and σ. A precise formulation is given in the next two lemmata.

LEMMA 3.7 *There is a real number* $h_0 > 0$ *with the following property: for any* $h > h_0$ $(h < -h_0)$*, for all* $\sigma > 0$ *and for any equilibrium state* $(\hat{u}, \hat{\chi})$ *of* $I[\cdot,\cdot,h,\sigma]$ *we have* $\hat{u} \equiv 0$ *and* $\hat{\chi} \equiv 0$ $(\hat{\chi} \equiv 1)$*.*

Proof. The idea is to find a real number $h_0 > 0$ such that for any $\sigma > 0$ and for any $(u, \chi) \in H \times M$

$$I[u,\chi,h_0,\sigma] \;\geq\; I[0,0,h_0,\sigma]\,. \tag{3.4}$$

Once (3.4) is established, $(0,0)$ is seen to be an equilibrium state of $I[\cdot,\cdot,h_0,\sigma]$ and the first assertion follows from Remark 3.6. The case $h < -h_0$ is treated in the same manner, where we have to increase h_0 if necessary. Thus, it remains to show (3.4) which is equivalent to

(3.5)

$$\int_\Omega \chi \Big[\big\langle (A^+ - A^-)\,\varepsilon(u), \varepsilon(u)\big\rangle - 2\big\langle A^+\xi^+ - A^-\xi^-, \varepsilon(u)\big\rangle + \big\langle A^+\xi^+, \xi^+\big\rangle$$
$$-\big\langle A^-\xi^-, \xi^-\big\rangle + h_0\Big]\,dx + \int_\Omega \big\langle A^-\varepsilon(u), \varepsilon(u)\big\rangle\,dx + \sigma\,\|u\|_H^p \;\geq\; 0\,.$$

We may estimate $(0 < \lambda < 1)$

$$\big\langle A^-\varepsilon(u), \varepsilon(u)\big\rangle + \chi\big\langle (A^+ - A^-)\,\varepsilon(u), \varepsilon(u)\big\rangle$$
$$\geq\; \big\langle A^-\varepsilon(u), \varepsilon(u)\big\rangle - \big|\big\langle (A^+ - A^-)\,\varepsilon(u), \varepsilon(u)\big\rangle\big|\,,$$
$$2\,\big|\big\langle A^\pm\xi^\pm, \varepsilon(u)\big\rangle\big| \;\leq\; \lambda\big\langle A^\pm\varepsilon(u), \varepsilon(u)\big\rangle + \frac{1}{\lambda}\big\langle A^\pm\xi^\pm, \xi^\pm\big\rangle\,,$$

thus (3.5) is implied by

$$
\begin{aligned}
&\int_\Omega \Big[\big\langle A^- \varepsilon(u), \varepsilon(u)\big\rangle - \big|\big\langle (A^+ - A^-)\,\varepsilon(u), \varepsilon(u)\big\rangle\big| \\
&\quad -\lambda \big\langle (A^+ + A^-)\,\varepsilon(u), \varepsilon(u)\big\rangle\Big]\, dx \\
&\quad + \int_\Omega \chi \Big[h_0 + \Big(1 - \frac{1}{\lambda}\Big)\big\langle A^+ \xi^+, \xi^+\big\rangle - \Big(\frac{1}{\lambda} + 1\Big)\big\langle A^- \xi^-, \xi^-\big\rangle\Big]\, dx \\
&\quad \geq \; 0\,.
\end{aligned} \tag{3.6}
$$

By (1.1) and (1.2) the first integral on the left–hand side of (3.6) is greater than or equal to

$$
\big(\nu - \mu - 2\,\lambda\,\nu^{-1}\big)\,\|u\|_X^2\,,
$$

hence positive if we choose λ sufficiently small. Increasing λ, if necessary, we finally let

$$
h_0 \;:=\; \Big(1 + \frac{1}{\lambda}\Big)\big\langle A^- \xi^-, \xi^-\big\rangle - \Big(1 - \frac{1}{\lambda}\Big)\big\langle A^+ \xi^+, \xi^+\big\rangle \;>\; 0\,.
$$

With this choice (3.6), hence (3.5), and in conclusion the lemma is valid. ■

Except for $h \neq \hat{h}$ the existence of two–phase equilibria requires also the boundedness of σ:

LEMMA 3.8 *There exits a real number $\sigma_0 > 0$ with the following property: for any $\sigma > \sigma_0$ and for any $h \in \mathbb{R}$ the functional $I[\cdot, \cdot, h, \sigma]$ admits only equilibria $(\hat{u}, \hat{\chi})$ satisfying $\hat{u} \equiv 0$.*

Proof. Recalling (3.1) and (3.2) one gets

$$
\begin{aligned}
\sigma\,\|\hat{u}\|_H^p \;&\leq\; 2\,|A^+\xi^+ - A^-\xi^-|\,|\Omega|^{\frac{1}{2}}\,R\,, \\
\text{i.e.} \qquad \sigma\,\|\hat{u}\|_X^p \;&\leq\; 2\,|A^+\xi^+ - A^-\xi^-|\,|\Omega|^{\frac{1}{2}}\,R\,\kappa^p\,,
\end{aligned}
$$

hence we may estimate

$$
\sigma^{\frac{1-p}{p}}\,\|\hat{u}\|_X^{1-p} \;\leq\; R' \;:=\; \Big(2\,|A^+\xi^+ - A^-\xi^-|\,|\Omega|^{\frac{1}{2}}\,R\,\kappa^p\Big)^{\frac{1-p}{p}}\,.
$$

If $\hat{u} \not\equiv 0$ is supposed, then (3.3) gives

$$\sigma^{\frac{1-p}{p}} \delta \sigma \;\leq\; R' \qquad \Leftrightarrow \qquad \sigma \;\leq\; (R'/\delta)^p \,,$$

thus the lemma is proved by letting $\sigma_0 := (R'/\delta)^p$. ■

As a last auxiliary result on the distribution of phases, a sufficient condition for the existence of two phase equilibria is given.

LEMMA 3.9 *If $\sigma > 0$ is sufficiently small, then $I[\cdot, \cdot, \hat{h}, \sigma]$ admits only equilibria $(\hat{u}, \hat{\chi})$ satisfying $\hat{u} \not\equiv 0$.*

Proof. Suppose by contradiction that there is a sequence $\{\sigma_n\}$ of positive real numbers, $\sigma_n \downarrow 0$ as $n \to \infty$, such that $I[\cdot, \cdot, \hat{h}, \sigma_n]$ admits a one–phase equilibrium state, i.e. $\hat{\chi}_n \equiv 0$ or $\hat{\chi}_n \equiv 1$ and, by Lemma 3.3, $\hat{u}_n \equiv 0$. Minimality implies for any $(u, \chi) \in H \times M$

$$I[u, \chi, \hat{h}, \sigma_n] \;\geq\; I[0, \hat{\chi}_n, \hat{h}, \sigma_n] \;=\; |\Omega| \, \langle A^- \xi^-, \xi^- \rangle \,.$$

Using the definition of $\hat{h}$ this can be rewritten as

$$\int_\Omega \chi \Big[\big\langle (A^+ - A^-) \, \varepsilon(u), \varepsilon(u) \big\rangle - 2 \big\langle \varepsilon(u), A^+ \xi^+ - A^- \xi^- \big\rangle \Big] \, dx$$
$$+ \int_\Omega \big\langle A^- \varepsilon(u), \varepsilon(u) \big\rangle \, dx + \sigma_n \, \|u\|_H^p \;\geq\; 0 \quad \text{for any} \;\; (u, \chi) \in H \times M \,.$$

If we replace u by $\sigma_n u$, divide through σ_n and pass to the limit $n \to \infty$, we get

$$- \int_\Omega \chi \, \big\langle \varepsilon(u), A^+ \xi^+ - A^- \xi^- \big\rangle \, dx \;\geq\; 0 \quad \text{for any} \;\; (u, \chi) \in H \times M \,.$$

In fact, equality is true since we may consider $-u$ instead of u. Let $\gamma = A^- \, \xi^- - A^+ \, \xi^+$, fix $x_0 \in \Omega$ and consider $\rho > 0$ such that $B_{2\rho}(x_0) \Subset \Omega$. Finally we choose $\chi = \mathbf{1}_{B_\rho(x_0)}$, $\varphi \in C_0^\infty(\Omega)$, $\varphi \equiv 1$ on $B_{2\rho}(x_0)$ and let $v_k(x) = e \, \varphi(x) \, x_k$ with $1 \leq k \leq d$, $e \in \mathbb{R}^d$. This choice implies on $B_{2\rho}(x_0)$

$$\varepsilon(v_k) \;=\; \frac{1}{2} \big(e^i \, \delta_{jk} + e^j \, \delta_{ik} \big)_{1 \leq i,j \leq d} \,,$$

hence we get

$$0 \;=\; \int_\Omega \chi\, dx\, \frac{1}{2}\left(\gamma_{ij}\, e^i\, \delta_{jk} + \gamma_{i,j}\, e^j\, \delta_{ik}\right) \;=\; |B_\rho(x_0)|\, \left(\gamma\, e\right)_k .$$

This gives the contradiction $\gamma = 0$ and the lemma is proved. ■

We finish this section with the following

LEMMA 3.10 *For any $h \in \mathbb{R}$ and for any real number $\sigma > 0$ we let*

$$I_1(\sigma, h) \;:=\; \inf_{(u,\chi)\in H\times M} I[u, \chi, h, \sigma] .$$

Then $I_1(\sigma, h)$ is a concave function, in particular, $I_1(\sigma, h)$ is continuous.

Proof. Note that for h and σ as above $I_1(\sigma, h)$ is well defined. Moreover, for any fixed $(u, \chi) \in H \times M$ the mapping $(h, \sigma) \mapsto I[u, \chi, h, \sigma]$ is a linear function in h and σ, hence concave. Since the infimum of a family of concave functions again is concave, the lemma is seen to be valid. ■

4 Proof of Theorem 1.1, i)–iii)

Step1. (Definition of the set B)
Note that by construction we have

$$I_1(\sigma, h) \;\le\; I_0(h) \quad \text{for any } \; h \in \mathbb{R}, \;\; \sigma > 0 . \tag{4.1}$$

Inequality (4.1) leads to the definition

$$B \;:=\; \left\{(\sigma, h) \in \mathbb{R}^+ \times \mathbb{R} :\; I_1(\sigma, h) < I_0(h)\right\} ,$$

and we observe that

$$(\sigma_0, h_0) \in B \;\;\Leftrightarrow\;\; I[\cdot, \cdot, h_0, \sigma_0] \;\text{ admits only two–phase equilibria } (\hat{u}, \hat{\chi}) .$$

By Lemma 3.9, B is known to be non–empty, moreover, B is seen to be open on account of $B = (I_0 - I_1)^{-1}(0, \infty)$ and the continuity of I_0, I_1. Finally, Lemma 3.7 and Lemma 3.8 prove B to be bounded. Given $\sigma_0 > 0$ let

$$L(\sigma_0) \;:=\; \left\{h \in \mathbb{R} :\; (\sigma_0, h) \in B\right\} .$$

LEMMA 4.1 *Either we have $L(\sigma_0) = \emptyset$ or there exist two uniquely defined real numbers $h^{\pm}(\sigma_0)$, $h^-(\sigma_0) < \hat{h} < h^+(\sigma_0)$, such that*

$$L(\sigma_0) \;=\; \big(h^-(\sigma_0), h^+(\sigma_0)\big)\,.$$

Proof. Suppose that $L(\sigma_0) \neq \emptyset$, i.e. there exists a real number $h \in \mathbb{R}$ such that $(\sigma_0, h) \in B$. Since B is open $L(\sigma_0)$ is also open, thus

$$L(\sigma_0) \;=\; \bigcup_{n=1}^{N} I_n\,, \qquad N \in \mathbb{N} \cup \{\infty\}\,,$$

where $I_n \neq \emptyset$ denote some open, bounded, mutually disjoint intervals. If we fix one of these intervals $I_n = (\alpha, \beta)$, then α, β do not belong to $L(\sigma_0)$, hence (σ_0, α), $(\sigma_0, \beta) \notin B$. This proves

$$\text{(4.2)} \qquad I_1(\sigma_0, \alpha) \;=\; I_0(\alpha)\,,\;\; I_1(\sigma_0, \beta) \;=\; I_0(\beta)\,,\;\; I_1(\sigma_0, h) \;<\; I_0(h)$$

for any $h \in (\alpha, \beta)$. Now we claim that $\alpha < \hat{h} < \beta$, which clearly gives the lemma. Suppose by contradiction that $\alpha \geq \hat{h}$. From $I_1(\sigma_0, \alpha) = I_0(\alpha)$ we see the existence of at least one one–phase equilibrium at (σ_0, α). The assumption $\alpha \geq \hat{h}$ gives

$$I_0(\alpha) \;=\; |\Omega|\,\langle A^-\xi^-, \xi^-\rangle \;=\; I[0, 0, \alpha, \sigma_0]\,,$$

hence the one–phase equilibrium with $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ exists for (σ_0, α). On the other hand, Remark 3.6 then proves that for $h > \alpha$ only one–phase equilibria with $\hat{\chi} \equiv 0$ exist which contradicts (4.2) and the lemma is proved since analogous arguments show the second inequality $\hat{h} < \beta$. ■

Step 2. (Definition of the functions $h^{\pm}(\sigma)$)
Following Lemma 4.1 we define for any $\sigma > 0$ satisfying $L(\sigma) \neq \emptyset$

$$h^+(\sigma) \;:=\; \sup L(\sigma)\,, \qquad h^-(\sigma) \;:=\; \inf L(\sigma)\,.$$

If $L(\sigma) = \emptyset$ then we let

$$h^+(\sigma) \;:=\; h^-(\sigma) \;:=\; \hat{h}\,.$$

Step 3. (Definition of the sets A and C)
The sets A and C are defined via

$$A \;:=\; \big\{(\sigma, h) : \sigma > 0\,,\;\; h > h^+(\sigma)\big\}\,,$$

$$C \;:=\; \big\{(\sigma, h) : \sigma > 0\,,\;\; h < h^-(\sigma)\big\}\,,$$

and we claim that for $(\sigma, h) \in A$ ($(\sigma, h) \in C$) the functional $I[\cdot,\cdot,h,\sigma]$ admits only one–phase equilibria $(\hat{u}, \hat{\chi})$ with $\hat{u} \equiv 0$ and $\hat{\chi} \equiv 0$ ($\hat{\chi} \equiv 1$). To verify our claim we assume $(\sigma, h) \in A$, hence $h > h^+(\sigma) \geq \hat{h}$. Recalling (4.2) we have $I_1(\sigma, h^+(\sigma)) = I_0(h^+(\sigma))$ and by Remark 3.6 $I[\cdot,\cdot,h,\sigma]$ admits only a one–phase equilibrium which on account of $h > \hat{h}$ is of type $\hat{\chi} \equiv 0$. The case $(\sigma, h) \in C$ is treated in the same way, and the claim is proved. Now let

$$A' \;:=\; \Big\{(\sigma,h) : \sigma > 0\,,\;\; h \geq \hat{h}\,,\;\; I_1(\sigma,h) = I_0(h) = \langle A^- \xi^-, \xi^- \rangle\, |\Omega| \Big\}\,.$$

It is easily seen that

$$A' \;=\; A \cup \operatorname{graph} h^+\,.$$

In fact, if $(\sigma, h) \in A'$, then we either have $h > h^+(\sigma)$ or $h = h^+(\sigma)$ since $h < h^+(\sigma)$ would imply two–phase equilibria which are excluded by the definition of A'. Thus the inclusion "$\subset$" is proved. The other inclusion follows from Lemma 3.4 b). In a similar way we define

$$C' \;=\; \Big\{(\sigma,h) : \sigma > 0\,,\;\; h \leq \hat{h}\,,\;\; I_1(\sigma,h) = I_0(h) = \big(\langle A^+ \xi^+, \xi^+ \rangle + h\big)\, |\Omega| \Big\}\,,$$

$$C' \;=\; C \cup \operatorname{graph} h^-\,.$$

LEMMA 4.2 *A' and C' are convex sets.*

Proof. Fix two points $(\sigma_i, h_i) \in A'$, $i = 1, 2$, a real number $0 \leq \tau \leq 1$, and let $\sigma_\tau := \tau\,\sigma_1 + (1-\tau)\,\sigma_2$, $h_\tau := \tau\,h_1 + (1-\tau)\,h_2$. Since σ_1, $\sigma_2 > 0$ and since h_1, $h_2 \geq \hat{h}$ the assertions $\sigma_\tau > 0$ and $h_\tau \geq \hat{h}$ are trivial, it remains to show

$$I_1(\sigma_\tau, h_\tau) \;=\; I_0(h_\tau) \;=\; |\Omega|\,\langle A^- \xi^-, \xi^- \rangle\,.$$

However, these equalities are known to be true for σ_i, h_i and since in addition I_1 is concave (see Lemma 3.10), we obtain

$$\begin{aligned} I_1(\sigma_\tau, h_\tau) \;&\geq\; \tau\, I_1(\sigma_1, h_1) + (1-\tau)\, I_1(\sigma_2, h_2) \\ &=\; \tau\, I_0(h_1) + (1-\tau)\, I_0(h_2) \;=\; |\Omega|\,\langle A^- \xi^-, \xi^- \rangle\,. \end{aligned}$$

On the other hand, $I_1(\sigma, h) \leq I_0(h)$ holds for any $h \in \mathbb{R}$, $\sigma > 0$. This together with $h_\tau \geq \hat{h}$ gives

$$I_1(\sigma_\tau, h_\tau) \;\leq\; I_0(h_\tau) \;=\; |\Omega|\,\langle A^- \xi^-, \xi^- \rangle\,.$$

This proves the convexity of A', C' is handled with anologous arguments. ∎

Step 4. (Properties of the functions $h^{\pm}(\sigma)$)

LEMMA 4.3 *The functions $h^{\pm}$ are bounded and depend continuously on $\sigma > 0$. Moreover, $h^+(\sigma)$ is convex on $(0,\infty)$, whereas $h^-(\sigma)$ is concave on $(0,\infty)$.*

Proof. In Step 1. it was shown that B is bounded, hence with Lemma 4.1 the functions $h^{\pm}$ are seen to be uniformly bounded on $(0,\infty)$. Thus we only have to prove that h^+ (h^-) is convex (concave) which will imply continuity. Now fix σ_1, $\sigma_2 > 0$, $0 \le \tau \le 1$, and observe that $(\sigma_i, h^+(\sigma_i)) \in A'$, $i = 1, 2$. In fact, $h^+(\sigma_i) \ge \hat{h}$ is proved in Lemma 4.1, and the existence of an one–phase equilibrium of type $\hat{\chi} \equiv 0$ follows from Lemma 3.4 b). Convexity of A' then yields

$$\big(\underbrace{\tau\,\sigma_1 + (1-\tau)\,\sigma_2}_{\tilde{\sigma}}, \underbrace{\tau\,h^+(\sigma_1) + (1-\tau)\,h^+(\sigma_2)}_{=:\tilde{h}}\big) \in A' .$$

Since $(\tilde{\sigma}, \tilde{h}) \in A'$ immediately gives (compare Step 3.) $\tilde{h} \ge h^+(\tilde{\sigma})$, we have proved the convexity of h^+:

$$\tau\,h^+(\sigma_1) + (1-\tau)\,h^+(\sigma_2) \;=\; \tilde{h} \;\ge\; h^+(\tilde{\sigma}) \;=\; h^+\big(\tau\,\sigma_1 + (1-\tau)\,\sigma_2\big) .$$

Using the same arguments h^- is seen to be concave and the lemma is verified. ∎

LEMMA 4.4 *There is a real number $\sigma^* > 0$ such that h^+ is strictly decreasing on $(0,\sigma^*)$, whereas h^- is strictly increasing on this intervall. On (σ^*,∞) both h^+ and h^- are equal to $\hat{h}$.*

Proof. By Lemma 3.9 we know that $h^-(\sigma) < \hat{h} < h^+(\sigma)$ if $\sigma \ll 1$ is sufficiently small. On the other hand, $\sigma \gg 1$ implies according to Lemma 3.8 $h^-(\sigma) = \hat{h} = h^+(\sigma)$. Hence, we may define

$$\sigma_+^* \;:=\; \inf\big\{\sigma > 0 : \; h^+ = \hat{h} \;\text{ on }\; (\sigma,\infty)\big\} .$$

Now assume by contradiction that h^+ is not strictly decreasing on $(0,\sigma_+^*)$, i.e. for some positive numbers $0 < \sigma_1 < \sigma_2 < \sigma_+^*$ we have $h^+(\sigma_1) \le h^+(\sigma_2)$. Together with this assumption, convexity of h^+ gives for any $\sigma > \sigma_2$

$$\frac{h^+(\sigma) - h^+(\sigma_2)}{\sigma - \sigma_2} \;\ge\; \frac{h^+(\sigma_2) - h^+(\sigma_1)}{\sigma_2 - \sigma_1} \;\ge\; 0 .$$

Since $\sigma_2 < \sigma_+^*$ implies $h^+(\sigma_2) > \hat{h}$, we obtain the contradiction $h^+(\sigma) \geq h^+(\sigma_2) > \hat{h}$ for any $\sigma > \sigma_2$. Up to now it is proved that h^+ is strictly decreasing on $(0, \sigma_+^*)$. Analogous considerations prove the existence of a real number $\sigma_-^* \in (0, \infty)$ such that $h^- \equiv \hat{h}$ for $\sigma \geq \sigma_-^*$ and such that h^- is strictly increasing on $(0, \sigma_-^*)$. It remains to verify $\sigma_+^* = \sigma_-^*$: to this purpose observe that by Lemma 4.1 $h^-(\sigma) \neq h^+(\sigma)$ implies $\hat{h} \in (h^-(\sigma), h^+(\sigma))$. If we assume that $\sigma_-^* < \sigma_+^*$, then we may find $\sigma \in (\sigma_-^*, \sigma_+^*)$ such that $(h^-(\sigma), h^+(\sigma)) \neq \emptyset$ and such that $h^-(\sigma) = \hat{h}$. This gives the contradiction $\hat{h} \notin (h^-(\sigma), h^+(\sigma))$. Again the case $\sigma_-^* > \sigma_+^*$ is excluded with the same arguments, and the proof of Lemma 4.4 is complete. ■

5 Equilibrium states of $\mathbf{I}[\cdot, \cdot, \mathbf{h}, \sigma]$ for points $(\sigma, \mathbf{h})$ on the graphs of $\mathbf{h}^{\pm}$

In this section we prove iv)–vii) of Theorem 1.1.
ad iv). Consider the case $0 < \sigma < \sigma^*$ and $h = h^+(\sigma)$. Letting $\sigma_n \equiv \sigma$ and by considering a sequence $\{h_n\}$ satisfying $h_n \uparrow h$ as $n \to \infty$ we may assume $(\sigma_n, h_n) \in B$ for n sufficiently large, hence there exists a sequence of two–phase equilibria $(\hat{u}_n, \hat{\chi}_n)$ of $I[\cdot, \cdot, h_n, \sigma_n]$. Since $\lim_{n\to\infty} h_n = h = h^+(\sigma) > \hat{h}$, Lemma 3.4 b) is applicable and $I[\cdot, \cdot, h^+(\sigma), \sigma]$ is seen to admit a two–phase equilibrium. On the other hand, now letting $\sigma_n \equiv \sigma$ and considering a sequence $\{h_n\}$, $h_n \downarrow h$ as $n \to \infty$, we have $(\sigma_n, h_n) \in A$ and the same reasoning proves the existence of a one–phase equilibrium, which on account of Remark 3.6 can only be of type $\hat{\chi} \equiv 0$.
ad v). We can apply the same arguments as used for iv) with obvious modifications.
ad vi). For $h = \hat{h}$ and $\sigma > \sigma^*$ we again apply Lemma 3.4 to find $(\hat{u}, \hat{\chi})$, $\hat{u} \equiv 0$, as an equilibrium state of $I[\cdot, \cdot, \hat{h}, \sigma]$. Here, Lemma 3.3 c) shows any characteristic function $\hat{\chi}$ to be admissible. Equilibrium states satisfying $\hat{u} \not\equiv 0$ are not possible: if we assume the existence of an equilibrium state $(\hat{u}_0, \hat{\chi}_0)$ of $I[\cdot, \cdot, \hat{h}, \sigma_0]$, $\sigma_0 > \sigma^*$, $\hat{u}_0 \not\equiv 0$, then we obtain for any $\sigma \in (\sigma^*, \sigma_0)$,

$$\begin{aligned} I_0(\hat{h}) &= I_1(\sigma, \hat{h}) \leq I[\hat{u}_0, \hat{\chi}_0, \hat{h}, \sigma] < I[\hat{u}_0, \hat{\chi}_0, \hat{h}, \sigma_0] \\ &= I_1(\sigma_0, \hat{h}) = I_0(\hat{h}), \end{aligned}$$

where we used the existence of equilibria of type $\hat{u} \equiv 0$ for the parameters $\sigma = \sigma_0$, $h = \hat{h}$.
ad vii). Finally the case $h = \hat{h}$ and $\sigma = \sigma^*$ has to be discussed. As in vi) equilibrium states of type $\hat{u} \equiv 0$, $\hat{\chi} \equiv$ arbitrary characteristic function, are found. The existence of a two–phase equilibrium state satisfying $\hat{u} \not\equiv 0$ is proved by considering a sequence $\{\sigma_n\}$, $\sigma_n \uparrow \sigma^*$ as $n \to \infty$, $h_n \equiv \hat{h}$, i.e. $(\sigma_n, \hat{h}) \in B$. By the definition of B we have $I_1(\sigma_n, \hat{h}) < I_0(\hat{h})$ and, as a consequence (compare Lemma 3.3 c)), $\hat{u}_n \not\equiv 0$ if $(\hat{u}_n, \hat{\chi}_n)$ denotes a corresponding equilibrium state of $I[\cdot, \cdot, \hat{h}, \sigma_n]$. With Lemma 3.4 a) assertion vii) holds and the whole theorem is proved. ■

6 Proof of Theorem 1.3

W.l.o.g. assume that $\hat{u} \not\equiv 0$. Then we have $\int_\Omega |\Delta \hat{u}|^2 \, dx > 0$ and letting $u_t := \hat{u} + t\varphi$, $t \in \mathbb{R}$, $\varphi \in C_0^\infty(\Omega; \mathbb{R}^d)$, minimality of $(\hat{u}, \hat{\chi})$ implies

$$
\begin{aligned}
0 &= \frac{d}{dt}_{|t=0} I[u_t, \hat{\chi}, h, \sigma] \\
&= 2 \int_\Omega \Big\langle \hat{\chi} A^+ \big(\varepsilon(\hat{u}) - \xi^+\big) + (1 - \hat{\chi}) A^- \big(\varepsilon(\hat{u}) - \xi^-\big), \varepsilon(\varphi) \Big\rangle \, dx \\
&\qquad + p\sigma \left(\int_\Omega |\Delta \hat{u}|^2 \right)^{\frac{p}{2} - 1} \int_\Omega \Delta \hat{u} : \Delta \varphi \, dx \, ,
\end{aligned}
$$

hence, letting $T = c\big(\hat{\chi} A^+ \big(\varepsilon(\hat{u}) - \xi^+\big) + (1 - \hat{\chi}) A^- \big(\varepsilon(\hat{u}) - \xi^-\big)\big)$ for a suitable real number $c > 0$, we obtain

$$
\int_\Omega \Delta \hat{u} : \Delta \varphi \, dx = \int_\Omega \nabla \varphi : T \, dx \quad \text{for all } \varphi \in C_0^\infty\big(\Omega; \mathbb{R}^d\big) \, . \tag{6.1}
$$

Now we abbreviate $U := \Delta \hat{u} \in L^2(\Omega; \mathbb{R}^d)$ and denote by U^ρ, T^ρ the standard mollifications of U and T, respectively, where $\rho > 0$ is chosen sufficiently small. Then (6.1) is valid for U^ρ, T^ρ in the following sense

$$
\begin{aligned}
\int_\Omega \nabla U^\rho : \nabla \varphi \, dx &= -\int_\Omega \nabla \varphi : T^\rho \, dx \, , \quad \varphi \in C_0^\infty\big(\Omega; \mathbb{R}^d\big) \, , \\
& \operatorname{dist}(\operatorname{spt} \varphi, \partial\Omega) > \rho \, .
\end{aligned} \tag{6.2}
$$

Since $\eta^2 U^\rho$, $\eta \in C_0^\infty(\Omega)$, $0 \leq \eta \leq 1$, is admissible in (6.2) for ρ sufficiently small, this implies

$$\begin{aligned} &\int_\Omega \eta^2 \left|\nabla U^\rho\right|^2 dx + 2\int_\Omega \eta \nabla\eta \otimes U^\rho : \nabla U^\rho \, dx \\ &\qquad = -\int_\Omega \eta^2 \nabla U^\rho : T^\rho \, dx - 2\int_\Omega \eta \nabla \eta \otimes U^\rho : T^\rho \, dx \, , \end{aligned}$$

hence, with the help of Young's inequality,

$$\int_\Omega \eta^2 \left|\nabla U^\rho\right|^2 dx \;\leq\; \tilde{c}(\eta) \left(\int_{\operatorname{spt}\eta} \left|U^\rho\right|^2 dx + \int_{\operatorname{spt}\eta} \left|T^\rho\right|^2 dx \right) .$$

This proves $\{U^\rho\}$ to be uniformly bounded in $W^1_{2,loc}(\Omega;\mathbb{R}^d)$ which, together with $U^\rho \to U$ in $L^2_{loc}(\Omega;\mathbb{R}^d)$ as $\rho \to 0$, gives $U \in W^1_{2,loc}(\Omega;\mathbb{R}^d)$. As a result we have the equation

$$\text{(6.3)} \qquad \int_\Omega \nabla U : \nabla\varphi \, dx \;=\; -\int_\Omega T : \nabla \varphi \, dx \quad \text{ for all } \varphi \in C_0^\infty\big(\Omega;\mathbb{R}^d\big) .$$

Now we apply the standard L^p–theory for weak solutions of "$\Delta v = \nabla T$" as well as again the Calderon–Zygmund regularity results. To be precise let us first consider the case $d = 2$. Here $\varepsilon(u) \in W^1_2(\Omega;\mathbb{R}^{d\times d})$ implies $T \in L^p(\Omega;\mathbb{R}^{d\times d})$ for any $p < \infty$. L^p–theory gives $\nabla U \in L^p_{loc}(\Omega;\mathbb{R}^{d\times d})$ (compare [GIA], Section 4.3, in particular p. 73), hence $\Delta u \in W^1_{p,loc}(\Omega;\mathbb{R}^d)$ for any $p < \infty$ and we obtain $\Delta u \in C^{0,\alpha}_{loc}(\Omega;\mathbb{R}^d)$ for any $\alpha \in (0,1)$. Finally, the assertion follows from the interior Schauder estimates (see [GIA], Theorem 3.6). Next we assume that $d \geq 3$ and let $s_l := 2d/(d-2l)$. Then it is easy to see that

$$\begin{aligned} & \hat{u} \in W^2_2\big(\Omega;\mathbb{R}^d\big) && \Rightarrow \; \varepsilon(\hat{u}) \in L^{s_1}\big(\Omega;\mathbb{R}^{d\times d}\big) \\ \Rightarrow \;& T \in L^{s_1}\big(\Omega;\mathbb{R}^{d\times d}\big) && \Rightarrow \; \nabla\big(\Delta\hat{u}\big) \in L^{s_1}_{loc}\big(\Omega;\mathbb{R}^{d\times d}\big) \\ \text{(6.4)} \qquad \Rightarrow \;& \Delta\hat{u} \in W^1_{s_1,loc}\big(\Omega;\mathbb{R}^d\big) && \Rightarrow \; \Delta\hat{u} \in L^{s_2}_{loc}\big(\Omega;\mathbb{R}^d\big) \\ \Rightarrow \;& \hat{u} \in W^2_{s_2,loc}\big(\Omega;\mathbb{R}^d\big) && \Rightarrow \; T \in L^{s_3}\big(\Omega;\mathbb{R}^{d\times d}\big) \\ & \ldots && \end{aligned}$$

This procedure stops if $d \leq 2l$. Thus, denote by l^* the maximum of all $l \in \mathbb{N}$ such that $d - 2l > 0$. Then s_{l^*} is well defined and satisfies $s_{l^*} \geq d$. In fact, the latter inequality is equivalent to $2 \geq d - 2l^*$ which is true on account of the maximality of l^*. Now assume that l^* is an even number. Then (6.4) implies for any $p < \infty$

$$\hat{u} \in W^2_{s_{l^*},loc}\big(\Omega;\mathbb{R}^d\big) \quad \Rightarrow \quad \varepsilon(\hat{u}) \in W^1_{d,loc}\big(\Omega;\mathbb{R}^{d\times d}\big)$$

$$\Rightarrow \quad \varepsilon(\hat{u}) \in L^p_{loc}\big(\Omega;\mathbb{R}^{d\times d}\big) \quad \Rightarrow \quad T \in L^p_{loc}\big(\Omega;\mathbb{R}^{d\times d}\big)\,,$$

thus $\Delta\hat{u} \in W^1_{p,loc}(\Omega;\mathbb{R}^d)$ for any $p < \infty$ (again compare [GIA], Section 4.3) and as a consequence $\Delta\hat{u} \in C^{0,\alpha}_{loc}(\Omega;\mathbb{R}^d)$ for all $0 < \alpha < 1$. Again the interior Schauder estimates (see [GIA], Theorem 3.6) prove the result. In the case that l^* is an odd number we conclude

$$\Delta\hat{u} \in W^1_{s_{l^*},loc}\big(\Omega;\mathbb{R}^d\big) \quad \Rightarrow \quad \Delta\hat{u} \in W^1_{d,loc}\big(\Omega;\mathbb{R}^d\big) \quad \Rightarrow \quad \Delta\hat{u} \in L^p_{loc}\big(\Omega;\mathbb{R}^d\big)$$

$$\Rightarrow \quad \hat{u} \in W^2_{p,loc}\big(\Omega;\mathbb{R}^d\big)\,,$$

which again is valid for any $p < \infty$, hence $\varepsilon(\hat{u}) \in L^p_{loc}(\Omega;\mathbb{R}^{d\times d})$ for any $p < \infty$ and we proceed as before, i.e. Theorem 1.3 is proved. ■

References

[BFO] Bildhauer, M., Fuchs, M., Osmolovskii, V., The effect of a surface energy term on the distribution of phases in an elastic medium with a two-well elastic potential.

[DA] Dacorogna, B., Direct methods in the calculus of variations. Appl. Math. Sciences 78, Springer, Berlin–Heidelberg–New York, 1989.

[GIA] Giaquinta, M., Introduction to regularity theory for nonlinear elliptic systems. Lectures in Mathematics ETH Zürich, Birkhäuser, Basel–Boston–Berlin 1993.

[GT] Gilbarg, D., Trudinger, N.S., Elliptic partial differential equations of second order. Grundlehren der math. Wiss. 224, second ed., revised third print., Springer, Berlin–Heidelberg–New York, 1998.

[GIU] Giusti, E., Minimal surfaces and functions of bounded variation. Birkhäuser, Boston–Basel–Stuttgart, 1984.

[GR] Grinfeld, M.A., The methods of continuum mechanics in the phase transition theory. Moskow 1990 (in Russian).

[KO] Kohn, R.V., The relaxation of a double–well energy. Continuum Mech. Thermodyn. 3 (1991), 193–236.

[KM] Kohn, R.V., Müller, S., Surface energy and microstructure in coherent phase transitions. Comm. Pure Appl. Math. 57 (1994), 405–432.

[MO] Morozov, N.F., Osmolovskii, V., The formulation and an existence theorem for a variational problem on phase transitions in continuous medium mechanics. J. Appl. Math. Mech. 58.5 (1994), 889–896.

[MU] Müller, S., Microstructures, phase transitions and geometry. Max–Planck–Institut für Mathematik in den Naturwissenschaften, preprint no. 3, Leipzig 1997.

[OS1] Osmolovskii, V.G., Phase transitions for models of an elastic medium with residual stress operators. Problemy matematicheskogo analiza 17 (1997), 153–191 (in Russian). Engl. Trans.: J. Math. Sciences 97, no. 4 (1999), 4280–4305.

[OS2] Osmolovskii, V.G., The variational problems on phase transition theory in the mechanics of continuum media. St. Petersburg 2000 (in Russian).

[SE] Seregin, G., J_p^1–quasiconvexity and variational problems on sets of solenoidal vector fields. Algebra and Analiz 11, 170–217, 1999. Engl. transl. in St. Petersburg Math. J. 11 (2000), No. 2, 337–373.