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Abstract

We consider local minimizers u: $\Omega \to \mathbb{R}^N$, Ω a domain in $\mathbb{R}^2$, of the variational integral $\int_\Omega f(\nabla u)dx$ with integrand f of upper (lower) growth rate q (s). We show using a lemma due to Frehse and Seregin that u has Hölder continuous first derivatives provided that $q < 2s$.

Regularity of vector-valued local minimizers of anisotropic variational integrals has been investigated by many authors in recent years, we refer to [AF2], [PS], [BF1], [BF2] and the references quoted therein, where it is shown that under slightly varying technical conditions on the energy density partial $C^{1,\alpha}$-regularity holds provided that the upper and lower growth rate do not differ too much. With the exception of the very special case of integrands just depending on the modulus of the gradient (see, for instance, [FM] or [Bi]) it seems to be an open problem if at least in two dimensions singular points can be excluded. In this short note we want to give a positive answer under quite general structure conditions.

To be precise, consider a bounded Lipschitz domain Ω in $\mathbb{R}^2$ and let $f : \mathbb{R}^{2N} \to [0, \infty)$ denote a function of class C^2 such that

$$\lambda(1 + |X|^2)^{\frac{s-2}{2}} |Y|^2 \;\leq\; D^2 f(X)(Y,Y) \;\leq\; \Lambda(1 + |X|^2)^{\frac{q-2}{2}} |Y|^2 \tag{1}$$

holds for all X, $Y \in \mathbb{R}^{2N}$ with positive constants λ, Λ. Here s and q are fixed exponents such that $1 < s < q < \infty$. ¿From (1) it follows that f is strictly convex, moreover, (1) implies the growth estimate (see [AF1] in case $s < 2$ or $q < 2$; for $s \geq 2$ the statement is immediate)

$$a\,|X|^s - b \;\leq\; f(X) \;\leq\; A\,\big(|X|^q + 1\big) \;\;\forall\, X \in \mathbb{R}^{2N} \tag{2}$$

for suitable constants a, b, $A > 0$.

Definition 1. *A function u from the local Sobolev space $W^1_{s,loc}(\Omega;\mathbb{R}^N)$ is termed a local minimizer of the variational integral $J[w] = \int_\Omega f(\nabla w)dx$ if and only if*

(a) $$\int_{\Omega'} f(\nabla u)\,dx \;<\; \infty \;\; \textit{for all } \Omega' \Subset \Omega$$

and

$$\text{(b)} \qquad \int_{\operatorname{spt}(u-v)} f(\nabla u)\,dx \;\le\; \int_{\operatorname{spt}(u-v)} f(\nabla v)\,dx$$
$$\textit{for all}\;\; v \in W^1_{s,loc}(\Omega;\mathbb{R}^N) \;\; \textit{such that}\;\; \operatorname{spt}(u-v) \Subset \Omega.$$

This definition is very natural since for boundary data $u_0 \in W^1_s(\Omega;\mathbb{R}^N)$ such that $J[u_0] < \infty$ the problem

$$J \;\to\; \min \quad \text{in } u_0 + \overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^N)$$

admits a unique solution. We have the following result:

THEOREM 1. *Let f satisfy condition (1), let u denote a local minimizer and assume that in addition to $1 < s < q$ we have*

$$\text{(3)} \qquad q < 2s.$$

Then $u \in C^{1,\alpha}(\Omega;\mathbb{R}^N)$ for any exponent $0 < \alpha < 1$.

As a model, our result applies to the variational integral

$$\int_\Omega \left(|\partial_1 u|^2 + \left(1 + |\partial_2 u|^2\right)^{\frac{q}{2}} \right) dx,$$

if we choose q in the intervall $[2,4)$. It would be interesting to know if condition (3) is optimal.

Another example covered by Theorem 1 is given by

$$\int_\Omega \left(\left(1 + |\nabla u|^2\right)^{\frac{1+\varepsilon}{2}} + |\partial_2 u|^2 \right) dx$$

for some $\varepsilon \in (0,1]$.

The **Proof of Theorem 1** is organized in several steps combining techniques of [BF1] with arguments due to Frehse and Seregin [FrS] which also turned out to be useful in the context of [FuS1], [FuS2]. In what follows we always assume that the hypotheses of Theorem 1 are satisfied.

Step 1. Approximation

W.l.o.g. we may assume that the disc $B_{2R} = B_{2R}(0)$ is compactly contained in Ω. Following [BF1], Section 2, we replace u by its mollification u_ε (with ε being small) and let

$$\begin{aligned} f_\varepsilon(Z) &= f(Z) + \delta(\varepsilon)\left(1+|Z|^2\right)^{\frac{q}{2}}, \quad Z \in \mathbb{R}^{2N}, \\ \delta(\varepsilon) &= \frac{1}{1+\varepsilon^{-1}+\|\nabla u_\varepsilon\|^{2q}_{L^q(B_{2R})}}. \end{aligned}$$

Moreover, we denote by v_ε the unique solution of the problem

$$\int_{B_{2R}} f_\varepsilon(\nabla w)\, dx \;\to\; \min \quad \text{in } u_\varepsilon + \overset{\circ}{W}{}^1_q(B_{2R};\mathbb{R}^N).$$

As in [BF1], Lemma 2.1, we have

$$v_\varepsilon \rightharpoondown u \;\text{ in } W^1_s(B_{2R};\mathbb{R}^N); \tag{4}$$

$$\delta(\varepsilon)\int_{B_{2R}} \left(1+|\nabla v_\varepsilon|^2\right)^{\frac{q}{2}} dx \;\to\; 0; \tag{5}$$

$$\int_{B_{2R}} f_{(\varepsilon)}(\nabla v_\varepsilon)\, dx \;\to\; \int_{B_{2R}} f(\nabla u)\, dx \quad \text{as } \varepsilon \to 0. \tag{6}$$

Note, that (4) is not explicitly stated in [BF1], only the weak convergence in the space $W^1_1(B_{2R};\mathbb{R}^N)$ is established, but by using inequality (2), the claim is immediate. A complete proof is also given in [BF2], Lemma 2.1, since at this stage the choice $r = q$ is admissible in [BF2]. Up to now we did not use condition (3). The bound on q is necessary if we want to quote [BF1], Lemma 3.4, with the choice $\mu = 2 - s$. Then we get

$$\exists\, t > q : \quad \|\nabla v_\varepsilon\|_{L^t(B_r)} \;\le\; c\,(r, R) \;<\; \infty \tag{7}$$

for any radius $r < 2R$ with constant c independent of ε. In particular, u is in the space $W^1_{t,loc}(B_{2R};\mathbb{R}^N)$. It should be noted that Lemma 3.4 of [BF1] actually gives (7) for any finite number t, see also Remark III. 3.7 of [Bi].

Step 2. The starting inequality

Consider a disc $B_{2r}(x_0) \subset B_{2R}$ and let $\eta \in C^1_0(B_{2r}(x_0))$ denote a cut-off function such that $\eta \equiv 1$ on $B_r(x_0)$, $0 \le \eta \le 1$, and $|\nabla \eta| \le \frac{c}{r}$. If $\Delta_h g$ is the difference quotient of a function g in direction e_s, $s = 1, 2$, then it was shown in [BF1], inequality (3.2), that

$$\begin{aligned} &\int \eta^2 \Delta_h\big(Df_\varepsilon(\nabla v_\varepsilon)\big) : \nabla \Delta_h v_\varepsilon\, dx \\ &\quad = -2\int \eta \Delta_h\big(Df_\varepsilon(\nabla v_\varepsilon)\big) : \big(\nabla\eta \otimes \Delta_h(v_\varepsilon - Qx)\big)\, dx \end{aligned} \tag{8}$$

is valid for any $Q \in \mathbb{R}^{2N}$. Moreover, the calculations in [BF1], proof of Lemma 3.1, show that we may pass to the limit $h \to 0$ with the result that (8) turns into (summation over $s = 1, 2$)

$$\begin{aligned}
&\int \eta^2 D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_s \nabla v_\varepsilon, \partial_s \nabla v_\varepsilon)\, dx \\
&\qquad \leq\ -2 \int \eta\, D^2 f_\varepsilon(\nabla v_\varepsilon)\big(\partial_s \nabla v_\varepsilon, \nabla\eta \otimes \partial_s[v_\varepsilon - Qx]\big)\, dx.
\end{aligned} \tag{9}$$

Next observe

$$\begin{aligned}
&\Big| D^2 f_\varepsilon(\nabla v_\varepsilon)\big(\partial_s \nabla v_\varepsilon, \nabla\eta \otimes \partial_s[v_\varepsilon - Qx]\big)\Big| \\
&\quad \leq\ \Big(D^2 f_\varepsilon(\nabla v_\varepsilon)(\partial_s \nabla v_\varepsilon, \partial_s \nabla v_\varepsilon)\Big)^{\frac{1}{2}} \\
&\qquad\qquad \Big(D^2 f_\varepsilon(\nabla v_\varepsilon)\big(\nabla\eta \otimes \partial_s[v_\varepsilon - Qx], \nabla\eta \otimes \partial_s[v_\varepsilon - Qx]\big)\Big)^{\frac{1}{2}}
\end{aligned}$$

and write

$$H_\varepsilon\ =\ \Big(D^2 f_\varepsilon(\nabla v_\varepsilon)\big(\partial_s \nabla v_\varepsilon, \partial_s \nabla v_\varepsilon\big)\Big)^{\frac{1}{2}},$$

which by Lemma 3.1 of [BF1] is a function of class $L^2_{loc}(B_{2R})$. ¿From (9) we get $\big(T_r(x_0) = B_{2r}(x_0) - B_r(x_0)\big)$

$$\begin{aligned}
\int_{B_r(x_0)} H_\varepsilon^2\, dx\ &\leq\ c\, r^{-1} \int_{T_r(x_0)} H_\varepsilon \sqrt{|D^2 f_\varepsilon(\nabla v_\varepsilon)|}\ |\nabla v_\varepsilon - Q|\, dx \\
&\leq\ c\, r^{-1} \bigg(\int_{T_r(x_0)} H_\varepsilon^2\, dx\bigg)^{\frac{1}{2}} \\
&\qquad\qquad \bigg(\int_{T_r(x_0)} \big|D^2 f_\varepsilon(\nabla v_\varepsilon)\big|\, |\nabla v_\varepsilon - Q|^2\, dx\bigg)^{\frac{1}{2}}.
\end{aligned} \tag{10}$$

¿From (1) and the definition of f_ε we deduce

$$|D^2 f_\varepsilon(\nabla v_\varepsilon)|\, |\nabla v_\varepsilon - Q|^2\ \leq\ c\,(1 + \delta(\varepsilon))\,(1 + |\nabla v_\varepsilon|^2)^{\frac{q-2}{2}} |\nabla v_\varepsilon - Q|^2.$$

Let us introduce the field $W(\xi) = (1 + |\xi|^2)^{\frac{q-2}{4}}\xi$, $\xi \in \mathbb{R}^{2N}$. Moreover, assume from now on that $q \geq 2$. Then, from [Gi], p. 151, we infer

$$(1 + |\xi|^2)^{\frac{q-2}{2}}\, |\xi - Q|^2\ \leq\ c\ |W(\xi) - W(Q)|^2, \tag{11}$$

and this is exactly the place where $q \geq 2$ is needed. For $q < 2$ the left-hand side of (11) has to be replaced by (see [CFM] Lemma 2.1) $(1 + |\xi|^2 +$

$|Q|^2)^{\frac{q-2}{2}}|\xi - Q|^2$ making the next calculations impossible. Now, using (11) and returning to (10), we arrive at

$$
\begin{aligned}
\int_{B_r(x_0)} H_\varepsilon^2\, dx \;\; &\le \;\; c\, r^{-1} \left(\int_{T_r(x_0)} H_\varepsilon^2\, dx \right)^{\frac{1}{2}} \\
&\qquad \left(\int_{T_r(x_0)} |W(\nabla v_\varepsilon) - W(Q)|^2\, dx \right)^{\frac{1}{2}}
\end{aligned}
\tag{12}
$$

with c independent of r, x_0 and ε (note that $\delta(\varepsilon) \le 1$ for ε sufficiently small). Since W is a diffeomorphism of $\mathbb{R}^{2N}$, we may choose Q in such a way that

$$
W(Q) \;=\; \fint_{T_r(x_0)} W(\nabla v_\varepsilon)\, dx
$$

which enables us to estimate the second integral on the right-hand side of (12) with the help of Sobolev-Poincaré's inequality, thus

$$
\int_{B_r(x_0)} H_\varepsilon^2\, dx \;\le\; c\; r^{-1} \left(\int_{T_r(x_0)} H_\varepsilon^2\, dx \right)^{\frac{1}{2}} \int_{T_r(x_0)} \Big| \nabla \big(W(\nabla v_\varepsilon) \big) \Big|\, dx. \tag{13}
$$

Note that the weak differentiability of $W(\nabla v_\varepsilon)$ is established in [GM] or [CA], references are given in Lemma 2.2, b), of [BF1]. Finally, we observe

$$
\begin{aligned}
\Big| \nabla \big(W(\nabla v_\varepsilon) \big) \Big| \;\; &\overset{*}{\le} \;\; c \left(1 + |\nabla v_\varepsilon|^2 \right)^{\frac{q-2}{4}} |\nabla^2 v_\varepsilon| \\
&= \;\; c \left(1 + |\nabla v_\varepsilon|^2 \right)^{\frac{s-2}{4}} |\nabla^2 v_\varepsilon| \left(1 + |\nabla v_\varepsilon|^2 \right)^{\frac{q-s}{4}} \\
&\le \;\; c \left(1 + |\nabla v_\varepsilon|^2 \right)^{\frac{s-2}{4}} |\nabla^2 v_\varepsilon| \left(1 + |\nabla v_\varepsilon|^2 \right)^{\frac{s}{4}}
\end{aligned}
$$

on account of $q < 2s$, and we may use inequality (1) to get

$$
\left(1 + |\nabla v_\varepsilon|^2 \right)^{\frac{s-2}{4}} |\nabla^2 v_\varepsilon| \;\le\; c\, H_\varepsilon.
$$

Letting $h_\varepsilon = \left(1 + |\nabla v_\varepsilon|^2 \right)^{\frac{s}{4}}$, (13) implies

$$
\int_{B_r(x_0)} H_\varepsilon^2\, dx \;\le\; c\, r^{-1} \left(\int_{T_r(x_0)} H_\varepsilon^2\, dx \right)^{\frac{1}{2}} \int_{T_r(x_0)} H_\varepsilon h_\varepsilon\, dx \tag{14}
$$

being valid for any disc $B_{2r}(x_0) \subset B_{2R}$.

REMARK 1. *The above inequality $*$ needs a technical comment. Let us write $W_\varepsilon := W(\nabla v_\varepsilon) = h_\varepsilon^{\frac{q-2}{s}} \nabla v_\varepsilon$. From Proposition 3.2 in [BF1] we infer $h_\varepsilon \in W^1_{2,loc}(B_{2R})$ together with*

$$\nabla h_\varepsilon \;=\; \frac{s}{2} |\nabla v_\varepsilon| \left(1 + |\nabla v_\varepsilon|^2\right)^{\frac{s}{4}-1} \nabla |\nabla v_\varepsilon| .$$

Since $n = 2$ and $h_\varepsilon \geq 1$, we see

$$h_\varepsilon^{\frac{q-2}{s}} \in W^1_{t,loc}(B_{2R}) \quad \textit{for any } t < 2,$$

and $v_\varepsilon \in W^2_{2,loc}(B_{2R}; \mathbb{R}^N)$ implies $\nabla v_\varepsilon \in L^p_{loc}(B_{2R}; \mathbb{R}^{2N})$ for any finite p. Observing also $h_\varepsilon \in L^p_{loc}(B_{2R})$, $p < \infty$, we clearly have $W_\varepsilon \in W^1_{1,loc}(B_{2R}; \mathbb{R}^{2N})$ together with

$$\begin{aligned} \partial_\alpha W_\varepsilon \;&=\; \partial_\alpha \big(h_\varepsilon^{\frac{q-2}{s}}\big) \nabla v_\varepsilon + h_\varepsilon^{\frac{q-2}{s}} \partial_\alpha \nabla v_\varepsilon \\ &=\; \frac{q-2}{s} h_\varepsilon^{\frac{q-2}{s}-1} \partial_\alpha h_\varepsilon \nabla v_\varepsilon + h_\varepsilon^{\frac{q-2}{s}} \partial_\alpha \nabla v_\varepsilon . \end{aligned}$$

Using the formula for $\partial_\alpha h_\varepsilon$ we have proved $*$.

Step 3. Application of the Frehse-Seregin lemma

Inequality (14) exactly corresponds to the hypotheses of Lemma 4.1 in [FrS], and we get: for any $p \geq 1$ and for any compact subdomain ω of B_{2R} there is a constant $K = K(\omega, p)$ such that

$$\int_{B_r(x_0)} H_\varepsilon^2 \, dx \;\leq\; K |\ln r|^{-p} \tag{15}$$

is true for any disc $B_r(x_0) \Subset \omega$. Note that K also depends on $\|H_\varepsilon\|_{L^2(\omega)}$ and $\|h_\varepsilon\|_{W^1_2(\omega)}$ but on account of [BF1], Lemma 3.1 (in combination with (5), (6) and (7)) and Proposition 3.5 with the choice $\mu = 2 - s$ these quantities stay bounded uniformly w.r.t. ε. Let $G_\varepsilon = (1 + |\nabla v_\varepsilon|^2)^{\frac{s-2}{4}} \nabla v_\varepsilon$. Recalling $h_\varepsilon \in W^1_{2,loc}(B_{2R})$ we see that

$$\left(1 + |\nabla v_\varepsilon|^2\right)^{\frac{s-2}{4}} \;=\; h_\varepsilon^{\frac{s-2}{s}} \;=\; h_\varepsilon^{1-\frac{2}{s}}$$

belongs to the same function space. v_ε is an element of $W^2_{2,loc}\big(B_{2R}; \mathbb{R}^{2N}\big)$ (see [BF1], Lemma 2.2, b), (i)), thus $G_\varepsilon \in W^1_{1,loc}(B_{2R}; \mathbb{R}^{2N})$, and for the derivative we get (see also Proposition 3.2 in [BF1]) $|\nabla G_\varepsilon| \le c(1+|\nabla v_\varepsilon|^2)^{\frac{s-2}{4}}|\nabla^2 v_\varepsilon|$ so that $|\nabla G_\varepsilon|^2 \le cH_\varepsilon^2$. Therefore (15) implies

$$\int_{B_r(x_0)} |\nabla G_\varepsilon|^2\,dx \;\le\; K\,|\ln r|^{-p}, \tag{16}$$

and if we choose $p > 2$ in (16), then the version of the Dirichlet-growth theorem given in [Fr], p.287, implies continuity of G_ε on ω with modulus of continuity independent of ε. In [BF1], Proposition 3.5, (iii), we showed $\nabla v_\varepsilon \to \nabla u$ almost everywhere on B_{2R}, therefore $G = (1+|\nabla u|^2)^{\frac{s-2}{4}}\nabla u$ is a continuous function. Since $\xi \mapsto (1+|\xi|^2)^{\frac{s-2}{4}}\xi$ is a homeomorphism (note that this field is proportional to the gradient of the strictly convex potential $(1+|\xi|^2)^{\frac{s+2}{4}}$), we finally get continuity of ∇u. Thus, the criterion for regular points stated in Lemma 4.1 of [BF1] (which works in case $n = 2$ just under the condition (3)) is satisfied everywhere which proves the claim of Theorem 1 in case $q \ge 2$.

Let us now look at the case that (1) is valid with exponents $1 < s < q < 2$. But then (1) also holds for the new choice $q = 2$, i.e. we have

$$\lambda\big(1+|X|^2\big)^{\frac{s-2}{2}}\,|Y|^2 \;\le\; D^2 f(X)(Y,Y) \;\le\; \Lambda|Y|^2,$$

and we may repeat all our calculations with q replaced by the exponent 2. Since $2 < 2s$, the appriopriate version of condition (3) holds which gives the result. $\square$

REMARK 2. *We like to remark that it is not necessary to refer to partial regularity theory, a direct proof based on inequality (14) can be obtained as in [FrS], Theorem 2.4.*

During our calculations condition (3) is needed to have uniform bounds for the quantities $\|H_\varepsilon\|_{L^2(\omega)}$ and $\|h_\varepsilon\|_{W_2^1(\omega)}$ on subdomains ω compactly contained in the disc $B_{2R}(0)$, which means that the constant K in inequality (15) and (16) does not depend on ε. Let us now look at the example

$$f(\xi) \;=\; |\xi|\,\ln(1+|\xi|) + |\xi_2|^2,\;\; \xi = (\xi_1,\xi_2) \in \mathbb{R}^{2N}. \tag{17}$$

Letting $g(\xi) = |\xi|\ln(1+|\xi|)$ we find

$$\frac{1}{1+|\xi|}\,|\eta|^2 \;\le\; D^2 g(\xi)(\eta,\eta) \;\le\; 2\frac{\ln(1+|\xi|)}{|\xi|}|\eta|^2,$$

hence

$$\lambda \left(1+|\xi|^2\right)^{-\frac{1}{2}} |\eta|^2 \ \le\ D^2 f(\xi)(\eta,\eta) \ \le\ \Lambda\, |\eta|^2 \ \text{ for all } \xi, \eta \in \mathbb{R}^{2N} \tag{18}$$

which means that formally (1) holds with $s=1$, $q=2$ and suitable positive constants λ, Λ. In contrast to Definition 1 local minimizers should now be located in the Orlicz-Sobolev space $W^1_{A,loc}(\Omega;\mathbb{R}^N)$, $A(t) = t\ln(1+t)$, compare, e.g. [Ad], and using the notation from [BF1] we see from (18) that f is of "(s,μ,q)-type" with $s=\mu=1$, $q=2$. But the "(s,μ,q)-condition" from [BF1] is violated in our example so that no information concerning the regularity of local minimizers (i.e. local higher integrability or $C^{0,\alpha}$-regularity of the gradient) can be deduced from [BF1]. Since we now are in the limit case of (3) and also of the (s,μ,q)-condition from [BF1], it is reasonable to impose further conditions on the data which might be sufficient for proving regularity. One natural possibility is to look at a boundary value problem with sufficiently regular boundary data.

THEOREM 2. *Suppose that* $u_0 \in W^2_2(\Omega;\mathbb{R}^N)$ *and let* $u \in u_0 + \overset{\circ}{W}{}^1_A(\Omega;\mathbb{R}^N)$ *denote the unique solution of*

$$J[w] \ =\ \int_\Omega f(\nabla w)\,dx \ \to\ \min \quad \textit{in } u_0 + \overset{\circ}{W}{}^1_A(\Omega;\mathbb{R}^N)$$

with f defined in (17). Then u is a continuously differentiable function.

The proof is similar to the proof of Theorem 1 but now we work with a global regularization: for $0<\delta<1$ define $f_\delta(\xi) = \frac{\delta}{2}|\xi|^2 + f(\xi)$ and let u_δ denote the unique solution of

$$\int_\Omega f_\delta(\nabla w)\,dx \ \to\ \min \ \text{ in } u_0 + \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^N).$$

We have (see, for instance, [FO])

$$u_\delta \ \overset{\delta\downarrow 0}{\rightharpoondown}\ u \ \text{ in } W^1_1(\Omega;\mathbb{R}^N) \tag{19}$$

and, what we need most

$$u_\delta \in W^2_2(\Omega;\mathbb{R}^N). \tag{20}$$

For a proof of (20) we quote standard techniques supposing for simplicity that a part Γ of $\partial\Omega$ is flat, e.g. $\Gamma \subset [x_2 = 0]$. If x_0 is a point in Γ, if η denotes a function in $C^1_0(B_{2r}(x_0))$ and if Δ_h is the difference quotient in direction e_1, then $\Delta_{-h}(\eta^2 \Delta_h[u_\delta - u_0])$ is admissible in the Euler-Lagrange equation for u_δ, and we find

$$\int \eta^2 \, |\Delta_h \nabla u_\delta|^2 \, dx \;\le\; c(\delta, u_0)$$

with c not depending on h. Thus $\partial_1\partial_1 u_\delta$, $\partial_1\partial_2 u_\delta \in L^2(\Omega; \mathbb{R}^N)$, and in order to control $\partial_2\partial_2 u_\delta$ we use the equation (recall that we already know $u_\delta \in W^2_{2,loc}(\Omega; \mathbb{R}^N)$)

$$\begin{aligned} A^{ij}_{\alpha\beta}\, \partial_\alpha \partial_\beta u^i_\delta \;&\equiv\; 0 \ \text{ a.e. on } \Omega, \quad j = 1, \ldots, N, \\ A^{ij}_{\alpha\beta} \;&:=\; \frac{\partial f_\delta}{\partial p^i_\alpha \, \partial p^j_\beta}(\nabla u_\delta). \end{aligned}$$

Multiplying

$$A^{ij}_{22}\, \partial_2\partial_2 u^i_\delta \;=\; -(A^{ij}_{12} + A^{ij}_{21})\, \partial_1 \partial_2 u^i_\delta - A^{ij}_{11} \partial_1\partial_1 u^i_\delta$$

with $\partial_2\partial_2 u^j_\delta$ and using the ellipticity of the coefficients we deduce $\partial_2\partial_2 u_\delta \in L^2(\Omega; \mathbb{R}^N)$. Let

$$H_\delta \;:=\; \Big(D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \partial_s \nabla u_\delta)\Big)^{\frac{1}{2}}.$$

Then we claim

$$\int_\Omega H^2_\delta \, dx \;\le\; c(u_0) \;<\; \infty \tag{21}$$

for a constant $c(u_0)$ depending also on the boundary data u_0 but being independent of δ. Let us sketch the proof of (21) for the special case $\Omega = B_1(0)$, the general situation can be reduced to this case by covering $\partial\Omega$ with open sets U such that $U \cap \Omega$ is star-shaped. Let $w := u_\delta - u_0 \in \overset{\circ}{W}{}^1_2 \cap W^2_2(B_1; \mathbb{R}^N)$ and set $w \equiv 0$ outside of B_1. We define for $0 < r < 1$

$$w_r(z) \;:=\; w(z/r), \quad |z| < 1,$$

thus $w_r(z) = 0$ if $|z| \ge r$. The differentiated form of the Euler equation for u_δ implies

$$\int_{B_1} D^2 f_\delta(\nabla u_\delta)(\partial_s \nabla u_\delta, \partial_s \nabla w_r)\, dz \;=\; 0,$$

and since by (20) $w_r \to w$ in $W^2_2(B_1; \mathbb{R}^N)$ as $r \to 1$, we find

$$\int_{B_1} D^2 f_\delta(\nabla u_\delta)\big(\partial_s \nabla u_\delta, \partial_s[\nabla u_\delta - \nabla u_0]\big)\, dx \;=\; 0,$$

and (21) follows from Young's inequality by observing that $|D^2 f_\delta(\xi)| \le c$. As before we let $h_\delta = (1+|\nabla u_\delta|^2)^{1/4}$ and get $\int_\Omega h_\delta^2 dx \le c < \infty$ as well as $|\nabla h_\delta|^2 \le c H_\delta^2$ so that by (21)

$$\sup_{0<\delta<1} \|h_\delta\|_{W_2^1(\Omega)} \; < \; \infty. \tag{22}$$

Now, the same calculations leading to (13) (choose $q=2$ there) imply

$$\int_{B_r(x_0)} H_\delta^2\, dx \;\le\; \frac{c}{r}\left(\int_{T_r(x_0)} H_\delta^2\, dx\right)^{\frac{1}{2}} \int_{T_r(x_0)} |\nabla^2 u_\delta|\, dx$$

where again we used the boundedness of $D^2 f_\delta$. From (18) it follows that

$$|\nabla^2 u_\delta| \;\le\; c\, h_\delta\, H_\delta,$$

thus we get the starting inequality for the Frehse-Seregin lemma:

$$\int_{B_r(x_0)} H_\delta^2\, dx \;\le\; \frac{c}{r}\left(\int_{T_r(x_0)} H_\delta^2\, dx\right)^{\frac{1}{2}} \int_{T_r(x_0)} h_\delta\, H_\delta\, dx, \quad B_{2r}(x_0) \Subset \Omega.$$

Let us fix $p>2$. Recalling (21) and (22) we deduce

$$\int_{B_r(x_0)} H_\delta^2\, dx \;\le\; K\,|\ln r|^{-p} \quad \text{or} \quad \int_{B_r(x_0)} |\nabla G_\delta|^2\, dx \;\le\; K\,|\ln r|^{-p}$$

for a constant K independent of δ. Here we have abbreviated

$$G_\delta \;=\; G(\nabla u_\delta), \quad G(\xi) \;=\; \left(1+|\xi|^2\right)^{-\frac{1}{4}}\xi, \quad \xi \in \mathbb{R}^{2N}$$

(note that $|\nabla G_\delta|^2 \;\le\; c H_\delta^2$ and $|G_\delta|^2 \;\le\; c\sqrt{1+|\nabla u_\delta|^2}$ which means $G_\delta \in W_2^1(\Omega;\mathbb{R}^{2N})$). The variant of the Dirichlet-growth theorem given by Frehse [Fr] implies uniform continuity of the functions G_δ, more precisely

$$\operatorname{osc}_{B_r(x_0)} G_\delta \;\le\; \tilde{K}\,|\ln r|^{1-\frac{p}{2}} \tag{23}$$

for a uniform constant $\tilde{K}$ and all discs $B_r(x_0) \subset \Omega$ such that $x_0 \in \tilde{\Omega}$ and $r \le \frac{1}{2}\mathrm{dist}(\tilde{\Omega}, \partial\Omega)$, where $\tilde{\Omega}$ is some fixed subdomain such that $\tilde{\Omega} \Subset \Omega$. ($\tilde{K}$ depends on $\mathrm{dist}(\tilde{\Omega}, \partial\Omega)$ but not on δ.) Let us cover the closure of $\tilde{\Omega}$ with a finite number L of discs $B_{r_i}(x_i)$ such that (see (23))

$$\operatorname{osc}_{B_{r_i}(x_i)} G_\delta \;\le\; 1 \;\text{ for all } 0<\delta<1,\; i=1,\dots,L.$$

Then $|G_\delta(x) - G_\delta(z)| \leq L$ for x, $z \in \tilde{\Omega}$ and we get from

$$|G_\delta(x)| \;\leq\; \left| \fint_{\tilde{\Omega}} G_\delta(z)\,dz \right| + \fint_{\tilde{\Omega}} |G_\delta(x) - G_\delta(z)|\,dz$$

the bound $|G_\delta(x)| \leq M < \infty$ for all $x \in \tilde{\Omega}$, $0 < \delta < 1$. Here we use $\sup_{0<\delta<1} \|G_\delta\|_{W^1_2(\Omega)} < \infty$ implying that $\fint_{\tilde{\Omega}} G_\delta(z)\,dz$ stays bounded. Thus we may apply Arcela's theorem to get $G_\delta \to: \tilde{G}$ uniformly for a continuous function $\tilde{G}$ on the closure of $\tilde{\Omega}$. Since G is a diffeomorphism of $\mathbb{R}^{2N}$, we have $\nabla u_\delta = G^{-1} \circ G_\delta \to G^{-1} \circ \tilde{G}$ everywhere on $\tilde{\Omega}$, and $G^{-1} \circ \tilde{G}$ is of class C^0. But from $\nabla u_\delta \rightharpoondown \nabla u$ in $L^1(\Omega; \mathbb{R}^{2N})$ it follows that $G^{-1} \circ \tilde{G} = \nabla u$ which proves the claim of Theorem 2. $\square$

REMARK 3. *Of course our arguments are valid not only for the particular example (17), we can consider any integrand $f \geq 0$ of class C^2 satisfying (18) and for which $f(\xi) \geq mA(|\xi|) - M$ with m, $M > 0$ and some N-function A is valid.*

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