Special Linear Series and Syzygies of Canonical Curves of Genus 9

An den Naturwissenschaftlich-Technischen Fakultäten
der Universität des Saarlandes
zur Erlangung des Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
vorgelegte Abhandlung

von

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geboren am 14. März 1978 in Regensburg

November 30, 2005

Abstract

In this thesis we give a complete description of the syzygies of irreducible, non-singular, canoncial curves C of genus 9. This includes a collection of all possible Betti tables for C. Moreover a direct correspondence between these Betti tables and the number and types of special linear series on C is given. Especially for $\mathrm{Cliff}(C) = 3$ the curve C is contained in determinantal surface Y on a 4-dimensional rational normal scroll $X \subset \mathbb{P}^8$ constructed from a base point free pencil of divisors of degree 5.

Contents

Deutsche Zusammenfassung. In der vorliegenden Arbeit werden kanonisch eingebettete Kurven $C \subset \mathbb{P}^{g-1}$ vom Geschlecht g behandelt. Wir betrachten die minimal freie Auflösung des Koordinatenrings S_C, wobei $S = \mathbb{k}[x_0, ..., x_{g-1}]$ den homogenen Koordinatenring des $\mathbb{P}^g$ bezeichnet. Diese Auflösung nimmt dann folgende Gestalt an:

$$\overbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & & & & & & & \\ \hline & \beta_{12} & \cdots & & & & & \\ \hline & & & & & \cdots & \beta_{g-3,g-1} & \\ \hline & & & & & & & 1 \\ \hline\end{array}}^{g-1}$$

$$\underbrace{\qquad\qquad\qquad}_{p}$$

wobei $p \leq \frac{g-3}{2}$ die Anzahl der führenden Nulleinträge in der dritten Zeile bezeichnet. Green′s Vermutung besagt nun, dass ein direkter Zusammenhang zwischen dieser Anzahl p und dem Auftreten spezieller Linearsysteme auf C besteht: Er vermutet, dass C genau dann Clifford Index p besitzt, wenn genau die ersten p Einträge in der dritten Zeile des obigen Tableaus Nulleinträge sind. Der Clifford Index eines Divisors D auf einer Kurve C ist definiert als

$$\text{Cliff}(D) = \deg D - 2(h^0\mathcal{O}_C(D) - 1)$$

und der Clifford Index von C als

$$\text{Cliff}(C) = \min\left\{\text{Cliff}(D) : D \in \text{Div}(C) \text{ effektiv mit } h^0\mathcal{O}_C(D), h^1\mathcal{O}_C(D) \geq 2\right\}$$

Green and Lazarsfeld bewiesen in [GL84], dass aus der Existenz spezieller Linearsysteme auf C die Existenz von Extrasyzygien folgt. Neuere Arbeiten von Hirschowitz und Ramanan in [HR98] (1998) und von Voisin in [V05] zeigen, dass Green's Vermutung fûr Kurven von ungeradem Geschlecht g und maximalem Clifford Index $\frac{g-1}{2}$. gilt. Darüber hinaus gilt Green's Vermutung auch allgemein für Kurven vom Geschlecht $g \leq 9$, was durch Ergebnisse von Mukai in [M95] (1995) und Schreyer in [S91] belegt wird.

Man kann nun einen Schritt weiter gehen und fragen, welche Betti Tableaus auftreten können im Falle einer irreduziblen, glatten, kanonischen Kurve C. Für Kurven vom Geschlecht $g \leq 8$ ist diese Fragestellung bereits in [S86] (1986) beantwortet worden. Basierend auf Computerberechnungen gibt Schreyer eine Liste von Betti Tableaus für Kurven vom Geschlecht 9, 10 and 11 in [S03] (2003) an, deren Vollständigkeit zu überprüfen ist. So vermutet der Autor bespielsweise, dass für Kurven vom Geschlecht 9 aus der Existenz von drei g^1_5′en bereits die eines g^2_7 folgt, was sich im Allgemeinen als nicht zutreffend erweist.

Das Ergebnis dieser Arbeit ist eine vollständige Liste von Betti Tableaus für glatte, irreduzible, kanonische Kurven vom Geschlecht $g = 9$:

general	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	-	-	-	-
2	-	-	-	-	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g_5^1$	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	4	-	-	-
2	-	-	-	4	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists$ two g_5^1	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	8	-	-	-
2	-	-	-	8	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists$ three g_5^1	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	12	-	-	-
2	-	-	-	12	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g_7^2$	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	24	-	-	-
2	-	-	-	24	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g_4^1$	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	75	24	5	-	-
2	-	-	5	24	75	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g_4^1 \times g_5^1$	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	75	44	5	-	-
2	-	-	5	44	75	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g_6^2$	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	90	64	20	-	-
2	-	-	20	64	90	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g_3^1$	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	70	105	84	35	6	-
2	-	6	35	84	105	70	21	-
3	-	-	-	-	-	-	-	1

Die Existenz mehrerer $g_5^1{}'$e muss dabei so interpretiert werden, dass einige der Linearsysteme auch doppelt oder dreifach zu zählen sind. Es stellt sich heraus, dass eine allgemeine Kurve im Stratum, welches durch diese Betti Zahlen gegeben ist, eine entsprechende Anzahl unterschiedlicher $g_5^1{}'e$ besitzt.
Bei der Berechnung der Betti Tableaus orientieren wir uns an der Vorgehensweise Schreyers in [S86]: Mit Hilfe eines g_d^1 auf C konstruieren wir einen $(d-1)$-dimensionalen Scroll $X \subset \mathbb{P}^{g-1}$, welcher die Kurve C enthält. Die Rulings auf dem Scroll schneiden dann auf C das entsprechende g_d^1 aus. Wir erhalten eine freie Auflösung von $\mathcal{O}_C$ als $\mathcal{O}_X$-Modul. Eine entsprechende Abbildungszylinderkonstruktion liefert dann eine (nicht unbedingt minimale) freie Auflösung von S_C. Hierbei treten schiefsymmetrische Matrizen ψ mit Einträgen aus globalen Schnitten von Vektorbündeln $\mathcal{O}_X(aH+bR)$, $a,b \in \mathbb{Z}$, auf, welche die freie Auflösung bestimmen. Schließlich sind die Ränge eventuell auftretender, nicht minimaler Abbildungen zu berechnen.
Von besonderem Interesse ist der Fall $\mathrm{Cliff}(C) = 3$: Im Falle der Existenz eines g_7^2 liegt die Kurve auf einer sogenannten Bordigatypfläche, deren Betti Tableau bereits das der Kurve festlegt. Existiert jedoch kein g_7^2, so kann man in jedem Fall eine determinantielle Fläche $Y \subset X$ auf dem Scroll angeben, welche die Kurve enthält: Hat C genau ein g_5^1 mit einfacher Multiplizität, so ist Y das Bild einer Aufblasung des $\mathbb{P}^2$ und im Falle der Existenz mehrerer $g_5^1{}'$e das Bild einer Aufblasung von $\mathbb{P}^1 \times \mathbb{P}^1$. Im Fall, dass ein g_5^1 mit höherer Multiplizität auftritt, erhalten wir Y als Bild einer Aufblasung der zweiten Hirzebruchfläche.

Im Kapitel "Summary" geben wird ein kurzen Ausblick auf die Behandlung der Fälle $g = 10$ und $g = 11$.

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Introduction

In this thesis we discuss curves $C \subset \mathbb{P}^{g-1}$ of geometric genus $g = 9$, which are embedded by the complete linear series $|\omega_C|$ associated to the canonical bundle ω_C. It is known that for $C \subset \mathbb{P}^n$ embedded by a very ample, complete linear series $|\mathcal{L}|$, properties of the homogeneous coordinate ring $S_C = S/I_C$, as its graded Betti numbers depend both on the curve itself and on $\mathcal{L}$ (cf. [E05]) . Here we denote by $S = \mathbb{k}[x_0, ..., x_n] = \operatorname{Sym} H^0(C, \mathcal{L})$ the homogenous coordinate ring of $\mathbb{P}^n$. In the case $\mathcal{L} = \omega_C$ we get an embedding of C in $\mathbb{P}^{g-1}$ if C is non-hyperelliptic and properties of S_C are intrinsic properties of C alone. For this reason one could ask, if it is possible to deduce geometric properties of C directly from the algebraic properties of S_C. Hilbert gave us a first answer to this problem by introducing the Hilbert polynomial, which contains information about the dimension and degree of the embedding. For more detailed information of the embedded curve C we have to go further: Hilbert showed that there exists a minimal free resolution of S_C as $S-$module. This resolution, especially its Betti table, contains further information, so we can ask in general which geometric properties are encoded in the Betti numbers. From the Castelnuovo-Mumford regularity of S_C and its Gorenstein property we know that the Betti table of the minimal free resolution of S_C has the following form

$g-1$

1							
	β_{12}	$\cdots$					
					$\cdots$	$\beta_{g-3,g-1}$	
							1

p

with a positive integer $p \leq \frac{g-3}{2}$, the number of leading zeros in the third row. Following Green we say that C fullfills N_p in this situation if there are at least p zeroes. Green conjectured that there exists a direct correspondance between the geometry of C and the N_p property:

Conjecture 0.0.1 *(Green, 1984) Let $C \subset \mathbb{P}^8$ be a smooth nonhyperelliptic curve over a field of characteristic 0 in its canonical embedding. Then*

$$\beta_{p,p+2} \neq 0 \Leftrightarrow C \textit{ has Clifford index } \operatorname{Cliff}(C) \leq p$$

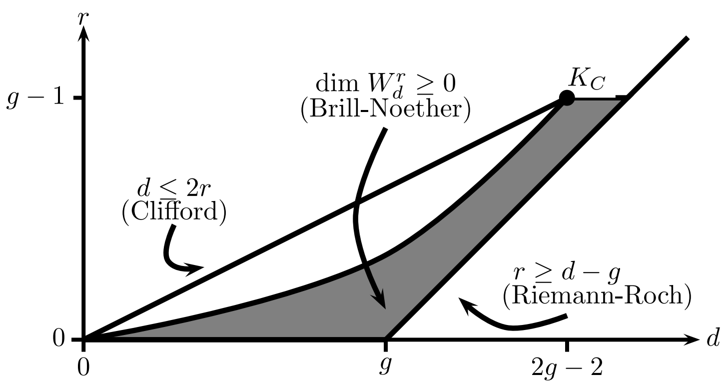
The Clifford index of an effective divisor D on C is defined as

$$\operatorname{Cliff}(D) = \deg D - 2(h^0\mathcal{O}_C(D) - 1)$$

and the Clifford index of C is

$$\operatorname{Cliff}(C) = \min\left\{\operatorname{Cliff}(D) : D \in \operatorname{Div}(C) \text{ effective with } h^0\mathcal{O}_C(D), h^1\mathcal{O}_C(D) \geq 2\right\}$$

For a better understanding of this definition, we state some results of Brill-Noether Theory, that studies the question whether there exists a certain special linear series $|D|$ on a curve C. $C_d^r \subset \operatorname{Div} C$ as usual denotes the variety of all divisors D of degree d that fulfill $r(D) = \dim |D| \geq r$. Further g_d^r denotes an element of $W_d^r = \{|D| : \deg D = d, r(D) \geq r\} \subset \operatorname{Pic}^d(C)$. Brill Noether Theory (cf. [ACGH85]) then gives us a lower bound of the dimension of this variety which is sharp for a generic curve. Together with the Riemann Roch Theorem which says that $r = \dim |D| \geq d - g$ and Clifford's Theorem we obtain the following picture:



The gray shaded area gives us bounds for which $\dim W_d^r \geq 0$ for a generic curve C of genus g. Clifford's Theorem says that for an effective Divisor D on C of degree $d \leq 2g-1$ we must have $r = h^0\mathcal{O}_C(D) - 1 \leq \frac{d}{2}$. If equality holds then either D is zero, or D is a canonical divisor, or C is hyperelliptic and D is linearly equivalent to a multiple of a hyperelliptic divisor.
In the last situation where C is hyperelliptic the canonical map

$$j : C \to \mathbb{P}(H^0(C, \omega_C)) = \mathbb{P}^{g-1}$$

is a $2:1$ map π onto a rational normal curve which is a $(g-1)-$uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^{g-1}$.

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "\pi"'] & \mathbb{P}^{g-1} \\ \mathbb{P}^1 \arrow[ur, hook, "v_{g-1}"'] & \end{tikzcd}$$

The sheaf $\pi_*\mathcal{O}_C$ is a rank two vector bundle $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(a)$ on $\mathbb{P}^1$ with an $a\in\mathbb{Z}^-$. As $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)\cong\mathcal{O}_C(D)$ we get $\pi^*\mathcal{O}_{\mathbb{P}^1}(2g-2)\cong\mathcal{O}_C(D)^{\otimes 2g-2}=\omega_C^{\otimes 2}$ and thus from the projection formula $\pi_*\pi^*\mathcal{O}_{\mathbb{P}^1}(2g-2)\cong\pi_*\omega_C^{\otimes 2}\cong\mathcal{O}_{\mathbb{P}^1}(2g-2)\otimes\pi_*\mathcal{O}_C$. Comparing dimensions of the global sections it follows that $a=-g-1$. Then $\Omega=\sum_{n\geq 0}H^0(C,\omega_C^{\otimes n})$ regarded as a module over the homogenous coordinate ring $S=\operatorname{Sym}H^0(C,\omega_C)$ of $\mathbb{P}^{g-1}$ is the module of global sections of the rank 2 vector bundle $j_*\mathcal{O}_C\cong\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-g-1)$ on the rational normal curve $\mathbb{P}^1\subset\mathbb{P}^{g-1}$. The Betti table for this rational normal curve is given as follows:

	0	1	2	$\cdots$	i	$\cdots$	g-2
0	1	-	-	-	-	-	-
1	-	$\binom{g-1}{2}$	$2\binom{g-1}{3}$	$\cdots$	$i\binom{g-1}{i+1}$	$\cdots$	$g-2$

and as $\mathcal{O}_{\mathbb{P}^1}(-2)\cong\ \omega_{\mathbb{P}^1}$ the curve C has the following Betti table:

	0	1	2	$\cdots$	g-4	g-3	g-2
0	1	-	-	-	-	-	-
1	-	$\binom{g-1}{2}$	$2\binom{g-1}{3}$	$\cdots$	$\cdots$	$\cdots$	$g-2$
2	$g-2$	$\cdots$	$\cdots$	$\cdots$	$2\binom{g-1}{3}$	$\binom{g-1}{2}$	-
3	-	-	-	-	-	-	1

For non hyperelliptic C the canonical map j is an embedding and Ω is the homogenous coordinate ring of $C\subset\mathbb{P}^{g-1}$ (cf. [N1880]). In this situation the definition of the Clifford index gives a natural measure of the speciality of a divisor D on C.

Green and Lazarsfeld proved in [GL84] that from the existence of special linear systems on C it follows the existence of exceptional syzygies. Due to recent works of Hirschowitz and Ramanan in [HR98] and Voisin in [V05], we know that Green's Conjecture holds for curves C of odd genus g with maximal Clifford index $\frac{g-1}{2}$.

Theorem 0.0.2 *(Hirschowitz-Ramanan-Voisin) Let C be a smooth curve C of genus $g=2k+1\geq 5$ with Betti number $\beta_{k,k+1}\neq 0$, then there exists a g^1_{k+1} on C.*

Moreover for curves of genus $g\leq 9$ Green's conjecture holds which follows from results of Mukai in [M95] and Schreyer in [S91]. From further results of Max Noether, Petri, Voisin and Schreyer we know that curves which fullfill N_p for $p\leq 2$ have special linear series of Clifford index p. A substantial progress has been done in [V01] 2001 and [V05] 2005, where Voisin proved the conjecture for general $k-$gonal curves of arbitrary genus.

Going one step further one might ask which Betti tables actually occur for irreducible, nonsingular curves C. For curves of genus $g\leq 8$ this is already done in [S86] (1986). Based on computational evidence, Schreyer gives a conjectural collection of Betti tables for curves of genus 9, 10 and 11 in [S03] (2003).

The result of this thesis is a complete table for smooth curves of genus 9. It turns out that the conjectural table in [S03] is correct with the exception that a curve C of genus 9 can even admit three linear systems of type g^1_5 (counted with multiplicities) and no g^2_7.

general

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	-	-	-	-
2	-	-	-	-	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g^1_5$

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	4	-	-	-
2	-	-	-	4	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists$ two g^1_5

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	8	-	-	-
2	-	-	-	8	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists$ three g^1_5

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	12	-	-	-
2	-	-	-	12	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g^2_7$

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	24	-	-	-
2	-	-	-	24	70	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g^1_4$

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	75	24	5	-	-
2	-	-	5	24	75	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g^1_4 \times g^1_5$

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	75	44	5	-	-
2	-	-	5	44	75	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g^2_6$

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	90	64	20	-	-
2	-	-	20	64	90	64	21	-
3	-	-	-	-	-	-	-	1

$\exists\, g^1_3$

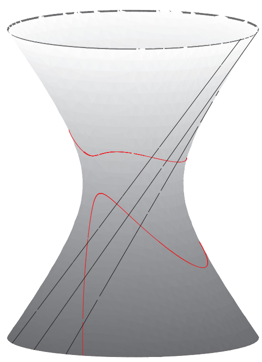
	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	70	105	84	35	6	-
2	-	6	35	84	105	70	21	-
3	-	-	-	-	-	-	-	1

The interpretation of the existence of several $g^{1\prime}_5 s$ has to be taken as a count with multiplicity. It is true that a general curve in the strata defined by these Betti numbers has that many $g^{1\prime}_5 s$ precisely.

To obtain the Betti numbers from a curve C that admits special divisors D of degree d as in Green's Conjecture we follow the approach in [S86]. We first construct a rational normal scroll $X \subset \mathbb{P}^{g-1}$ from a pencil $(D_\lambda)_\lambda$ of divisors $D_\lambda \sim D$:

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \bar{D}_\lambda \subset \mathbb{P}^{g-1}$$

where $\bar{D}_\lambda$ denotes the linear span of D_λ.



The vanishing ideal I_X of this scroll is given by the 2×2 minors of a $2 \times f$ matrix, $f = h^0\mathcal{O}_C(K_C - D)$, with linear entries in S. The minimal free resolution of the homogenous coordinate ring S_X takes the following simple form:

1				
	β_{12}	$\cdots$		

$g-d+1$

and as C is contained in X the syzygies of S_X are also syzygies of S_C. Further there exists a corresponding $\mathbb{P}^{d-2}$-bundle $\mathbb{P}(\mathcal{E})$ of degree $d-1$ over $\mathbb{P}^1$, which is a desingularization of X. With $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(e_{d-1})$ we say that X is of type $S(e_1, ..., e_{d-1})$. The type of a scroll constructed in this way can be determined by the values $h^0\mathcal{O}_C(iD)$ for $i \in \mathbb{N}$. Schreyer gives in [S86] a resolution of $\mathcal{O}_C$ in terms of $\mathcal{O}_X$-modules. The minimal free resolution of these modules in $\mathbb{P}^{g-1}$ are given by complexes $\mathcal{C}^i$. Then Schreyer shows in [S86] that a mapping cone construction leads to a free resolution of $C \subset \mathbb{P}^{g-1}$. Unfortunately this resolution can contain non minimal maps, so it remains to determine the ranks of them.

In Chapter 3 we repeat the results from [S86] for the case of trigonal and tetragonal curves. In the main part of the thesis we focus on pentagonal curves $C \subset \mathbb{P}^8$ of genus 9. We distinguish two cases with $\mathrm{Cliff}(C) = 3$:

I. (*C admits a g^2_7*) If there exists a g^2_7 on C then we get a plane model $C' \subset \mathbb{P}^2$ of C having six double points as only singularities. Projection from each of them leads to a g^1_5. The image $S \subset \mathbb{P}^8$ of the blowup of $\mathbb{P}^2$ in these 6 points under the adjoint series is a Bordiga surface $S' \subset \mathbb{P}^8$ which is smooth iff C' has no infinitely near double points. It has only isolated rational singularities that come from contraction of a strict transform E'_i of a point p_i that admits one infinitely near double point p_j. The minimal free resolution for S' determines the Betti table for C.

II. (*C admits no g^2_7*) Now we assume that C has no g^2_7. Then the scroll X constructed from one of the existing $g^1_5 = |D|$ turns out to be of type $S(2,1,1,1)$, $S(2,2,1,0)$ or $S(3,1,1,0)$. We define the multiplicity $m_{|D|}$ of the linear system $|D|$ to be equal to one, two or three depending on that type. Later on we will show that there also exists a geometric interpretation that justifies this definition: Let $k = \sum_{|D| \sim g^1_5} m_{|D|}$ be the total number of all g^1_5 (counted with multilplicities), then there exists a local one parameter family $(C_\lambda)_\lambda$ with $C_0 = C$ and C_λ a curve with Clifford index 3 that has exactly k ordinary g^1_5. Following the approach of Schreyer in [S86] we consider a representation of $C \subset \mathbb{P}(\mathcal{E})$ as vanishing locus of the Pfaffians of a 5×5 skew symmetric matrix ψ with entries in $\mathbb{P}(\mathcal{E})$ obtained from the structure theorem for Gorenstein ideals of codimension 3. Let H denote the class of a hyperplane section and R that of a ruling on

X, then a closer examination of all possible types for ψ leads to the following complete table:

Table for curves with a g^1_5 but no g^2_7

Special Linear Series	**Determinantal Surface Y** $C \subset Y \subset X \subset \mathbb{P}^8$	**Type of matrix ψ**
$\exists!\ g^1_5$ with $m_{g^1_5} = 1$	$\mathbb{P}^2$ blown-up in 9 doublepoints and 1 triple point of a g^2_8	$\begin{pmatrix} 0 & H & H & H & H \\ & 0 & H-R & H-R & H-R \\ & & 0 & H-R & H-R \\ & & & 0 & H-R \\ & & & & 0 \end{pmatrix}$ on $S(2,1,1,1)$
$\exists$ exactly two g^1_5 with $m_{g^1_5} = 1$	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 7 double points of a $g^1_5 \times g^1_5$	$\begin{pmatrix} 0 & H & H & H & H \\ & 0 & H-R & H-R & H-R \\ & & 0 & H-R & H-R \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$ on $S(2,1,1,1)$
$\exists$ exactly three g^1_5 with $m_{g^1_5} = 1$	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 7 double points $p_1, \ldots, p_7$ of a $g^1_5 \times g^1_5$ but $p \in C \smallsetminus \{ p_1, \ldots, p_7\}$ is base point of $\vert(2,2) - p_1 - \ldots - p_7\vert$	$\begin{pmatrix} 0 & H & H & H & H \\ & 0 & 0 & H-R & H-R \\ & & 0 & H-R & H-R \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$ on $S(2,1,1,1)$
$\exists!\ g^1_5$ with $m_{g^1_5} = 2$	$P_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$ blown-up in 7 double points $p_1, \ldots, p_7$	$\begin{pmatrix} 0 & H+R & H & H & H \\ & 0 & H & H & H-R \\ & & 0 & H-R & H-2R \\ & & & 0 & H-2R \\ & & & & 0 \end{pmatrix}$ on $S(2,2,1,0)$
$\exists!\ g^1_5$ with $m_{g^1_5} = 3$	$P_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$ blown-up in 7 double points $p_1, \ldots, p_7$ lying on a rational curve of class $A + B$	$\begin{pmatrix} 0 & H+R & H & H & H \\ & 0 & H & H & H-R \\ & & 0 & H-R & H-2R \\ & & & 0 & H-2R \\ & & & & 0 \end{pmatrix}$ on $S(3,1,1,0)$
$\exists!\ g^1_5$ with $m_{g^1_5} = 1$ and $\exists!\ g^1_5$ with $m_{g^1_5} = 2$	$\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 7 double points $p_1, \ldots, p_7$ of a $g^1_5 \times g^1_5$ lying on a rational curve of type $(2,1)$	$\begin{pmatrix} 0 & H & H & H & H \\ & 0 & H-R & \mathbf{F} & H-R \\ & & 0 & H-R & \mathbf{F} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$ on $S(2,1,1,1)$ with an entry $\mathbf{F} \sim H - R$

Here $Y \subset X$ is a surface on the scroll X given by the 2×2 minors of a matrix

$$\omega \sim \begin{pmatrix} H - a_1 R & H - a_2 R & H - a_3 R \\ H - (a_1 + k)R & H - (a_2 + k)R & H - (a_3 + k)R \end{pmatrix}$$

with entries in $\mathbb{P}(\mathcal{E})$.

The non minimal maps in the mapping cone construction, representing a free resolution of $C \subset \mathbb{P}^8$, are given by certain submatrices of ψ. Calculating ranks in the individual cases, it turns out that the entries $\beta_{35} = \beta_{45}$ in the Betti table for C are given by $4k$ with k the number of all g^1_5 counted with multiplicities.

Acknowledgement. I'm grateful to the DFG for financial support during my studies. I want to thank Dr. Stavros Papadakis who spent a lot of his time giving answer to my questions. I benefitted a lot from a multitude of discussions with him. I'm also grateful to Janko Boehm and Michael Kunte who read this text and pointed out a lot of things that needed to be fixed. Dominik Dietrich has put a lot of effort into the layout. If it looks well then it is due to him. Thank you.
Special thanks go to Prof. Dr. Frank-Olaf Schreyer who suggested the problem to me. I want to thank him for his guidance, encouragement and patience. This thesis is inspired by his ideas.

Notation Throughout the text $\mathbb{k}$ denotes an algebraically closed field of characteristic 0. $S = \mathbb{k}[x_0, ..., x_n]$ denotes the homogeneous coordinate ring of $\mathbb{P}^n$ and $\mathfrak{m} = (x_0, ..., x_n) \subset S$ its maximal ideal. For two divisors D and D' on a variety V we write $D \sim D'$ iff they are linear equivalent.

1

Background

1.1 Syzygies

A projective variety $X \subset \mathbb{P}^n$ can be described by its vanishing ideal $I_X \subset S := \Bbbk[x_0, ..., x_n]$. We denote the corresponding homogeneous coordinate ring by $S_X := S/I_X$. Because of the Noetherian property of S, there exists a finite number of generators of I_X. These generators can also have certain relations which can be described as a finitely generated module over S. Then we consider the relations of these relations and so on. Hilbert famous Syzygy Theorem says that this process stops after finitely many steps:

Theorem 1.1.1 *(Hilbert Syzygy Theorem) Any finitely generated graded S-module M has a finite graded free resolution*

$$0 \leftarrow M \overset{\varphi_0}{\leftarrow} F_0 \overset{\varphi_1}{\leftarrow} F_1 \leftarrow ...F_{m-1} \overset{\varphi_m}{\leftarrow} F_m \leftarrow 0$$

with free S-modules F_i, $i = 0, ..., m \leq n + 1$.

Proof. [E05] sect.2B. ■

Unfortunately a free resolution of S_X has not to be minimal. But if we choose a minimal set of generators in each step above then we obtain a minimal free resolution, i.e. every map φ_i, $i = 0, ..., m$, has no degree zero part or equivalently it is not possible to seperate a trivial subcomplex

$$0 \leftarrow S(-d) \overset{\cong}{\leftarrow} S(-d) \leftarrow 0$$

To give a formal definition for this property we use the standard notation $\mathfrak{m}$ to denote the homogeneous maximal ideal $(x_0, ..., x_n) \subset S$:

Definition 1.1.2 *A complex of graded S-modules*

$$... \leftarrow F_{i-1} \overset{\varphi_i}{\leftarrow} F_i \leftarrow ...$$

is called minimal if for each i the image of φ_i is contained in $\mathfrak{m}F_{i-1}$.

Given a resolution of S_X, we obtain a minimal free resolution by canceling trivial subcomplexes and in consequence we have the following important property

Theorem 1.1.3 *Every minimal free resolution of a graded S-module M is unique up to an isomorphism of complexes inducing the identity map on M.*

Proof. [E05] Section 1. ■

According to the last theorem, it follows the important fact that for each minimal free resolution of a finitely generated graded $S-$module the number of generators of each degree $j \in \mathbb{Z}$ required for the free modules F_i is the same in every minimal free resolution. We call these numbers β_{ij} the Betti numbers of M.

Definition 1.1.4 *(Betti numbers) Let M be a finitely generated, graded S-module and*

$$0 \leftarrow M \overset{\varphi_0}{\leftarrow} F_0 \overset{\varphi_1}{\leftarrow} F_1 \leftarrow ...F_{m-1} \overset{\varphi_m}{\leftarrow} F_m \leftarrow 0$$

a minimal free resolution of M with free modules $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$ *, then we call the numbers* β_{ij} *the syzygy numbers or graded Betti numbers of M.*

For $X \subset \mathbb{P}^n$ a projective variety we call the Betti numbers of X those of the homogenous coordinate ring S_X. Given a set of generators of the vanishing ideal of X, it is possible to determine a minimal free resolution of S_X by Gröbner Basis algorithms (implemented in Macaulay2, Singular,...) after finitely many steps. We use the Macaulay notation to write down the Betti table of such a resolution:

	0	1	...	m-1	m
0	β_{00}	β_{11}	....	$\beta_{m-1,m-1}$	$\beta_{m,m}$
1	β_{01}	β_{12}		$\beta_{m-1,m}$	$\beta_{m,m+1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
r	$\beta_{0,r}$	$\beta_{1,r+1}$	....	$\beta_{m-1,m+r-1}$	$\beta_{m,m+r}$

Example 1.1.5 *(complete intersection in $\mathbb{P}^3$)* A complete intersection of two hypersurfaces of degree 2 in $\mathbb{P}^3$ is an elliptic curve $C \subset \mathbb{P}^3$. I_C is generated by exactly two quadratic forms $q_1, q_2 \in S_2$ and the only relation between them is the Koszul relation $-q_2 \cdot q_1 + q_1 \cdot q_2 = 0$, hence the minimal free resolution of S_C is given by

$$0 \leftarrow S_C \leftarrow S \leftarrow S(-2) \oplus S(-2) \leftarrow S(-4) \leftarrow 0$$

and has the following Betti table

	0	1	2
0	1	-	-
1	-	2	-
2	-	-	1

Example 1.1.6 *(The Koszul complex in general)* Let S be a ring and $N \cong S^{n+1}$ a free $S-$module of rank $n+1$. For $x = (x_0, ..., x_n) \in N$ we define the Koszul complex to be the complex

$$K(x) : 0 \to R \to N \to \wedge^2 N \to ... \to \wedge^i N \stackrel{d_x}{\to} \wedge^{i+1} N \to ...$$

where d_x sends an element a to $x \wedge a$. Notice that $\wedge^0 N \cong S$. In particular for $S = \Bbbk[x_0, ..., x_n]$ the graded homogeneous polynomial ring in $\mathbb{P}^n$ and $x = (x_0, ..., x_n)$ we get a graded free resolution of $\Bbbk$ as $S-$module:

$$K(x): \ 0 \to S(-n-1) \to S^{n+1}(-n) \to S^{\binom{n+1}{2}(-n+1)} \to ... \to S^{\binom{n+1}{i}}(-n+i-1) \stackrel{d_x}{\to}$$

$$\stackrel{d_x}{\to} S^{\binom{n+1}{i+1}}(-n+i) \to ... \to \wedge^{n+1} S^{n+1} \cong S \to \Bbbk \to 0$$

as the cokernel of $\wedge^n N \to \wedge^{n+1} N$ is isomorphic to $S/(x_0, ..., x_n) \cong \Bbbk$ (cf. [E95] page 428).

We derive further properties of the Betti numbers:

Theorem 1.1.7 *Let M be a finitely generated, graded S-module and*

$$0 \leftarrow M \stackrel{\varphi_0}{\leftarrow} F_0 \stackrel{\varphi_1}{\leftarrow} F_1 \leftarrow ...F_{m-1} \stackrel{\varphi_m}{\leftarrow} F_m \leftarrow 0$$

a free resolution of M with free modules $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$. Then the Hilbert function H_M is

$$H_M(d) = \textstyle\sum_{i=0}^m (-1)^i \sum_j \beta_{ij} \displaystyle\binom{n+d-j}{n}$$

Proof. [E05] Corollary 1.2 ■

It is a direct consequence of the theorem above that for sufficently large d, the Hilbert function becomes polynomial:

Theorem 1.1.8 *There exists a polynomial P_M (called the Hilbert polynomial) such that, if M has a free resolution as above, $P_M(d) = H_M(d)$ for $d \geq \max_{i,j}\{\beta_{ij} - n\}$.*

Proof. For $d \geq \max_{i,j}\{j - n : \beta_{ij} \neq 0\}$ we get $n + d - j \geq 0$ in every binomial coefficient $\binom{n+d-j}{n}$, thus it becomes polynomial in d. ■

In consequence, we are able to calculate the Hilbert function and the Hilbert polynomial from the Betti numbers. In our definition of the graded Betti numbers β_{ij} of a minimal free resolution of an $S-$module M, we used the important fact, that these numbers are uniquely determined by M and do not depend on the minimal resolution.

There is a further way of introducing the Betti numbers β_{ij} and the space $F_{i,j}$ of $(i-1)$-th syzygies of degree j of the module S_X as

$$F_{i,j} = \mathrm{Tor}_i^S(S_X, \mathbb{k})_j$$

These two definitions coincide as proved in [E95] Exercise A 3.18 page 639. The advantage of this second definition of the Betti numbers is, that $\mathrm{Tor}_i^S(S_X, \mathbb{k})_j$ can be computed as well by a free resolution of S_X as by one of $\mathbb{k}$ in terms of free $S-$modules, which is given by the Koszul komplex (cf. Example 1.1.6). But it is not possible in general to obtain the complete Betti table only from the values of the Hilbert function. Therefore we need more information of an $S-$module M such as the Castelnuovo-Mumford regularity :

Definition 1.1.9 *(Castelnuovo-Mumford regularity) The Castelnuovo-Mumford regularity of M is given by*

$$\operatorname{reg} M = \max(i - j, \beta_{ij} \neq 0)$$

For X a projective variety we define $\operatorname{reg} X := \operatorname{reg} I_X$ *with I_X the vanishing ideal of X.*

The first property of the regularity of a module M is that it gives us a rather good lower boundary for the natural numbers d such that the Hilbertfunction $H_M(d)$ agrees with the Hilbert polynomial $P_M(d)$:

Theorem 1.1.10 *Let M be a finitely generated graded module over the polynomial ring $S = \mathbb{k}[x_0, ..., x_n]$, then*

1) $H_M(d) = P_M(d)$ for all $d \geq \operatorname{reg} M + \operatorname{pd} M - n$

2) For M Cohen-Macaulay the boundary in 1) is sharp.

Proof. [E05] Theorem 4.2. ■

The following theorem helps us to calculate the regularity in terms of the vanishing of cohomology groups. It says that for a projective variety X, $\operatorname{reg} I_X$ can be obtained as the minimal value r_0, such that I_X is $r-$regular for all $r \geq r_0$:

Theorem 1.1.11 *Let $X \subset \mathbb{P}^n$ be projective variety and r_0 the minimal number with*

$$H^i(\mathbb{P}^n, \mathcal{I}_X(r - i)) = 0 \text{ for all } i > 0 \text{ and all } r \geq r_0$$

Then $\operatorname{reg} X = \operatorname{reg} I_X = r_0$.

Proof. [E95] Exercise 20.20. ■

According to the definition of $\operatorname{reg} M$ the regularity gives the number of rows in the Betti table of M with nonzero entries. The number of columns with nonzero entries is determined by the projective dimension of M :

Definition 1.1.12 *(projective dimension) For M an $S-$module, the projective dimension* $\operatorname{pd} M$ *is the minimal length of a projective resolution of M.*

There is a direct correspondence of the projective dimension of a module M and the length $\operatorname{depth}(\mathfrak{m}, M)$ of a maximal M-sequence in $\mathfrak{m}$, $\mathfrak{m}$ the maximal ideal of S, given by the the Auslander-Buchsbaum formula:

Theorem 1.1.13 *(Auslander-Buchsbaum formula) Let S be a graded ring with maximal ideal $\mathfrak{m}$ and M a finitely generated $S-$module with* $\operatorname{pd} M < \infty$*. Then*

$$\operatorname{pd} M = \operatorname{depth}(\mathfrak{m}, S) - \operatorname{depth}(\mathfrak{m}, M)$$

Proof. [E95] Theorem 19.9 and Exercise 19.8, page 479 and 489. ■

For $S = \Bbbk[x_0, ..., x_n]$ we get $\operatorname{depth}(\mathfrak{m}, M) = n+1$, hence $\operatorname{depth}(\mathfrak{m}, M)$ and $\operatorname{pd} M$ can directly be obtained from the Betti table of M.
For $X \subset \mathbb{P}^n$ a projective variety, the definition of being arithmetically Cohen-Macaulay says that $\operatorname{depth}(\mathfrak{m}, S_X) = \dim R_X$, hence X is arithmetically Cohen-Macaulay if and only if

$$\max\{i | \beta_{ij} \neq 0 \text{ for at least one } j\} = \operatorname{codim} X$$

It will turn out that a canonical curve C is always arithmetically Cohen Macaulay. Further we will show that C is even arithmetically Gorenstein.

Definition 1.1.14 *Let $X \subset \mathbb{P}^n$ be a projective, arithmetically Cohen-Macaulay variety and* $\operatorname{codim} X = c$*. Then we call X arithmetically Gorenstein if and only if there exists an $n \in \mathbb{Z}$ with*

$$\operatorname{Ext}^c(S_X, S) = S_X(n)$$

If F is a minimal free resolution of S_X then the minimal resolution of $\operatorname{Ext}^c(S_X, S)$ can be obtained as the dual F^* of F. Now it is an easy consequence that for X arithmetically Gorenstein, F has to be selfdual, i.e. for the Betti numbers we get

$$\beta_{ij} = \beta_{c-i,n-j}$$

with c, n as in the definition above.
In Example 1.1.5 we have seen that for an elliptic curve $C \subset \mathbb{P}^3$ given as complete intersection of two quadrics, its homogenous coordinate ring S_C has the following symmetric Betti table

	0	1	2
0	1	-	-
1	-	2	-
2	-	-	1

Hence from the results above it follows that C is arithmetically Gorenstein.

1.2 Canonical Curves and Green's Conjecture

As we have remarked above we want to show that a canonical curve $C \subset \mathbb{P}^{g-1}$ is arithmetically Gorenstein. Further we will see that $\operatorname{reg} C = 3$ and as the Hilbert function of S_C can easily be calculated from the values $h^0(C, mK_C)$, $m \in \mathbb{N}$ and K_C a canonical divisor on C, we get a first approximation for the Betti table of S_C.
Let C be a smooth, non hyperelliptic curve and K_C its canonical divisor, then we can embed C in $\mathbb{P}^{g-1}$ canonically:

$$\varphi_{|K_C|} : C \stackrel{|K_C|}{\to} \mathbb{P}^{g-1} = \mathbb{P}H^0(C, \mathcal{O}_C(K_C))$$

We denote by I_C the vanishing ideal of $C \subset \mathbb{P}^{g-1}$ and S_C its homogeneous coordinate ring. The following theorem due to Max Noether says that $C \subset \mathbb{P}^{g-1}$ is embedded projectively normal and thus C is arithmetically Cohen-Macaulay:

Theorem 1.2.1 *(Max Noether) If C is non hyperelliptic then*

$$\Omega = \textstyle\sum_{m \geq 0} H^0(C, \mathcal{O}_C(K_C)^{\otimes m})$$

is the homogenous coordinate ring of $C \subset \mathbb{P}^{g-1}$. It follows that $H^1(C, \mathcal{I}_C(m)) = 0$ for all $m \geq 0$.

Proof. [ACGH85] page 117. ■

Now as an easy consequence of Max Noether's Theorem and Theorem 1.1.10 we obtain the following corollary:

Corollary 1.2.2 *Let $C \subset \mathbb{P}^{g-1}$ be the canonical model of a non hyperelliptic curve of genus $g \geq 3$, then the Hilbert function H_{S_C} takes the following values:*

$$H_{S_C}(d) = \begin{cases} 0 & \text{if } d < 0 \\ 1 & \text{if } d = 0 \\ g & \text{if } d = 1 \\ (2d-1)(g-1) & \text{if } d > 0 \end{cases}$$

In particular $\beta_{1,2}(S_C)$, the number of quadratic generators of I_C, is $\binom{g-1}{2}$ and $\operatorname{reg} S_C = 3$.

Proof. ([E05] Corollary 9.4.) From Max Noether's Theorem we already know that S_C is arithmetically Cohen-Macaulay and $(S_C)_d \cong H^0(C, \mathcal{O}_C(K_C)^{\otimes d}) = H^0(C, \mathcal{O}_C(dK_C))$. Therefore the values for H_{S_C} follow from the Riemann-Roch Theorem. From the Cohen Macaulay property we obtain the existence of a regular sequence on C consisting of 2 linear forms l_1, l_2. The regularity of S_C is the same as that of $S_C/(l_1, l_2)$. The Hilbert function of this last module has values $(1, g-2, g-2, 1)$, and thus applying Theorem 1.1.10 we have $\operatorname{reg} S_C = \operatorname{reg} S_C/(l_1, l_2) = 3$. ■

The Gorenstein property of S_C is a direct consequence of $\omega_C = \mathcal{O}_C(1)$. From the collected results we obtain that the Betti table of C has exactly $\operatorname{codim} C+1 = g-1$ columns and $\operatorname{reg} C + 1 = 4$ rows. Further it is symmetric and $\beta_{ii} = 0$ for all $i > 0$ as C is not contained in any hyperplane. Therefore the Betti table of C looks like:

$g-1$

1							
	β_{12}	$\cdots$					
					$\cdots$	$\beta_{g-3,g-1}$	
							1

with $\beta_{i,i+1} - \beta_{i-1,i+1} = i \cdot \binom{g-2}{i+1} - (g-1-i) \cdot \binom{g-2}{i-2}$, which we know from the values of the Hilbert function H_{S_C}. Due to recent results of Voisin in [V01] and [V05], for the generic curve over a field $\Bbbk$ of characteristic 0, the Betti numbers $\beta_{i-1,i+1}$ become zero for $i = 0, ..., \left\lfloor \frac{g-3}{2} \right\rfloor$. Especially in the case of our interest the Betti table of a generic curve of genus 9 looks as follows:

Example 1.2.3 *For $C \subset \mathbb{P}^{g-1}$ a generic curve of genus $g = 9$ the Betti table of S_C has the following form:*

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	-	-	-	-
2	-	-	-	-	70	64	21	-
3	-	-	-	-	-	-	-	1

The question that arises is which Betti numbers β_{ij} can be nonzero? A classical result of Petri is the following theorem:

Theorem 1.2.4 *(Petri) Let I_C be the homogenous ideal of a non hyperelliptic canonical curve $C \subset \mathbb{P}^{g-1}$, then I_C is generated by $\binom{g-1}{2}$ quadrics except the two cases where*

1) C is trigonal, i.e. there exists a g^1_3 or

2) C is isomorphic to a plane quintic ($g = 6$ and C has a g^2_5)

In these two exceptional cases the quadrics contained in I_C generate a rational normal scroll in 1) and the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ in 2).

Proof. [S-D73] ■

Thus we obtain $\beta_{1j} = 0$ for $j > 2$ except in the two exceptional cases 1) and 2). This result suggests that the values β_{ij} that are nonzero correspond to special linear systems on the curve C. Green's conjecture, if shown to be true, would give an exact answer to this question:

Conjecture 1.2.5 *(Green, 1984) Let $C \subset \mathbb{P}^{g-1}$ be a smooth non hyperelliptic curve over a field of characteristic 0 in its canonical embedding. Then*

$$\beta_{p,p+2} \neq 0 \Leftrightarrow C \text{ has Clifford index } \operatorname{Cliff}(C) \leq p$$

The Clifford index of an effective divisor D on C is defined as

$$\operatorname{Cliff}(D) = \deg D - 2r(D)$$

and the Clifford index of C is

$$\operatorname{Cliff}(C) = \min\{\operatorname{Cliff}(D) : D \in \operatorname{Div}(C) \text{ effective with } h^i\mathcal{O}_C(D) \geq 2 \text{ for } i = 1,2\}$$

As we have remarked in the introduction the conjecture is already shown to be true for the " $\Leftarrow$ " direction and also in several cases, especially for genus $g = 9$, for the other direction. We will give a complete list of all possible Betti tables for irreducible, nonsingular, canonical curves C of genus $g = 9$. For $\operatorname{Cliff}(C) \leq 2$ the results in [S86] can be applied to obtain these tables. This is done in Chapter 3. It remains to examine the case, where $\operatorname{Cliff}(C) = 3$. Then there exists a g^1_5, g^2_7, g^3_9 or g^4_{11}. The Brill Noether duals to g^3_9 and g^4_{11} are of type g^2_7 or g^1_5 correspondingly. From a g^2_7 we get a plane model of C of degree 7 that has exactly 6 double points. Then g^1_5 on C can be obtained from projection from one of the double points (cf. Theorem 4.3.1). Thus for a canonical curve C of genus 9 to be pentagonal is equivalent to $\operatorname{Cliff}(C) = 3$.

For $(D_\lambda)_\lambda$ a base point free pencil of divisors of degree, we consider the variety swept out by the linear spans of these divisors:

$$X = \bigcup\nolimits_{D \in g^1_5} \bar{D} \subset \mathbb{P}^8$$

This is a rational normal scroll of dimension 4. We apply the results of [S86] to obtain a minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}$-module:

1) Resolve $\mathcal{O}_C$ as an $\mathcal{O}_X$-module by direct sums of line bundles on X :

$$F_* : \qquad 0 \to \mathcal{O}_X(-5H + 3R) \to \sum_{i=1}^{5} \mathcal{O}_X(-3H + b_iR) \xrightarrow{\psi}$$

$$\xrightarrow{\psi} \sum_{i=1}^{5} \mathcal{O}_X(-2H + a_iR) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

2) Take the resolution $\mathcal{C}^b(a)$, $a, b \in \mathbb{Z}$, of each of these line bundle as $\mathcal{O}_{\mathbb{P}^8}$-module. Then a mapping cone construction leads to a (not necessarily minimal) resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}$-module:

$$\left[\left[\mathcal{C}^3(-5) \to \sum_i \mathcal{C}^{b_i}(-3)\right] \to \sum_i \mathcal{C}^{a_i}(-2)\right] \to \mathcal{C}^0$$

3) The non minimal parts of the mapping cone are related to submatrices of ψ. Hence to determine the ranks of these maps we have to study ψ. Then we obtain a minimal free resolution.

We will see that the matrix ψ and the ranks of the non minimal maps in the mapping cone are related to a certain number of special linear systems of type g^1_5 or g^2_7.

We will provide some results on scrolls and give a description how to manage the steps 1) and 2).

1.3 Scrolls in general

Let $\mathcal{E} = \mathcal{O}(e_1) \oplus ... \oplus \mathcal{O}(e_d)$, $e_1 \geq ... \geq e_d \geq 0$, be a globally generated, locally free sheaf of rank d on $\mathbb{P}^1$ and let

$$\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$$

the corresponding $\mathbb{P}^{d-1}$-bundle. For $f = \sum_{i=1}^{d} e_i \geq 2$ consider the image of $\mathbb{P}(\mathcal{E})$ in $\mathbb{P}^r = \mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$:

$$j : \mathbb{P}(\mathcal{E}) \to X \subset \mathbb{P}^r \quad , r = f + d - 1$$

Then we call X a rational normal scroll of type $S(e_1, ..., e_d)$. X is a non-degenerate, irreducible variety of minimal degree

$$\deg X = f = r - d + 1 = \operatorname{codim} X + 1$$

in $\mathbb{P}^r$.
If all $e_i > 0$ then X is smooth and $j : \mathbb{P}(\mathcal{E}) \to X$ an isomorphism. Otherwise X is singular and $j : \mathbb{P}(\mathcal{E}) \to X$ a resolution of singularities. The singularities of X are rational, i.e.

$$j_* \mathcal{O}_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_X, \quad R^i j_* \mathcal{O}_{\mathbb{P}(\mathcal{E})} = 0 \text{ for } i > 0$$

Therefore there is no problem to replace X by $\mathbb{P}(\mathcal{E})$ for most cohomological considerations, even if X is singular.

Remark 1.3.1 *(Geometric description of scrolls)* A rational normal scroll X of type $S(e_1, ..., e_d)$ admits the following geometric description: Consider the $e_i - th$ Veronese embedding of $\mathbb{P}^1$:

$$\gamma_i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^{e_i}$$

then the image Γ_{e_i} is a rational normal curve of degree e_i. Now we can embed the projective spaces $\mathbb{P}^{e_i}$ in $\mathbb{P}^r$, $r = d + f - 1 = \sum_{i=1}^{d}(e_i + 1) - 1$, as linearly independent subspaces of $\mathbb{P}^r$. Identifying the curves Γ_{e_i} with a common $\mathbb{P}^1$ we can take the linear span of corresponding points which leads us to the scroll X. If some of the $e_i = 0$ the image Γ_{e_i} under the mapping γ_i is a single point, thus X becomes a cone with center spanned by all points Γ_{e_i} where $e_i = 0$.

Now let us examine the Picard group of $\mathbb{P}(\mathcal{E})$. First we denote the hyperplane class $H = [j^*\mathcal{O}_{\mathbb{P}^r}(1)]$ and the ruling $R = [\pi_*\mathcal{O}_{\mathbb{P}^1}(1)]$. Then the following theorem contains the needed results:

Theorem 1.3.2 *(Picard Group of $\mathbb{P}(\mathcal{E})$)*
1) $\operatorname{Pic}\mathbb{P}(\mathcal{E}) = \mathbb{Z}H \oplus \mathbb{Z}R$
2) $H^d = f$, $H^{d-1}.R = 1$ *and* $R^2 = 0$
3) $K_X = -dH + (f-2)R$
4) There exist basic sections $\varphi_i \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H - e_iR))$ *and* $s, t \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R))$, *such that every section* $\psi \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR))$ *can be identified with a homogeneous polynomial*

$$\psi = \textstyle\sum_\alpha p_\alpha(s,t)\varphi_1^{\alpha_1}...\varphi_d^{\alpha_d}$$

of degree $a = \alpha_1 + ... + \alpha_d$ *in the* $\varphi_i's$ *and coefficients homogeneous polynomials* p_α *of degree*

$$\deg p_\alpha = \alpha_1 e_1 + ... + \alpha_d e_d + b$$

5) For $b \geq -1$ *the dimension* $h^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR))$ *does not depend on the type* $S(e_1, ..., e_d)$ *of the scroll but only on its degree* f :

$$h^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR)) = f\binom{a+d-1}{d} + (b+1)\binom{a+d-1}{d-1}$$

Especially $h^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H - R)) = f$.

Proof. For 1), 2) and 3) see [S86] 1.2.-1.7.. Applying the Leray spectral sequence we can calculate the cohomology of a line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR)$:

$$H^i(\mathbb{P}^1, R^j\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR)) \Rightarrow H^{i+j}(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR))$$

especially for $a \geq 0$ and $i = j = 0$ we obtain for the global section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR)$:

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR)) \cong H^0(\mathbb{P}^1, (S_a\mathcal{E})(b))$$

Then there exist basic sections $\varphi_i \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H - e_iR))$ obtained from the inclusion of the $i^{th}-$summand

$$\mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}(-e_i) \cong \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(H - e_iR)$$

Denote the global generators of $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R))$ by s and t, then there is a natural way to identify a section $\psi \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR))$ with a polynomial in the way as claimed. The formula for the dimension $h^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR))$ is a direct consequence of this representation. ∎

Example 1.3.3 *(X a scroll of type $S(2,1,1,1)$ in $\mathbb{P}^8$)* Let X be a scroll of type $S(2,1,1,1)$ and $\mathbb{P}(\mathcal{E})$ the corresponding $\mathbb{P}^3-$bundle over $\mathbb{P}^1$, then there exist basic sections $\varphi_0 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-2R))$ and $\varphi_1, \varphi_2, \varphi_3 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R))$. Especially the hyperplane sections $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H))$ of the scroll are generated by $x_0 = s^2\varphi_0$, $x_1 = st\varphi_0$, $x_2 = t^2\varphi_0$, $x_3 = s\varphi_1$, $x_4 = t\varphi_1, ..., x_8 = t\varphi_3$. It follows that the 2×2 minors of the 2×5 matrix

$$\Phi = \begin{pmatrix} x_0 & x_1 & x_3 & x_5 & x_7 \\ x_1 & x_2 & x_4 & x_6 & x_8 \end{pmatrix}$$

vanish on the scroll X. Moreover the following theorem says that they even generate its vanishing ideal.

Theorem 1.3.4 *Let X be a scroll of type $S(e_1, ..., e_d)$ and*

$$\Phi = \begin{pmatrix} x_{10} \; x_{11} \; .. \; x_{1e_1-1} & \dots & x_{d0} \; .. \; x_{de_d-1} \\ x_{11} \; x_{12} \; .. \; x_{1e_1} & \dots & x_{d1} \; .. \; x_{de_d} \end{pmatrix}$$

be the $2 \times f$ matrix given by the multiplication map

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R)) \otimes H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R)) \to H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H))$$

i.e. $x_{ij} = t^j s^{e_i - j}\varphi_i$, then the vanishing ideal I_X of X is generated by the 2×2 minors of Φ.

Proof. [S86] 1.6. ■

As we have already remarked at the end of the last section, we want to resolve the line bundles $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bR)$ in terms of $\mathcal{O}_{\mathbb{P}^r}-$modules. From our definition of a scroll X we have

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H)) \cong H^0(X, \mathcal{O}_X(H)) \cong H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$$

If we further denote

$$F = H^0\mathcal{O}(H-R) \otimes \mathcal{O}_{\mathbb{P}^r} \cong \mathcal{O}^f_{\mathbb{P}^r} \quad \text{and} \quad G = H^0\mathcal{O}(R) \otimes \mathcal{O}_{\mathbb{P}^r} \cong \mathcal{O}^2_{\mathbb{P}^r}$$

then the multiplication map

$$G \otimes F \to \mathcal{O}_{\mathbb{P}^r}(1)$$

in the theorem above induces a map

$$\Phi : F \otimes \mathcal{O}_{\mathbb{P}^r}(-1) \to G^* \otimes \mathcal{O}_{\mathbb{P}^r} \cong G$$

Now define the complexes $\mathcal{C}^b$ by

$$\mathcal{C}^b_j = \begin{cases} \bigwedge^j F \otimes S_{b-j}G \otimes \mathcal{O}_{\mathbb{P}^r}(-j) & \text{for} \quad 0 \le j \le b \\ \bigwedge^{j+1} F \otimes D_{j-b-1}G^* \otimes \mathcal{O}_{\mathbb{P}^r}(-j-1) & \text{for} \quad j \ge b+1 \end{cases}$$

and the differential map

$$\mathcal{C}_j^b \to \mathcal{C}_{j-1}^b$$

by the multiplication with $\Phi \in H^0(F^* \otimes G \otimes \mathcal{O}_{\mathbb{P}^r}(1))$ for $j \neq b+1$ and $\wedge^2\Phi \in H^0(\wedge^2 F^* \otimes \wedge^2 G \otimes \mathcal{O}_{\mathbb{P}^r}(2))$ for $j = b+1$.
E.g. For $j \leq b$ the differential of a term

$$f_1 \wedge ... \wedge f_j \otimes g \in H^0(\mathcal{C}_j^b),$$

with $f_i \in H^0(F \otimes \mathcal{O}_{\mathbb{P}^r}(-1))$, $g \in H^0(S_{b-j}G)$ is given by

$$f_1 \wedge ... \wedge f_j \otimes g \to \sum_{i=1}^{j}(-1)^i f_1 \wedge ... \wedge \hat{f}_i \wedge ... \wedge f_j \otimes \Phi(f_i) \cdot g$$

Theorem 1.3.5 *$\mathcal{C}^b(a)$ for $b \geq -1$ is the minimal free resolution of $\mathcal{O}_X(aH + bR)$ as an $\mathcal{O}_{\mathbb{P}^r}-$module.*

Proof. [S86] 2.2. ■

It follows that the resolution of $\mathcal{O}_X$ is given by $\mathcal{C}^0$. This is the well known Eagon-Northcott complex with Betti table as follows:

1				
	$\binom{f}{2}$	$2\binom{f}{3}$	$\cdots$	$(f-1)\binom{f}{f}$

f

1.4 Scrolls constructed from varieties

In the next step, for a linearly normal embedded smooth variety $V \subset \mathbb{P}^r$ and a pencil of divisors $(D_\lambda)_\lambda$ on V, we construct a scroll $X \subset \mathbb{P}^r$ such that $V \subset X$ and further the pencil $(D_\lambda)_\lambda$ is cut out on V by the class of a ruling R on X. Assume that D is a divisor on V with $h^0(V, \mathcal{O}_V(D)) \geq 2$ and $h^0(V, \mathcal{O}_V(H-D)) = f \geq 2$. Furthermore let $G \subset H^0(V, \mathcal{O}_V(D))$ be the 2 dimensional subspace that defines the pencil of divisors $(D_\lambda)_\lambda$, then from the multiplication map

$$G \otimes H^0(V, \mathcal{O}_V(H-D)) \to H^0(V, \mathcal{O}_V(H))$$

we obtain a $2 \times f$ matrix Φ with linear entries whose 2×2 minors vanish on V. It turns out that the variety $X \subset \mathbb{P}^r$ defined by these minors is a scroll of degree f, such that $(D_\lambda)_\lambda$ is cut out by the class of a ruling R on X. Geometrically X can be obtained as the union of the linear spans of D_λ:

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \bar{D}_\lambda \subset \mathbb{P}^r$$

Remark 1.4.1 *(Scroll X constructed from a g^1_d on a canonical curve C)* If $|D|$ is a base point free complete linear system of type g^1_d on C, then from the geometric version of Riemann-Roch we get

$$\dim \bar{D} = \deg D - \dim |D| - 1 = d - 2$$

and therefore the scroll X constructed from $|D|$ is $(d-1)$-dimensional. Let φ_{K_C} denote the canonical map

$$\varphi_{K_C} : C \to \mathbb{P}^{g-1}$$

and

$$\varphi_{|D|} : C \to \mathbb{P}^1$$

the map which corresponds to $|D|$. Then

$$\mathcal{E} := (\varphi_{|D|})_* \mathcal{O}_C(1)$$

is a locally free sheaf $\mathcal{O}_{\mathbb{P}^1}(e_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(e_{d-1})$ and $\mathbb{P}(\mathcal{E})$ the $\mathbb{P}^{d-2}$-bundle corresponding to the scroll X. $\mathbb{P}(\mathcal{E})$ is a desingularisation of X and we obtain the following diagram:

$$\begin{tikzcd}
\mathbb{P}(\mathcal{E}) \arrow[r, "\sigma"] \arrow[d] & X \arrow[r] & \mathbb{P}^{g-1} \\
\mathbb{P}^1 & C \arrow[l, "\varphi_{|D|}"'] \arrow[u, hook] \arrow[ur, hook, "\varphi_{K_C}"'] &
\end{tikzcd}$$

The type $S(e_1, ..., e_d)$ of a scroll X constructed from a basepoint free pencil of divisors D_λ on a variety V can be determined by considering the following partition of $r+1$:

$$\begin{aligned}
d_0 &= h^0(V, \mathcal{O}_V(H)) - h^0(V, \mathcal{O}_V(H-D)) \\
d_1 &= h^0(V, \mathcal{O}_V(H-D)) - h^0(V, \mathcal{O}_V(H-2D)) \\
&\vdots \\
d_i &= h^0(V, \mathcal{O}_V(H-iD)) - h^0(V, \mathcal{O}_V(H-(i+1)D)) \\
&\vdots
\end{aligned}$$

In [S86] 2.5. the author shows that X is a $d_0 - 1$ $(= d-2$ for $V = C \subset \mathbb{P}^{g-1}$ and $(D_\lambda)_\lambda = g^1_d$) dimensional scroll of type $S(e_1, ..., e_{d_0})$ and the numbers e_i are given by the dual partition:

$$e_i = \#\{j | d_j \geq i\} - 1$$

Example 1.4.2 *(Canonical Curve $C \subset \mathbb{P}^3$)* Let $C \subset \mathbb{P}^3$ be a canonical curve of degree 3 and genus 4. Then C admits a divisor D of degree 3 with $\dim |D| = 1$. From the geometric version of Riemann-Roch $\dim \bar{D} = 1$, thus the pencil of divisors $(D_\lambda)_\lambda$ is cut out by trisecants on C. We denote the class of hyperplane sections in $\mathbb{P}^3$ by H, that cuts out the canonical divisors on C. Then we get

$$d_0 = h^0(C, \mathcal{O}_C(K_C)) - h^0(C, \mathcal{O}_C(K_C - D)) = 4 - 2 = 2$$

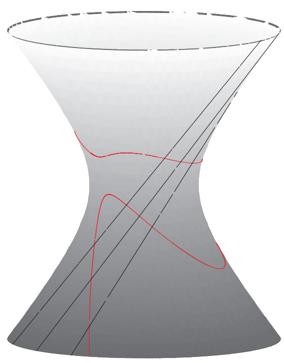
$$d_1 = h^0(C, \mathcal{O}_C(K_C - D)) - h^0(C, \mathcal{O}_C(K_C - 2D)) = 2 - h^0(C, \mathcal{O}_C(H|_C - 2D))$$

$$d_2 = h^0(C, \mathcal{O}_C(K_C - 2D)) - h^0(C, \mathcal{O}_C(K_C - 3D)) = h^0(C, \mathcal{O}_C(K_C - 2D))$$

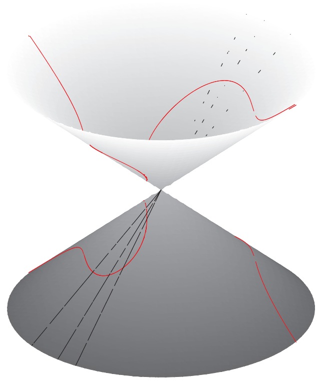
and

$$d_i = 0 \text{ for } i > 2 \text{ as } \deg(K_C - iD) = 6 - 3i < 0$$

The value $h^0(C, \mathcal{O}_C(K_C - D))$ gives the number of hyperplane sections passing through the points of D. As they all lie on a line, it follows that there exist exactly two of them, and they intersect exactly in this line. If one of the hyperplanes H_0 is tangent to C in all the points of D, then $h^0(C, \mathcal{O}_C(K_C - D)) = h^0(C, \mathcal{O}_C(H|_C - 2D)) = 1$, otherwise we get $h^0(C, \mathcal{O}_C(K_C - 2D)) = 0$. In summary we obtain a 2-dimensional scroll X of type $S(1,1)$ or $S(2,0)$ depending on whether there exists such a special hyperplane H_0 or not. In the general case the corresponding $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and C lies on a smooth quadric surface.



Whereas in the more special case we have $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) = P_2$, the second Hirzebruch surface, where the scroll X becomes singular. It is the cone over a conic in $\mathbb{P}^2$.



Our aim was to determine a minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^{g-1}}-$module. We have mentioned that we want to do this in two steps. First we consider a resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module. Then we apply Theorem 1.3.5 that leads to a mapping cone construction, which gives us a (not necessarily) minimal resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^{g-1}}-$module:

Theorem 1.4.3 *Let C be a canonical curve $C \subset \mathbb{P}^{g-1}$ of genus g that admits a base point free g^1_d. Further let X be the scroll constructed from the g^1_d as above and $\mathbb{P}(\mathcal{E})$ the corresponding $\mathbb{P}^{d-2}-$bundle. Then*
1) $C \subset \mathbb{P}(\mathcal{E})$ has a resolution F_* of type

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-dH + (f-2)R) \to \textstyle\sum_{k=1}^{\beta_{d-2}} \mathcal{O}_{\mathbb{P}(\mathcal{E})}((-d+2)H + b_k R) \to$$

$$\to \textstyle\sum_{k=1}^{\beta_1} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + a_k R) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_C \to 0$$

with $\beta_i = \frac{i(d-2-i)}{d-1}\binom{d}{i+1}$ and $a_k, b_k \in \mathbb{Z}$, $a_k + b_k = f-2$, $a_1 + ... + a_{\beta_{d-2}} =$ $2g-12$.
2) F_* is selfdual :

$$Hom(F_*, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-dH + (f-2)R)) \cong F_*$$

3) If all $b_k \geq -1$ then an iterated mapping cone

$$\left[... \left[\mathcal{C}^{(f-2)}(-d) \to \sum_{k=1}^{\beta_{d-2}} \mathcal{C}^{b_k}(-d+2) \right] \to ... \right] \to \mathcal{C}^0$$

is a (not necessarily minimal) resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}-$module.

Proof. [S86] Corollary 4.4 and 6.7. ■

From the existence of certain linear series on a canonical curve C one derives a nonsingular model $C' \subset S$ on a smooth rational surface S. Then the adjoint series $|K_S + C'|$ embedds C canonically. If $|K_S + C'|$ is even base point free on S, we can consider the image S' under the map given by $|K_S + C'|$. In some cases we have an analogue of the theorem above for the surface S'. Then from a minimal free resolution of S' it is possible to deduce information about the Betti numbers for C. We write down the most important results as can be found in [S86] Chapter 5:

For H a sufficiently positive divisor on a rational surface S the image S' of $j : S \to \mathbb{P}H^0(S, \mathcal{O}_S(H)) = \mathbb{P}^r$ can be described as a subvariety of a scroll X. It will turn out that in the cases of our interest S' is a determinantal surface on the scroll X, i.e. its vanishing ideal is given by the 2×2 minors of a $2 \times d$ matrix

$$\omega \sim \begin{pmatrix} H - a_1 R & ... & H - a_d R \\ H - (a_1 + k)R & ... & H - (a_d + k)R \end{pmatrix}$$

with entries in X. From this representation it is possible to obtain a free resolution of $\mathcal{O}_{S'}$ as $\mathcal{O}_X$-module.

This can be seen as follows: Let us start with a rational ruled surface

$$\pi : P_k := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^1 \ , \ k \geq 0$$

and consider $V := S$ the surface obtained from P_k via a sequence of blowups:

$$\sigma : S \to P_k$$

By abuse of notation we denote the hyperplane class and the ruling of P_k by A and B and also their pullbacks to S. Further $E = \bigcup_i E_i$ denotes the exceptional divisor of σ. Let

$$H \sim dA + eB - \textstyle\sum_i \lambda_i E_i$$

be a divisor on S with base point free complete linear series $|H|$ and consider the map

$$j : S \to \mathbb{P}H^0(S, \mathcal{O}_S(H)) = \mathbb{P}^r$$

with image $S' \subset \mathbb{P}^r$.
Now suppose that

1. $h^0(\mathcal{O}_S(H - B)) \geq 2$
2. $H^1(\mathcal{O}_S(kH - B)) = 0$ for $k \geq 1$ and
3. the map $S_k H^0 \mathcal{O}_S(H) \to H^0 \mathcal{O}_S(kH)$ is surjective.

Then we apply our construction from above to obtain a $(d+1)$-dimensional rational normal scroll

$$X = \bigcup_{B_\lambda \in |B|} \bar{B}_\lambda$$

Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ denote the corresponding $\mathbb{P}^d$-bundle and S'' the strict transform of S' in $\mathbb{P}(\mathcal{E})$. Blowing up S further we may assume that $S \to S'$ factors through S'' :

$$\begin{tikzcd}
 & S \arrow[r] \arrow[dl] \arrow[dr] \arrow[ddr, "\pi_s"'] & S' \arrow[r, hook] & X \arrow[r, hook] & \mathbb{P}^r \\
P_k \arrow[drr, "\pi_{P_k}"'] & & S'' \arrow[u] \arrow[r] \arrow[d] & \mathbb{P}(\mathcal{E}) \arrow[u] \arrow[dl, "\pi_{\mathbb{P}(\mathcal{E})}"] & \\
 & & \mathbb{P}^1 & &
\end{tikzcd}$$

If the conditions 1. – 3. are fulfilled, then we can describe the syzygies of $\mathcal{O}_{S''}$ in terms of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-modules:

Theorem 1.4.4 *$\mathcal{O}_{S''}$ has an $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module resolution of type*

$$0 \to \sum_{k=1}^{\beta_{d-1}} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-dH + b_k^{(d-1)}R) \to ...$$

$$... \to \sum_{k=1}^{\beta_1} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + b_k^{(1)}R) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_{S''} \to 0$$

with $\beta_i = i\binom{d}{i+1}$.

Proof. [S86] 5.1-5.5. ■

Especially in the cases of our interest, where we have $d = 3$, the resolution above is of the following type:

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H + b_1^{(2)}R) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H + b_2^{(2)}R) \to$$

$$\xrightarrow{\omega} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + b_1^{(1)}R) \oplus ... \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + b_3^{(1)}R) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_{S''} \to 0$$

Further the map ω is given by a 2×3 matrix

$$\omega \sim \begin{pmatrix} H - a_1 R & ... & H - a_3 R \\ H - (a_1 + k)R & ... & H - (a_3 + k)R \end{pmatrix}$$

with entries in $\mathbb{P}(\mathcal{E})$ and the 2×2 minors of ω generate the vanishing ideal of $S'' \subset \mathbb{P}(\mathcal{E})$ (cf. [S86] 5.5.).

The following important theorem due to Schreyer ([S86] Theorem 5.7) states that there also exists a partial converse of this result: Let $X \subset \mathbb{P}^r$ be a scroll of dimension $d+1$ and $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ the corresponding $\mathbb{P}^d$-bundle. Further let $S'' \subset \mathbb{P}(\mathcal{E})$ denote the irreducible surface defined by the 2×2 minors of a matrix

ω on $\mathbb{P}(\mathcal{E})$ as above, then the image S' of S'' in $\mathbb{P}^r$ can be obtained as the image of a blowup of $P_k = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}$ defined by a complete linear system H as above. We denote by A and B the class of a hyperplane and a ruling on P_k. Then with

$$a = a_1 + ... + a_d \quad \text{and} \quad f = \deg X$$

we have:

Theorem 1.4.5 *$S' \subset \mathbb{P}^r$ is the image of P_k under a rational map defined by a subseries of*

$$H^0(P_k, \mathcal{O}_{P_k}(dA + (f - dk - a)B)$$

which has

$$\delta = df - \frac{d(d+1)}{2}k - (d+1)a$$

assigned base points. Furthermore, if $S' \subset X \subset \mathbb{P}^r$ contains a canonical curve C of genus $r+1$, then the ruling of X cuts on C a g^1_{d+2} and the strict transform C' of C in P_k is a divisor of class

$$C' \sim (d+2)A + (f - (d+1)k - a + 2)B$$

and arithmetic genus

$$p_a C' = r + 1 + \delta.$$

Proof. [S86] Theorem 5.7. ■

We will use this important theorem to show that a curve C has a certain model on a blowup of P_k and therefore that there exist further special linear systems on C. We have already mentioned above that we are also interested in the converse, i.e. we start with a model $C' \subset P_k$ of C and want to give a description of the image S' of P_k under the mapping defined by the adjoint series. For this reason we have to show that this series is base point free and fulfills the conditions $1. - 3.$ from above. We treat this problem in the next chapter.

2

Ampleness of the Adjoint series

Let $C' \subset X$ be a curve on a surface X, that has only singularities in the points $p_1, ..., p_s$ with multiplicity 2. After blowing up these singularities, we get a smooth curve $C \subset S = \tilde{X}(p_1, ..., p_m)$. Now we are interested if certain linear systems on S, especially the adjoint series $|K_S + C|$, are $i-$very ample on S for $i = 0, 1$. There we call a linear system $|L|$ on a smooth projective surface $0-$very ample if it is base point free and $1-$very ample if it is very ample. We follow the approach of Roland Weinfurtner in his PhD thesis [W] that makes use of Reider's Theorem (cf. [R88]) in a modified version:

Theorem 2.0.1 *(Modified version of Reider's Theorem) Let L be a line bundle on a projective surface X and $L^2 \geq 5+4i$. If $|K_X + L|$ is not $i-$very ample then there exists an effective divisor D on X with $L - 2D$ $\mathbb{Q}-$effective ($\exists n \in \mathbb{Z}^+$: $n(L - 2D)$ is effective) and a $0-$cycle Z of degree $\leq i + 1$, where $|K_X + L|$ is not $i-$very ample and the following inequality holds:*

$$D.(L - D) \leq i + 1$$

Proof. [BFS89] Theorem 2.1. and [W] Theorem 1.2. ∎

If C' has singularities in the points $p_1, ..., p_s$, an iterated blowup of X in $p_1, ..., p_s$ gives a desingularisation of C' :

$$S = \tilde{X}(p_1, ..., p_s) \overset{\sigma_s}{\to} ... \overset{\sigma_2}{\to} \tilde{X}(p_1) \overset{\sigma_1}{\to} X$$

Let σ be the composition of the blowups σ_i and $E_j := \sigma_s^*(\sigma_{s-1}^*...(\sigma_j^{-1}(p_j))...)$ the total transform of the exceptional divisor of the blowup of the point p_j on S. Further C denotes the strict transform of C'. Then from the adjunction formula we get

$$\omega_S(C) \otimes \mathcal{O}_C \cong \omega_C$$

and therefore

$$K_C \sim (K_S + C)|_C$$

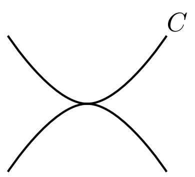
For $X = \mathbb{P}^2$ or P_2 let H denote the class of a hyperplane section and by abuse of notation also its pullback to S. R denotes the class of a ruling on P_2. In the case $X = \mathbb{P}^1 \times \mathbb{P}^1$ we have $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \cdot (0,1) \oplus \mathbb{Z} \cdot (1,0)$ with factor classes $(0,1)$ and $(1,0)$. Again by abuse of notation we also denote its pullbacks to S by $(0,1), (1,0)$. Then intersection theory gives the following theorem:

Theorem 2.0.2 *The Picard group $Pic(S)$ is generated by the pullback $\sigma^* Pic(X)$ of the Picard group of X and the exceptional divisors E_j as defined above. Further*

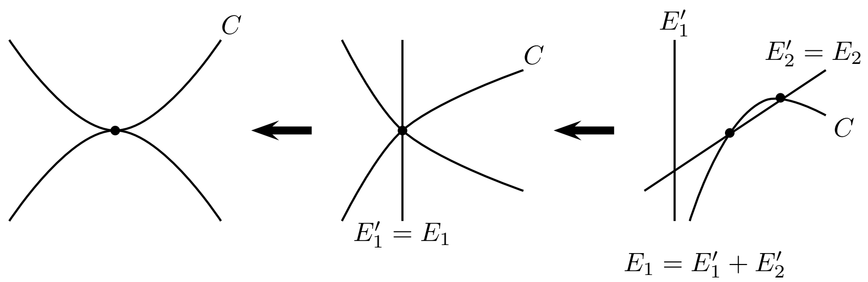
1) $E_j^2 = -1$
2) $E_i.E_j = 0$ *for* $i \neq j$
3) $\Gamma.E_j = 0$ *for* $\Gamma \in \sigma^* Pic(X)$
4) a) $H^2 = 1$ *for* $X = \mathbb{P}^2$
b) $H^2 = 2$, $H.R = 0$ *and* $R.R = 0$ *for* $X = P_2$
c) $(0,1).(0,1) = (1,0).(1,0) = 0$ *and* $(1,0).(0,1) = 1$ *for* $X = \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. [Hs77] V.3.2. ■

If C' has only ordinary nodes, then all exceptional curves E_j are irreducible and intersect the strict transform C of C' in exactly two points. For char $\mathbb{k} \neq 2$, a singularity of multiplicity 2 is always of analytical type $y^2 - x^k = 0$ with $k \geq 2$. If there exists such a singularity with $k \geq 4$, for example if C has a tacnode:



then blowing up X we get the following picture:



where the strict transform $\tilde{C}$ of C under the first blowup σ_1 still has a double point lying on the exceptional divisor E_1. Blowing up in this double point again, we get a smooth curve C that intersects the total transforms E_1 and E_2 in exactly two points. In this situation E_1 decomposes into the strict transforms E_1' and E_2' and $E_2 = E_2'$ is irreducible. Therefore it is useful to state the following definition:

Definition 2.0.3 *(infinitely near points) A point p_j is called infinitely near to another point p_i if and only if p_j lies on the strict transform of the exceptional divisor in the blowup of p_i.*

We assumed that C' has only double points as singularities. Then for the strict transform $C^{(i)} \subset \tilde{X}(p_1, ..., p_i)$ of C' under the blowup $\sigma^{(i)} := \sigma_i \circ ... \circ \sigma_1$ there are several possibilities: One is that $C^{(i)}$ meets the exceptional divisor $E^{(i)} := \sigma_i^{-1}(p_i)$ transversally in two distinct points. The second is that $C^{(i)}$ meets $E^{(i)}$ in one point P, with $C^{(i)}$ nonsingular in P, but $C^{(i)}$ and $E^{(i)}$ having intersection multiplicity 2 at P. The third possibility is that $C^{(i)}$ has a double point P on $E^{(i)}$ (see the example above). In this case $E^{(i)}$ must pass through P in a direction not equal to any tangent direction of P, since $C^{(i)}.E^{(i)} = 2$. It follows that for each point p_i there exists at most one further point p_j, $j > i$, that lies infinitely near to p_i. From our definition of the total transforms E_i above, it turns out that $E_i = \sigma_s^{-1}(\sigma_{s-1}^{-1}...(\sigma_j^{-1}(p_j))...)$.
At any time the total transforms contain exactly the strict transforms as components and they are all irreducible if and only if none of the points p_j is infinitely near to another point p_i. To be more precisely the strict transforms E_i' are inductively given by

$$\begin{aligned}
E_s' &= E_s \\
E_{s-1}' &= E_{s-1} - \delta_{s-1,s} E_s \\
E_{s-2}' &= E_{s-2} - \delta_{s-2,s-1} E_{s-1} - \delta_{s-2,s} E_{s-2} \\
&\vdots \\
&\vdots \\
E_1' &= E_1 - \textstyle\sum_{i=2}^{s} \delta_{1,i} E_i
\end{aligned}$$

with

$$\delta_{i,s} = \begin{cases} 1 \text{ if } p_s \text{ is lying on } E_i' \text{ in } \tilde{X}_{s-1} \\ 0 \text{ else} \end{cases}$$

Therefore we obtain a decomposition $E_i = \sum_{k=i}^{s} \delta_k E_k'$, $\delta_k = 0$ or 1, of E_i as sum of irreducible components. In our situation, this decomposition is given as follows: If there exists a maximal chain of points $p_i, ..., p_{i+k_i}$, such that p_{i+k+1} lies infinitely near to p_{i+k} for $k = 0, ..., k_i - 1$ and p_i does not lie infinitely near to any other double point p_j, then

$$\begin{aligned} E_i &= E_i' + ... + E_{i+k}' \\ E_{i+1} &= E_{i+1}' + ... + E_{i+k}' \\ &\vdots \\ E_{i+k} &= E_{i+k}' \end{aligned}$$

Especially in the case where p_i is an ordinary double point of C' we have $E_i = E_i'$. Further, we see that each strict transform E_i' can be written in terms of the total transforms, and $E_i' = E_i$ iff E_i is irreducible and $E_i' = E_i - E_{i+1}$ iff p_{i+1} lies infinitely near to p_i.

The strict transform C of C' is a divisor of class

$$C \sim \sigma^* C' - \textstyle\sum_{i=1}^s 2E_i$$

and therefore the adjoint series is given by an effective divisor linear equivalent to $K_S + C \sim \sigma^*\Delta - \sum_{i=1}^s E_i$, $\Delta \in \mathrm{Pic(X)}$. Let $L' \sim \sigma^*\Sigma - \sum_{i=1}^s \lambda_i E_i$, $\Sigma \in \mathrm{Pic(X)}$ be an effective divisor on S, then assuming that $L^2 \geq 5 + 4i$, $L := L' - K_S$, Reider's Theorem says that $|L'|$ is $i-$very ample if we cannot find an effective divisor $D \sim \sigma^*\Gamma - \sum_{i=1}^s \alpha_i E_i$, $\Gamma \in \mathrm{Pic(X)}$ on S, such that $L - 2D$ is $\mathbb{Q}$-effective and $D.(L-D) \geq 2+i$. Under the assumption $0 \leq \lambda_1 \leq ... \leq \lambda_s \leq 2$, it is possible to restrict to a very manageable set of divisors D. We start with a divisor D as above that fullfills $D.(L-D) < 2+i$, then after several modification steps we get a divisor $\tilde{D}$ with $\alpha_i = 0$ or 1, such that $L - 2\tilde{D}$ is also $\mathbb{Q}$-effective and $\tilde{D}.(L-\tilde{D}) \leq D.(L-D) < 2+i$:

Lemma 2.0.4 *Let X be a surface, $L' \sim \sigma^*\Sigma - \sum_{i=1}^s \lambda_i E_i$, $\Sigma \in \mathrm{Pic(X)}$, $0 \leq \lambda_1 \leq ... \leq \lambda_s \leq 2$, an effective divisor on the iterated blowup $S = \tilde{X}(p_1, ..., p_s)$ as above and $L^2 = (L' - K_S)^2 \geq 5 + 4i$. If there exists no effective divisor $D \sim \sigma^*\Gamma - \sum_{i=1}^s \alpha_i E_i$, $\Gamma \in \mathrm{Pic(X)} \setminus \{0\}$, $\alpha_i \in \{0,1\}$, or $D \sim E_i'$ on S, such that $L - 2D$ is $\mathbb{Q}$-effective and $D.(L-D) \leq 1+i$, then $|L'|$ is i-very ample on S ($i = 0,1$).*

Proof. According to Reider's Theorem we have to assure that there exists no divisor $D \sim \sigma^*\Gamma - \sum_{i=1}^s \alpha_i E_i$, $\Gamma \in \mathrm{Pic(X)}$ on S, such that $L - 2D$ is $\mathbb{Q}$-effective and $D.(L-D) \leq 1+i$. For $\Gamma \sim 0$ the only effective divisors are sums of the strict transforms E_i'. The calculation in (i) and (ii) below shows that we can restrict to a single E_i'.

For $\Gamma \not\sim 0$, we start with an effective divisor $D \sim \sigma^*\Gamma - \sum_{i=1}^s \alpha_i E_i$, $\Gamma \in \mathrm{Pic(X)}$ on S, such that $L - 2D$ is $\mathbb{Q}$ effective and $D.(L-D) \leq 1+i$. In the first step we will show that we can restrict to those divisors D with $\alpha_i \geq 0$.

1st Modification step: If some of the coefficients α_i are negativ, then we aim to modify D to an effective divisor $\tilde{D} \sim \sigma^*\Gamma - \sum_{i=1}^s \tilde{\alpha}_i E_i$ with $\tilde{\alpha}_i \geq 0$ for all i. In each step we have to assure that $\tilde{D}.(L-\tilde{D}) \leq D.(L-D)$. This is done in the following way: If none of the α_i is negative, there is nothing to do. Otherwise let j be the maximal index with $\alpha_j < 0$. If E_j is irreducible we have $E_j = E'_j$ and in this situation we set $\tilde{D} := D - E_j$. In the case where E_j is reducible we have $E'_j = E_j - E_{j+1}$. Here we consider $\tilde{D} := D - (E_j - E_{j+1})$. Because of $E'_j.D = \alpha_j < 0$ the strict transform E'_j is a component of D, hence $\tilde{D}$ stays effective. The second condition is satisfied if

(i)
$$\tilde{D}.(L-\tilde{D}) \leq D.(L-D)$$

$$\begin{aligned}
&\Leftrightarrow (D-E_j).(L-D+E_j) \leq D.(L-D)\\
&\Leftrightarrow 2D.E_j + 1 - L.E_j \leq 0\\
&\Leftrightarrow 2\alpha_j - (\lambda_j+1) + 1 = 2\alpha_j - \lambda_j \leq 0
\end{aligned}$$

and

(ii)
$$\tilde{D}.(L-\tilde{D}) \leq D.(L-D)$$

$$\begin{aligned}
&\Leftrightarrow (D-(E_j-E_{j+1})).(L-D+(E_j-E_{j+1})) \leq D.(L-D)\\
&\Leftrightarrow 2D.(E_j-E_{j+1}) + 2 - L.(E_j-E_{j+1}) \leq 0\\
&\Leftrightarrow 2(\alpha_j-\alpha_{j+1}) + (\lambda_j-\lambda_{j+1}) + 2 \leq 0.
\end{aligned}$$

As $\lambda_i \geq 0$ the condition (i) is always fulfilled and because for $\lambda_j \leq \lambda_{j+1}$ also condition (i) is given. Now the following sequence of modification steps leads us to an effective divisor $\tilde{D} \sim \sigma^*\Gamma - \sum_{i=1}^{s}\tilde{\alpha}_i E_i$ with $\tilde{\alpha}_i \geq 0$ for all i : If $E_j = E'_j$ we can substract the divisor E'_j $|\alpha_j|$ times to get a divisor $\tilde{D} \sim \sigma^*\Gamma - \sum_{i=1}^{s}\tilde{\alpha}_i E_i$ with $\alpha_i \geq 0$ for $i \leq j$.
In the case $E'_j = E_j - E_{j+1}$ the situation is a little more complicated: For $\alpha_{j+1} \geq |\alpha_j|$ we can substract E'_j $\alpha_{j+1}+1$ times to get a divisor $\tilde{D}$ with $\alpha_i \geq 0$ for $i \leq j$. Otherwise $\tilde{D} \sim D - (\alpha_{j+1}+1)E'_j$ has a negative coefficient $\tilde{\alpha}_{j+1} = \alpha_{j+1} - (\alpha_{j+1}+1) = -1$. Now we have to apply some correction steps to assure that $\tilde{D}$ has non negative coefficients α_i for $i > j$:

a) E_{j+1} irreducible: We substract E_{j+1} to get an effective divisor $\tilde{D}$ with $\alpha_i \geq 0$ for all $i > j$, such that we can go on with our procedure.

b) $E'_{j+1} = E_{j+2} - E_{j+1}$ reducible: We substract E'_{j+1} first to get $\alpha_{j+1} \geq 0$. If α_{j+2} becomes negative in this step, we apply one further correction step of type a) or b) and so on. This chain of steps stops with the last exceptional divisor $E_s = E'_s$, which is irreducible.

Now we continue with the divisor $D = \tilde{D}$ which has non negative coefficients

α_i for $i \geq j-1$. After finitely many steps we obtain an effective divisor $\tilde{D}$ with only non negative α_i.

2nd Modification step: The following consideration shows that we can even restrict to those divisors D with $\alpha_i = 0$ or 1 : Starting with a divisor D with non negative α_i, we consider $\tilde{D}$ with $\tilde{\alpha}_i = \min(\alpha_i, 1)$. $\tilde{D}$ is again effective, as we have only added multiples of the exceptional divisors E_i. It remains to show that $(D+E_i).(L-(D+E_i)) \leq D.(L-D)$ for all i with $\alpha_i \geq 2$, which is true for $\lambda_i \leq 2$ because of

$$\begin{aligned}(D+E_i).(L-(D+E_i)) &= D.(L-D) + 1 + L.E_i - 2E_i.D \\ &= D.(L-D) + 2 + \lambda_i - 2\alpha_i = \\ &= D.(L-D) + \lambda_i - 2(\alpha_i - 1) \leq D.(L-D)\end{aligned}$$

■

Now assume that $|L'|$ is base point free and $d = \dim |L'| \geq 3$, then we can consider its image S' under the morphism defined by this complete linear system

$$\varphi : S \stackrel{|L|}{\rightarrow} S' \subset \mathbb{P}^{d-1}$$

Our aim is to show that for certain linear systems the image S' is an arithmetically Cohen-Macaulay surface. In the case where $X = \mathbb{P}^2$ and $L' \sim dH - \sum_{i=1}^{s} E_i$ these surfaces are called Bordiga surfaces. Weinfurtner gives in his PhD thesis [W] a complete description of them. We are interested in the following linear systems:

2.1 Adjoint linear series on blowups of $\mathbb{P}^2$

As we have mentioned before we have to examine the case where C is an irreducible, smooth, canonical curve of genus 9 that admits a plane model $C' \subset \mathbb{P}^2 = X$ of degree $d = 7$. C' has singularities of multiplicity 2 in exactly $s = \binom{d-1}{2} - 9 = 6$ points $p_1, ..., p_6$. Blowing up these singularities

$$\sigma : S = \tilde{X}(p_1, ..., p_6) \stackrel{\sigma_6}{\rightarrow} ... \stackrel{\sigma_2}{\rightarrow} \tilde{X}(p_1) \stackrel{\sigma_1}{\rightarrow} \mathbb{P}^2$$

we can assume that C is the strict transform of C' and

$$C \sim 7H - \sum_{i=1}^{6} 2E_i$$

its canonical system is cut out by the complete linear series $|L'| \sim |C + K_S| = |4H - \sum_{i=1}^{s} E_i|$

Theorem 2.1.1 *Let S be the iterated blowup of 6 points $p_1, ..., p_6$ on $\mathbb{P}^2$ and $C \sim 7H - \sum_{i=1}^{6} 2E_i$ an irreducible, nonsingular curve on S, then the adjoint linear series $|C + K_S| = |4H - \sum_{i=1}^{s} E_i|$ is base point free. It is very ample on S if and only if none of the points p_i lies infinitely near to another one. The image $S' \subset \mathbb{P}^8$ of S under the morphism defined by $|C + K_S|$ is arithmetically Cohen-Macaulay. Furthermermore if two of the points p_i are infinitely near, then S' has only isolated singularities that are contractions of strict transforms $E_i' \sim E_i - E_{i+1}$.*

Proof. For $L := L' - K_S \sim C$ we get $L^2 \geq 49 - 24 = 25 \geq 5 + 2 \cdot 1$, hence the first condition of Lemma 2.0.4 for $|L'|$ to be i-very ample ($i = 0, 1$) is fulfilled. We call an effective divisor D *$i-$critical* if $L - 2D$ is $\mathbb{Q}-$effective and $D.(C - D) < 2 + i$. Then according to Lemma 2.0.4 it remains to consider D to be one of the following three types:

(a) $D \sim E_k$ for a $k \in \{1, ..., 6\}$.
(b) $D \sim E_k - E_{k+1}$ $k \in \{1, ..., 6\}$.
(c) $D \sim eH - \sum_{i=1}^{6} \alpha_i E_i$ with $e \geq 1$ and $\alpha_i = 0$ or 1.

As $E_k.(C - E_k) = 3 \geq 2 + i$ there exists no $i-$critical divisor of type (a). If two points p_k and p_{k+1} are infinitely near, then there exists an effective divisor of type (b) and because of $(E_k - E_{k+1}).(C - (E_k - E_{k+1})) = 2$, this divisor is $1-$critical. For a divisor of type (c) to be $i-$critical we can restrict to $e \leq 3$ as $L - 2D$ has to be $\mathbb{Q}-$effective. Let $\delta = \#\{\alpha_i : \alpha_i = 1\}$, then

$$D.(C - D) = e(7 - e) - \delta < 3$$

if and only if $e = 1$ and $\delta > 3$. In this situation we would get $D.C \leq -1$ and therefore D and C have a common component, which contradicts to our assumption that C is irreducible. In summary, there exists at most 1-critical effective divisors D, which is exactly the case if two points p_k and p_l are infinitely near. According to Lemma 2.0.4, the map

$$\varphi : S \overset{|C+K_S|}{\to} S' \subset \mathbb{P}^8$$

defined by the adjoint series is base point free. The following arguments show that φ is even very ample outside the finite union $\Delta = \bigcup_\lambda D_\lambda$ of all $1-$critical critical divisors (being of type $E_k - E_{k+1}$). With $U := S \backslash \Delta$, this is exactly the case iff $|L' - E_p|$ is base point free for all $p \in U$, where E_p denotes the exceptional divisor of the blowup of $p \in S$. We want to apply Lemma 2.0.4 to $|L' - E_p|$. This is possible as $(7H - \sum_{i=1}^{6} 2E_i - 2E_p)^2 = 49 - 28 \geq 5$. A 0-critical divisors $D = D' - \alpha_p E_p$ for $|L' - E_p|$, $\alpha_p = 0$ or 1, has to satisfy the following inequality:

$$\begin{aligned} D.(L - D - 2E_p) \leq 1 &\Rightarrow (D' - \alpha_p E_p).(L - D' - (2 - \alpha_p)E_p) \leq 1 \\ &\Rightarrow D'.(L - D') - \alpha_p(2 - \alpha_p) \leq 1 \end{aligned}$$

Because of $D'.(L-D') \geq 2$ this is only possible if $\alpha_p = 1$ and $D.(L'-D) = 2$. But then $p \in D$ and D is a divisor of type $E_k - E_{k+1}$, which was excluded.

Thus the effective divisors of type $E_k - E_{k+1}$ are contracted by the morphism φ, whereas φ is very ample on U. The surface S' is called a Bordiga surface. It is smooth iff none of the points p_i is infinitely near to another one and it has isolated singularities otherwise.

It remains to show that S' is arithmetically Cohen-Macaulay. For this purpose we aim to find a smooth hyperplane section on S', that is arithmetically Cohen-Macaulay. As a direct consequence S' is then arithmetically Cohen-Macaulay, too (see Lemma 2.1.3 below). S' has only isolated singularities and therefore we can find a smooth hyperplane section Γ applying Bertini's Theorem. Γ is obtained from an element of the linear system $|L'| = \left|4H - \sum_{j=1}^{6} E_j\right|$ on $S = \tilde{\mathbb{P}}^2(p_1, ..., p_6)$. Hence we get $\deg(\Gamma) = (4H - \sum_{j=1}^{6} E_j)^2 = 10$ and $g(\Gamma) = \binom{4-1}{2} = 3$. It follows that $\mathcal{L} := \mathcal{O}_\Gamma(1)$ is very ample on Γ. From the exact sequence

$$0 \to H^0(S', \mathcal{O}_{S'}) \to H^0(S', \mathcal{O}_{S'}(1)) \to H^0(\Gamma, \mathcal{O}_\Gamma(1)) \to H^1(S', \mathcal{O}_{S'}) = 0$$

(S' is rational) it follows that Γ is embedded projectively normal by $\mathcal{L}$. Then the two following lemmas show that Γ and therefore also S' is arithmetically Cohen-Macaulay. ■

Lemma 2.1.2 *Let $X \subset \mathbb{P}^n$ be a d-dimensional connected variety, then X is arithmetically Cohen Macaulay if and only if*
1) $H^1(\mathbb{P}^n, \mathcal{I}_X(m)) = 0$ for all $m \in \mathbb{Z}$ and
2) $H^i(\mathbb{P}^n, \mathcal{O}_X(m)) = 0$ for all $i \neq 0, d$ and $m \in \mathbb{Z}$

Proof. [E95] page 472 Ex. 18.16 ■

Then the following well known lemma can be obtained from this properties:

Lemma 2.1.3 *Let $X^d \subset \mathbb{P}^n$, $d \geq 2$, be an irreducible variety of dimension d, then X is arithmetically Cohen-Macaulay iff there exists a hyperplane section $\Gamma = \mathbb{P}^{n-1} \cap X \subset \mathbb{P}^n$ that is arithmetically Cohen-Macaulay.*

Proof. " $\Leftarrow$ " Consider the exact sequence $(S1^*)$

$$0 \to \mathcal{I}_X(m-1) \to \mathcal{I}_X(m) \to \mathcal{I}_\Gamma(m) \to 0$$

then $H^1(\Gamma, \mathcal{I}_\Gamma(m)) = 0$ for all $m \in \mathbb{Z}$ as Γ is arithmetically Cohen-Macaulay. It follows that

$$h^1(X, \mathcal{I}_X(m-1)) \geq h^1(X, \mathcal{I}_X(m))$$

From the exact sequence $(S2^*)$

$$0 \to \mathcal{I}_X(m) \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_X(m) \to 0$$

we obtain the corresponding long exact sequence $(LS2^*)$

$$\begin{aligned} ... &\to H^0(X,\mathcal{O}_X(m)) \to H^1(X,\mathcal{I}_X(m)) \to H^1(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(m)) = 0 \to \\ &\to H^1(X,\mathcal{O}_X(m)) \to H^2(X,\mathcal{I}_X(m)) \to ... \end{aligned}$$

and from

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_\Gamma \to 0$$

the long exact sequence

$$0 \to H^0(X,\mathcal{O}_X(-1)) \to H^0(X,\mathcal{O}_X) \to H^0(\Gamma,\mathcal{O}_\Gamma)$$

As X is irreducible and reduced we get $H^0(X,\mathcal{O}_X) \cong \Bbbk$, and thus $H^0(X,\mathcal{O}_X) \hookrightarrow H^0(\Gamma,\mathcal{O}_\Gamma)$ is the restriction of constant functions on X to C. Further we have $H^0(X,\mathcal{O}_X(-1)) = 0$ and $H^0(X,\mathcal{O}_X(m)) = 0$ for all negative values m. Together with $h^1(X,\mathcal{I}_X(m-1)) \geq h^1(X,\mathcal{I}_X(m))$ we conclude that $H^1(X,\mathcal{I}_X(m)) = 0$ for all m. This shows 1) in the lemma above. For 2) we look at the sequence $(LS2^*)$ again: To determine the group $H^2(X,\mathcal{I}_X(m))$ consider the long exact cohomology sequence

$$... \to 0 = H^1(\Gamma,\mathcal{I}_\Gamma(m)) \to H^2(X,\mathcal{I}_X(m-1)) \to H^2(X,\mathcal{I}_X(m)) \to ...$$

and use Serre's vanishing theorem that says that $H^2(X,\mathcal{I}_X(m)) = 0$ for $m \gg 0$ to see that $H^2(X,\mathcal{I}_X(m)) = 0$ for all m. Applying this result to $LS2^*$ leads to $H^1(X,\mathcal{O}_X(m)) = 0$ for all $m \in \mathbb{Z}$.

" $\Rightarrow$ " 2) is trivially fulfilled, so that it remains to show that $H^1(\Gamma,\mathcal{I}_\Gamma(m)) = 0$ for all $m \in \mathbb{Z}$. Consider the long exact sequence corresponding to $(S1^*)$:

$$... \to H^1(X,\mathcal{I}_X(m)) \to H^1(\Gamma,\mathcal{I}_\Gamma(m)) \to H^2(\Gamma,\mathcal{I}_\Gamma(m-1) \to ...$$

The first group is 0 as X is arithmetically Cohen-Macaulay and $H^2(\Gamma,\mathcal{I}_\Gamma(m-1) = 0$, which follows from

$$0 = H^1(X,\mathcal{O}_X(m-1)) \to H^2(X,\mathcal{I}_X(m-1)) \to H^2(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(m-1)) = 0$$

thus $H^1(\Gamma,\mathcal{I}_\Gamma(m)) = 0$. ■

Remark 2.1.4 For C' an irreducible plane curve of degree 7 having exactly one triple point p_0 and three doublepoints $p_1,...,p_3$ as only singularities, we consider the blowup in these singularities and the strict transform $C \sim 7H - 3E_0 - \sum_{i=1}^3 2E_i$ of C'. Then the adjoint linear series $|L'| = \left|4H - 2E_0 - \sum_{j=1}^3 E_j\right|$ is base point free (see [W] page 35/36 with the same methods as above) on $S = \tilde{P}^2(p_0,...,p_3)$ if there exists no effective divisor $D \sim H - E_0 - \sum_{i=1}^3 E_i$. But this follows easily from $D.C < 0$ and the irreducibility of C. Therefore we obtain a morphism

$$\varphi : S \overset{|C+K_S|}{\to} S' \subset \mathbb{P}^8$$

from $|L'|$ and φ is even very ample outside the union of all effective divisors D of the following types:

$H - E_0 - \sum_{j\in\Delta}\alpha_j E_j$ with $a_j = 0$ or 1 and $|\Delta| = \#\{\alpha_i : \alpha_i = 1\} = 2$.
$E_l - E_{l+1}$ for $l \in \{1, 2, 3\}$.

The existence of effective divisors D of these types are given in the situation where one triple point and two of the double points are lying on a line or two of the double points are infinitely near. Outside the finite union of these critical divisors φ is very ample and thus the image S' of S under the morphism φ is a surface with only isolated singularities, such that we can find a smooth hyperplane section Γ applying Bertini's Theorem. Γ is obtained from an element of the linear system $|L'| = \left|4H - 2E_0 - \sum_{j=1}^{3} E_j\right|$ on $S = \tilde{P}^2(p_0, ..., p_3)$. Thus we get $\deg(\Gamma) = (4H - 2E_0 - \sum_{j=1}^{3} E_j)^2 = 9$ and $g(\Gamma) = \binom{4-1}{2} = 3$. It follows that $L := \mathcal{O}_\Gamma(1)$ is very ample on Γ and Γ is embedded by L projectively normal, hence Γ is arithmetically Cohen-Macaulay. Then the surface S', which is called a Castelnuovo surface is arithmetically Cohen-Macaulay, too.
Furthermore it is possible to generalize our above results, especially in the case where C is an irreducible plane curve that has a singular point p_1 of multiplicity $m \geq 3$ and further double points as only singularities. We remark that in the situation where C has a quadruple point, it is possible to obtain an effective divisor of class $E_1 - E_2 - E_3$. This happens exactly in the case where two double points are lying infinitely near to the quadruple point. Then one has to check if this divisor is $i - critical$.

2.2 Adjoint linear series on blowups of $\mathbb{P}^1 \times \mathbb{P}^1$

In the section about pentagonal curves we consider canonical curves C of genus 9 that admit two distinct linear systems of type g^1_5. From these linear systems we get a model $C' \subset \mathbb{P}^1 \times \mathbb{P}^1 =: X$ of C, which is a divisor of class $(5, 5)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. C' has $s := p_a(C') - g(C) = 4 \cdot 4 - 9 = 7$ double points $p_1, ..., p_7$ as only singularities. Blowing up X in these singular points we can assume that C is the strict transform of C' on the blowup

$$\sigma : S = \tilde{X}(p_1, ..., p_7) \xrightarrow{\sigma_7} ... \xrightarrow{\sigma_2} \tilde{X}(p_1) \xrightarrow{\sigma_1} \mathbb{P}^1 \times \mathbb{P}^1$$

Then $C \sim (5, 5) - \sum_{i=1}^{7} 2E_i$ and its canonical series is cut out by $|L'| = |K_S + C| = \left|(3, 3) - \sum_{i=1}^{7} E_i\right|$. As in the previous section we want to apply Lemma 2.0.4 again to show that $|L'|$ is $i-$very ample:

Theorem 2.2.1 *Let S be the iterated blowup of 7 points $p_1, ..., p_7$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $C \sim (5, 5) - \sum_{i=1}^{7} 2E_i$ an irreducible, nonsingular curve on S, then the adjoint linear series $|C + K_S| = \left|(3, 3) - \sum_{i=1}^{7} E_i\right|$ is base point free. It is very* *ample on S if and only if none of the points p_i lies infinitely near to another one. Furthermore the image $S' \subset \mathbb{P}^8$ of S under the morphism defined by $|C + K_S|$ is arithmetically Cohen-Macaulay. If there exist two infinitely near double points then S' has isolated singularities that are contractions of strict transforms $E'_i \sim E_i - E_{i+1}$.*

Proof. For $L := L' - K_S \sim C$ we get $L^2 = 50 - 28 > 5 + 4i$ for $i = 0, 1$ and from Theorem 2.0.4 it follows the $i-$very ampleness of $|L'|$ iff there exists no i-critical effective divisor D, such that $L - 2D$ is $\mathbb{Q}-$effective and $D.(C - D) < 2 + i$. As above for $X = \mathbb{P}^2$, we have to distinguish three types for D :

(a) $D \sim E_k$ for a $k \in \{1, ..., 7\}$.
(b) $D \sim E_k - E_{k+1}$ for $k \in \{1, ..., 6\}$.
(c) $D \sim (a, b) - \sum_{i=1}^{7} \alpha_i E_i$ with $\alpha_i = 0$ or 1 and $(a, b) \in \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)^*$

A divisor of type (a) cannot be $i-$critical as $E_k.(C - E_k) = 3 \geq 2 + i$. Further there exists an effective divisor of type (b) exactly if two of the points $p_1, ..., p_7$ are infinitely near. In this situation every such divisor, which can be written as difference of two total transforms E_k and E_{k+1} is 1-critical but not $0-$critical because of $(E_k - E_{k+1}).(C - (E_k - E_{k+1})) = 2$. Now let $D \sim (a, b) - \sum_{i=1}^{7} \alpha_i E_i$ be a $1-$critical divisor, then from the condition that $C - 2D$ is $\mathbb{Q}$-effective we can restrict to $a, b \leq 2$. As above we denote $\delta = \#\{\alpha_i : \alpha_i = 1\}$, then the condition $D.(C - 2D) \leq 2$ transforms into:

$$
\begin{aligned}
2 &\geq ((a, b) - \textstyle\sum_{i=1}^{7} \alpha_i E_i).((5 - a, 5 - b) - \sum_{i=1}^{7} (2 - \alpha_i) E_i) = \\
&= a(5 - b) + b(5 - a) - \textstyle\sum_{i=1}^{7} \alpha_i (2 - \alpha_i) = 5a + 5b - 2ab - \delta = \\
&= D.C + \delta - 2ab \geq 0 + \delta - 2ab \\
&\Rightarrow \delta \leq 2ab + 2
\end{aligned}
$$

We remark that the inequality $D.C \geq 0$ is a consequence of the irreducibility of C. Therefore we get

$$D.(C - D) = 5a + 5b - 2ab - \delta \geq 5a + 5b - 4ab - 2$$

In the case $a = 0$ or $b = 0$ we obtain $D.(C - D) \geq 5 - 2 = 3$. For $a = 1$ we have $D.(C - D) \geq 5 + b - 2 \geq 3$ and the same for $b = 1$. If $a = 2$ then $D.(C - D) \geq 10 - 3b - 2 \geq 5$ for $b = 0, 1$. In the case $a = b = 2$ we use the inequality $D.(C - D) \geq 5a + 5b - 2ab - \delta \geq 12 - \delta \geq 5$. The same holds for $b = 2$. In summary we have seen that there exists no 0-critical divisor and the only $1-$critical are exactly the divisors of type (b). Now with the same arguments as in the "$\mathbb{P}^2$ case" it follows that $|L'|$ is base point free and even very ample outside the finite union of divisors of type (b). Further the image $S' \subset \mathbb{P}^8$ of S under the morphism φ defined by $|L'|$ has only isolated singularities. Due to Bertini's Theorem we can find a smooth hyperplane section $\Gamma \in \left|(3, 3) - \sum_{i=1}^{7} E_i\right|$ with $\deg(\Gamma) = ((3, 3) - \sum_{j=1}^{7} E_i)^2 = 11$ and $g(\Gamma) = 2 \cdot 2 = 4$. It follows that $\mathcal{L} := \mathcal{O}_\Gamma(1)$

is very ample on Γ. Then the same arguments as above show that Γ is embedded by $\mathcal{L}$ projectively normal, hence Γ and therefore also S' is arithmetically Cohen-Macaulay. ∎

2.3 Adjoint linear series on blowups of P_2

Later on we will see that from a g^1_5 of higher multiplicity (for the definition we refer to Chapter 4) on a canonical curve C of genus 9, we get a model $C' \subset P_2 = \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}) =: X$ of C, that is a divisor of class $5H$ on P_2. C' has $p_a(C') - g(C) = 4 \cdot 4 - 9 = 7$ double points $p_1, ..., p_7$ as only singularities. In the case of our interest C' has no intersection with the exceptional divisor $E \sim H - 2R$ on P_2. Blowing up X in the singular points we can assume that C is the strict transform of C' in the blowup

$$\sigma : S = \tilde{X}(p_1, ..., p_7) \xrightarrow{\sigma_7} ... \xrightarrow{\sigma_2} \tilde{X}(p_1) \xrightarrow{\sigma_1} P_2$$

Then $C \sim 5H - \sum_{i=1}^{7} 2E_i$ and its canonical series is cut out by $|L'| = |K_S + C| = \left|3A - \sum_{i=1}^{7} E_i\right|$. The following theorem can be obtained in analogous manner as the main theorems in the last two sections:

Theorem 2.3.1 *Let S be the iterated blowup of 7 points $p_1, ..., p_7$ on P_2 and $C \sim 5H - \sum_{i=1}^{7} 2E_i$ an irreducible, nonsingular curve on S, then the adjoint linear series $|C + K_S| = \left|3H - \sum_{i=1}^{7} E_i\right|$ is base point free. The image $S' \subset \mathbb{P}^8$ of S under the morphism defined by $|C + K_S|$ is arithmetically Cohen-Macaulay and the only singularities are contractions of the exceptional divisor $E \sim H - 2R$ and strict transforms $E'_i \sim E_i - E_{i+1}$ in the case where p_{i+1} lies infinitely near p_i.*

Proof. With $L := L' - K_S \sim C$ we have $L^2 = 50 - 28 > 5 + 4i$ for $i = 0, 1$. From Theorem 2.0.4 it follows the $i-$very ampleness of $|L'|$ iff there exists no i-critical effective divisor D, such that $L - 2D$ is $\mathbb{Q}-$effective and $D.(C - D) < 2 + i$. As above for $X = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, we have to distinguish three types for D :

(a) $D \sim E_k$ for a $k \in \{1, ..., 7\}$.
(b) $D \sim E_k - E_{k+1}$ for $k \in \{1, ..., 6\}$
(c) $D \sim aH + bR - \sum_{i=1}^{7} \alpha_i E_i$ with $\alpha_i = 0$ or 1 and $aH + bR \in \mathrm{Pic}(F_2)^*$, thus $a \geq 0$ and $b \geq -2a$.

In complete analogue to our arguments above, a divisor of type (a) cannot be $i-$critical and an effective divisor of type (b) exists if and only if two of the points $p_1, ..., p_7$ are infinitely near. In this situation $E_k - E_{k+1}$ is 1-critical but not 0$-$critical. Now let $D \sim aH + bR - \sum_{i=1}^{7} \alpha_i E_i$ be a 1$-$critical divisor, then from the condition that $C - 2D$ is $\mathbb{Q}$-effective we can restrict to $a \leq 2$ and $b \leq \left\lfloor \frac{5-a}{2} \right\rfloor$. Let $\delta = \#\{\alpha_i : \alpha_i = 1\}$, then from $D.(C - D) \leq 2$ we conclude:

$$\begin{aligned} 2 &\geq (aH + bR - \textstyle\sum_{i=1}^{7}\alpha_i E_i).((5-a)H - bR - \sum_{i=1}^{7}(2-\alpha_i)E_i) = \\ &= 2a(5-a) + b(5-a) - ab - \textstyle\sum_{i=1}^{7}\alpha_i(2-\alpha_i) = \\ &= 10a + 5b - \delta - 2a(a+b) = D.C + \delta - 2a(a+b) \geq \delta - 2a(a+b) \\ &\Rightarrow \delta \leq 2a(a+b) + 2 \end{aligned}$$

Therefore it follows that

$$D.(C - D) = 10a + 5b - 2a(a+b) - \delta \geq 10a + 5b - 4a(a+b) - 2$$

In the case $a = 0$ we obtain $D.(C - D) \geq 5b - 2 \geq 3$ and for $a = 1$ we have $D.(C - D) \geq 4 + b \geq 2$ and $D.(C - D) = 2 \Leftrightarrow D \sim H - 2R$. If $a = 2$ then $D.(C - D) \geq 2 - 3b \geq 5$ for $b < 0$. It remains to discuss the cases $a = 2$ and $b = 0$ or 1 : Here we use the inequality $D.(C - D) = 10a + 5b - 2a(a+b) - \delta \geq 12 + b - \delta \geq 5$. In summary we have seen that there exists no 0-critical divisor and the only 1−critical are exactly the divisors of type (b) and the exceptional divisor $E \sim H - 2R$ on P_2. From the same arguments as in the "$\mathbb{P}^2$ case" it follows that $|L'|$ is base point free and even very ample outside the finite union of divisors of type (b) and the exceptional divisor E. Further the image $S' \subset \mathbb{P}^8$ of S under the morphism φ defined by $|L'|$ has only isolated singularities. Due to Bertini's Theorem we can find a smooth hyperplane section $\Gamma \in \left| 3H - \sum_{i=1}^{7} E_i \right|$ with $\deg(\Gamma) = (3H - \sum_{i=1}^{7} E_i)^2 = 11$ and $g(\Gamma) = 2 \cdot 2 = 4$. It follows that $\mathcal{L} := \mathcal{O}_\Gamma(1)$ is very ample on Γ and that Γ is embedded by $\mathcal{L}$ projectively normal, hence Γ and therefore also S' is arithmetically Cohen-Macaulay. ■

In the situation where C' has infinitely near double points we will show in Section 4.6.2 that it is possible to separate these points, i.e. there exists a one paramter family of curves C'_λ having only ordinary nodes. For this purpose we will provide a further result here. We recall that E'_i denotes the strict transform of the point p_i in the iterated blowup of all singular points. Now let p_7 be infinitely near to p_6 and

$$\sigma' : S^{(6)} = \tilde{X}(p_1, ..., p_6) \xrightarrow{\sigma_6} ... \xrightarrow{\sigma_2} \tilde{X}(p_1) \xrightarrow{\sigma_1} P_2$$

the iterated blowup in the points $p_1, ..., p_6$. The following theorem then states that the linear system $|L'| = \left| 5H - \sum_{i=1}^{6} 2E_i \right|$ is very ample on $S^{(6)} \backslash (\bigcup_{i=1}^{5} E'_i \cup E)$. With $L := L' - K_S \sim 7H - \sum_{i=1}^{6} 3E_i$ we get $L^2 = 98 - 54 - 1 > 5$ and therefore the first condition to apply Lemma 2.0.4 to $|L'|$ is fulfilled.

Theorem 2.3.2 *Let S be a surface as in Theorem 2.3.1 above. Then the complete linear series $|L'| := \left| 5H - \sum_{i=1}^{6} 2E_i \right|$ is very ample on $S \backslash (\bigcup_{i=1}^{5} E'_i \cup E)$.*

Proof. The base point freeness of $|L'|$ on $S^{(6)}$ follows if there exists no 0-critical effective divisor D. As above we have to distinguish three types for D :

(a) $D \sim E_k$ for a $k \in \{1, ..., 6\}$.
(b) $D \sim E_k - E_{k+1}$ for $k \in \{1, ..., 5\}$
(c) $D \sim aH + bR - \sum_{i=1}^{6} \alpha_i E_i$ with $\alpha_i = 0$ or 1 and $aH + bR \in \mathrm{Pic}(P_2)^*$, thus $a \geq 0$ and $b \geq -2a$.

It is easy to check that for every divisor D of type (a) or (b) we get $D.(L-D) \geq 2$. Assume that $D \sim aH + bR - \sum_{i=1}^{7} \alpha_i E_i$ is a $0-$critical divisor, then from the condition that $L - 2D$ is $\mathbb{Q}$-effective we can restrict to $a \leq 3$ and $b \leq \left\lfloor \frac{7-a}{2} \right\rfloor \leq 3$. On $S^{(6)}$ the curve C is a divisor of class $5H - \sum_{i=1}^{6} 2E_i$ that has exactly one double point as only singularity. Let $\delta = \#\{\alpha_i : \alpha_i = 1\}$, then we conclude:

$$\begin{aligned} D.(L-D) &= 2a(7-a) + b(7-a) - ab - \textstyle\sum_{i=1}^{6} \alpha_i (3 - \alpha_i) = \\ &= 14a + 7b - 2\delta - 2a(a+b) = D.C + 4a + 2b - 2a(a+b) \geq \\ &\geq 2(2a+b) - 2a(a+b) \end{aligned}$$

For $a = 0$ we get $D.(L-D) \geq 2b \geq 2$ and for $a = 1 : D.(L-D) \geq 2$. In the cases $a = 2, 3$ we must have $-6 \leq -2a \leq b \leq 2$. Thus if $a = 2$ and $b < 0$ then $D.(L-2D) \geq -2b \geq 2$ and for $a = 2$, $b \geq 0$ we get $D.(L-2D) = D.C - 2b = 10a + 3b - 2\delta \geq 20 - 12 = 8$. In the situation $a = 3, b \leq -2$ we obtain $D.(L-2D) \geq -6 - 4b \geq 2$ and for $a = 3, b \geq -1$ this inequality also holds because of $D.(L-2D) = 10a + 3b - 2\delta \geq 30 - 3 - 2\delta \geq 15$. In summary we have seen that there exists no 0-critical divisor, hence $|L'|$ is base point free on $S^{(6)}$.

Now let $p \in S^{(6)} \backslash (\bigcup_{i=1}^{5} E'_i \cup E)$ be an arbritrary point. The very ampleness of $|L'|$ on $S^{(6)} \backslash (\bigcup_{i=1}^{5} E'_i \cup E)$ follows if $|L' - E_p| = \left| 5H - \sum_{i=1}^{6} 2E_i - E_p \right|$ is base point free on the blowup $S_p := \tilde{S}^{(6)}(p)$ with exceptional divisor E_p. With $L_p := L' - K_{S_p} \sim 7H - \sum_{i=1}^{6} 3E_i - 2E_p$ we still have $L_p^2 = 98 - 54 - 8 > 5$, such that we can use Lemma 2.0.4 again. A 0-critical divisors D for $|L' - E_p|$ has to satisfy the following inequality:

$$D.(L' - D - E_p) \leq 1 \Rightarrow D.(L' - D) - E_p.D \leq 1$$

Because of $E_p.D \leq 1$ and $D.(L' - D) \geq 2$ this is only possible if $E_p.D = 1$ and $D.(L' - D) = 2$. Then $p \in D$ with D a strict transform $E'_k = E_k - E_{k+1}$ or D the exceptional divisor E, which was excluded. ∎

3

Curves C with Clifford index Cliff(C)≤2

3.1 Trigonal Curves

If C is a canonical curve of genus 9 and $\text{Cliff}(C) = 1$, then it follows the existence of a g^r_{2r+1} with $r \in \{1, ..., 6\}$. As the Brill Noether dual of a g^r_{2r+1} is of type g^{7-r}_{15-2r}, we can restrict to the existence of a g^1_3, g^2_5 or g^3_7. The existence of a g^2_5 is not possible as a plane curve of degree 5 has geometric genus less or equal than $\binom{4}{2} = 6 < 9$. Further Castelnuovo inequality gives a boundary for the genus of space curves of degree $d \geq 3$ (cf. Theorem IV 6.4 in [Hs77]): According to this theorem we must have $g = 9 \leq \frac{1}{4}(d^2 - 1) - d + 1$ for odd degrees d. As $d = 7$ does not fullfill this inequality we can omit the possibility of the existence of a g^3_7. Hence, it follows:

Corollary 3.1.1 *Let $C \subset \mathbb{P}^8$ be a canonical curve of genus 9 that has Clifford index* $\text{Cliff}(C) = 1$*, then C is trigonal.*

We will repeat the main results as can be found in [S86] Section 6.1.: A trigonal canonical curve of genus g is contained in a 2−dimensional scroll $X \subset \mathbb{P}^{g-1}$:

$$X = \bigcup_{D \in g^1_3} \bar{D} \subset \mathbb{P}^{g-1}$$

of type $S(e_1, e_2)$ and degree $f = e_1 + e_2 = g - 2$ (see Section 1.4). Because of $\deg(K_C - nD) = 2g - 2 - 3n < 0$ for $n > \frac{2g-2}{3}$, for the values e_i we get the following bounds:

$$\frac{2g-2}{3} \geq e_1 \geq e_2 \geq \frac{g-4}{3}$$

Further C is a divisor of class $3H - (f - 2)R$ on X (cf. Theorem 1.4.3). The mapping cone

$$\mathcal{C}^{f-2}(-3) \to \mathcal{C}^0$$

is a minimal resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module and therefore the scroll X and hence g^1_3 is uniquely determined by C for a trigonal curve. For $g(C) = 9$ we get the minimal free resolution of $\mathcal{O}_C$ from the following mapping cone construction with $F = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H - R) \cong \mathcal{O}^f_{\mathbb{P}^8}$ and $G = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R) \cong \mathcal{O}^2_{\mathbb{P}^8}$ (cf. Section 1.3):

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+5R) & \to & \mathcal{O}_{\mathbb{P}(\mathcal{E})} & \to & \mathcal{O}_C & \to & 0\\
 & & \uparrow & & \uparrow & & & & \\
 & & S_5G(-3) & & \mathcal{O}_{\mathbb{P}^8} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & F\otimes S_4G(-4) & & \wedge^2 F(-2) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \wedge^2 F\otimes S_3G(-5) & & \wedge^3 F\otimes D_1G^*(-3) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \wedge^3 F\otimes S_2G(-6) & & \wedge^4 F\otimes D_2G^*(-4) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \wedge^4 F\otimes G(-7) & & \wedge^5 F\otimes D_3G^*(-5) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \wedge^5 F(-8) & & \wedge^6 F\otimes D_4G^*(-6) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \wedge^7 F(-10) & & \wedge^7 F\otimes D_5G^*(-7) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

There exists no non-minimal map, thus it follows

Theorem 3.1.2 *Let $C \subset \mathbb{P}^8$ be an irreducible, canonical and non hyperelliptic curve of genus 9 that admits a g^1_3. Then the Betti table for C is given as follows:*

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	70	105	84	35	6	-
2	-	6	35	84	105	70	21	-
3	-	-	-	-	-	-	-	1

3.2 Tetragonal Curves

For a canonical curve C of genus 9, from the property of having Clifford index 2 it follows the existence of a g^1_4. This can be seen as follows: From the existence of a g^2_6 we get a plane model of C that has exactly $\binom{5}{2} - 9 = 1$ doublepoint as only singularity. Projection from this point leads to a g^1_4. In the case where C admits a g^3_8, we get a space model C' for C of degree $l = 8$. If C' has a singular point, projection from this point leads to a g^2_d with $d \leq 6$. Otherwise according to a theorem of Castelnuovo (cf. [Hs77] IV Theorem 6.4), it follows from $g(C') = 9 = \frac{1}{4}l^2 - l + 1$ that C' is contained in a quadric surface $Q \subset \mathbb{P}^3$. In the situation where $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ is nonsingular, C' must be a divisor of type $(4,4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Projection along each factor of $\mathbb{P}^1 \times \mathbb{P}^1$ leads to two different $g^1_4{}'s$ on C. For Q a singular quadric cone over an elliptic curve E, C' is of class $C' \sim 4A$, A the class of a hyperplane section on Q. The rulings $|R|$ on Q cut out a g^1_4 on C. It remains to remark that for a g^4_{10} or a g^5_{12} the Brill Noether dual is of type g^2_6 or g^1_4 respectively.

Corollary 3.2.1 *Let $C \subset \mathbb{P}^8$ be a canonical curve of genus 9 with Clifford index* $\mathrm{Cliff}(C) = 2$, *then C is tetragonal.*

Now applying the main ideas, which can be found in [S86] Section 6.2-6.6, to a canonical, tetragonal curve C of genus $g = 9$ leads to the following results: Constructing X as above from the linear system of type g^1_4 on C, we see that C is contained in a 3−dimensional rational normal scroll of type $S(e_1, e_2, e_3)$ with

$$4 = \frac{2g-2}{4} \geq e_1 \geq e_2 \geq e_3 \geq 0$$

and degree $f = e_1 + e_2 + e_3 = g - 3 = 6$. We denote by $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ the corresponding $\mathbb{P}^2$−bundle. According to Theorem 1.4.3 C is given as complete intersection of two divisors

$$Y \sim 2H - b_1 R \quad , \quad Z \sim 2H - b_2 R$$

on X with

$$b_1 + b_2 = f - 2 = 4$$

and $b_1 \geq b_2$. In [S86] Section 6.3. the author verifies that

$$5 = f - 1 \geq b_1 \geq b_2 \geq -1$$

and even $b_1 \leq f - 2 = 4$ for genus $g \neq 0$. Therefore we can apply Theorem 1.4.3 to get a minimal free resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{g-1}}$-module via an iterated mapping cone:

$$[\mathcal{C}^{f-2}(-4) \to \mathcal{C}^{b_1}(-2) \oplus \mathcal{C}^{b_2}(-2)] \to \mathcal{C}^0$$

The Betti table for the minimal free resolution is determined by the values for b_1 and b_2 and vice versa. X and hence the g^1_4 is uniquely determined by C unless $b_1 \geq f - 2 = 4$: Consider the map

$$\mathcal{O}^{\beta_{g-4,g-3}}(-g+3) \to \mathcal{O}^{\beta_{g-5,g-4}}(-g+4)$$

in the resolution of C, then X is given as support of the cokernel of its dual.

Theorem 3.2.2 *Let $C \subset \mathbb{P}^8$ be an irreducible, nonsingular, canonical curve of genus 9 with* $\mathrm{Cliff}(C) = 2$, *then*
a) C admits a g^2_6 or a g^3_8 exactly if $(b_1, b_2) = (4, 0)$. The Betti table for C then takes the following form

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	90	64	20	-	-
2	-	-	20	64	90	64	21	-
3	-	-	-	-	-	-	-	1

b) C admits a g^1_4, a base point free linear system of Clifford index 3 and no g^2_6 or g^3_8 exactly if $(b_1, b_2) = (3,1)$ with Betti table for C as follows

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	75	44	5	-	-
2	-	-	5	44	75	64	21	-
3	-	-	-	-	-	-	-	1

c) C admits a g^1_4 and no g^2_6, g^3_8 and no base point free linear system of Clifford index 3 iff $(b_1, b_2) = (2,2)$. The Betti table for C is then given by

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	75	24	5	-	-
2	-	-	5	24	75	64	21	-
3	-	-	-	-	-	-	-	1

Proof. From the conditions for (b_1, b_2) we have to distinguish three different cases for $g = 9$:
1. $(b_1, b_2) = (4, 0)$: From the mapping cone construction above we get the following Betti table for $C \subset \mathbb{P}^8$:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	90	64	20	-	-
2	-	-	20	64	90	64	21	-
3	-	-	-	-	-	-	-	1

The fibres of $Y' \subset \mathbb{P}(\mathcal{E})$ over $\mathbb{P}^1$ are conics. If all these fibres are degenerate the g^1_4 is composed by an elliptic or hyperelliptic involution

$$C \overset{2:1}{\to} E \overset{2:1}{\to} \mathbb{P}^1$$

and Y is a birational ruled surface over E with a rational curve $\tilde{E}$ of double points. As C is assumed to be nonsingular we must have $\tilde{E} \cap Z = \varnothing$ and therefore

$$0 = \tilde{E}.Z = \tilde{E}.2H = 2 \deg \tilde{E} \Rightarrow \deg \tilde{E} = 0$$

A general divisor Γ of class $H - \frac{b_2}{2}R$ intersects Y in a smooth curve isomorphic to E, so the geometric genus is given by

$$2p_a E - 2 = \Gamma.Y.(f - 2 - b_1 - \frac{b_2}{2})R = b_2 = 0$$

(cf. [S86] Example 3.6). Thus E is an elliptic curve that can be embedded as a plane cubic. Then the composition $C \overset{2:1}{\to} E \hookrightarrow \mathbb{P}^2$ gives a g^2_6 on C.

In the situation where a general fibre of Y is a nonsingular conic, the number of singular fibres is given by

$$\delta = 2f - 3b_1 = 0$$

hence there exists no singular fibre. Further Y admits a determinantal presentation. It is obtained from the determinant of a matrix ψ with entries on X as indicated below

$$\begin{pmatrix} H - a_1R & H - a_2R \\ H - (a_1 + k)R & H - (a_2 + k)R \end{pmatrix}$$

with $a_1, a_2 \in \mathbb{Z}$, $k \in \mathbb{N}$ and $a_1 + a_2 + k = b_1 = 4$. Then Y may be identified with the image of $P_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1})$ under a rational map defined by a base point free linear series. The composition

$$C \subset Y \to P_k \to H^0(P_k, \mathcal{O}_{P_k}(A))) \cong \mathbb{P}^{k+1}$$

defines a linear series of degree $b_2 + 2 + 2(k + 1) = A.C'$ (cf. Theorem 1.4.5), hence of Clifford index 2. For $k = 0$ we get a further g^1_4 and therefore a g^3_8 from $g^1_4 \times g^1_4$. In the case $k = 1$, Y is a Del-Pezzo surface, precisely $\mathbb{P}^2$ blown up in one point p, and C the strict transform of a plane model C' of degree 6 with a doublepoint in p. We easily check that for $k \geq 3$ the determinant of the matrix above becomes reducible, which can be excluded as Y is irreducible. It remains to consider the case $k = 2$, where C' is a divisor of class $4A$ on the quadric cone Y, A the hyperplane class of Y. Then the g^1_4 is cut out by the class of a ruling R on Y and $|(A|_{C'})| = |(2R|_{C'})|$ is a linear series of type g^3_8.

Conversely in the case where C admits a g^3_8, we get a space model C' of C of degree $l = 8$. If C' has a singular point, projection from this point leads to a g^2_d with $d \leq 6$. Otherwise C' is contained in a quadric surface $Q \subset \mathbb{P}^3$. In the situation where $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ is nonsingular, C' must be a divisor of type $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Projection along each factor of $\mathbb{P}^1 \times \mathbb{P}^1$ leads to two different g^1_4 on C, such that we get $(b_1, b_2) = (4, 0)$ as otherwise the g^1_4 is uniquely determined. For Q a singular quadric cone, C' is of class $C' \sim 4A$, A the class of a hyperplane section on Q. Then the rulings $|R|$ on Q cut out a g^1_4 on C and because of $(A - 2R)|_{C'} \sim 0$, we get

$$K_{C'} \sim 2A|_{C'} \sim 4R|_{C'} \Rightarrow |K_C| = 4g^1_4$$

The adjoint series $|2A|$ defines a mapping

$$\varphi : Q \to \mathbb{P}^8$$

with image Q'. This surface is given on X by the determinant of a matrix ψ of type

$$\begin{pmatrix} H - a_1R & H - a_2R \\ H - (a_1 + 2)R & H - (a_2 + 2)R \end{pmatrix}$$

on X with $a_1, a_2 \in \mathbb{Z}$, $a_1 + a_2 = f - 4 = 2$ (cf. Theorem 1.4.5 and Section 6.4. in [S86]). The scroll X is of type $S(4, 2, 0)$ and therefore $a_1, a_2 \leq 2$ as Q'

is irreducible. The case $a_1 = a_2 = 1$ can also be excluded as in this situation it is possible to make one of the $H - 3R$ entries zero, thus Q' would become reducible. It follows that $a_1 = 2$, $a_2 = 0$ and therefore $Q' = Y \sim 2H - 4R \Rightarrow (b_1, b_2) = (4, 0)$.

It remains to show that in the situation where C admits a plane model C' of degree 6, we also get $(b_1, b_2) = (4, 0)$: C' has exactly $\binom{5}{2} - 9 = 1$ doublepoint p as only singularity. Blowing up $\mathbb{P}^2$ in p we get a surface S with exceptional divisor E_p. Then the image under the adjoint series $|3H - E_p|$ is a Del-Pezzo surface $S' \subset \mathbb{P}^8$ of degree $g - 3 = 6$, that is arithmetically Cohen-Macaulay. Its projective dimension $R_{S'}$ is $6 = 8 - \dim S'$ and further $\operatorname{reg} R_{S'} = 2$. According to Theorem 1.1.10 the Hilbert function $H_{R_{S'}}$ of $R_{S'}$ is given by the Hilbert polynomial $P_{S'}$ of S' calculated in [Hs77] Chapter V, Exercise 1.2.:

$$P_{S'}(n) = \frac{1}{2}an^2 + bn + c$$

with $a = (K_S + C)^2 = (3H - E_p)^2 = 9 - 1 = 8$, $b = \frac{1}{2}(K_S + C)^2 + 1 - g(K_S + C) = 4 + 1 - 1 = 4$ and $c = 1$. Therefore we obtain

$$H_{R_{S'}}(n) = P_{S'}(n) = 4n^2 + 4n + 1 \text{ for all } n \in \mathbb{N}$$

The minimal free resolution F of $R_{S'}$ over $R = \Bbbk[x_0, ..., x_n]$ reduces modulo (y_1, y_2, y_3) to a minimal free resolution of $R'_{S'} := R_{S'}/(y_1, y_2, y_3)R_{S'}$ over $R/(y_1, y_2, y_3)R \cong R' := \Bbbk[x'_0, ..., x'_5]$ with (y_1, y_2, y_3) being a $R_{S'}$ sequence of linear polynomials in $x_0, ..., x_8$. The corresponding Betti numbers β_{ij} stay the same. Thus the Hilbert function of $R'_{S'}$ can be obtained by succesively dividing out y_1, y_2 and y_3, hence it has values $(1, 6, 1)$ and $H_{R'_{S'}}(n) = 0$ for $n \geq 3$. We consider a free resolution of $\Bbbk$ with free R' modules which is given by the Koszul complex of length 6:

$$0 \leftarrow \Bbbk \leftarrow R' \leftarrow R'^6 \leftarrow R'^{15} \leftarrow R'^{20} \leftarrow R'^{15} \leftarrow R'^6 \leftarrow R' \leftarrow 0$$

The Betti numbers $\beta_{ij} = \dim Tor_i^{R'}(\Bbbk, R'_{S'})_j$ can be calculated by tensoring the complex above with $R'_{S'}$:

$$0 \leftarrow \underbrace{R'_{S'}}_{M^{(0)}} \xleftarrow{\varphi_1} \underbrace{R'^6 \otimes R'_{S'}}_{M^{(1)}} \xleftarrow{\varphi_2} \underbrace{R'^{15} \otimes R'_{S'}}_{M^{(2)}} \leftarrow ... \xleftarrow{\varphi_6} \underbrace{R'_{S'}}_{M^{(6)}} \leftarrow 0$$

Taking into acount the graduation we get the following format:

$$M^{(0)} \longleftarrow M^{(1)} \longleftarrow M^{(2)} \longleftarrow M^{(3)} \longleftarrow M^{(4)} \longleftarrow M^{(5)} \longleftarrow M^{(6)}$$

$$\begin{tikzcd}
1 & 6 \arrow[dl] & 15 \arrow[dl] & 20 \arrow[dl] & 15 \arrow[dl] & 6 \arrow[dl] & 1 \arrow[dl] \\
6 & 36 \arrow[dl] & 90 \arrow[dl] & 120 \arrow[dl] & 90 \arrow[dl] & 36 \arrow[dl] & 6 \arrow[dl] \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{tikzcd}$$

where the arrows stand for the maps $\varphi_k^{(l)} : M_l^{(k)} \to M_l^{(k-1)}$, which give a decomposition of φ_k in the parts of degree l and the numbers in the format are given by $c_{kl} := \dim M_l^{(k)}$. As S' is not contained in any hyperplane we get $\beta_{11} = 0$ and therefore $\beta_{m,m} = 0$ for $m = 1, ..., 6$. The dual F^* of F is a free resolution of $\omega_{S'}$ (up to a shift of degrees) and therefore we obtain from Green's Linear Syzygy Theorem (see [E05] Theorem 7.1) that the length n of the linear strand of F^* satisfies $n \leq \beta_{68} - 1 = 1 - 1 = 0$, hence $\beta_{m,m+2} = 0$ for $m = 0, ..., 5$. It follows that the Betti table for S' takes the following form:

	0	1	2	3	4	5	6
0	1	-	-	-	-	-	-
1	-	20	64	90	64	20	-
2	-	-	-	-	-	-	1

From the exact sequence

$$0 \to \mathcal{O}_{S'}(-C) \to \mathcal{O}_{S'} \to \mathcal{O}_C \to 0$$

and $\mathcal{O}_{S'}(-C) \cong \omega_{S'}$ the minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}$-module is given as the mapping cone of the minimal free resolutions of $\mathcal{O}_{S'}$ and $\omega_{S'}$ as every map is minimal:

$$\begin{array}{ccccccccc}
0 & \to & \omega_{S'} & \to & \mathcal{O}_{S'} & \to & \mathcal{O}_C & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^8}(-2)^1 & & \mathcal{O}_{\mathbb{P}^8} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^8}(-4)^{20} & & \mathcal{O}_{\mathbb{P}^8}(-2)^{20} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^8}(-5)^{64} & & \mathcal{O}_{\mathbb{P}^8}(-3)^{64} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^8}(-6)^{90} & & \mathcal{O}_{\mathbb{P}^8}(-4)^{90} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^8}(-7)^{64} & & \mathcal{O}_{\mathbb{P}^8}(-5)^{64} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^8}(-8)^{20} & & \mathcal{O}_{\mathbb{P}^8}(-6)^{20} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^8}(-10) & & \mathcal{O}_{\mathbb{P}^8}(-8) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Therefore we obtain the Betti table for C :

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	90	64	20	-	-
2	-	-	20	64	90	64	21	-
3	-	-	-	-	-	-	-	1

We have shown that a tetragonal curve C of genus 9 admits a g^2_6 or a g^3_8 exactly if its betti table looks as above, which is exactly the case for $(b_1, b_2) = (4, 0)$. In the next step we will give the Betti table for a tetragonal curve C that admits no g^2_6, g^3_8 or a base point free g^r_d of Clifford index 3 :

2. Let $g^1_4 = |D|$ with an effective divisor D of degree 4 on C. We can omit the case $h^0(C, \mathcal{O}_C(2D)) = 4$ as this would give a $g^3_8 = |2D|$. Then $F \sim K_C - 2D$ gives a g^2_8. If $|F|$ has base points we can deduce a g^2_d with $d \leq 7$, that we have excluded. In the situation where $F \sim 2D$ we must have $h^0(C, \mathcal{O}_C(K_C - 3D)) = 2$ and $h^0(C, \mathcal{O}_C(K_C - 4D)) = 1$, hence the scroll X is of type $S(4, 1, 1)$. Then $(b_1, b_2) \neq (3, 1)$ as the defining equation of $Y \sim 2H - 3R$ would contain a factor $\varphi_0 \in H^0(X, \mathcal{O}_X(H - 4R))$, which contradicts that $C \subset Y$ is irreducible. It follows that $(b_1, b_2) = (2, 2)$ in this case, hence C has Betti table as follows

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	24	5	-	-
2	-	-	5	24	70	64	21	-
3	-	-	-	-	-	-	-	1

Now we assume that $F \not\sim 2D$, then from $|F|$ we get a plane model $C' \subset \mathbb{P}^2$ of degree 8 for C. C' does not have any triple points as projection from such a point would give a base point free g^1_5. If C' has double points as only singularities then from a base point free g^1_6 obtained from projection from one of these double points and the g^1_4 we get a space model $C'' \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ of C. C'' is a divisor of class $(4, 6)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, thus it has arithmetic genus $p_a(C'') = 3 \cdot 5 = 15 > 9$. C'' has no singularity of multiplicity 3 or higher as projection from such a point would give a g^2_d with $d \leq 7$. It follows that C'' has exactly $15 - 9 = 6$ double points as only singularities hence projection from one of them leads to a g^2_8 with a quadruple point and 6 double points. Thus we can assume the existence of a plane model C' of C with a quadruple point p_0 and 6 double points $p_1, ..., p_6$ as only singularities. After blowing up $\mathbb{P}^2$ in these points

$$\sigma : S := \tilde{\mathbb{P}}^2(p_0, ..., p_6) \to \mathbb{P}^2$$

with exceptional divisors $E_0, ..., E_6$ and hyperplane class H, we can assume that C is the strict transform of C', i.e. $C \sim 8H - 4E_0 - \sum_{i=1}^{6} E_i$. Then the canonical system is cut out on C by the adjoint series $|L'| = \left| 5H - 3E_0 - \sum_{i=1}^{6} E_i \right|$. With the methods of Chapter 2 we can show that this series is base point free and the image $S' \subset \mathbb{P}^8$ of S under the morphism defined by $|L'|$ has only isolated singularities and is arithmetically Cohen-Macaulay. Therefore the projective dimension of its homogenous coordinate ring $R_{S'}$ is $6 = 8 - \dim S'$ and further $\operatorname{reg} R_{S'} = 2$. According to Theorem 1.1.10 the Hilbertfunction $H_{R_{S'}}$ of $R_{S'}$ is given by the Hilbert polynomial $p_{S'}$ of S' calculated in [Hs77] Chapter V, Exercise 1.2.:

$$P_{S'}(n) = \frac{1}{2}an^2 + bn + c$$

with $a = (K_S + C)^2 = (5H - 3E_0 - \sum_{i=1}^{6} E_i)^2 = 10$, $b = \frac{1}{2}(K_S + C)^2 + 1 - g(K_S + C) = 3$ and $c = 1$. Therefore we obtain

$$H_{R_{S'}}(n) = P_{S'}(n) = 5n^2 + 3n + 1 \text{ for all } n \in \mathbb{N}$$

Now we consider the same approach as in **1.** to obtain the Betti table for C : As in the previous consideration the minimal free resolution F of $R_{S'}$ over R reduces modulo to a minimal free resolution of $R'_{S'} := R_{S'}/(y_1, y_2, y_3)R_{S'}$ over $R/(y_1, y_2, y_3)R \cong R' := \mathbb{k}[x'_0, ..., x'_5]$. The corresponding Hilbert function of $R'_{S'}$ has values $(1, 6, 3)$ and $H_{R'_{S'}}(n) = 0$ for $n \geq 3$. Tensoring the Koszul complex of length 6 with $R'_{S'}$ a similar argumentation as in **1.** shows that $\beta_{m,m} = 0$, $m = 1, ..., 6$ and $\beta_{m,m+2} = 0$, $m = 0, ..., 3$ for the Betti numbers of S'. The linear strand of the minimal free resolution of $R_{S'}$ is a subcomplex of the minimal free resolution of R_C, thus we must have $\beta_{6,7} = 0$ as Cliff$(C) = 2$. The Betti table of the minimal free resolution of $R_{S'}$ then takes the following form

	0	1	2	3	4	5	6
0	1	-	-	-	-	-	-
1	-	18	52	60	24	5	-
2	-	-	-	-	15	12	3

and from the exact sequence

$$0 \to \mathcal{O}_{S'}(-C) \to \mathcal{O}_{S'} \to \mathcal{O}_C \to 0$$

the minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}$-module is given as the mapping cone of the minimal free resolutions of $\mathcal{O}_{S'}$ and $\omega_{S'}$:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	24	5	-	-
2	-	-	5	24	70	64	21	-
3	-	-	-	-	-	-	-	1

Therefore we have shown that for a tetragonal curve C of genus 9 that admits a g^1_4 and no g^2_6, g^3_8 or a further g^r_d of Clifford index 3, its Betti table looks as above, which is exactly the case for $(b_1, b_2) = (2, 2)$.

3. $(b_1, b_2) = (3, 1)$: In this situation C has Betti table as follows

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	75	44	5	-	-
2	-	-	5	44	75	64	21	-
3	-	-	-	-	-	-	-	1

According to our results in **1.** and **2.**, a curve C that is given as complete intersection of divisors of type $Y \sim 2H - 3R$, $Z \sim 2H - R$ on the scroll X

admits a base point free g^r_d of Clifford index 3. It remains to show that from the existence of such a linear system it follows that $(b_1, b_2) = (3, 1)$. From a base point free linear system of Clifford index 3 we always get a g^1_5 or a g^2_7 (The Brill Noether dual of a g^3_9 or g^4_{11} is a g^2_7 or g^1_5 respectively!). A g^2_7 cannot have a singularity with higher multiplicity than 3 as in this case projection from this singular point would lead to g^1_d with $d \leq 3$. The g^2_7 cannot have exactly two triple points as then from the existence of two different $g^1_4{}'s$ we would deduce that $(b_1, b_2) = (4, 0)$ as in **1.**, hence the existence of a g^2_6 or a g^3_8. Thus the g^2_7 has exactly 6 double points or one triple point and 3 double points. Except in the case where three double points are infinitely near a triple point, projection from one of the double points leads to a g^1_5. Therefore C admits a $g^1_4 \times g^1_5$ with the mentioned exception. In the case where C admits a $g^1_4 \times g^1_5$ but no g^2_6 or g^3_8, we consider the image $C' \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ of degree 9 under the morphism obtained from the $g^1_4 \times g^1_5$. Then C' is a divisor of class $(4, 5)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with arithmetic genus $p_a(C') = 12$. Further C' has exactly three double points as only singularities, as projection from a singular point of higher multiplicity would lead to a g^2_d with $d \leq 6$. Then projection from one of the double points gives a plane model $C'' \subset \mathbb{P}^2$ of degree 7 with one triple point p_0 and 3 double points p_1, p_2 and p_3. Thus in general we can assume that we have a g^2_7 with one triple point p_0 and 3 (possibly infinitely near) double points $p_1, ..., p_3$. Blowing up $\mathbb{P}^2$ in the these points

$$\sigma : S = \tilde{\mathbb{P}}^2(p_0, ..., p_3) \to \mathbb{P}^2$$

with exceptional divisors $E_0, ..., E_3$, we can consider $C \sim 7H - 3E_0 - \sum_{i=1}^3 2E_i$ to be the strict transform of C'', H denoting the hyperplane class on $\mathbb{P}^2$ and by abuse of notation also on S. The canonical series is then cut out by the adjoint series $|L'| := \left|4H - 2E_0 - \sum_{i=1}^3 2E_i\right|$, which is base point free on S (cf. Corollary 2.1.1). The image S' of S under the morphism given by $|L'|$ is then a Castelnuovo surface and it is even arithmetically Cohen-Macaulay. Therefore the projective dimension $R_{S'}$ is $6 = 8 - \dim S'$ and further $\operatorname{reg} R_{S'} = 2$. According to Theorem 1.1.10 the Hilbertfunction $H_{R_{S'}}$ of $R_{S'}$ is given by the Hilbert polynomial $p_{S'}$ of S' calculated in [Hs77] Chapter V, Ex. 1.2.:

$$P_{S'}(n) = \frac{1}{2}an^2 + bn + c$$

with $a = (K_S + C)^2 = (4H - 2E_0 - \sum_{i=1}^3 2E_i)^2 = 9$, $b = \frac{1}{2}(K_S + C)^2 + 1 - g(K_S + C) = 5$ and $c = 1$. Therefore we obtain

$$H_{R_{S'}}(n) = P_{S'}(n) = 4n^2 + 4n + 1 \text{ for all } n \in \mathbb{N}$$

Now we consider the same approach as in **1.** and **2.** to obtain the Betti table for C : The minimal free resolution F of $R_{S'}$ over R reduces to a minimal free resolution of $R'_{S'} := R_{S'}/(y_1, y_2, y_3)R_{S'}$ over $R/(y_1, y_2, y_3)R \cong R' := \mathbb{k}[x'_0, ..., x'_5]$. The Hilbertfunction of $R'_{S'}$ has values $(1, 6, 2)$ and $H_{R'_{S'}}(n) = 0$ for $n \geq 3$. Tensoring the Koszul complex of length 6 with $R'_{S'}$ a similar argumentation as in **1.** shows that $\beta_{m,m} = 0$, $m = 1, ..., 6$ and $\beta_{m,m+2} = 0$, $m = 0, ..., 4$ for the Betti numbers of S'. The linear strand of the minimal free resolution of $R_{S'}$ is a subcomplex of the minimal free resolution of R_C, thus we must have $\beta_{6,7} = 0$ as Cliff$(C) = 2$. The Betti table of the minimal free resolution of $R_{S'}$ then takes the following form

	0	1	2	3	4	5	6
0	1	-	-	-	-	-	-
1	-	19	58	75	44	5	-
2	-	-	-	-	-	6	2

and from the exact sequence

$$0 \to \mathcal{O}_{S'}(-C) \to \mathcal{O}_{S'} \to \mathcal{O}_C \to 0$$

the minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}$-module is given as the mapping cone of the minimal free resolutions of $\mathcal{O}_{S'}$ and $\omega_{S'}$:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	75	44	5	-	-
2	-	-	5	44	75	64	21	-
3	-	-	-	-	-	-	-	1

Therefore we have shown that a tetragonal curve C of genus 9 admits a g^1_4, a base pont free linear system of Clifford index 3 and no g^2_6 or g^3_8 exactly if its Betti table looks as above, which is exactly the case for $(b_1, b_2) = (3, 1)$. ■

4

Curves C with Clifford index Cliff(C)=3

4.1 Results

Let $C \subset \mathbb{P}^8$ be a canonical curve of genus 9 and Clifford index 3, then there exists a g^r_{2r+3} with $r \in \{1, ..., 6\}$. Its Brill Noether dual is of type g^{5-r}_{13-2r}, thus for a curve of genus 9 it remains to consider the case where C has a g^1_5 or a g^2_7. A plane curve of degree 7 has arithmetic genus $\binom{6}{2} = 15$, therefore it must have singular points. If it has a singular point of multiplicity $d \geq 3$ then projection from this point leads to a g^1_{7-d} which is a linear system of Clifford index less or equal to 2, a contradiction. Therefore the g^2_7 has exactly six double points as only singularities. Projection from one of them gives a g^1_5. We conclude:

Corollary 4.1.1 *Let $C \subset \mathbb{P}^8$ be a canonical curve of genus 9 with Clifford index* $\mathrm{Cliff}(C) = 3$. *Then C is pentagonal.*

In the following section we first want to give some examples of pentagonal curves that exist. Afterwards we will consider the special case where C admits a g^2_7. It turns out that such$C$ has the following Betti table (cf. Theorem 4.3.2):

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	24	-	-	-
2	-	-	-	24	70	64	21	-
3	-	-	-	-	-	-	-	1

The corresponding plane model $C' \subset \mathbb{P}^2$ of degree 7 has exactly 6 (possibly infinitely near) double points. Hence we deduce 6 (possible infinitely near) $g^1_5{}'s$ by projection from these double points. It turns out that these are exactly all linear series of this type (cf. Theorem 4.3.1).

In the main part of this thesis, we focus on pentagonal curves C, where none of the appearing $g^1_5{}'s$ can be obtained from a g^2_7. Starting with D an effective divisor on C, such that $|D|$ is of type g^1_5, we construct the corresponding 4−dimensional scroll X of type $S(e_1, ..., e_4)$ from the complete linear system $|D|$

(cf. Section 1.4). Applying Riemann-Roch one easily determines the following possibilities for D that can occur (cf. Theorem 4.4.1):

$$\begin{aligned}
&(1)\ \ h^0(C,\mathcal{O}_C(2D)) = 3 \text{ and } h^0(C,\mathcal{O}_C(3D)) = 7\\
&(2)\ \ h^0(C,\mathcal{O}_C(2D)) = 4 \text{ and } h^0(C,\mathcal{O}_C(3D)) = 7\\
&(3)\ \ h^0(C,\mathcal{O}_C(2D)) = 4 \text{ and } h^0(C,\mathcal{O}_C(3D)) = 8
\end{aligned}$$

Then the scroll X is of type $S(2,1,1,1)$, $S(2,2,1,0)$ or $S(3,1,1,0)$ respectively. In this situation the results of Sections 4.5.2-4.6.2 justify the definition of the *multiplicity* $m_{|D|}$ of $|D|$:

Definition 4.1.2 *Let C be an irreducible, smooth, canonical curve of genus 9 and $\mathrm{Cliff}(C) = 3$. Further assume that C has no g^2_7. Then for each linear system $|D|$ of type g^1_5 on C we define its multiplicity $m_{|D|}$ to be equal to i if and only if D fullfills (i).*

Let $k = \sum_{|D|=g^1_5} m_{|D|}$ be the number of all appearing g^1_5, counted with multiplicities, then one of the main results of this thesis is that $k \leq 3$ if $\mathrm{Cliff}(C) = 3$ and C has no g^2_7. This exactly occurs if and only if C has Betti table as follows:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	4k	-	-	-
2	-	-	-	4k	70	64	21	-
3	-	-	-	-	-	-	-	1

For $k = 1$ there exists a plane model for C of degree 8 with exactly one triple point and 9 double points as only singularities. Furthermore in the case $k \geq 2$, we obtain a space model of C on a quadric surface $Y \subset \mathbb{P}^3$. In the case where C has only $g^1_5{}'s$ of multiplicity one, the quadric Y is smooth otherwise it is a cone over a conic. We will also show that in all cases where C has a g^1_5 with multiplicity 2 or 3, there exists a local one-parameter family of curves C_t with $C_0 = C$ and C_t having the correspondent number of distinct g^1_5 for $t \neq 0$, such that this case can be seen as a specialization of those where all g^1_5 are different (cf. Section 4.6.2).

4.2 Examples of pentagonal curves

We have remarked above that for an irreducible, canonical curve C of genus 9 and Clifford index 3 it is possible to have a certain number $k \in \{1,2,3\}$ of $g^1_5{}'s$ (counted with multiplicities) or even a g^2_7. We give examples of such curves over a field of finite characteristic p. Applying the main results of this work and taking into account the semicontinuity of the Betti numbers, we get that each of these curves has the right number of $g^1_5{}'s$.

Example 4.2.1 *(Canonical Curve of genus 9 with exactly one g^1_5)* Let us start with the construction of a curve C that has exactly one g^1_5. From the Brill Noether number

$$\varrho(9,8,2) = 9-(2+1)\cdot(9-8+2) = 0$$

we deduce the existence of a plane model C' of degree 8. If C' has a triple point, projection from this point would lead to a g^1_5. We consider a curve C' that has exactly one triple point q and 9 double points $p_1,...,p_9$ in general position. Then we determine its normalisation $C \subset \mathbb{P}^8$ given as the image of C' under the adjoint series $\left|5H-2q-\sum_{i=1}^{9} p_i\right|$. Obviously C has at least one g^1_5 which is obtained from projection from the triple point in the plane model. The Betti table for C then takes the following form

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	4	-	-	-
2	-	-	-	4	70	64	21	-
3	-	-	-	-	-	-	-	1

According to our main results C has exactly one g^1_5 of multiplicity one and $\text{Cliff}(C) = 3$. The following theorem says that every pentagonal curve C of genus 9 admits a plane model of degree 8 with a triple point.

Theorem 4.2.2 *a) Let C be a canonical curve of genus 9 with* $\text{Cliff}(C) = 3$ *that admits no g^2_7, then there exists a g^2_8 on C with at least one triple point.*
b) If C has exactly one g^1_5 (of multiplicity one) then the g^2_8 in a) has exactly one triple point and 9 double points.

Proof. We start with the second claim. Let C admit a plane model C' of degree 8 with two triple points p_1, p_2. If p_1 and p_2 are not infinitely near, projection from each of them gives two distinct g^1_5, so it remains to discuss the case where p_1 and p_2 are infinitely near. Blowing up $\mathbb{P}^2$ in the triple points p_1, p_2 and the double points $q_1,...,q_6$:

$$\sigma : S \to \mathbb{P}^2$$

with exceptional divisors $E_{p_1}, E_{p_2}, E_{q_1},...,E_{q_6}$ and hyperplane class H, we can assume that C is the strict transform of C' :

$$C \sim 8H - 3E_{p_1} - 3E_{p_2} - \textstyle\sum_{i=1}^{6} 2E_{q_i}$$

The canonical system is cut out by the adjoint series

$$|A_C| = |5H - 2E_{p_1} - 2E_{p_2} - \textstyle\sum_{i=1}^{6} E_{q_i}|$$

As p_2 lies infinitely near to p_1 we obtain $E_{p_1}|_C = E_{p_2}|_C$. For the complete linear series $g^1_5 = |D| = |(H-E_{p_1})|_C| = |(H-E_{p_2})|_C|$ it follows that

$$K_C - 2D = 3H|_C - E_{p_1}|_C - E_{p_2}|_C - \textstyle\sum_{i=1}^{6} E_{q_i}|_C$$

and because of $h^0(S, \mathcal{O}_S(3H - E_{p_1} - E_{p_2} - \sum_{i=1}^{6} E_{q_i})) \geq 10 - 8 = 2$ we get $h^0(C, \mathcal{O}_C(K_C - 2D)) = 2$ (as $\mathrm{Cliff}(C) = 3$), thus $m_{|D|} \geq 2$. This proves b). It remains the question if there always exists a g^2_8 with a triple point. We know that there always exists a g^2_8 on C. This series is base point free as we have assumed that $\mathrm{Cliff}(C) = 3$ and C admits no g^2_7. Let H be an effective divisor of this system, then $h^0(C, \mathcal{O}_C(H - D)) = 0$ or 1. For $h^0(C, \mathcal{O}_C(H - D)) = 1$ it follows the existence of points q_1, q_2 and q_3 on C with

$$H - D \sim q_1 + q_2 + q_3$$

Thus $|H|$ has a triple point. In the case $h^0(C, \mathcal{O}_C(H - D)) = 0$ we can apply the base point free pencil trick to obtain

$$h^0(C, \mathcal{O}_C(H + D)) \geq h^0(C, \mathcal{O}_C(H)) \cdot h^0(C, \mathcal{O}_C(D)) - h^0(C, \mathcal{O}_C(H - D)) = 6$$

Riemann-Roch then says that $h^0(C, \mathcal{O}_C((K_C - H) - D)) \geq 1$ and equality holds. The g^2_8 given by the divisor $L := K_C - H$ is then base point free and has a triple point. The first property follows from the assumption that $\mathrm{Cliff}(C) = 3$ and C admits no g^2_7 and the second from the existence of points q_1, q_2 and q_3 on C with $L - D \sim K_C - D \sim q_1 + q_2 + q_3$ ■

Counting dimensions we have $2 \cdot 10$ possibilities to choose 10 points in $\mathbb{P}^2$ and a parameter space of plane curves of degree 8 passing one of the points with multiplicity 3 and the others with multiplicity 2 that has projective dimension $\binom{10}{2} - 1 - 6 - 3 \cdot 9$. Taking into account the projective transformations on $\mathbb{P}^2$ we obtain the dimension for the subscheme $\mathcal{H}_{(1,5)} \subset \mathcal{M}_9$ of all pentagonal curves of genus 9:

$$\begin{aligned}\dim \mathcal{H}_{(1,5)} &= 2 \cdot 10 + \binom{10}{2} - 1 - 6 - 3 \cdot 9 - \dim PGL(3) = 23 = \\ &= \dim \mathcal{M}_9 - 1\end{aligned}$$

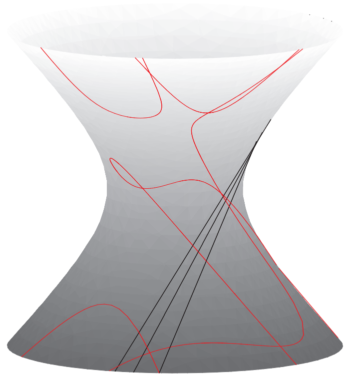
Example 4.2.3 *(Canonical curves with exactly two different ordinary $g^1_5{}'s$)* We want to construct a curve C that has at least two different $g^1_5{}'s$ and no g^2_7. From these special linear systems we obtain a space model $C' \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ given as the image of the natural mapping:

$$C \overset{g^1_5 \times g^1_5}{\to} \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$$

This model in $\mathbb{P}^3$ has exactly $7 = p_a(C') - g(C) = 4 \cdot 4 - 9$ (possibly infinitely near) double points $p_1, ..., p_7$, as otherwise projection from a singular point of higher multiplicity would lead to a g^2_d with $d \leq 7$. For a general curve C having two g^1_5 we would expect that all double points are distinct. Then C' is a divisor of type $(5, 5)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and there exists two $g^1_5{}'s$ which are cut out by the complete linear systems $|(1, 0)|$ and $|(0, 1)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We remark that projection from one of the double points gives a plane model of C that has exactly two triple points. The following example of a curve in this family is calculated over $\mathbb{Q}$:

Curve of type $(5,5)$ *on* $\mathbb{P}^1 \times \mathbb{P}^1$ *with 7 double points as only singularities*

$I_C = ideal(x^2+y^2-z^2-w^2,\ w^3x^2-\frac{70}{3}w^2x^3-\frac{21}{5}wx^4-\frac{49}{9}x^5+3w^4y-\frac{28}{5}w^3y^2-\frac{698428568345581244477}{288489565246275000}w^2y^3+$
$\frac{90215896038289435403}{14424478262313750}wy^4-\frac{68220455180173043}{250860491518500}y^5-6w^4z-7w^3xz+14w^2x^2z-\frac{7}{3}wx^3z-\frac{35}{2}x^4z+\frac{49}{5}w^3yz+$
$\frac{232419262058128 80277}{1492187406446250}w^2y^2z-\frac{2819953624015928 9641}{17309373914776 50}wy^3z+\frac{447595968691098406 39}{2163671739347062 5}y^4z+\frac{49}{3}w^3z^2-$
$\frac{474932490888828 9073}{5769791304925500}w^2xz^2-\frac{830773482885513 41}{1923263768308500}wx^2z^2+\frac{874350643541572 7}{836201638395 00}x^3z^2+\frac{946387415490901362 67}{2257744423666500 0}w^2yz^2+$
$\frac{877466317906635532537}{259640608721647500}wy^2z^2-\frac{554169997124749656173}{259640608721647500}y^3z^2-\frac{738738135418747700 27}{432734347869412 50}w^2z^3-\frac{487977728485206293}{2404079710385625}wxz^3+$
$\frac{114591427898711159}{5769791304925500}x^2z^3+\frac{233221574263914713 02}{72122391311568 75}wyz^3-\frac{1476196213947530908 43}{259640608721647500}y^2z^3-\frac{237127767153385632 88}{64910152180411875}wz^4-$
$\frac{482495537563012499 83}{259640608721647500}xz^4+\frac{367432235132098263 73}{173093739147765000}yz^4+\frac{585938920391961381 7}{129820304360823750}z^5)$



We choose 7 points $p_1, ..., p_7$ on $\mathbb{P}^1 \times \mathbb{P}^1$ in general position and a divisor C' of type $(5,5)$ that passes all these points with multiplicity 2. The canonical series is then cut out by the linear system $\left|(3,3) - \sum_{i=1}^{7} p_i\right|$ and the normalisation C has at least two different g^1_5. Calculating the Betti table for C gives

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	8	-	-	-
2	-	-	-	8	70	64	21	-
3	-	-	-	-	-	-	-	1

Thus it follows that C has exactly two different $g^1_5{}'s$ of multiplicity one and $\text{Cliff}(C) = 3$ (cf. Section 4.5.9).

It will turn out later (cf. Theorem 4.5.6) that for a general curve C' of this family there exist exactly these two $g^1_5{}'s$.
To see that the property of having two $g^1_5{}'s$ gives an independent condition in the subscheme $\mathcal{H}_{(1,5)} \subset \mathcal{M}_9$ of all pentagonal curves of genus 9 , we count the

dimension of the family of plane curves of degree 8 with 2 triple points and 6 double points as only singularities:

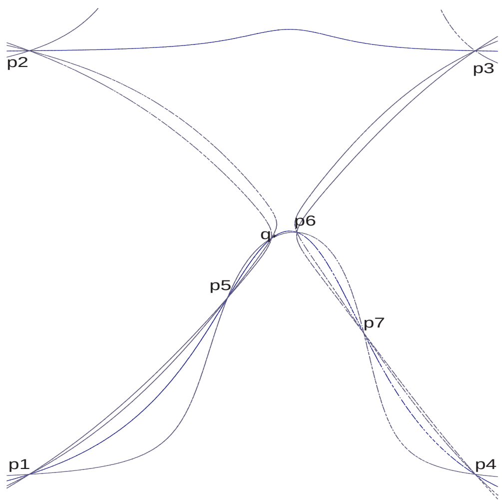
$$2\cdot 8+\binom{10}{2}-1-2\cdot 6-3\cdot 6-\dim PGL(3)=22=\dim\mathcal{M}_9-2$$

Example 4.2.4 *(Canonical curves with exactly three different ordinary $g^1_5{}'s$)* The question arises if it is possible for C to have an additional g^1_5 without admitting a g^2_7. If we start with a plane curve of degree 8 with 3 triple points and 3 double points as only singularities, then these curves fail to have the right number of $g^1_5{}'s$. The reason for this is that the conics passing through the three triple points cut out a g^2_7 on C. However if there exists a curve with exactly three $g^1_5{}'s$ then we would expect that the condition of having a third g^1_5 is of codimension one in the variety of all curves with two g^1_5. Therefore we have examined several curves in this family, which have been randomly constructed over a finite field $\mathbb{F}_p, p\neq 3$. We observed that in approximately one out of p examples the Betti table of the minimal free resolution of S_C looks like

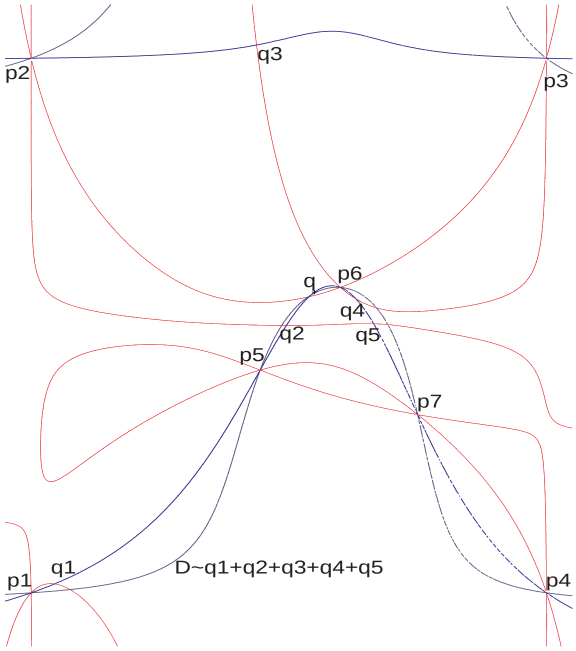
	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	12	-	-	-
2	-	-	-	12	70	64	21	-
3	-	-	-	-	-	-	-	1

A closer examination in these cases led us to the assumption that they exactly occur if the following geometric situation on $\mathbb{P}^1\times\mathbb{P}^1$ is given: From $\dim\left|(2,2)-\sum_{i=1}^7 p_i\right|=2$, it follows the existence of a pencil $(D_\lambda)_\lambda$ of divisors of type $(2,2)$ on $\mathbb{P}^1\times\mathbb{P}^1$ passing through the points $p_1,...,p_7$. Let q denote the basepoint of the linear system $\left|(2,2)-\sum_{i=1}^7 p_i\right|$:

Pencil of divisors of type $(2,2)$ on $\mathbb{P}^1\times\mathbb{P}^1$ passing through $p_1,...,p_7$



It turns out that $q \in C'$ if and only if we get a further g^1_5. This g^1_5 is then cut out on C by the pencil $(D_\lambda)_\lambda$:



For the plane model $C'' \subset \mathbb{P}^2$ of C obtained from projection from one of the double points, this third g^1_5 is cut out by the pencil of cubics passing through the triple points s_1, s_2 and the double points $r_1, ..., r_6$ of C''. The base point p of $|3H - s_1 - s_2 - r_1 - ... - r_6|$ then lies on C. Therefore the family of canonical curves of genus 9 that admit exactly three different $g^1_5{}'s$ has dimension $\dim \mathcal{M}_9 - 3 = 21$.

From Theorem 4.5.6 we know that there cannot exist a fourth g^1_5 on C if C has no g^2_7 and Theorem 4.5.7 states that this g^1_5 is different from the two others if and only if

$$\dim \left|(2,1) - \textstyle\sum_{i=1}^{7} p_i\right|, \dim \left|(1,2) - \textstyle\sum_{i=1}^{7} p_i\right| = 0$$

i.e. there exists no divisor of class $(2,1)$ or $(1,2)$ passing through the points $p_1, ..., p_7$.

We will see in the following sections that it is possible that some of the $g^1_5{}'s$ become equal and that in these cases the multiplicity of these linear systems takes the corresponding value. Furthermore the Betti table stays the same as in the situation where all g^1_5 are different (for details see Section 4.6.2).

Example 4.2.5 *(Canonical curve with a g^2_7)* We complete this section with the most special case for a curve C of $\mathrm{Cliff}(C) = 3$ where C admits a g^2_7. Here the construction is straightforward: The g^2_7 must have exactly 6 double points. We start with $p_1, ..., p_6 \in \mathbb{P}^2$ in general position and $C' \subset \mathbb{P}^2$ a curve of degree 7 passing through these points with multiplicity 2. The canonical series of C' is then cut out by quartics passing through the points $p_1, ..., p_6$. Let $C \subset \mathbb{P}^8$ denote its canonical image. The Betti table for C takes the following form:

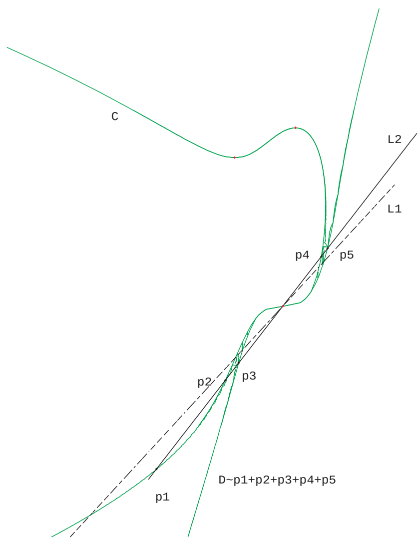
	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	24	-	-	-
2	-	-	-	24	70	64	21	-
3	-	-	-	-	-	-	-	1

It follows from our results that C has $\mathrm{Cliff}(C) = 3$. In special cases the double points can also lie infinitely near. It can even happen that C' has an $A_{11}-$singularity as only singularity. Using a work of Lossen ([L99] Chapter 1) we constructed the following example C' given by the affine equation:

$$8y^2 - 16x^3y - 8x^2y^3 + 2xy^5 - y^7 + 8x^6 + 8x^5y^2 - 8x^3y^3 = 0$$

that has such a special singularity at the origin:

Plane curve C of degree 7 with an A_{11}-singularity at the origin



4.3 The g^2_7 case

Let C be a canonical curve of genus 9 with $\mathrm{Cliff}(C) = 3$ that admits a plane model $C' \subset \mathbb{P}^2$ of degree 7. Then C' has exactly 6 double points $p_1, ..., p_6$ as singularities. We will show that all $g^1_5{}'s$ on C' are given by projection from one of these double points:

Theorem 4.3.1 *Let C be a canonical curve with* $\mathrm{Cliff}(C) = 3$ *that admits a g^2_7. Then every g^1_5 is obtained from projection from one of the double points of the g^2_7.*

Proof. Let H be an effective divisor of the g^2_7 and $|D|$ be a base point free linear series of type g^1_5, then $h^0(C, \mathcal{O}_C(H - D)) = 0$ or 1. From the base point free pencil trick we obtain

$$h^0(C, \mathcal{O}_C(H + D)) \geq h^0(C, \mathcal{O}_C(H)) \cdot h^0(C, \mathcal{O}_C(D)) - h^0(C, \mathcal{O}_C(H - D)) =$$
$$= 6 - h^0(C, \mathcal{O}_C(H - D))$$

Hence because of $\mathrm{Cliff}(C) = 3$ we must have $h^0(C, \mathcal{O}_C(H + D)) \leq 5$, thus $h^0(C, \mathcal{O}_C(H - D)) = 1$. It follows the existence of points q_1 and q_2 on C with

$$H - D \sim q_1 + q_2$$

Therefore $|D|$ is obtained from projection from a double point. ■

A dimension count shows that the subscheme $\mathcal{H}_{(2,7)}$ of all $C \in \mathcal{M}_9$ with a g^2_7 has the expected codimension $-\varrho(9, 7, 2) = 3$ in $\mathcal{M}_9$: We can choose 6 points in general position and a curve of degree 7 passing through these points with multiplicity 2. Therefore we obtain a parameter space of dimension

$$2 \cdot 6 + \binom{7+2}{2} - 1 - 3 \cdot 6 - \dim PGL(3) = 21 = \dim \mathcal{M}_9 - 3$$

It is remarkable that the dimension of the stratum of all curves with a g^2_7 equals to that of the pentagonal curves that admit no g^2_7 but exactly three $g^1_5{}'s$.

Theorem 4.3.2 *Let $C \subset \mathbb{P}^8$ be a smooth, irreducible, canonical curve of genus 9 that has a special linear series of type g^2_7 and* $\mathrm{Cliff}(C) = 3$. *Then C has the following Betti table:*

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	24	-	-	-
2	-	-	-	24	70	64	21	-
3	-	-	-	-	-	-	-	1

Proof. Let us consider the blowup of $\mathbb{P}^2$ in the singular points of C' :

$$\sigma : S \to \mathbb{P}^2$$

Further E_i. denotes the exceptional divisors and H the class of a hyperplane on $\mathbb{P}^2$ and by abuse of notation also its pullback to S. Then we can assume that C is the strict transform of C':

$$C \sim 7H - \sum_{i=1}^{6} 2E_i$$

The adjoint series can be obtained by using the adjunction formula:

$$K_S + C \sim -3H + \sum_{i=1}^{6} E_i + 7H - \sum_{i=1}^{6} 2E_i \sim 4H - \sum_{i=1}^{6} E_i$$

We first want to focus on the image $S' \subset \mathbb{P}^8$ of the map φ defined by the adjoint series

$$\varphi : S \to S' \subset \mathbb{P}^8$$

Let $R = \Bbbk[x_0, ..., x_8]$ and $R_{S'}$ denotes the homogenous coordinate ring of $\mathbb{P}^8$ and S' respectively. From Corollary 2.1.1 we already know that S' is arithmetically Cohen-Macaulay, thus the projective dimension $R_{S'}$ is $6 = 8 - \dim S'$ and $\operatorname{reg} R_{S'} = 2$. Now we consider the same approach as in the proof of Theorem 3.2.2: The Hilbert function $H_{R_{S'}}$ of $R_{S'}$ is given by the Hilbert polynomial

$$P_{S'}(n) = \frac{1}{2}an^2 + bn + c$$

with $a = (K_S + C)^2 = (4\sigma^* H - \sum_{i=1}^{6} E_i)^2 = 16 - 6 = 10$, $b = \frac{1}{2}(K_S + C)^2 + 1 - g(K_S + C) = 5 + 1 - 3 = 3$ and $c = 1$. Therefore, we obtain

$$H_{R_{S'}}(n) = P_{S'}(n) = 5n^2 + 3n + 1 \text{ for all } n \in \mathbb{N}$$

The minimal free resolution F of $R_{S'}$ over R reduces to a minimal free resolution of $R'_{S'} := R_{S'}/(y_1, y_2, y_3)R_{S'}$ over $R/(y_1, y_2, y_3)R \cong R' := \Bbbk[x'_0, ..., x'_5]$ with (y_1, y_2, y_3) being an $R_{S'}$ sequence of linear polynomials in $x_0, ..., x_8$. The Hilbertfunction of $R'_{S'}$ has values $(1, 6, 3)$ and $H_{R'_{S'}}(n) = 0$ for $n \geq 3$. Tensoring $R'_{S'}$ with the Koszul complex of length 6:

$$0 \leftarrow \underbrace{R'_{S'}}_{M^{(0)}} \xleftarrow{\varphi_1} \underbrace{R'^6 \otimes R'_{S'}}_{M^{(1)}} \xleftarrow{\varphi_2} \underbrace{R'^{15} \otimes R'_{S'}}_{M^{(2)}} \leftarrow ... \xleftarrow{\varphi_6} \underbrace{R'_{S'}}_{M^{(6)}} \leftarrow 0$$

and taking into acount the graduation we get the following format:

$$\begin{tikzcd}
M^{(0)} & M^{(1)} \arrow[l] & M^{(2)} \arrow[l] & M^{(3)} \arrow[l] & M^{(4)} \arrow[l] & M^{(5)} \arrow[l] & M^{(6)} \arrow[l] \\
1 & 6 \arrow[dl] & 15 \arrow[dl] & 20 \arrow[dl] & 15 \arrow[dl] & 6 \arrow[dl] & 1 \arrow[dl] \\
6 & 36 \arrow[dl] & 90 \arrow[dl] & 120 \arrow[dl] & 90 \arrow[dl] & 36 \arrow[dl] & 6 \arrow[dl] \\
3 & 18 & 45 & 60 & 45 & 18 & 3
\end{tikzcd}$$

where the arrows stand for the maps $\varphi_k^{(l)} : M_l^{(k)} \to M_l^{(k-1)}$, which give a decomposition of φ_k in the parts of degree l, and the numbers in the format are given by $c_{kl} := \dim M_l^{(k)}$. As S' is not contained in any hyperplane we get $\beta_{11} = 0$ and therefore $\beta_{m,m} = 0$ for $m = 1, ..., 6$. The dual F^* of F is a free resolution of $\omega_{S'}$ (up to a shift of degrees) and therefore we obtain from Green's Linear Syzygy Theorem (cf. [E05] Theorem 7.1) that the length n of the linear strand of F^* satisfies $n \leq \beta_{68} - 1 = 3 - 1 = 2$, hence $\beta_{m,m+2} = 0$ for $m = 0, ..., 3$. The linear strand of the minimal free resolution of $R_{S'}$ is a subcomplex of the minimal free resolution of R_C, hence we must have $\beta_{56} = \beta_{67} = 0$ as $\mathrm{Cliff}(C) = 3$. It follows that the Betti table for S' takes the following form

	0	1	2	3	4	5	6
0	1	-	-	-	-	-	-
1	-	18	52	60	24	-	-
2	-	-	-	-	10	12	3

From the exact sequence

$$0 \to \mathcal{O}_{S'}(-C) \to \mathcal{O}_{S'} \to \mathcal{O}_C \to 0$$

and $\mathcal{O}_{S'}(-C) \cong \omega_{S'}$ the minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}$-module is given as the mapping cone of the minimal free resolutions of $\mathcal{O}_{S'}$ and $\omega_{S'}$ as every map is minimal, thus we obtain the Betti table for C as claimed. ∎

In this section we have shown that for a canonical curve $C \subset \mathbb{P}^8$ of genus 9 and Clifford index 3 that admits a g^2_7, every g^1_5 can be obtained from projection from one of the doublepoints of the g^2_7. Therefore in general, if none of the doublepoints lies infinitely near to another, then there exist exactly 6 different $g_5^{1\prime}s$ on C. The Betti table of C is uniquely determined in this case:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	24	-	-	-
2	-	-	-	24	70	64	21	-
3	-	-	-	-	-	-	-	1

In the following considerations we concentrate on those pentagonal canonical curves that admit no g^2_7.

4.4 Pentagonal Curves and Scrolls

Let C be a pentagonal curve of genus 9 that admits no g^2_7. The variety swept out by the linear spans of these divisors is a 4-dimensional rational normal scroll

$$X = \bigcup_{D \in g^1_5} \overline{D} \subset \mathbb{P}^8$$

At first we determine all different possible types $S(e_1, e_2, e_3, e_4)$ of X which can occur: These are $S(2,1,1,1)$, $S(3,1,1,0)$ and $S(2,2,1,0)$. Then we resolve $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^8}$-module as described in Chapter 1 to get certain conditions on $(a_1, ..., a_5)$ in this resolution. The number of different $(a_1, ..., a_5)$ can be reduced to the remaining three cases $(a_1, ..., a_5) = (2,1,1,1,1)$, $(2,2,1,1,0)$ and $(2,2,2,0,0)$. We will then only focus on the case, where X is of type $S(2,1,1,1)$. For $(a_1, ..., a_5) = (2,2,2,0,0)$ or $(2,2,1,1,0)$, it turns out that C has a linear system of type g^1_4 or a plane model of degree 7 respectively. The case $(a_1, ..., a_5) = (2,1,1,1,1)$ will then be discussed in full detail in Section 4.6. We will show that it is possible for C to have one, two or even three different $g^1_5{}'s$. In Section 4.6.2 it will turn out that $X \simeq S(2,2,1,0)$ or $X \simeq S(3,1,1,0)$ occur as specializations of the general case, when two or three different linear systems of type g^1_5 become infinitely near.

Theorem 4.4.1 *Let C be a smooth, irreducible, canonical curve of genus 9 with a base point free complete pencil $g^1_5 = |D|$ and* $\mathrm{Cliff}(C) = 3$. *Then the 4-dimensional rational normal scroll X swept out by the linear spans of these divisors is of type $S(2,1,1,1)$, $S(2,2,1,0)$ or $S(3,1,1,0)$.*

Proof. We mention that $H|_C \sim K_C$ and $R|_C \sim D$. According to Section 1.4 the type $S(e_1, e_2, e_3, e_4)$ of the scroll X can be determined by considering the following partition of 9 :

$$d_i = h^0(C, \mathcal{O}_C(K_C - iD)) - h^0(C, \mathcal{O}_C(K_C - (i+1)D)),\ i = 0, ..., 3$$

The numbers e_i are given by

$$e_i = \#\{j | d_j \geq i\} - 1$$

Applying Riemann-Roch we get

$$h^0(C, \mathcal{O}_C(K_C - D)) = 8 - \deg D + h^0(C, \mathcal{O}_C(D)) = 5$$

and

$$h^0(C, \mathcal{O}_C(K_C - 2D) - h^0(C, \mathcal{O}_C(2D)) = 9 - 1 - \deg 2D = -2$$
$$\Rightarrow h^0(C, \mathcal{O}_C(K_C - 2D)) = h^0(C, \mathcal{O}_C(2D)) - 2 \geq 1$$

because of $h^0(C, \mathcal{O}_C(D)) = 2$. If $h^0(C, \mathcal{O}_C(K_C - 2D)) \geq 3$ we would get a g^2_6 as $\deg(K_C - 2D) = 6$, so that we can reduce to $1 \leq h^0(C, \mathcal{O}_C(K_C - 2D)) \leq 2$.

As $h^0(C, \mathcal{O}_C(K_C - 2D)) > h^0(C, \mathcal{O}_C(K_C - 3D))$ we have $0 \leq h^0(C, \mathcal{O}_C(K_C - 3D)) \leq 1$ and $h^0(C, \mathcal{O}_C(K_C - kR)) = 0$ for $k \geq 4$ because of $\deg(K_C - kD) < 0$. Therefore we conclude that there are exactly the three different possible types for the scroll as given above. ∎

In following sections we will only focus on the case where X is of tye $S(2,1,1,1)$. According to Theorem 1.4.3 the resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module is given by

$$F_* : \qquad 0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-5H + 3R) \to \sum_{i=1}^{5} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H + b_i R) \xrightarrow{\psi}$$

$$\xrightarrow{\psi} \sum_{i=1}^{5} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + a_i R) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_C \to 0$$

We consider all possible values for a_i, b_i in F_*:

Theorem 4.4.2 *If C is an irreducible, nonsingular, canonical curve of genus 9 and* $\mathrm{Cliff}(C) = 3$ *that admits no g^2_7 then the possible values $a_1, ..., a_5$, $b_1, ..., b_5 \in \mathbb{Z}$ in F_* are:*

$$(a_1, ..., a_5) = (2,1,1,1,1), \ (2,2,1,1,0) \ \textit{or} \ (2,2,2,0,0)$$

Proof. Without loss of generality we assume $a_1 \geq ... \geq a_5$. The calculation of the numbers e_i shows that

$$\frac{2g-2}{5} \geq e_1 \geq e_2 \geq e_3 \geq e_4 \geq 0$$

and $f = e_1 + e_2 + e_3 + e_4 = 5$. The complex F_* is selfdual and $a_i + b_i = f - 2 = 3$, $a_1 + ... + a_5 = 2g - 12 = 6$ (cf. Theorem 1.4.3). From the structure theorem for Gorenstein ideals in codimension 3 (see [BE77]) we obtain further information from the complex F_* above: The matrix ψ is skew-symmetric and its 5 Pfaffians generate the ideal of C in $\mathbb{P}(\mathcal{E})$, i.e. form the entries of

$$\sum_{i=1}^{5} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + a_i R) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}$$

Thus C is determined by the entries of ψ. If one of the nondiagonal entries is zero, say $\psi_{45} = \psi_{54} = 0$, then C is contained in the determinantal surface Y defined by the matrix

$$\omega \sim \begin{pmatrix} \psi_{14} & \psi_{24} & \psi_{34} \\ \psi_{15} & \psi_{25} & \psi_{35} \end{pmatrix}$$

since in this case the 2×2 minors of that matrix are among the Pfaffians of ψ. With C also Y is irreducible. A general fibre of $Y \subset \mathbb{P}(\mathcal{E})$ over $\mathbb{P}^1$ is a twisted cubic. Furthermore we see that none of the entries of this 2×3 matrix above can be made zero by row and column operations if C is irreducible. We distinguish the three cases $X \simeq S(2,1,1,1)$, $X \simeq S(3,1,1,0)$ and $X \simeq S(2,2,1,0)$:

For $X \simeq S(2,1,1,1)$, let us assume that $a_1 \geq 3$, then the Pfaffian $\psi \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H - a_1R))$ can be written as a sum of products of global sections in $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R)$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-2R)$. As each summand must contain a factor in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-2R))$ and this vector space is generated by exactly one global section φ_0, φ_0 is a factor of the Pfaffian, which contradicts that C is irreducible.

In the case $X \simeq S(3,1,1,0)$, a similar argument shows that for $a_1 \geq 3$ the unique global section $\varphi_0 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-3R))$ must be a factor in the Pfaffian of type $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H - a_1R))$.

It remains to exclude the cases where $a_5 \leq -1$. This is only possible for $(a_1,...,a_5) = (2,2,2,1,-1)$ or $(a_1,...,a_5) = (2,2,2,2,-2)$. Therefore the last column of ψ only contains entries in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-mR))$ with $m \geq 2$, hence at least two of these entries can be made zero by suitable row and column operations as $h^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-2R)) \leq 2$. This contradicts that the Pfaffians are irreducible.

Now we consider the case $X \simeq S(2,2,1,0)$: If $a_5 \leq -1$ then setting $k = a_4 - a_5 \geq 0$ we obtain

$$\begin{aligned} a_5 + 4(a_5+k) = a_5 + 4a_4 \leq \sum_{i=1}^{5} a_i = 6 < 7 \leq 4 - 3a_5 = 4(1-a_5) + a_5 \\ \Rightarrow a_4 = a_5 + k < 1 - a_5 = (3-a_5) - 2 \Rightarrow b_5 - a_4 \leq 3 \end{aligned}$$

and therefore the entry ψ_{45} of the matrix ψ vanishs. It follows that the Pfaffians of ψ include the $2\times 2-$minors of a $2\times 3-$matrix ω with entries indicated as below:

$$\begin{pmatrix} H-(3-(a_5+a_1))R & H-(3-(a_5+a_2))R & H-(3-(a_5+a_3))R \\ H-(3+k-(a_5+a_1))R & H-(3+k-(a_5+a_2))R & H-(3+k-(a_5+a_3))R \end{pmatrix}$$

Then C is contained in a determinantal surface $Y \subset \mathbb{P}^8$ given by the minors of ω on $\mathbb{P}(\mathcal{E})$. If we apply Theorem 1.4.5 we see that Y is the image of $P_k := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1})$ under a rational map defined by a subseries of

$$H^0(P_k, \mathcal{O}_{P_k}(3A + (5-3k-a)B))$$

with hyperplane class A and ruling B on P_k where $a := 3-(a_5+a_1)+3-(a_5+a_2)+3-(a_5+a_3) = 9-3a_5-(6-2a_5-k) = 3+k-a_5$. Then, the strict transform C' of C in P_k is a divisor of class

$$C' \sim 5A + (7-4k-a)B = 5A + (4+a_5-5k)B$$

It follows that the hyperplanes on P_k cut out a $g^{k+1}_{4+a_5}$ on C' which has Clifford index less than 2, a contradiction. For $a_4 = a_5 = 0$ the same argument as above leads to the existence of a g^1_4 on C. Thus, for $(a_1,...,a_5)$ with $a_1 \geq 3$, there is only the possibility $(a_1, ..., a_5) = (3, 1, 1, 1, 0)$ left. In this case, the matrix ψ is of type

$$\psi \sim \begin{pmatrix} 0 & H+R & H+R & H+R & H \\ & 0 & H-R & H-R & H-2R \\ & & 0 & H-R & H-2R \\ & & & 0 & H-2R \\ & & & & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 0 & H+R & H+R & \mathbf{H+R} & \mathbf{H} \\ & 0 & H-R & \mathbf{H-R} & \mathbf{H-2R} \\ & & 0 & \mathbf{H-R} & \mathbf{H-2R} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

hence C is contained in a determinantal surface Y given by the 2×2 minors of

$$\omega \sim \begin{pmatrix} H+R & H-R & H-R \\ H & H-2R & H-2R \end{pmatrix}$$

on $\mathbb{P}(\mathcal{E}) \simeq S(2, 2, 1, 0)$. Y is the image of $P_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$ under a rational map defined by a subseries of

$$H^0(P_1, \mathcal{O}_{P_1}(3A + B))$$

and the image C' of C in P_1 is a divisor of class

$$C' \sim 5A + 2B$$

(cf. Theorem 1.4.5) from which we deduce the existence of a g^2_7 cut out by the hyperplane sections.

Then the given possibilities for $(a_1, ..., a_5)$ remain. ∎

For the rest of this section we will examine the two cases $(a_1, ..., a_5) = (2, 2, 2, 0, 0)$ and $(a_1, ..., a_5) = (2, 2, 1, 1, 0)$, where it will turn out that either we get a linear system of type g^1_4 or a plane model of C of degree 7 respectively.

Theorem 4.4.3 *Let C be a curve given by the Pfaffians of a matrix ψ with entries on a scroll of type $S(2, 1, 1, 1)$ as above. Then*

a) For $(a_1, ..., a_5) = (2, 2, 2, 0, 0)$ the curve C has Clifford index 2. There exists a special linear series of type g^1_4 and the Betti table for C has the following form:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	44	5	-	-
2	-	-	5	44	70	64	21	-
3	-	-	-	-	-	-	-	1

b) For $(a_1, ..., a_5) = (2, 2, 1, 1, 0)$ the curve C has Clifford index 3 and there exists a plane model of degree 7. The Betti table for C is given by:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	24	-	-	-
2	-	-	-	24	70	64	21	-
3	-	-	-	-	-	-	-	1

Proof. a) For $(a_1, ..., a_5) = (2, 2, 2, 0, 0)$ the matrix ψ in the resolution F_* of $\mathcal{O}_C$ on the corresponding $\mathbb{P}^3-$bundle $\mathbb{P}(\mathcal{E})$ is of type

$$\psi \sim \begin{pmatrix} 0 & H+R & H+R & H-R & H-R \\ & 0 & H+R & H-R & H-R \\ & & 0 & H-R & H-R \\ & & & 0 & H-3R \\ & & & & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 0 & H+R & H+R & \mathbf{H-R} & \mathbf{H-R} \\ & 0 & H+R & \mathbf{H-R} & \mathbf{H-R} \\ & & 0 & \mathbf{H-R} & \mathbf{H-R} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

Applying Theorem 1.4.5 again, we see that C is contained in a determinantal surface Y, given by the minors of the 2×3 matrix

$$\omega = \begin{pmatrix} \psi_{14} & \psi_{24} & \psi_{34} \\ \psi_{15} & \psi_{25} & \psi_{35} \end{pmatrix} \sim \begin{pmatrix} H-R & H-R & H-R \\ H-R & H-R & H-R \end{pmatrix}$$

with $(H-R)$-entries on $\mathbb{P}(\mathcal{E})$. Therefore Y is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in 3 points. The strict transform C' of C in Y is a divisor of type $(5, 4)$, thus we get a g^1_4 from projection onto the second factor of $\mathbb{P}^1 \times \mathbb{P}^1$. From Theorem 3.2.2, we expect for C to admit the following Betti table:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	44	5	-	-
2	-	-	5	44	70	64	21	-
3	-	-	-	-	-	-	-	1

Considering the corresponding mapping cone construction (cf. Theorem 1.4.3) and calculating the ranks of the non minimal maps leads to the same Betti table, thus it follows that C has Clifford index $\mathrm{Cliff}(C) = 2$, and C admits a $g^1_4 \times g^1_5$, but no g^2_6 or g^3_8.

b) In the case $(a_1, ..., a_5) = (2, 2, 1, 1, 0)$ the matrix ψ has the following form:

$$\psi \sim \begin{pmatrix} 0 & H+R & H & H & H-R \\ & 0 & H & H & H-R \\ & & 0 & H-R & H-2R \\ & & & 0 & H-2R \\ & & & & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & H+R & H & \mathbf{H} & \mathbf{H-R} \\ & 0 & H & \mathbf{H} & \mathbf{H-R} \\ & & 0 & \mathbf{H-R} & \mathbf{H-2R} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

As one of the $H - 2R$ entries can be made to zero, we can assume that $\psi_{45} = 0$. It follows from Theorem 1.4.5 that C is contained in a determinantal surface Y given by the 2×2 minors of the matrix

$$\begin{pmatrix} \psi_{14} & \psi_{24} & \psi_{34} \\ \psi_{15} & \psi_{25} & \psi_{35} \end{pmatrix} \sim \begin{pmatrix} H & H & H-R \\ H-R & H-R & H-2R \end{pmatrix}$$

on $\mathbb{P}(\mathcal{E})$. Y is a blowup of $\mathbb{P}^2$ in 6 points. The strict transform C' of C in Y is a divisor of type $C' \sim 5A + 2B$, hence there exists a g^2_7 cut out by the class of a hyperplane divisor A. Calculating the ranks of the non minimal maps in the corresponding mapping cone, it turns out that the Betti table of the minimal free resolution of $\mathcal{O}_C$ is the same as determined in Theorem 4.3.2, hence C has Clifford index 3. $\blacksquare$

So far we have seen that in the case, where C is a smooth, irreducible, canonical curve of Clifford index 3 that is contained in a scroll of type $S(2, 1, 1, 1)$, for any possible configuration $(a_1, ..., a_5) \neq (2, 1, 1, 1, 1)$ the curve C has a plane model of degree 7. In the next section we will concentrate on the "general case" $(a_1, ..., a_5) = (2, 1, 1, 1, 1)$. We will show that it is possible for C to have exactly one, two or even three different linear series of type g^1_5 and that they correspond to syzygies in the minimal free resolution of $\mathcal{O}_C$.

4.5 Curves with ordinary g_5^1

In this section we will formulate our main theorems. We have seen that for a pentagonal curve $C \subset X$ with $\mathrm{Cliff}(C) = 3$, that admits no g_7^2 and is contained in a scroll X of type $S(2,1,1,1)$, there exists exactly one type for the matrix ψ, whose Pfaffians generate the vanishing ideal of C on the scroll X. The following theorems give the correspondence between the possible number of different $g_5^1{}'s$ and the Betti numbers for $C \subset \mathbb{P}^8$.

4.5.1 Types of ψ and non minimal maps

For $(a_1, ..., a_5) = (2,1,1,1,1)$, the resolution of C on the associated $\mathbb{P}^3-$bundle $\mathbb{P}(\mathcal{E})$ is of type

$$F_* : \quad 0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-5H+3R) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+R) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+2R)^{\oplus 4} \xrightarrow{\psi}$$

$$\xrightarrow{\psi} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+2R) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+R)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_C \to 0$$

with ψ a skew-symmetric matrix with entries as indicated below:

(*)

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & \mathbf{0} & \mathbf{H-R} & \mathbf{H-R} & \mathbf{H-R} \\ & & \mathbf{0} & \mathbf{H-R} & \mathbf{H-R} \\ & & & \mathbf{0} & \mathbf{H-R} \\ & & & & \mathbf{0} \end{pmatrix}$$

We have already mentioned that the vanishing ideal of $C \subset \mathbb{P}(\mathcal{E})$ is given by the Pfaffians of ψ. As we assumed that C is irreducible, at most one entry in each row and column of ψ can be made to zero by suitable row and column operations. Especially if one of the $(H-R)$-entries is zero, we can assume $\psi_{45} = 0$, then none of the remaining ones, except ψ_{23}, can be made zero. In the case where $\psi_{45} = \psi_{23} = 0$ the global sections $\psi_{24}, \psi_{25}, \psi_{34}, \psi_{35} \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R))$ are linear independent as otherwise C contains a proper one dimensional component, given by the vanishing locus of these sections, hence C cannot be irreducible. In the following theorems we will show that the $H-R$ entries that can be made to zero exactly correspond to different additional linear systems of type g_5^1. It turns out that up to conjugation there occur 4 different types for the $4 \times 4-$submatrix $\widetilde{\psi}$ of ψ with entries in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R))$.

Lemma 4.5.1 *Let C be an irreducible smooth pentagonal curve C of genus 9 that is contained in a scroll of type $S(2,1,1,1)$. If C is given by the Pfaffians of a matrix ψ with entries as in (*) then ψ is conjugated to one of the following types:*

A

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & f_1 & f_2 & f_3 \\ & & 0 & f_4 & f_5 \\ & & & 0 & f_1 \\ & & & & 0 \end{pmatrix}$$

B

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & f_1 & f_2 & f_3 \\ & & 0 & f_4 & f_5 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

C

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & 0 & f_2 & f_3 \\ & & 0 & f_4 & f_5 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

D

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & f_1 & f_2 & f_3 \\ & & 0 & f_4 & f_2 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

with linear independent $f_1, ..., f_5 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R))$.

Proof. As we have already mentioned, the case

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & 0 & f_2 & f_3 \\ & & 0 & f_4 & f_2 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

cannot occur since in this situation C contains a proper one dimensional component on $\mathbb{P}(\mathcal{E})$ given by the vanishing locus of the sections $f_2, ..., f_4 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R))$, so C cannot be irreducible. It remains to remark that D is a special case of B, where $f_2, ..., f_5$ are linear dependent and $f_1 \notin \langle f_2, ..., f_5 \rangle$. Therefore, we get

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & f_1 & f_2 & f_3 \\ & & 0 & f_4 & f_5 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

with one of the entries $f_2, ..., f_5$ being a linear combination of the others. It follows, that ψ has the form as given above. ■

To obtain the minimal free resolution of $C \subset \mathbb{P}^8$, we have to determine the rank of the only non-minimal map in the corresponding mapping cone (MC) of F_* which arise from the $4 \times 4-$submatrix $\widetilde{\psi}$ of ψ:

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+2R)^{\oplus 4} & \longrightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+R)^{\oplus 4} \\
\uparrow & & \uparrow \\
----|---- & ---------- & ----|---- \\
S_2G(-3)^{\oplus 4} & \longrightarrow & G(-2)^{\oplus 4} \\
\uparrow & & \uparrow \\
F \otimes G(-4)^{\oplus 4} & \longrightarrow & F(-3)^{\oplus 4} \\
\uparrow & & \uparrow \\
\wedge^2 F(-5)^{\oplus 4} & \xrightarrow{\alpha} & \wedge^3 F(-5)^{\oplus 4} \\
\uparrow & & \uparrow \\
\wedge^4 F(-7)^{\oplus 4} & \longrightarrow & \wedge^4 F \otimes DG^*(-6)^{\oplus 4} \\
\uparrow & & \uparrow \\
DG^*(-8)^{\oplus 4} & \longrightarrow & D_2G^*(-7)^{\oplus 4}
\end{array}$$

Identifying $\psi_{ij} \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R)) \cong H^0(\mathbb{P}^8, F)$, $i, j = 2, ..., 5$, and $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R)) \cong H^0(\mathbb{P}^8, G)$ (cf. page 18), α is given by the wedge product with the corresponding section:

$$\alpha : \wedge^2 F^{\oplus 4} \xrightarrow{\wedge \widetilde{\psi}} \wedge^3 F^{\oplus 4}$$

The following lemma answers the question of how the rank of α depends on the type of ψ in Lemma 4.5.1:

Lemma 4.5.2 *According to the 4 different types of ψ in Lemma 4.5.1, we get:*

a) rank $\alpha = 40 \Leftrightarrow \psi \sim A$
b) rank $\alpha = 36 \Leftrightarrow \psi \sim B$
c) rank $\alpha = 32 \Leftrightarrow \psi \sim C$ *or* $\psi \sim D$

If C is given by the Pfaffians of a matrix ψ, then the Betti table for C has the following form

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	a	-	-	-
2	-	-	-	a	70	64	21	-
3	-	-	-	-	-	-	-	1

with $a = 44 - \text{rank } \alpha$

Proof. a) The matrix ψ has the following form:

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & f_1 & f_2 & f_3 \\ & & 0 & f_4 & f_5 \\ & & & 0 & f_1 \\ & & & & 0 \end{pmatrix}$$

with linear independent entries $f_1, ..., f_5 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R)) \cong H^0(\mathbb{P}^8, F)$. An easy calculation (c.f. Appendix 6.1) shows, that $\ker \alpha = 0$.
b) A similar calculation (c.f. Appendix 6.1) as in a) shows that the kernel of α is given by:

$$\ker \alpha = \left\langle \begin{pmatrix} 0 \\ 0 \\ f_2 \wedge f_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_3 \wedge f_5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ f_3 \wedge f_5 \\ f_2 \wedge f_5 + f_3 \wedge f_4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ f_2 \wedge f_5 + f_3 \wedge f_4 \\ f_2 \wedge f_4 \end{pmatrix} \right\rangle$$

i.e. rank $\alpha = 36$.
c) For $\psi \sim D$, $\ker \alpha$ is generated by 8 elements (cf. Appendix 6.1):

$$\ker \alpha = \left\langle \begin{matrix} \begin{pmatrix} 0 \\ 0 \\ f_2 \wedge f_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_2 \wedge f_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ f_2 \wedge f_3 \\ f_3 \wedge f_4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ f_3 \wedge f_4 \\ f_2 \wedge f_4 \end{pmatrix}, \\ \begin{pmatrix} f_2 \wedge f_3 \\ 0 \\ 0 \\ f_1 \wedge f_3 \end{pmatrix}, \begin{pmatrix} 0 \\ f_2 \wedge f_4 \\ f_1 \wedge f_4 \\ 0 \end{pmatrix}, \begin{pmatrix} f_2 \wedge f_4 \\ f_3 \wedge f_4 \\ 0 \\ f_1 \wedge f_4 \end{pmatrix}, \begin{pmatrix} f_3 \wedge f_4 \\ f_2 \wedge f_3 \\ f_1 \wedge f_3 \\ 0 \end{pmatrix} \end{matrix} \right\rangle$$

and in the in the case $\psi \sim C$ we obtain (cf. Appendix 6.1)

$$\ker \alpha = \left\langle \begin{matrix} \begin{pmatrix} 0 \\ f_4 \wedge f_5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_3 \wedge f_5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ f_3 \wedge f_5 \\ f_3 \wedge f_4 + f_2 \wedge f_5 \end{pmatrix}, \begin{pmatrix} -f_4 \wedge f_5 \\ f_3 \wedge f_4 - f_2 \wedge f_5 \\ 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 0 \\ 0 \\ f_2 \wedge f_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ f_3 \wedge f_4 + f_2 \wedge f_5 \\ f_2 \wedge f_4 \end{pmatrix}, \begin{pmatrix} f_2 \wedge f_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} f_2 \wedge f_5 - f_3 \wedge f_4 \\ f_2 \wedge f_3 \\ 0 \\ 0 \end{pmatrix} \end{matrix} \right\rangle$$

■

Remark 4.5.3 For char($\mathbb{k}$) $\neq 3$ the rank of α is determined by the matrix ψ as in the above lemma. In the case char($\mathbb{k}$) $= 3$ we obtain further elements in $\ker(\alpha)$ for ψ of type A and B. To be more precisely in this special situation we get $\dim(\ker\alpha) = 2$ and 6 for $\psi \sim A$ and $\psi \sim B$ respectively (cf. Appendix 6.1). In the situation where C is given by the pfaffians of a matrix ψ as in $(*)$ on a scroll X of type $S(2,2,1,0)$ or $S(3,1,1,0)$ the Betti table for C can be calculated in the same way as for $S(2,1,1,1)$, i.e. the only non minimal map has rank depending on the type of ψ as in Lemma 4.5.2.

We have provided the basic information to calculate the Betti tables for curves C, that lie on a scroll of type $S(2,1,1,1)$ and have one, two or three different $g^1_5{}'s$. It remains to assign the different cases to the different possible types for the matrix ψ. This will be done in the next section.

4.5.2 Geometric interpretation

Lemma 4.5.2 says that the Betti table of the minimal free resolution of $\mathcal{O}_C$ is determined up to the entries $\beta_{35} = \beta_{45}$, which can be equal to $4, 8$ or 12. We have seen that for $\beta_{35} = 8$ or $\beta_{35} = 12$ the matrix ψ is of type B, C or D in Lemma 4.5.2, i.e. at least one $(H-R)$-entry of ψ can be made to zero by suitable row and column operations. If we apply Theorem 1.4.5, it follows that C is contained in a surface Y given by the 2×2 minors of a matrix

$$\omega \sim \begin{pmatrix} H & H-R & H-R \\ H & H-R & H-R \end{pmatrix}$$

on $\mathbb{P}(\mathcal{E})$. Y is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in 7 points. The image C' of C in $\mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of type $(5,5)$. The existence of at least two different $g^1_5{}'s$ that are cut out by the factor classes $|(1,0)|$ and $|(0,1)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$ follows. The next theorem shows that even the converse is true, i.e. $\beta_{35} = 4$ if and only if C has exactly one g^1_5 :

Theorem 4.5.4 *An irreducible, nonsingular canonical curve C with Clifford index 3 that admits no g^2_7 has exactly one ordinary linear system of type g^1_5 if and only if it C is given by the pfaffians of matric ψ is of type A on a scroll X of type $S(2,1,1,1)$. The unique g^1_5 is cut out by the class of a ruling R on X. The minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}-$module has the following Betti table*

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	4	-	-	-
2	-	-	-	4	70	64	21	-
3	-	-	-	-	-	-	-	1

Proof. It remains to prove that in the case where C has an additional $g^1_5 = |D|$, obtained from an effective divisor D of degree 5 on C, we get $\beta_{35} = \beta_{45} \neq 4$. Now let us assume this case, then there is a map

$$C \overset{|R|_C| \times |D|}{\rightarrow} C' \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$$

C' is a divisor of type $(5,5)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, hence $p_a(C') = 4 \cdot 4 = 16$. Because of $p_a(C') - g(C') = 16 - 9 = 7$ the space model C' of C has certain singularities. If C' has a singular point with multiplicity 3, then projection from this point leads to a g^2_7 in case of a triple point or a special linear system of lower Clifford index, which is not possible as in this situation we would get a different Betti table. It follows that C' has exactly 7 double points $p_1, ..., p_7$. Let S be the surface, which is obtained from $X := \mathbb{P}^1 \times \mathbb{P}^1$ after blowing up the singularities of C':

$$\sigma : S = \tilde{X}(p_1, ..., p_7) \rightarrow X = \mathbb{P}^1 \times \mathbb{P}^1$$

We denote $A \sim (1,0)$ and $B \sim (0,1)$ the two factor classses of $P_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and by abuse of notation also their pullbacks to S. E_i denotes the total transforms of the point p_i for $i = 1, ..., 7$. We can assume that C is the strict transform of C' on S:

$$C \sim 5A + 5B - \sum_{i=1}^{7} 2E_i$$

We consider the rational map

$$\varphi : S \rightarrow S' \subset \mathbb{P}^8$$

defined by the adjoint series

$$V = H^0(S, \omega_S(C)) = H^0(S, \mathcal{O}_S(3A + 3B - \textstyle\sum_{i=1}^{7} E_i))$$

which is base point free according to Corollary 2.2.1. We want to apply our results on page 23 to show that the variety

$$X = \bigcup_{B_\lambda \in |B|} \bar{B}_\lambda \subset \mathbb{P}^8$$

is a 4-dimensional rational normal scroll. Therefore we have to check the following conditions for $H = 3A + 3B - \sum_{i=1}^{7} E_i$:

1. $h^0(\mathcal{O}_S(H - B)) \geq 2$
2. $H^1(\mathcal{O}_S(kH - B)) = 0$ for $k \geq 1$ and
3. the map $S_k H^0 \mathcal{O}_S(H) \rightarrow H^0 \mathcal{O}_S(kH)$ is surjective

The first condition is trivial because of $h^0(\mathcal{O}_S(H - B)) \geq 12 - 7 = 5$ and 3. follows from Corollary 2.2.1 where S' turns out to be arithmetically Cohen

Macaulay. It remains to examine the second condition. Consider the exact sequence

$$0 \to \mathcal{O}_S(kH - B) \to \mathcal{O}_S(kH) \to \mathcal{O}_B(kH|_B) \to 0$$

and the corresponding long exact sequence of cohomology groups

$$\begin{aligned} 0 \to H^0\mathcal{O}_S(kH - B) \to H^0\mathcal{O}_S(kH) \xrightarrow{\delta_k} H^0\mathcal{O}_B(kH|_B) \to \\ \to H^1\mathcal{O}_S(kH - B) \to H^1\mathcal{O}_S(kH) = 0 \end{aligned}$$

We first show that $H^0\mathcal{O}_S(H) \xrightarrow{\delta_1} H^0\mathcal{O}_B(H|_B) \cong H^0\mathcal{O}_{\mathbb{P}^1}(H.B) = H^0\mathcal{O}_{\mathbb{P}^1}(3)$ is surjective. If $h^0(\mathcal{O}_S(H - B)) = d > 5$ then the complete linear system $|(H - B)|_C|$ would be of type g^{d-1}_{11} and therefore $\mathrm{Cliff}(C) \leq 2$. It follows that $h^0(\mathcal{O}_S(H - B)) = 5$ and thus $\dim \delta \geq 9 - 5 = 4 = h^0\mathcal{O}_{\mathbb{P}^1}(3)$, so δ_1 is surjective. As further consequence we also obtain that $H^0\mathcal{O}_S(kH) \xrightarrow{\delta_k} H^0\mathcal{O}_B(kH|_B)$ is surjective (the image of B under φ is a rational normal curve), hence $H^1\mathcal{O}_S(kH - B) = 0$.

Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ denote the corresponding $\mathbb{P}^3$-bundle and S'' the strict transform of S' in $\mathbb{P}(\mathcal{E})$. Then $\mathbb{P}(\mathcal{E})$ is of type $S(2, 1, 1, 1)$ an Theorem 1.4.4 tells us that $\mathcal{O}_{S''}$ has an $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module resolution of type

$$\begin{aligned} &F_* : \\ &0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H + 2R)^{\oplus 2} \xrightarrow{\omega} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + R)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H + 2R) \to \\ &\to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_{S''} \to 0 \end{aligned}$$

where ω is given by a matrix

$$\omega \sim \begin{pmatrix} H - a_1R & H - a_2R & H - a_3R \\ H - a_1R & H - a_2R & H - a_3R \end{pmatrix}$$

with entries in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H - a_iR))$, $a_i \in \mathbb{Z}$ for $i = 1, 2, 3$. From Theorem 1.4.5. we obtain certain conditions on the numbers a_i :

$$5 - 3 \cdot 0 - (a_1 + a_2 + a_3) = 3 \Rightarrow a_1 + a_2 + a_3 = 5 - 3 = 2$$

As S' is irreducible, we must have $a_i \leq 1$ for all i and therefore, assuming $a_1 \geq a_2 \geq a_3$:

$$\omega \sim \begin{pmatrix} H & H - R & H - R \\ H & H - R & H - R \end{pmatrix}$$

The corresponding mapping cone

$$
\begin{array}{ccccccc}
0 & \rightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+2R)^{\oplus 2} & \longrightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+R)^{\oplus 2} & \oplus \\
 & & \uparrow & & \uparrow & \\
 & & -----|----- & - & -----|----- & - \\
 & & & & & \\
 & & S_2G(-3)^{\oplus 2} & \longrightarrow & G(-2)^{\oplus 2} & \oplus \\
 & & \uparrow & & \uparrow & \\
 & & F \otimes G(-4)^{\oplus 2} & \longrightarrow & F(-3)^{\oplus 2} & \oplus \\
 & & \uparrow & & \uparrow & \\
 & & \wedge^2 F(-5)^{\oplus 2} & \xrightarrow{\gamma} & \wedge^3 F(-5)^{\oplus 2} & \oplus \\
 & & \uparrow & & \uparrow & \\
 & & \wedge^4 F(-7)^{\oplus 2} & \longrightarrow & \wedge^4 F \otimes DG^*(-6)^{\oplus 2} & \oplus \\
 & & \uparrow & & \uparrow & \\
 & & DG^*(-8)^{\oplus 2} & \longrightarrow & D_2G^*(-7)^{\oplus 2} & \oplus
\end{array}
$$

$$
\begin{array}{ccccc}
\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+2R) & \longrightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})} & \rightarrow & 0 \\
\uparrow & & \uparrow & & \\
----|---- & & ----|---- & & \\
 & & & & \\
S_2G(-2)^{\oplus 2} & \longrightarrow & \mathcal{O}_{\mathbb{P}^8} & & \\
\uparrow & & \uparrow & & \\
F \otimes G(-3)^{\oplus 2} & \longrightarrow & \wedge^2 F(-2) & & \\
\uparrow & & \uparrow & & \\
\wedge^2 F(-4)^{\oplus 2} & \longrightarrow & \wedge^3 F \otimes DG^*(-3) & & \\
\uparrow & & \uparrow & & \\
\wedge^4 F(-6)^{\oplus 2} & \longrightarrow & \wedge^4 F \otimes D_2G^*(-4) & & \\
\uparrow & & \uparrow & & \\
DG^*(-7)^{\oplus 2} & \longrightarrow & D_3G^*(-5) & &
\end{array}
$$

gives us a (not necessarily) minimal free resolution of $\mathcal{O}_{S'}$. We calculate the rank of the only non minimal map (cf. Appendix 6.2)

$$\gamma : \wedge^2 F(-5)^{\oplus 2} \xrightarrow{\alpha} \wedge^3 F(-5)^{\oplus 2}$$

which is obtained from the matrix $\begin{pmatrix} \omega_{12} & \omega_{13} \\ \omega_{22} & \omega_{23} \end{pmatrix} \sim \begin{pmatrix} H-R & H-R \\ H-R & H-R \end{pmatrix}$:

$$\gamma : \begin{pmatrix} f_1 \wedge f_2 \\ f_3 \wedge f_4 \end{pmatrix} \rightarrow \begin{pmatrix} f_1 \wedge f_2 \wedge \omega_{12} + f_3 \wedge f_4 \wedge \omega_{22} \\ f_1 \wedge f_2 \wedge \omega_{13} + f_3 \wedge f_4 \wedge \omega_{23} \end{pmatrix}$$

with $f_1, ..., f_3 \in H^0F$. The matrix $\begin{pmatrix} \omega_{12} & \omega_{13} \\ \omega_{22} & \omega_{23} \end{pmatrix}$ has full rank since S is irreduciblew. A calculation shows that $\dim \ker \gamma = 4$:

$$\ker \gamma = \langle \begin{pmatrix} \omega_{12} \wedge \omega_{13} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \omega_{22} \wedge \omega_{23} \end{pmatrix}, \begin{pmatrix} \omega_{12} \wedge \omega_{23} + \omega_{22} \wedge \omega_{13} \\ \omega_{12} \wedge \omega_{13} + \omega_{22} \wedge \omega_{23} \end{pmatrix}, \begin{pmatrix} \omega_{12} \wedge \omega_{13} + \omega_{22} \wedge \omega_{23} \\ \omega_{12} \wedge \omega_{23} + \omega_{22} \wedge \omega_{13} \end{pmatrix} \rangle$$

Therefore, the Betti table of the minimal free resolution of $\mathcal{O}_{S'}$ is given by

	0	1	2	3	4	5	6
0	1	-	-	-	-	-	-
1	-	17	46	45	8	-	-
2	-	-	-	4	25	18	4

C is contained in S', hence the Betti number β_{45} in the minimal free resolution of $\mathcal{O}_C$ has to be greater or equal than 8, which contradicts $\beta_{34} = \beta_{45} = 4$. ■

Remark 4.5.5 In the proof of Theorem 4.5.4 it turns out that for a pentagonal curve C on a scroll X of type $S(2,1,1,1)$ that has two different g^1_5, there exists a determinantal surface $S' \subset X$ that contains C. This surface has Betti table as follows

$$\begin{matrix} 1 & & & & & & \\ & 17 & 46 & 45 & 8 & & \\ & & & 4 & 25 & 18 & 4 \end{matrix}$$

Considering the exact sequence

$$0 \to \mathcal{O}_{S'}(-C) \to \mathcal{O}_{S'} \to \mathcal{O}_C \to 0$$

a mapping cone construction gives a (not necessarily minimal) free resolution for C : As $\mathcal{O}_{S'}(-C) \cong \omega_{S'}$ the Betti table for the minimal free resolution of $\omega_{S'}$ as $\mathcal{O}_{\mathbb{P}^8}$−module is given by

$$\begin{matrix} 4 & 18 & 25 & 4 & & & \\ & & 8 & 45 & 46 & 17 & \\ & & & & & & 1 \end{matrix}$$

Then the Betti table for C can be obtained as sum of these two Betti tables, except the values for $\beta_{35} = \beta_{45}$ which are determined by the rank of the non minimal map "‘in the middle"’:

$$\begin{matrix} 1 & & & & & & \\ & 17 & 46 & 45 & 8 & & \\ & & & 4 & 25 & 18 & 4 \end{matrix} + \begin{matrix} 4 & 18 & 25 & 4 & & & \\ & & 8 & 45 & 46 & 17 & \\ & & & & & & 1 \end{matrix}$$

$$= \begin{matrix} 1 & & & & & & \\ & 21 & 64 & 70 & \beta_{45} & & \\ & & & \beta_{35} & 70 & 64 & 21 \\ & & & & & & & 1 \end{matrix}$$

In this situation, we know that $8 \leq \beta_{35} = \beta_{45} \leq 12$. In the case of a trigonal and tetragonal curves the Betti table for C could always be obtained from that of a determinantal surface S' that contains C (cf. Theorems 3.1.2 and 3.2.2): It is given as the direct sum of the Betti tables for $\mathcal{O}_{S'}$ and $\omega_{S'}$. However, in Theorem 4.5.9 we will see, that for a pentagonal curve with exactly two different g^1_5 this property fails for pentagonal curves. In this situation we get $\beta_{35} = \beta_{45} = 8$. In Theorem 4.2.2 we considered pentagonal curves C with exactly one g^1_5 of multiplicity one. It turns out that C admits a g^2_8 with exactly one triple point and 9 double points as only singularities. Blowing up $\mathbb{P}^2$ in the singular points we get a surface S and with the methods of Chapter 2 and Theorem 1.4.5 we can see that its image $S' \subset X \subset \mathbb{P}^8$ under the adjoint series on S has a determinantal representation, i.e. on the corresponding $\mathbb{P}^3$-bundle $\mathbb{P}(\mathcal{E})$ the surface S' is given by the 2×2 minors of a matrix

$$\omega \sim \begin{pmatrix} H - a_1 R & H - a_2 R & H - a_3 R \\ H - (a_1 + 1)R & H - (a_2 + 1)R & H - (a_3 + 1)R \end{pmatrix}$$

with $a_1 + a_2 + a_3 = 0$.

The next step is to examine the case where C has at least two different special linear series of type g^1_5. As in the proof of Theorem 4.5.4, we consider the space model $C' \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ of C given by $|D_1|$ and $|D_2|$:

$$C \overset{|D_1|\times|D_2|}{\to} C' \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$$

With the notations as above C' is a divisor of type $(5,5) \sim 5A+5B$ on $\mathbb{P}^1 \times \mathbb{P}^1 \cong P_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$ and we can assume that C' has exactly 7 (possibly infinitely near) double points since otherwise C has a g^2_7 or a special linear series of lower Clifford index:

Blowing up the singularities of C' we can assume that C is given as strict transform of C':

$$C \sim 5A + 5B - \textstyle\sum_{i=1}^{7} 2E_i$$

and $K_C \sim (3A+3B-\sum_{i=1}^{7} E_i)|_C$. Then the two different linear systems of type g^1_5 are given by the divisors $D_1 \sim A|_C$ and $D_2 \sim B|_C$. In the next step, we will examine whether it is possible for C to have a third g^1_5 and how such a special linear series can be obtained:

Theorem 4.5.6 *Let S be the iterated blowup in 7 (possibly infinitely near) points $p_1, ..., p_7$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $C \sim 5A+5B-\sum_{i=1}^{7} 2E_i$ an irreducible, nonsingular curve with* $\mathrm{Cliff}(C) = 3$, *that admits no g^2_7. Then C has two different g^1_5 given by the divisors $D_1 \sim A|_C$ and $D_2 \sim B|_C$. If C admits a third $g^1_5 = |D|$, different from $|D_1|$ and $|D_2|$, then there exists a point $p \in C$ with $D \sim K_C - D_1 - D_2 - p$.*

Proof. Let us assume that there exists a further $g^1_5 = |D|$, D an effective divisor on C. From the base point free pencil trick we get a long exact sequence:

$$0 \to H^0(C, \mathcal{O}_C(D_1 - D_2)) \to H^0(C, \mathcal{O}_C(D_1)) \otimes H^0(C, \mathcal{O}_C(D_2)) \to$$
$$\to H^0(C, \mathcal{O}_C(D_1 + D_2)) \to H^1(C, \mathcal{O}_C(D_1 - D_2)) \to ...$$

As $D_1 \nsim D_2$ it follows that $h^0(C, \mathcal{O}_C(D_1 - D_2)) = 0$ and therefore

$$h^0(C, \mathcal{O}_C(D_1 + D_2)) \geq h^0(C, \mathcal{O}_C(D_1)) \cdot h^0(C, \mathcal{O}_C(D_2)) = 4$$

Hence $h^0(C, \mathcal{O}_C(D_1+D_2)) = 4$ as for $h^0(C, \mathcal{O}_C(D_1+D_2)) > 4$ the linear system $|D_1 + D_2|$ would have Clifford index

$$i = \deg(D_1 + D_2) - 2(h^0(C, \mathcal{O}_C(D_1 + D_2)) - 1) \leq 2$$

Applying the base point free pencil trick again, we obtain the exact sequence

$$0 \to H^0(C, \mathcal{O}_C(D_1 + D_2 - D)) \to H^0(C, \mathcal{O}_C(D_1 + D_2)) \otimes H^0(C, \mathcal{O}_C(D)) \to$$
$$\to H^0(C, \mathcal{O}_C(D_1 + D_2 + D)) \to H^0(C, \mathcal{O}_C(D_1 + D_2 - D)) \to ...$$

from which we deduce

$$h^0(C, \mathcal{O}_C(D_1+D_2+D)) \geq 4{\cdot}2 - h^0(C, \mathcal{O}_C(D_1+D_2-D)) = 8 - h^0(C, \mathcal{O}_C(D_1+D_2-D))$$

The first step will be to exclude $h^0(C, \mathcal{O}_C(D_1 + D_2 - D)) \geq 2$, where we must have at least two distinct effective divisors

$$D^*, D^{**} \sim D_1 + D_2 - D$$
$$\Rightarrow (A + B)|_C \sim D_1 + D_2 \sim D + D^*,\ D + D^{**} > 0$$

The divisors $D + D^*$ and $D + D^{**}$ are effective divisors and linear equivalent to $D_1 + D_2$. We already know, that $h^0(C, \mathcal{O}_C(D_1 + D_2)) = 4$ and therefore every effective divisor, linear equivalent to $D_1 + D_2$ can be written as $(h|_C)$ with an element $h \in H^0(S, \mathcal{O}_S(A+B))$. Let $h^*, h^{**} \in H^0(S, \mathcal{O}_S(A+B))$ with $D^* + D = (h^*|_C)$ and $D^{**} + D = (h^{**}|_C)$. It follows that $H_1 := (h^*)$ and $H_2 := (h^{**})$ must have at least the 5 points of D in common. As $(A+B).(A+B) = 2 < 5$, this is only possible, if they contain a common divisor $\Gamma \sim aA + bB - \sum_{i=1}^{7} \lambda_i E_i$. We assume that Γ is maximal in the sense that $H_1 - \Gamma$ and $H_2 - \Gamma$ have no further component. Then we must have $0 \leq (H_1-\Gamma).(H_2-\Gamma) = 2(1-a)(1-b) - \sum_{i=1}^{7} \lambda_i^2$. For $(a, b) = (0, 0)$ this is only possible if $\Gamma \sim E_i + E_j$, $\Gamma \sim E_i - E_j$ or $\Gamma \sim E_i$ for distinct $i, j \in \{1, ..., 7\}$. If $\Gamma \sim E_i + E_j$ or $\Gamma \sim E_i - E_j$ then we get $(H_1 - \Gamma).(H_2 - \Gamma) = 0$ and therefore $(H_1 - \Gamma)|_C$ and $(H_2 - \Gamma)|_C$ have no common points. Then $D^* - D^{**} =(H_1 - \Gamma)|_C - (H_2 - \Gamma)|_C$ is not possible as $(H_1-\Gamma).C = (H_2-\Gamma).C \geq 6$. For $\Gamma \sim E_i$ the correspondance $D^* - D^{**} =(H_1 - \Gamma)|_C - (H_2 - \Gamma)|_C$ says that $(H_1 - \Gamma)|_C$ and $(H_2 - \Gamma)|_C$ must have at least $(H_1-\Gamma).C-5 = 3$ points in common, which contradicts to $(H_1-\Gamma).(H_2-\Gamma) = 1$. Thus we deduce the existence of a common component $l \in H^0(S, \mathcal{O}_S(A))$ or

$l \in H^0(S, \mathcal{O}_S(B))$. We consider the first case where $h^* = l \cdot l^*$ and $h^{**} = l \cdot l^{**}$ with $l \in H^0(S, \mathcal{O}_S(A))$ and $l^*, l^{**} \in H^0(S, \mathcal{O}_S(B))$. Now we conclude that $(l^*|_C) = D^*$ and $(l^{**}|_C) = D^{**}$ and therefore $D_1 + D_2 - D \sim D_2 \Rightarrow D \sim D_1$, a contradiction. For $l \in H^0(S, \mathcal{O}_S(B))$ the same argument holds. It remains to show that it is not possible for $\mathcal{O}_C(D_1 + D_2 - D)$ to have exactly one global section. From $\dim |D| = 1$ we obtain two different, effective divisors $D^*, D^{**} \sim D$. As

$$h^0(C, \mathcal{O}_C(D_1+D_2-D)) = h^0(C, \mathcal{O}_C(D_1+D_2-D^*)) = h^0(C, \mathcal{O}_C(D_1+D_2-D^{**})) = 1$$

there exist effective divisors $\bar{D}^*, \bar{D}^{**}$ on C with $D_1 + D_2 - D^* \sim \bar{D}^*$ and $D_1 + D_2 - D^{**} \sim \bar{D}^{**}$. We get two effective divisors

$$\tilde{D}^* := D^* + \bar{D}^* \sim D_1 + D_2$$
$$\tilde{D}^{**} := D^{**} + \bar{D}^{**} \sim D_1 + D_2$$

linear equivalent to $D_1 + D_2$ and in consequence $\tilde{D}^* = (h^*|_C)$, $\tilde{D}^{**} = (h^{**}|_C)$ with $h^*, h^{**} \in H^0(S, \mathcal{O}_S(A+B))$. Because of $h^0(C, \mathcal{O}_C(D_1+D_2-D)) = 1$, the effective divisors $\bar{D}^*, \bar{D}^{**}$ have to be equal. Therefore

$$(h^*|_C) - (h^{**}|_C) = D^* - D^{**}$$

and the same argument as above leads to $D \sim D_1$ or $D \sim D_2$, hence we get a contradiction. The inequality

$$h^0(C, \mathcal{O}_C(D_1 + D_2 + D)) \geq 8 - h^0(C, \mathcal{O}_C(D_1 + D_2 - D)) = 8$$

becomes an equality since from Riemann-Roch we obtain

$$\begin{aligned} h^0(C, \mathcal{O}_C(D_1 + D_2 + D)) &= h^0(C, \mathcal{O}_C(K_C - D_1 - D_2 - D) + 7 \\ &\leq 7 + \deg(K_C - D_1 - D_2 - D) = 8 \end{aligned}$$

(cf. [Hs77] page 298 Ex. 1.5.). It follows that $h^0(C, \mathcal{O}_C(K_C - D_1 - D_2 - D) = 1$, hence the existence of a point $p \in C$ with

$$p \sim K_C - D_1 - D_2 - D$$

so

$$D \sim K_C - D_1 - D_2 - p \sim (2A + 2B)|_C - \sum_{i=1}^{7} E_i|_C - p.$$

■

Given an irreducible pentagonal curve $C \sim 5A + 5B - \sum_{i=1}^{7} 2E_i$ on a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$, having two different g_5^1, given by $D_1 \sim A|_C$ and $D_2 \sim B|_C$, and no g_7^2, in the case where C has a third $g_5^1 = |D|$ this linear system is uniquely obtained from a divisor

$$D \sim K_C - D_1 - D_2 - p \sim (2A + 2B)|_C - \sum_{i=1}^{7} E_i|_C - p$$

with $p \in C$. In the proof of the above theorem, we have seen that $h^0(C, K_C - D_1 - D_2) = 2$, hence there exist two global generators $g_1, g_2 \in H^0(S, \mathcal{O}_S(2A + 2B - \sum_{i=1}^7 E_i))$. Geometrically, the complete linear system $|2A+2B-\sum_{i=1}^7 E_i|$ is given by a pencil $(Q_\lambda)_\lambda$ of conics passing through the points $p_1, ..., p_7$ (cf. picture on page 56). Conversely if D is an effective divisor linear equivalent to $(2A+2B)|_C - \sum_{i=1}^7 E_i|_C - p$, we get $(2A+2B)|_C - \sum_{i=1}^7 E_i|_C \sim D + p$. Thus $|D|$ is cut out by the pencil $(Q_\lambda)_\lambda$. As $(2A+2B).(2A+2B) = 8$ we get $h^0(C, \mathcal{O}_{C'}(K_C - D_1 - D_2 - p)) = 2$, hence $\dim |D| = 1$ if and only if p is the basepoint of the linear system $\left|2A + 2B - \sum_{i=1}^7 E_i\right|$. In general we would expect for D to give a different linear system of type g_5^1 from $|D_1|$ and $|D_2|$. In the following theorem, we will see that $D \sim D_1$ or $D \sim D_2$ can occur in special configurations. For this purpose let us denote the global sections of $\mathcal{O}_S(A)$ and $\mathcal{O}_S(B)$ by λ, μ and s, t respectively and w.l.o.g. $(\lambda|_C) = D_1$ and $(s|_C) = D_2$, then $H^0(C, \mathcal{O}_C(D_1)) = \langle \lambda|_C, \mu|_C \rangle$ and $H^0(C, \mathcal{O}_C(D_2)) = \langle s|_C, t|_C \rangle$. We consider the natural maps

$$H^0(S, \mathcal{O}_S(A)) \otimes H^0(S, \mathcal{O}_S(2A+2B-\sum_{i=1}^7 E_i)) \stackrel{\delta_1}{\to} H^0(S, \mathcal{O}_S(3A+2B-\sum_{i=1}^7 E_i))$$

and

$$H^0(S, \mathcal{O}_S(B)) \otimes H^0(S, \mathcal{O}_S(2A+2B-\sum_{i=1}^7 E_i)) \stackrel{\delta_2}{\to} H^0(S, \mathcal{O}_S(2A+3B-\sum_{i=1}^7 E_i))$$

Then it turns out that $\ker\ \delta_i = 0$ if and only if D is not linear equivalent to D_i for $i = 1, 2$:

Theorem 4.5.7 *Let C be given as in Theorem 4.5.6 and $D \sim (2A+2B)|_C - \sum_{i=1}^7 E_i|_C - p$ with p the unique base point of $\left|2A + 2B - \sum_{i=1}^7 E_i\right|$ on C. Then, for $i = 1, 2$, we get $m_{|D|} \in \{1, 2\}$ and*

$$\ker\ \delta_i \neq 0 \Leftrightarrow D \sim D_i \Leftrightarrow m_{|D|} = 2$$

Proof. If $D \sim D_1 \Leftrightarrow (2A+2B)|_C - \sum_{i=1}^7 E_i|_C - p \sim A|_C \Leftrightarrow (A+2B)|_C \sim \sum_{i=1}^7 E_i|_C + p$ then it follows the existence of an element $\gamma \in H^0(C, \mathcal{O}_C((A + 2B)|_C - \sum_{i=1}^7 E_i|_C))$ and therefore $H^0(C, \mathcal{O}_C((2A+2B)|_C - \sum_{i=1}^7 E_i|_C)) = \langle \lambda|_C \cdot \gamma, \mu|_C \cdot \gamma \rangle$. We can assume that $(g_1|_C) = (\lambda|_C \cdot \gamma)$ and $(g_2|_C) = (\mu|_C \cdot \gamma)$. Hence $(\mu \cdot g_1 - \lambda \cdot g_2)|_C = 0$ and taking into account that

$$\begin{aligned} H^0(C, \mathcal{O}_C((2A+3B)|_C - \textstyle\sum_{i=1}^7 E_i|_C)) &= H^0(C, \mathcal{O}_C(K_C - A|_C)) = \\ &= H^0(S, \mathcal{O}_S(2A+3B-\textstyle\sum_{i=1}^7 E_i))|_C \end{aligned}$$

we conclude that $\mu \cdot g_2 - \lambda \cdot g_1 = 0 \Rightarrow \ker\ \delta_1 = \langle \mu \otimes g_2 - \lambda \otimes g_1 \rangle \neq 0$. Moreover the divisors $Q_1, Q_2 \sim 2A+2B-\sum_{i=1}^7 E_i$ given by g_1 and g_2 must have a common component Γ of type $A + 2B - \sum_{i=1}^7 E_i$. Then Γ passes through the point p.

Conversely from ker $\delta_1 \neq 0$ we deduce the existence of $\tilde{\lambda}, \tilde{\mu} \in H^0(S, \mathcal{O}_S(A))$ with $\tilde{\lambda} \cdot g_1 = \tilde{\mu} \cdot g_2$, hence $(g_2|_C) - (g_1|_C) = (\tilde{\mu}|_C) - (\tilde{\lambda}|_C) \Rightarrow D \sim D_2$. For $i = 2$ the statement follows in analogous way.

For the second equivalence we assume that $D \sim D_1$, then according to the approach above, we must have a curve Γ of type $A + 2B - \sum_{i=1}^7 E_i$ that passes through the point p. Thus $h^0(C, \mathcal{O}_C(K_C - 2D_1)) = h^0(C, \mathcal{O}_C((A + 3B)|_C - \sum_{i=1}^7 E_i|_C)) = 2$ and applying Riemann Roch, it follows $h^0(C, \mathcal{O}_C(2D_1) = 4$. The same holds for D_2. For the other direction, let us assume that $h^0(C, \mathcal{O}_C(2D_1) = 4$ equivalently $h^0(C, \mathcal{O}_C((A + 3B)|_C - \sum_{i=1}^7 E_i|_C) = h^0(C, \mathcal{O}_C(K_C - 2D_1)) = 2$. Hence the existence of two different effective divisors K_1 and K_2 of type $(A + 3B - \sum_{i=1}^7 E_i)|_C$ on C follows. The following argument shows that they are cut out by a pencil of divisors of class $A + 3B - \sum_{i=1}^7 E_i$ as above: Without loss of generalisation we can assume that (λ) is irreducible and that K_1 is cut out by an effective divisor $\Sigma_1 \in H^0(S, \mathcal{O}_S(A + 3B - \sum_{i=1}^7 E_i)) \neq 0$, then

$$K_1 + (\lambda^2|_C) = \Sigma_1|_C + 2(\lambda|_C) \sim K_C$$

and

$$K_2 + 2(\lambda|_C) \sim K_C$$

hence we obtain

$$K_2 = \Omega|_C - 2(\lambda|_C)$$

with an effective divisor $\Omega \sim 3A + 3B - \sum_{i=1}^7 E_i$. Because of $\deg K_2 = 6$ and $\deg \Omega|_C = 16$, the divisors Ω and $\Lambda := 2(\lambda)$ must intersect in 10 points in the case where (λ) is not contained in Ω. As $2A.\Omega = 6$ this is not possible. Thus $\Gamma = (\lambda)$ or $2(\lambda)$ is a common divisor of Ω and Λ, i.e. $\Omega - \Gamma$ and $\Lambda - \Gamma$ are effective. After substracting (λ) once we get $K_2 = (\Omega - (\lambda))|_C - (\lambda|_C)$ and as $\deg(\Omega - (\lambda))|_C = 11$, $A.(\Omega - (\lambda)) = 3$ the curves $(\Omega - (\lambda))$ and (λ) must contain a further common component (λ). It follows the existence of an effective divisor $\Sigma_2 := \Omega - 2(\lambda) \sim A + 3B - \sum_{i=1}^7 E_i$, that is distinct from Σ_1. Now from $\Sigma_1.\Sigma_2 = 6 - 7 = -1 < 0$ it follows that they contain a maximal common divisor $\Gamma \sim aA + bB - \sum_{i=1}^7 \lambda_i E_i$, $a, b \in \mathbb{N}$ and $\lambda_i \in \mathbb{Z}$, in the sense that $\Sigma_1 - \Gamma$ and $\Sigma_2 - \Gamma$ contain no further common components. Here we can assume that Γ is not a union of components of type E_i or $E_i - E_j$, as from substracting such a component the intersection product $(\Sigma_1 - \Gamma).(\Sigma_2 - \Gamma) < \Sigma_1.\Sigma_2 < 0$ would stay negative. We distinguish the different cases for Γ :

- $a = 1, b = 0 \Rightarrow \Sigma_i = \Gamma + \Omega_i$ with $\Omega_i \sim 3B + \sum_{i=1}^7 (\lambda_i - 1)E_i$ for $i = 1, 2$. Thus we obtain $\Omega_1.\Omega_2 = -\sum_{i=1}^7 (\lambda_i - 1)^2 \leq 0$ and $\Omega_1.\Omega_2 = 0 \Leftrightarrow \lambda_1 = ... = \lambda_7 = 1 \Leftrightarrow \Gamma \sim A - \sum_{i=1}^7 E_i$, which gives a contradiction as $\Gamma.C < 0$ and C was assumed to be irreducible.

$a = 0, b = 1 \Rightarrow \Sigma_i = \Gamma + \Omega_i$ with $\Omega_i \sim A + 2B + \sum_{i=1}^7 (\lambda_i - 1)E_i$ for $i = 1, 2$. Because of $\Omega_1.\Omega_2 = 4 - \sum_{i=1}^7 (\lambda_i - 1)^2 \geq 0$ at least three of the λ_i are equal to one, we assume this for $\lambda_1, ..., \lambda_3$. But then

$$\begin{aligned}\Gamma.C = 5 - \sum_{i=1}^{7} 2\lambda_i = 5 + \sum_{i=1}^{7}(\lambda_i - 1)^2 - \sum_{i=1}^{7}(1 + \lambda_i^2) \leq \\ \leq 9 - \sum_{i=1}^{7}(1 + \lambda_i^2) < 0\end{aligned}$$

which contradicts that C is irreducible.

$a = 0$, $b = 2 \Rightarrow \Sigma_i = \Gamma + \Omega_i$ with $\Omega_i \sim A + B + \sum_{i=1}^{7}(\lambda_i - 1)E_i$ for $i = 1, 2$. Then $0 \leq \Omega_1.\Omega_2 = 2 - \sum_{i=1}^{7}(\lambda_i - 1)^2$ and therefore $\lambda_i = 1$ for at least five distinct i, we assume $i = 1, ..., 5$. Then consider a divisor L of type $(0, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ that passes through one of the points $p_1, ..., p_5$. Its strict transform $L^* := \sigma^* L \sim B - \sum_{i=1}^{7} \mu_i E_i$, $\mu_i \in \mathbb{N}$ and $\mu_{i_0} \geq 1$ for at least one $i_0 \in \{1, ..., 5\}$, has to be a component of Γ as $\Gamma.L^* < 0$. Therefore Γ factors into two components Γ_1 and Γ_2 with one of them, we assume Γ being of type $B - \sum_{i=1}^{7} \mu_i E_i$, $\mu_i \in \{0, 1\}$ and $\#\{i : \mu_i = 1\} \geq 3$. Therefore $\Gamma_1.C \leq 5 - 6 < 0$, a contradiction.

$a = 1$, $b = 1 \Rightarrow \Sigma_i = \Gamma + \Omega_i$ with $\Omega_i \sim 2B + \sum_{i=1}^{7}(\lambda_i - 1)E_i$ for $i = 1, 2$.From $0 \leq \Omega_1.\Omega_2 = -\sum_{i=1}^{7}(\lambda_i - 1)^2 \leq 0$ it follows, that $\lambda_1 = ... = \lambda_7 = 1$, which is not possible as we would get $\Gamma.C = 10 - 14 < 0$.

$a = 1$, $b = 2 \Rightarrow \Sigma_i = \Gamma + \Omega_i$ with $\Omega_i \sim B + \sum_{i=1}^{7}(\lambda_i - 1)E_i$ for $i = 1, 2$. As $0 \leq \Omega_1.\Omega_2 = -\sum_{i=1}^{7}(\lambda_i - 1)^2$ we must have $\lambda_1 = ... = \lambda_7 = 1$, thus the curve Γ is of class $A+2B-\sum_{i=1}^{7} E_i$. In this situation it follows that $D \sim D_1$ as above.

$a = 0$, $b = 3 \Rightarrow \Sigma_i = \Gamma + \Omega_i$ with $\Omega_i \sim A + \sum_{i=1}^{7}(\lambda_i - 1)E_i$ for $i = 1, 2$, hence we get $\lambda_1 = ... = \lambda_7 = 1$. Then Γ is an effective divisor of type $3B - \sum_{i=1}^{7} E_i$ which intersects C in one further point q. Let $Q_j := (g_j) \sim 2A+2B-\sum_{i=1}^{7} E_i$, $j = 1, 2$, denote two of the conics that span the pencil $(Q_\lambda)_\lambda$, then $\Gamma.Q_j = 6 - 7 < 0$ and thus it follows the existence of a maximal common component Δ_j of Γ and Q_j. For $\Delta_1 \sim B - \sum_{i=1}^{7} \tilde{\lambda}_i E_i$, $\tilde{\lambda}_i \in \mathbb{Z}$, we obtain $0 \geq (Q_1 - \Delta_1).(\Gamma - \Delta_1) = 4 - \sum_{i=1}^{7}(\tilde{\lambda}_i - 1)^2$ and therefore we can assume that $\tilde{\lambda}_1 = ... = \tilde{\lambda}_3 = 1$ and $\tilde{\lambda}$ non negative for all i. However this contradicts to the irreducibility of C because of $\Delta_1.C = 5 - \sum_{i=1}^{7} 2\tilde{\lambda}_i \leq 9 - \sum_{i=1}^{7}(1 + \lambda_i^2) < 0$. The same argument shows that $\Delta_1 \sim B - \sum_{i=1}^{7} \tilde{\lambda}_i E_i$ is not possible. It remains to discuss the case where $\Delta_j \sim 2B - \sum_{i=1}^{7} \tilde{\lambda}_i E_i$: Here we get $0 \geq (Q_1 - \Delta_1).(\Gamma - \Delta_1) = 2 - \sum_{i=1}^{7}(\tilde{\lambda}_i - 1)^2$ and therefore we can assume $\tilde{\lambda}_1 = ... = \tilde{\lambda}_5 = 1$. Now from $0 \leq \Delta_1.C = 10 - \sum_{i=1}^{7} 2\tilde{\lambda}_i \leq 12 - \sum_{i=1}^{7}(1 + \lambda_i^2)$ it follows that $\tilde{\lambda}_5 = \tilde{\lambda}_6 = 0$, hence Δ_1 is an effective divisor of type $2B - \sum_{i=1}^{5} E_i$. In the case $(a, b) = (0, 2)$ we have already excluded this possibility.

It remains to show that $h^0(C, \mathcal{O}_C(K_C - 3D_1)) \neq 1$. For $h^0(C, \mathcal{O}_C(K_C - 3D_1)) =$

1 we have $h^0(C, \mathcal{O}_C(K_C - 2D_1)) = 2$ and therefore with the considerations above there exists an effective divisor $\Gamma \sim A + 2B - \sum_{i=1}^{7} E_i$. Let $L \sim K_C - 3D_1$ be an effective divisor on C, then we get the following representation

$$(A + 3B - \sum_{i=1}^{7} E_i)|_C \sim L + (\lambda|_C) = \Gamma|_C + (s'|_C)$$

and

$$L + (\mu|_C) = \Gamma|_C + (t'|_C)$$

with $s', t' \in H^0(S, \mathcal{O}_S(B))$. We conlcude $(\lambda|_C) - (\mu|_C) = (s'|_C) - (t'|_C)$ and thus $(\lambda|_C) \sim (s'|_C) \Rightarrow D_1 \sim D_2$, a contradiction.

For $i = 2$ the claim can be proven in an analogous way. ∎

It is time for a short overview of what we have done in this chapter: We have proven that a pentagonal curve C, which is given by the Pfaffians of a matrix ψ on a scroll of type $S(2,1,1,1)$ as in **(*)**, has exactly one special linear series of type g^1_5 if and only if ψ is of type A as in Lemma 4.5.1. Then the entry $\beta_{46} = \beta_{56}$ of the Betti table for C equals to 4. Furthermore we have seen that if a pentagonal curve C that admits no g^2_7 and has two different $g^1_5{}'s$, then at least one of them has multiplicity one, hence the scroll constructed from $|D|$ is of type $S(2,1,1,1)$. Therefore, in this case we can assume that C is given by the Pfaffians of a matrix ψ as in **(*)**. Moreover, it turns out that if C has a third g^1_5, then this linear series is uniquely determined. In some cases it becomes equal to one of the two others. Now it remains to give the correspondence between these cases and the different types of the matrix ψ which gives us the Betti table for C.

Theorem 4.5.8 *Let C be an irreducible, smooth, canonical curve of genus 9 with* $\mathrm{Cliff}(C) = 3$ *that admits no g^2_7. Then*

a) C has exactly three different $g^1_5{}'s$ if and only if C is given by the Pfaffians of a matrix ψ of type C in Lemma 4.5.1 on a scroll $X \simeq S(2,1,1,1)$.

b) C has exactly two different linear systems $|D_1|, |D_2|$ of type g^1_5 with $m_{|D_1|} = 2$ and $m_{|D_2|} = 1$ if and only if C is given by the Pfaffians of a matrix ψ of type D.

In both cases the minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}$-module has the following Betti diagram:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	12	-	-	-
2	-	-	-	12	70	64	21	-
3	-	-	-	-	-	-	-	1

Proof. a) Let us first assume that C different g_5^1, then we already know that C is given by the Pfaffians of a matrix ψ of type B, C or D (cf. Lemma 4.5.1) on a scroll $X \simeq S(2,1,1,1)$. As we have seen before, there exists a space model C' of C on $\mathbb{P}^1 \times \mathbb{P}^1$, having exactly 7 (possibly infinitely near) double points $p_1, ..., p_7$ as only singularities. We can assume that C is given as strict transform of C' in the blowup in the points $p_1, ..., p_7$ (cf. page 78), i.e. $C \sim 5A + 5B - \sum_{i=1}^{7} 2E_i$ and $K_C \sim (3A+3B)|_C - \sum_{i=1}^{7} E_i|_C$. From Theorem 4.5.6, we already know that the three different linear series of type g_5^1 are given by divisors $D_1 \sim A|_C$, $D_2 \sim B|_C$ and $D \sim (2A+2B)|_C - \sum_{i=1}^{7} E_i|_C - p$ with $p \in C$ the basepoint of $\left|2A + 2B - \sum_{i=1}^{7} E_i\right|$. The global sections of $\mathcal{O}_S(A)$, $\mathcal{O}_S(B)$ and $H^0(S, \mathcal{O}_S(2A+2B-\sum_{i=1}^{7} E_i))$ are denoted as in the theorem above. $C \subset S$ is given by the Pfaffians of the matrix ψ regarding the entries of ψ as global sections of vectorbundles on S. As the 2×2 minors of the submatrix

$$\omega = \begin{pmatrix} \psi_{14} & \psi_{24} & \psi_{34} \\ \psi_{15} & \psi_{25} & \psi_{35} \end{pmatrix}$$

vanish on S, the two rows of ω are linear dependent on S. Therefore we get

$$\psi \sim \begin{pmatrix} 0 & h_1 & h_2 & \lambda \cdot \varphi & \mu \cdot \varphi \\ & 0 & \gamma & \lambda \cdot g_1 & \mu \cdot g_1 \\ & & 0 & \lambda \cdot g_2 & \mu \cdot g_2 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

with $\varphi \in H^0(S, \mathcal{O}_S(2A+3B-\sum_{i=1}^{7} E_i))$ and $\gamma \in H^0(S, \mathcal{O}_S(3A+2B-\sum_{i=1}^{7} E_i))$. Recall that $H^0(C, \mathcal{O}_C(D_1)) = \langle \lambda|_C, \mu|_C \rangle$, $H^0(C, \mathcal{O}_C(D_2)) = \langle s|_C, t|_C \rangle$, $H^0(C, \mathcal{O}_C(D + p)) = \langle g_1|_C, g_2|_C \rangle$ The curve C is given by the equation

$$\delta = h_1 g_2 - h_2 g_1 + \varphi\gamma \in H^0(S, \mathcal{O}_S(5A + 5B - \sum_{i=1}^{7} 2E_i)$$

Then δ also vanishes at p as $p \in C$. As $h_1 g_2 - h_2 g_1$ vanishs at p, too, the same holds for $\varphi\gamma \in H^0(S, \mathcal{O}_S(5A + 5B - \sum_{i=1}^{7} 2E_i))$, hence γ or φ has to vanish at p. As we assumed for $|D|$ to be different from $|D_1|$, Theorem 4.5.7 tells us, that $\varphi \in \langle s \cdot g_1, t \cdot g_1, s \cdot g_2, t \cdot g_2 \rangle$ or $\gamma \in \langle \lambda \cdot g_1, \mu \cdot g_1, \lambda \cdot g_2, \mu \cdot g_2 \rangle$. In the case $\varphi \in \langle s \cdot g_1, t \cdot g_1, s \cdot g_2, t \cdot g_2 \rangle$ the entry $\psi_{14} = \lambda \cdot \varphi$ could be made to zero by suitable row and column operations, hence we get a contradiction to our assumption that C is irreducible. Therefore we must have $\psi_{23} = \gamma \in \langle \lambda \cdot g_1, \mu \cdot g_1, \lambda \cdot g_2, \mu \cdot g_2 \rangle$, so this entry of ψ can be made zero. This proves one direction.

For the other direction let C be given by the Pfaffians of

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & 0 & H-R & H-R \\ & & 0 & H-R & H-R \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

on a scroll $\mathbb{P}(\mathcal{E})$ of type $S(2,1,1,1)$. Then the 5 Pfaffians are given by the 2×2 minors of

$$\omega_1 = \begin{pmatrix} \psi_{14} & \psi_{24} & \psi_{34} \\ \psi_{15} & \psi_{25} & \psi_{35} \end{pmatrix} \sim \begin{pmatrix} H & H-R & H-R \\ H & H-R & H-R \end{pmatrix}$$

and

$$\omega_2 = \begin{pmatrix} -\psi_{12} & \psi_{24} & \psi_{25} \\ -\psi_{13} & \psi_{34} & \psi_{35} \end{pmatrix} \sim \begin{pmatrix} H & H-R & H-R \\ H & H-R & H-R \end{pmatrix}$$

Each of the minors of ω_1 and ω_2 define a surface Y_1 and Y_2 respectively with $C = Y_1 \cap Y_2$. Both surfaces are blowups of $\mathbb{P}^1 \times \mathbb{P}^1$ with C being a divisor of type $5A + 5B$ on each of them (cf. Theorem 1.4.5). Then, the two linear systems of type g^1_5 cut out by global sections of $\mathcal{O}_{Y_1}(B)$ and $\mathcal{O}_{Y_2}(B)$ are the same. The other two $g^1_5{}'s$, given by $A|_C$ on Y_1 and Y_2 respectively, have to be different as the surfaces Y_1 and Y_2 are different. This proves the existence of three different $g^1_5{}'s$.

b) The two different linear series of type g^1_5 are given by divisors $D_1 \sim A|_C$ and $D_2 \sim B|_C$. Then according to the last theorem we can assume that the scroll constructed from $|D_2|$ is of type $S(2,1,1,1)$. We have also shown that a third g^1_5 is obtained from a divisor $D \sim (2A+2B)|_C - \sum_{i=1}^{7} E_i|_C - p$ with $p \in C$ the basepoint of $\left|2A + 2B - \sum_{i=1}^{7} E_i\right|$. Furthermore Theorem 4.5.7 tells us that $D \not\sim D_2$ and $D \sim D_1 \Leftrightarrow \dim \langle \lambda \cdot g_1, \mu \cdot g_1, \lambda \cdot g_2, \mu \cdot g_2 \rangle \leq 3$. The only possibility for $|D_1|$ to have multiplicity 2 is therefore given exactly in the case, where $D \sim D_1$. This is equivalent to that the four entries $\psi_{24}, \psi_{25}, \psi_{34}$ and ψ_{24} span a three dimensional vector space, thus ψ is of type D. The claim according to the Betti table is a direct consequence of Lemma 4.5.2. ■

It remains to consider the case where C has exactly two different linear systems of type g^1_5, each of them with multiplicity one. The following theorem is a direct conclusion from the above theorems:

Theorem 4.5.9 *Let C be an irreducible, smooth, canonical curve of genus 9 with* Cliff$(C) = 3$, *that admits no g^2_7. Then C has exactly two different g^1_5, each of them with multiplicity one, if and only if it is given by the Pfaffians of a matrix ψ of type B on a scroll of type $S(2,1,1,1)$. The minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}-$module has the following Betti table*

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	8	-	-	-
2	-	-	-	8	70	64	21	-
3	-	-	-	-	-	-	-	1

4.6 Infinitely near $g_5^{1\prime}s$

In the last sections, we have seen that for C, an irreducible, smooth, canonical curve of genus 9, it is possible to have one, two or even three different special linear series of type g_5^1, such that at least one of the $g_5^{1\prime}s$ has multiplicity one. We have shown that in each case C is given by the Pfaffians of a corresponding matrix ψ on a scroll of type $S(2,1,1,1)$. Moreover we have calculated the Betti tables for $C \subset \mathbb{P}^8$. In this section we concentrate on the remaining case where C has Clifford index 3, admits no g_7^2, and has exactly one $g_5^1 = |D|$ of multiplicity 2 or 3. In this situation the scroll

$$X = \bigcup_{D_\lambda \in |D|} \bar{D}_\lambda \subset \mathbb{P}^8$$

constructed from $|D|$ is a 4-dimensional rational scroll of type $S(2,2,1,0)$ or $S(3,1,1,0)$ depending on if $m_{|D|} = 2$ or $m_{|D|} = 3$ respectively (cf. Theorem 4.4.1). Then Theorem 4.6.3 says that also in this situation there exists a unique representation of C by the Pfaffians of a skew symmetric matrix ψ on the scroll X. It is of type

$$\psi \sim \begin{pmatrix} 0 & H+R & H & H & H-R \\ & 0 & H & H & H-R \\ & & 0 & H-R & H-2R \\ & & & 0 & H-2R \\ & & & & 0 \end{pmatrix}$$

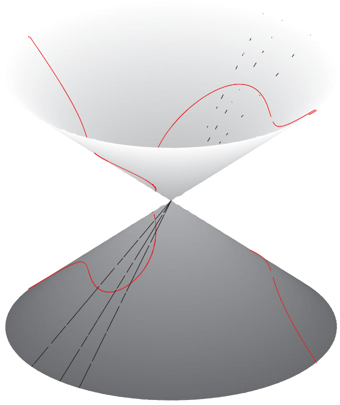
and the Betti table for C can be obtained by considering the non minimal maps in the corresponding mapping cone construction. Moreover C is contained in a determinantal surface $Y \subset X$, that is obtained as image of a blowup of a cone in $\mathbb{P}^3$: We focus on this question first. A necessary condition for $m_{|D|} \geq 2$ is $h^0(C, \mathcal{O}_C(2D)) = 4$. We consider the space model C' of C obtained from the complete linear series $|2D|$.

Lemma 4.6.1 *Let $X \subset \mathbb{P}^8$ be the scroll given by a special linear series $|D|$ of type g_5^1 of a canonical curve C of genus 9 as above, i.e. $m_D \geq 2$. Then there exists a space model C' of C lying on the cone $Y \subset \mathbb{P}^3$ over the irreducible conic $x_1^2 - x_0x_2$ in $\mathbb{P}^3$.*

Proof. We have remarked that X is of type $S(2,2,1,0)$ or $S(3,1,1,0)$ if and only if $h^0(C, \mathcal{O}_C(2D)) = 4$. Let $r, s \in H^0(C, \mathcal{O}_C(D))$ denote the global generators of $\mathcal{O}_C(D)$, then we get three global sections $x_0,\ x_1,\ x_2 \in H^0(C, \mathcal{O}_C(2D))$ by $x_0 := r^2,\ x_1 := s \cdot r$ and $x_2 := s^2$. Further there exists an element $x_3 \in H^0(C, \mathcal{O}_C(2D)) \setminus \langle x_0, x_1, x_2 \rangle$. Considering the space model C' of C, which is given by the complete linear system $|2D|$:

$$C \overset{|2D|}{\to} C' \subset \mathbb{P}^3$$

we see that C' is contained in the surface $Y \subset \mathbb{P}^3$, given by the equation $x_1^2 - x_0x_2 = 0$, i.e. Y is a cone over a conic in $\mathbb{P}^2$



As $|D|$ was assumed to be base point free C' does not pass the vertex of the cone. ■

4.6.1 Resolution and Representation

In the foregoing lemma we have seen that there exists a space model $C' \subset Y \subset \mathbb{P}^3$ of C that is contained in the singular cone $Y \subset \mathbb{P}^3$ over the irreducible conic $x_1^2 - x_0x_2$ in $\mathbb{P}^2$. Consider

$$\pi : P_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^1$$

the corresponding $\mathbb{P}^1-$bundle with hyperplane class A and ruling B, that represents a desingularisation of the cone. The exceptional divisor of this blowup is an effective divisor of class $A - 2B$. The strict transform $C'' \subset P_2$ of C' is then given as divisor of class $5A$ as C' does not pass the vertex of the cone Y. From $p_a(C') = (5-1)^2 = 16$ it follows that C'' has certain singularities on P_2 that cannot be of multiplicity 3 or higher, as projection from such a point would lead to a g_d^2 with $d \leq 7$. Thus C'' has exactly 7 (possible infinitely near) double points $p_1, ..., p_7$ on P_2. The g_5^1 is then cut out by the class of a ruling B. We consider the iterated blowup

$$\sigma : S = \tilde{P}_2(p_1, ..., p_7) \to P_2$$

in the singular points of C''. We can assume that C is the strict transform of C'' under the blowup σ and $K_C \sim (3A - \sum_{i=1}^{7} E_i)|_C$. Consider the morphism

$$\varphi : S \to S' \subset \mathbb{P}^8$$

defined by the adjoint series

$$H^0(S, \omega_S(\sigma^* C'')) = H^0(S, \mathcal{O}_S(3A - \textstyle\sum_{i=1}^7 E_i))$$

According to Corollary 2.3.1 the adjoint linear series is base point free and the surface S' is arithmetically Cohen-Macaulay. Now we want to apply our results on page 1.4 to show that the variety

$$X = \bigcup_{B_\lambda \in |B|} \bar{B}_\lambda \subset \mathbb{P}^8$$

is a 4-dimensional rational normal scroll. We have to check the following conditions for $H = 3A - \sum_{i=1}^7 E_i$:

1. $h^0(\mathcal{O}_S(H - B)) \geq 2$
2. $H^1(\mathcal{O}_S(kH - B)) = 0$ for $k \geq 1$ and
3. the map $S_k H^0 \mathcal{O}_S(H) \to H^0 \mathcal{O}_S(kH)$ is surjective.

The first condition is trivial because of $h^0(\mathcal{O}_S(H-B)) \geq 12 - 7 = 5$. Condition 3. follows as S' is arithmetically Cohen-Macaulay. It remains to examine the second condition. Consider the exact sequence

$$0 \to \mathcal{O}_S(kH - B) \to \mathcal{O}_S(kH) \to \mathcal{O}_B(kH|_B) \to 0$$

and the corresponding long exact sequence of cohomology groups

$$\begin{aligned} 0 \to H^0 \mathcal{O}_S(kH - B) \to H^0 \mathcal{O}_S(kH) &\xrightarrow{\delta_k} H^0 \mathcal{O}_B(kH|_B) \to \\ \to H^1 \mathcal{O}_S(kH - B) \to H^1 \mathcal{O}_S(kH) &= 0 \end{aligned}$$

We first show that $H^0 \mathcal{O}_S(H) \xrightarrow{\delta_1} H^0 \mathcal{O}_B(H|_B) \cong H^0 \mathcal{O}_{\mathbb{P}^1}(H.B) = H^0 \mathcal{O}_{\mathbb{P}^1}(3)$ is surjective. If $h^0(\mathcal{O}_S(H - B)) = d > 5$ then the complete linear system $|(H - B)|_C|$ would be of type g_{11}^{d-1} and therefore $\mathrm{Cliff}(C) \leq 2$. It follows that $h^0(\mathcal{O}_S(H - B)) = 5$ and thus $\dim \delta \geq 9 - 5 = 4 = h^0 \mathcal{O}_{\mathbb{P}^1}(3)$, so δ_1 is surjective. As a further consequence we also obtain that $H^0 \mathcal{O}_S(kH) \xrightarrow{\delta_k} H^0 \mathcal{O}_B(kH|_B)$ is surjective (the image of B under φ is a rational normal curve), hence $H^1 \mathcal{O}_S(kH - B) = 0$. Let $\mathbb{P}(\mathcal{E})$ be the corresponding $\mathbb{P}^3-$bundle to the scroll X, then according to Theorem 1.4.4 S' can be given by the 2×2 minors of a matrix of type

$$\omega \sim \begin{pmatrix} H - a_1 R & H - a_2 R & H - a_3 R \\ H - (a_1 + 2)R & H - (a_2 + 2)R & H - (a_3 + 2)R \end{pmatrix}$$

on $\mathbb{P}(\mathcal{E})$ with $a_1, a_2, a_3 \in \mathbb{Z}$.

From Theorem 1.4.5, we obtain certain conditions on the numbers a_i (we denote $f = \deg X = 5$, $d = 3$ and $a = a_1 + a_2 + a_3$) :

$$K_C \sim (3A - \textstyle\sum_{i=1}^7 E_i)|_C = (3A + (f - d \cdot k - a)B - \sum_{i=1}^7 E_i)|_C \Rightarrow a = -1$$

If X is of type $S(2,2,1,0)$, it follows that $a_i \leq 0$ for all $i = 1,2,3$ since otherwise some of the minors of ω would vanish or become reducible. In the case, where $X \simeq S(3,1,1,0)$, we must have $a_i \leq 1$ for all i for the same reasons. Moreover we cannot have $a_1 = a_2 = 1$ in the case $X \simeq S(3,1,1,0)$, since we would get a common factor $\varphi_0 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-3R))$ of al minors. We conclude:

Corollary 4.6.2 *Let $X \subset \mathbb{P}^8$ be the scroll constructed from a special linear series $|D|$ of type g^1_5, $m_{|D|} \geq 2$, on an irreducible, nonsingular, canonical curve $C \subset \mathbb{P}^8$ of genus 9 with* $\mathrm{Cliff}(C) = 3$, *that admits no g^7_2. With $\mathbb{P}(\mathcal{E})$ denoting the corresponding $\mathbb{P}^3-$bundle, C is contained in a determinantal surface Y given by the minors of a 2×3 matrix of type*

a)

$$\omega \sim \begin{pmatrix} H & H & H+R \\ H-2R & H-2R & H-R \end{pmatrix}$$

if $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O})$.

b)

$$\omega \sim \begin{pmatrix} H-R & H+R & H+R \\ H-3R & H-R & H-R \end{pmatrix}$$

if $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O})$.

In Section 4.4 we mentioned that on $\mathbb{P}(\mathcal{E})$, the vanishing ideal of C is given by the Pfaffians of the matrix ψ that occurs in the free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module:

$$F_* : \qquad 0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-5H+3R) \to \sum_{i=1}^{5} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+b_iR) \xrightarrow{\psi}$$

$$\xrightarrow{\psi} \sum_{i=1}^{5} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+a_iR) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_C \to 0$$

Theorem 4.4.2 then shows that we can restrict to 3 different possibilities for $(a_1, ..., a_5)$ and from the proposition above, we can deduce further information:

Theorem 4.6.3 *Let C be a curve as given in Corollary 4.6.2, then the matrix ψ has the following form:*

$$\psi \sim \begin{pmatrix} 0 & H+R & H & H & H-R \\ & 0 & H & H & H-R \\ & & 0 & H-R & H-2R \\ & & & 0 & H-2R \\ & & & & 0 \end{pmatrix}$$

Proof. In Theorem 4.4.2 we have seen that there are only three possible types for ψ, namely $(a_1, ..., a_5) = (2,1,1,1,1)$, $(2,2,1,1,0)$ or $(2,2,2,0,0)$. From Corollary 4.6.2 above, it turns out that C is contained in a determinantal surface Y given by the 2×2 minors of a 2×3 matrix ω with global sections on $\mathbb{P}(\mathcal{E})$. Therefore the 2×2 minors of ω, which are elements in $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H - \tilde{a}_i R))$, $i = 1,2,3$, are contained in the ideal generated by the Pfaffians $\rho_i \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H - a_i R))$, $i = 1, ..., 5$, of ψ. Now $(a_1, ..., a_5) = (2,1,1,1,1)$ and $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O})$ can be omitted, as in this case the two minors of ω, which are elements of $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H - 2R))$, are equal to a multiple of $\rho_1 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2H - 2R))$, which is not possible as Y is irreducible and non degenerate.

The following arguments show that the case $(a_1, ..., a_5) = (2,1,1,1,1)$ does not occur for $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O})$, too: In this case C is given by the Pfaffians of

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & H-R & H-R & H-R \\ & & 0 & H-R & H-R \\ & & & 0 & H-R \\ & & & & 0 \end{pmatrix}$$

We already know that C is contained in a determinantal surface Y given by the minors of

$$\omega \sim \begin{pmatrix} H & H & H+R \\ H-2R & H-2R & H-R \end{pmatrix}$$

on $\mathbb{P}(\mathcal{E})$. We first calculate the Betti table for $Y \subset \mathbb{P}^8$: Theorem 1.4.4 tells us that $\mathcal{O}_Y$ has an $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module resolution of type

$$F_* \ : 0 \qquad \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+R) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+3R) \xrightarrow{\omega}$$

$$\xrightarrow{\omega} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+R)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+2R) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_Y \to 0$$

The corresponding mapping cone

$$
\begin{array}{ccccccc}
0 & \to & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+3R) & \oplus & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+R) & \longrightarrow & \\
 & & \uparrow & & \uparrow & & \\
 & & -----|----- & & -----|----- & - & \\
 & & S_3G(-3) & \oplus & G(-3) & \longrightarrow & \\
 & & \uparrow & & \uparrow & & \\
 & & F\otimes S_2G(-4) & \oplus & F(-4) & \longrightarrow & \\
 & & \uparrow & & \uparrow & & \\
 & & \wedge^2F\otimes G(-5) & \oplus & \wedge^3F(-6) & \xrightarrow{\gamma} & \\
 & & \uparrow & & \uparrow & & \\
 & & \wedge^3F(-6) & \oplus & DG^*\otimes\wedge^4F(-7) & \longrightarrow & \\
 & & \uparrow & & \uparrow & & \\
 & & \wedge^5F(-8) & \oplus & D_2G^*(-8) & \longrightarrow &
\end{array}
$$

$$
\begin{array}{ccccccc}
\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+R)^{\oplus 2} & \oplus & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+2R) & \longrightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})} & \to & 0\\
\uparrow & & \uparrow & & \uparrow & & \\
----|---- & & ----|---- & & ----|---- & & \\
G(-2)^{\oplus 2} & \oplus & S_2G(-2)^{\oplus 2} & \longrightarrow & \mathcal{O}_{\mathbb{P}^8} & & \\
\uparrow & & \uparrow & & \uparrow & & \\
F(-3)^{\oplus 2} & \oplus & F\otimes G(-3)^{\oplus 2} & \longrightarrow & \wedge^2F(-2) & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\wedge^3F(-5)^{\oplus 2} & \oplus & \wedge^2F(-4)^{\oplus 2} & \longrightarrow & \wedge^3F\otimes DG^*(-3) & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\wedge^4F\otimes DG^*(-6)^{\oplus 2} & \oplus & \wedge^4F(-6)^{\oplus 2} & \longrightarrow & \wedge^4F\otimes D_2G^*(-4) & & \\
\uparrow & & \uparrow & & \uparrow & & \\
D_2G^*(-7)^{\oplus 2} & \oplus & DG^*(-7)^{\oplus 2} & \longrightarrow & D_3G^*(-5) & &
\end{array}
$$

gives us a (not necessarily) minimal free resolution of $\mathcal{O}_Y$. We calculate the rank of the non minimal map

$$\gamma : \wedge^2F\otimes G(-5) \xrightarrow{\alpha} \wedge^3F(-5)^{\oplus 2}$$

which is obtained from the submatrix $\begin{pmatrix} \omega_{22} & \omega_{23} \end{pmatrix} \sim \begin{pmatrix} H-2R & H-2R \end{pmatrix}$:

$$\gamma : f_1\wedge f_2\otimes g \to \begin{pmatrix} f_1\wedge f_2\wedge g\omega_{22} \\ f_1\wedge f_2\wedge g\omega_{23}\end{pmatrix}$$

with $f_1,...,f_4 \in H^0(\mathbb{P}(\mathcal{E}),\mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R)) \cong H^0(\mathbb{P}^8,\mathcal{O}^5_{\mathbb{P}^8})$ and $g_1,g_2 \in H^0(\mathbb{P}(\mathcal{E}),\mathcal{O}_{\mathbb{P}(\mathcal{E})}(R)) \cong H^0(\mathbb{P}^8,\mathcal{O}^2_{\mathbb{P}^8})$. It turns out that $\dim\ker\gamma = 4$:

$$
\begin{aligned}
&\ker\gamma = \langle s\omega_{22}\wedge s\omega_{23}\otimes s,\ t\omega_{22}\wedge t\omega_{23}\otimes t,\\
&s\omega_{22}\wedge t\omega_{23}\otimes s + t\omega_{22}\wedge s\omega_{23}\otimes s + s\omega_{22}\wedge s\omega_{23}\otimes t,\\
&t\omega_{22}\wedge t\omega_{23}\otimes s + s\omega_{22}\wedge t\omega_{23}\otimes t + t\omega_{22}\wedge s\omega_{23}\otimes t\rangle
\end{aligned}
$$

hence the Betti table for $Y \subset \mathbb{P}^8$ is given by

	0	1	2	3	4	5	6
0	1	-	-	-	-	-	-
1	-	17	46	45	8	-	-
2	-	-	-	4	25	18	4

The linear strand of the minimal free resolution for Y is a subcomplex of that for $C \subset Y$. It follows that the Betti number $\beta_{45}(C)$ in the minimal free resolution of $\mathcal{O}_C$ has to be at least 8. Now if C is given by the Pfaffians of a matrix

$$\psi \sim \begin{pmatrix} 0 & H & H & H & H \\ & 0 & H-R & H-R & H-R \\ & & 0 & H-R & H-R \\ & & & 0 & H-R \\ & & & & 0 \end{pmatrix}$$

then from Lemma 4.5.2 and Remark 4.5.3 we already know that $\beta_{45}(C) \geq 8$ is only possible if one of the $(H-R)$-entries of ψ can be made zero by suitable row and column operations. But then we obtain a second special linear series of type g_5^1 that is different from the one we have started with. This contradicts to our assumption that C has only one g_5^1.
Let us now consider the case $(a_1, ..., a_5) = (2, 2, 2, 0, 0)$: For $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O})$, ψ is of type

$$\psi \sim \begin{pmatrix} 0 & H+R & H+R & H-R & H-R \\ & 0 & H+R & H-R & H-R \\ & & 0 & H-R & H-R \\ & & & 0 & H-3R \\ & & & & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & H+R & H+R & \mathbf{H-R} & \mathbf{H-R} \\ & 0 & H+R & \mathbf{H-R} & \mathbf{H-R} \\ & & 0 & \mathbf{H-R} & \mathbf{H-R} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

It follows that C is contained in a determinantal surface Y given by the minors of

$$\omega \sim \begin{pmatrix} H-R & H-R & H-R \\ H-R & H-R & H-R \end{pmatrix}$$

on $\mathbb{P}(\mathcal{E})$. According to Theorem 1.4.5. Y is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in 3 points. The direct image C' of C in $\mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of type $(5, 4)$, thus we get a g_4^1 from projection onto the second factor of $\mathbb{P}^1 \times \mathbb{P}^1$, which we have excluded.

It remains to examine the same possibility for $(a_1, ..., a_5)$ if $X \simeq S(3, 1, 1, 0)$: After suitable row and column operations, one of the $(H-R)$-entries in the last column of ψ can be made to zero. Assuming $\psi_{15} = 0$, our matrix takes the following form

$$\psi \sim \begin{pmatrix} 0 & H+R & H+R & \varphi' & 0 \\ & 0 & H+R & \varphi'' & \varphi_2 \\ & & 0 & \varphi_2 & \varphi_1 \\ & & & 0 & \varphi_0 \\ & & & & 0 \end{pmatrix}$$

with linear independent, global sections $\varphi_0 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-3R))$, $\varphi_1, \varphi_2 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R))$ and $\varphi', \varphi'' \in \langle \varphi_1, \varphi_2 \rangle$. We apply Theorem 1.4.5 again to see that C is contained in a determinantal surface Y given by the matrix

$$\omega := \begin{pmatrix} \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{25} & \psi_{35} & \psi_{45} \end{pmatrix} \sim \begin{pmatrix} H+R & H+R & \varphi' \\ \varphi_2 & \varphi_1 & \varphi_0 \end{pmatrix}$$

on $\mathbb{P}(\mathcal{E})$. The image of this surface in $\mathbb{P}^8$ is given as the image of $P_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)+\mathcal{O}_{\mathbb{P}^1})$ under a rational map defined by a subseries $H^0(P_2, \mathcal{O}_{P_2}(3A))$ which has 7 assigned base points. The image C' of C in P_2 is given by the Pfaffians of a matrix $\tilde{\psi}$ regarding the entries of ψ as global sections of vector bundles in P_2. As the two rows of ω are linear dependend, we get

$$\tilde{\psi} \sim \begin{pmatrix} 0 & \lambda g_2 & \lambda g_1 & \varphi' = \lambda g_0 & 0 \\ & 0 & 3A+B & \varphi'' & \mu g_2 \\ & & 0 & \mu g_2 & \mu g_1 \\ & & & 0 & \mu g_0 \\ & & & & 0 \end{pmatrix}$$

with elements $\lambda \in H^0(P_2, \mathcal{O}_{P_2}(A))$, $\mu \in H^0(P_2, \mathcal{O}_{P_2}(A-2B))$ and $g_1, g_2 \in H^0(P_2, \mathcal{O}_{P_2}(2A+B))$, $g_0 \in H^0(P_2, \mathcal{O}_{P_2}(2A-B))$. Because of $\varphi', \varphi'' \in \langle \varphi_1, \varphi_2 \rangle = \langle \mu g_1, \mu g_2 \rangle$, μ is a common factor of the last two columns, hence it is a common factor of all Pfaffians of $\tilde{\psi}$. This contradicts that C is assumed to be irreducible. It remains only one possibility for $(a_1, ..., a_5)$, i.e. $(a_1, ..., a_5) = (2, 2, 1, 1, 0)$. ■

Theorem 4.6.4 *Let C be an irreducible, nonsingular, canonical curve of genus 9 with* $\mathrm{Cliff}(C) = 3$ *that admits no g^2_7. If C has exactly one g^1_5 with multiplicitiy two and no further g^1_5 then it is given by the Pfaffians of a matrix ψ*

$$\psi \sim \begin{pmatrix} 0 & H+R & H & H & H-R \\ & 0 & H & H & H-R \\ & & 0 & H-R & H-2R \\ & & & 0 & H-2R \\ & & & & 0 \end{pmatrix}$$

on a scroll of type $S(2,2,1,0)$. The minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}-$modul has the following Betti table:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	8	-	-	-
2	-	-	-	8	70	64	21	-
3	-	-	-	-	-	-	-	1

Proof. Let $\mathbb{P}(\mathcal{E})$ be the corresponding $\mathbb{P}^3$-bundle of type $S(2,2,1,0)$ constructed from a unique $g^1_5 = |D|$ of multiplicity 2, then C is given by the Pfaffians of a matrix ψ as in the above theorem. By suitable row and column operations, ψ has the following form:

$$\psi \sim \begin{pmatrix} 0 & H+R & H & H & 0 \\ & 0 & H & H & \varphi_2 \\ & & 0 & \varphi & \varphi_1 \\ & & & 0 & \varphi_0 \\ & & & & 0 \end{pmatrix}$$

with independent global sections $\varphi_0, \varphi_1 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-2R))$ and $\varphi_2 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R))$. We can assume that $\varphi = 0$ or $\varphi = \varphi_2$. If $\varphi = 0$, then C is contained in the determinantal surface Y given by the 2×2 minors of

$$\omega := \begin{pmatrix} \psi_{31} & \psi_{32} & \psi_{35} \\ \psi_{14} & \psi_{24} & \psi_{54} \end{pmatrix} \sim \begin{pmatrix} H & H & H-2R \\ H & H & H-2R \end{pmatrix}$$

We apply Theorem 1.4.5 to see that Y is a blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in 7 points and the direct image $C' \subset \mathbb{P}^1 \times \mathbb{P}^1$ of C is a divisor of type $(5,5)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore C would have two different special linear series of type g^1_5, which contradicts our assumptions. It follows that $\varphi = \varphi_2$. Now we want to calculate the Betti table for the minimal free resolution of $\mathcal{O}_C$. We have to look at the non minimal maps in the corresponding mapping cone

$$\begin{array}{ccccccc}
\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+R)^{\oplus 2} & \oplus & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+2R)^{\oplus 2} & \oplus & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+3R) & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & \\
-----|----- & - & ----|---- & - & -----|----- & ----- \\
G(-3)^{\oplus 2} & \oplus & S_2G(-3)^{\oplus 2} & \oplus & S_3G(-3) & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & \\
F(-4)^{\oplus 2} & \oplus & F \otimes G(-4)^{\oplus 2} & \oplus & S_2G \otimes F(-4) & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & \\
\wedge^3F(-6)^{\oplus 2} & \oplus & \wedge^2F(-5)^{\oplus 2} & \oplus & \wedge^2F \otimes G(-5) & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & \\
\wedge^4F \otimes DG^*(-7)^{\oplus 2} & \oplus & \wedge^4F(-7)^{\oplus 2} & \oplus & \wedge^3F(-6) & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & \\
D_2G^*(-8)^{\oplus 2} & \oplus & DG^*(-8)^{\oplus 2} & \oplus & \wedge^5F(-8) & \longrightarrow
\end{array}$$

$$\begin{array}{cccccc}
\longrightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H+2R)^{\oplus 2} & \oplus & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H+R)^{\oplus 2} & \oplus & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2H) \\
 & \uparrow & & \uparrow & & \uparrow \\
----- & -----|----- & & -----|----- & - & -----|----- \\
\longrightarrow & S_2G(-2)^{\oplus 2} & \oplus & G(-2)^{\oplus 2} & \oplus & \mathcal{O}(-2) \\
 & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & F \otimes G(-3)^{\oplus 2} & \oplus & F(-3)^{\oplus 2} & \oplus & \wedge^2F(-4) \\
 & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & \wedge^2F(-4)^{\oplus 2} & \oplus & \wedge^3F(-5)^{\oplus 2} & \oplus & \wedge^3F \otimes DG^*(-5) \\
 & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & \wedge^4F(-6)^{\oplus 2} & \oplus & \wedge^4F \otimes DG^*(-6)^{\oplus 2} & \oplus & \wedge^4F \otimes D_2G^*(-6) \\
 & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & \wedge^5F \otimes DG^*(-7)^{\oplus 2} & \oplus & D_2G^*(-7)^{\oplus 2} & \oplus & \wedge^5F \otimes D_3G^*(-7)
\end{array}$$

which are given by the following maps:

$$\alpha: \; F(-4)^{\oplus 2} \quad \oplus \quad F\otimes G(-4)^{\oplus 2} \quad \oplus \quad S_2G\otimes F(-4) \quad \to \wedge^2 F(-4)$$

$$\begin{pmatrix} f_1 \\ f_2 \\ g_1\otimes f_3 \\ g_2 \otimes f_4 \\ g_3^{(2)}\otimes f_5 \end{pmatrix} \to f_2\wedge\varphi_2 + g_1\varphi_1\wedge f_3 + g_2\varphi_0\wedge f_4$$

its dual

$$\alpha^*: \wedge^3 F(-6) \to \quad \wedge^4 F\otimes DG^*(-6)^{\oplus 2} \quad \oplus \quad \wedge^4 F\otimes D_2G^*(-6) \quad \oplus \quad \wedge^4 F(-6)^{\oplus 2}$$

and

$$\beta: \; \wedge^2 F\otimes G(-5) \quad \oplus \quad \wedge^2 F(-5)^{\oplus 2} \quad \to \quad \wedge^3 F(-5)^{\oplus 2} \quad \oplus \quad \wedge^3 F\otimes DG^*(-5)$$

$$\begin{pmatrix} f_1\wedge f_2 \\ f_3\wedge f_4 \\ g\otimes f_5\wedge f_6 \end{pmatrix} \to \begin{pmatrix} -\varphi_2\wedge f_3\wedge f_4 - g\varphi_1\wedge f_5\wedge f_6 \\ \varphi_2\wedge f_1\wedge f_2 - g\varphi_0\wedge f_5\wedge f_6 \\ (s\varphi_1\wedge f_1\wedge f_2\otimes s^* + t\varphi_1\wedge f_1\wedge f_2\otimes t^* + \\ s\varphi_0\wedge f_3\wedge f_4\otimes s^* + t\varphi_0\wedge f_3\wedge f_4\otimes t^*) \end{pmatrix}$$

with $f_1,..,f_6\in H^0F$, $g,g_1,g_2\in H^0G$, $g_3^{(2)}\in H^0S_2G$ and $G=\langle s,t\rangle$, $H^0DG^*=\langle s^*,t^*\rangle$. Trivially α is surjective as every element in $\wedge^2F(-4)$ is a sum of elements of type $f\wedge\varphi_2$, $g\varphi_1\wedge f$ and $g\varphi_0\wedge f$ with $f\in F$ and $g\in G$. Thus, if C is given by a matrix ψ as above, then we get $\beta_{24}=0$. A calculation of $\ker\beta$ gives (cf. Appendix 6.3)

$$\ker\beta = \left\langle \begin{pmatrix} 0 \\ 0 \\ t\otimes(t\varphi_0\wedge t\varphi_1) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ s\otimes(s\varphi_0\wedge s\varphi_1) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ s\otimes(t\varphi_0\wedge t\varphi_1) + t\otimes(t\varphi_0\wedge s\varphi_1 + s\varphi_0\wedge t\varphi_1) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ s\otimes(t\varphi_0\wedge s\varphi_1 + s\varphi_0\wedge t\varphi_1) + t\otimes(s\varphi_0\wedge s\varphi_1) \end{pmatrix} \right\rangle$$

Therefore, the Betti table for C is given as follows

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	8	-	-	-
2	-	-	-	8	70	64	21	-
3	-	-	-	-	-	-	-	1

■

It remains to determine the Betti table for C if C lies on a scroll of type $S(3,1,1,0)$, i.e. we must have $h^0(C,\mathcal{O}(K_C-3D))=1$ for the unique $g_5^1=|D|$ on C :

Theorem 4.6.5 *Let C be an irreducible, smooth, canonical curve of genus 9 with* $\mathrm{Cliff}(C)=3$*, that admits no g_7^2. If C has exactly one g_5^1 with multiplicitiy 3 then it is given by the Pfaffians of a matrix ψ*

$$\psi \sim \begin{pmatrix} 0 & H+R & H & H & H-R \\ & 0 & H & H & H-R \\ & & 0 & H-R & H-2R \\ & & & 0 & H-2R \\ & & & & 0 \end{pmatrix}$$

on a scroll of type $S(3,1,1,0)$. The minimal free resolution of $\mathcal{O}_C$ as $\mathcal{O}_{\mathbb{P}^8}-$module has the following Betti diagram:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	12	-	-	-
2	-	-	-	12	70	64	21	-
3	-	-	-	-	-	-	-	1

Proof. Let X be the scroll X of type $S(3,1,1,0)$ constructed from the unique $g_5^1=|D|$ and $\mathbb{P}(\mathcal{E})$ the corresponding $\mathbb{P}^3$-bundle, then C is given by the Pfaffians of a matrix ψ as above. By suitable row and column operations, ψ gets the following form:

$$\psi \sim \begin{pmatrix} 0 & H+R & H & H & H-R \\ & 0 & H & H & \varphi' \\ & & 0 & \varphi & t\varphi_0 \\ & & & 0 & s\varphi_0 \\ & & & & 0 \end{pmatrix}$$

with global sections $\varphi_0 \in H^0(\mathbb{P}(\mathcal{E}),\mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-3R))$ and $\varphi,\varphi' \in H^0(\mathbb{P}(\mathcal{E}),\mathcal{O}_{\mathbb{P}(\mathcal{E})}(H-R))$. As in the proof of Theorem 4.6.4 *varphi* cannot be made zreo by suitable row and column operations. The calculation of the minimal free resolution of $\mathcal{O}_C$ by a mapping cone construction can be done in an analogous way as in the case where X is of type $S(2,2,1,0)$. The only difference in this case is, that the non minimal maps are given by

$$\alpha:\ F(-4)^{\oplus 2} \quad \oplus \quad F\otimes G(-4)^{\oplus 2} \quad \oplus \quad S_2G\otimes F(-4) \quad \to \wedge^2 F(-4)$$

$$\begin{pmatrix} f_1 \\ f_2 \\ g_1\otimes f_3 \\ g_2\otimes f_4 \\ g_3^{(2)}\otimes f_5 \end{pmatrix} \to f_2\wedge\varphi' + g_1 t\varphi_0\wedge f_3 + g_2 s\varphi_0 \wedge f_4$$

its dual

$$\alpha^* : \wedge^3 F(-6) \to \wedge^4 F \otimes DG^*(-6)^{\oplus 2} \quad \oplus \quad \wedge^4 F \otimes D_2 G^*(-6) \quad \oplus \quad \wedge^4 F(-6)^{\oplus 2}$$

and

$$\beta : \wedge^2 F \otimes G(-5) \quad \oplus \quad \wedge^2 F(-5)^{\oplus 2} \quad \to \quad \wedge^3 F(-5)^{\oplus 2} \quad \oplus \quad \wedge^3 F \otimes DG^*(-5)$$

$$\begin{pmatrix} f_1 \wedge f_2 \\ f_3 \wedge f_4 \\ g \otimes f_5 \wedge f_6 \end{pmatrix} \to \begin{pmatrix} -\varphi \wedge f_3 \wedge f_4 - gt\varphi_0 \wedge f_5 \wedge f_6 \\ \varphi \wedge f_1 \wedge f_2 - gs\varphi_0 \wedge f_5 \wedge f_6 \\ (st\varphi_0 \wedge f_1 \wedge f_2 \otimes s^* + t^2\varphi_0 \wedge f_1 \wedge f_2 \otimes t^* + \\ s^2\varphi_0 \wedge f_3 \wedge f_4 \otimes s^* + st\varphi_0 \wedge f_3 \wedge f_4 \otimes t^*) \end{pmatrix}$$

with $f_1, .., f_6 \in H^0 F$, $g, g_1, g_2 \in H^0 G$, $g_3^{(2)} \in H^0 S_2 G$ and $G = \langle s, t \rangle$, $H^0 DG^* = \langle s^*, t^* \rangle$. φ' cannot be obtained from a linear combination of $s^2\varphi_0, st\varphi_0, t^2\varphi_0$, as then φ_0 would be a factor of one of the Pfaffians, thus it is a direct consequence that α is surjective. For β a calculation shows (cf. Appendix 6.4), that

$$\ker\beta = \left\langle \begin{array}{c} \begin{pmatrix} st\varphi_0 \wedge t^2\varphi_0 \\ 0 \\ t \otimes (t^2\varphi_0 \wedge \varphi) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ t \otimes (st\varphi_0 \wedge t^2\varphi_0) \end{pmatrix}, \begin{pmatrix} s^2\varphi_0 \wedge t^2\varphi_0 \\ st\varphi_0 \wedge t^2\varphi_0 \\ s \otimes (t^2\varphi_0 \wedge \varphi) + 2t \otimes (st\varphi_0 \wedge \varphi) \end{pmatrix}, \\ \begin{pmatrix} 0 \\ 0 \\ s \otimes (st\varphi_0 \wedge t^2\varphi_0) + t \otimes (s^2\varphi_0 \wedge t^2\varphi_0) \end{pmatrix}, \begin{pmatrix} 0 \\ s^2\varphi_0 \wedge st\varphi_0 \\ s \otimes (s^2\varphi_0 \wedge \varphi) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ s \otimes (s^2\varphi_0 \wedge st\varphi_0) \end{pmatrix}, \\ \begin{pmatrix} s^2\varphi_0 \wedge st\varphi_0 \\ s^2\varphi_0 \wedge t^2\varphi_0 \\ 2s \otimes (st\varphi_0 \wedge \varphi) + t \otimes (s^2\varphi_0 \wedge \varphi) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ s \otimes (s^2\varphi_0 \wedge t^2\varphi_0) + t \otimes (s^2\varphi_0 \wedge st\varphi_0) \end{pmatrix} \end{array} \right\rangle$$

Therefore the Betti table for C is given as follows:

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	21	64	70	12	-	-	-
2	-	-	-	12	70	64	21	-
3	-	-	-	-	-	-	-	1

■

4.6.2 Deformation

In the last two sections we have determined the Betti table for a canonical curve $C \subset \mathbb{P}^8$ of genus 9 that has exactly $k = 1, 2$ or 3 special linear series of type g^1_5 (counted with multiplicities):

$$\begin{array}{cccccccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 21 & 64 & 70 & 4k & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 4k & 70 & 64 & 21 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

Our definition of the multiplicity of a g_5^1 was rather technically, so it remains to give a geometric explanation for this procedure. In the situation where C has three different ordinary g_5^1, we have seen that in some cases it is possible for the third one to become equal to one of the two others. In this sitiuation its multiplicity rises to 2. We want to show that in any case where the curve $C_0 := C$ has a $g_5^1 = |D|$, D an effective divisor of degree 5 on C, with higher multiplicity, there exists a local one parameter family $(C_\lambda)_{\lambda\in\mathbb{A}^1}$ of curves in $\mathcal{M}_9$, such that C_λ has the corresponding number of different g_5^1 with ordinary multiplicity for $\lambda \neq 0$.

According to Theorem 4.6.1 we can assume that C has a space model C' given as the image of C under the map defined by the complete linear series $|2D|$. Then C' is contained in the cone $Y \subset \mathbb{P}^3$ over the irreducible conic $x_0x_2 = x_1^2$ in $\mathbb{P}^2$ and has exactly 7 (possibly infinitely near) double points p_i, $i = 1, ..., 7$, as only singularities. C' does not pass through the vertex of Y. The corresponding $\mathbb{P}^1-$bundle $P_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$ gives a resolution of the singularity of Y. Let A denote the class of a hyperplane divisor and B that of a ruling on P_2. The curve $C' \subset P_2$ is given by an element $\tilde{F}_0 \in |5A - 2p_1 - ... - 2p_7|$ on P_2. We denote Q_0 the defining equation of Y in $\mathbb{P}^3$ and H the hyperplane divisor in $\mathbb{P}^3$. Then A is the pullback of $H|_Y$ and thus $C' \subset \mathbb{P}^3$ is the complete intersection of a quintic $\Gamma = V(F_0)$ given by an $F_0 \in |5H|$ and the cone $Y = V(Q_0)$. Blowing up P_2 in the points $p_1, ..., p_7$, we can assume that C is a divisor of type $5A - \sum_{i=1}^{7} 2E_i$ on $S := \tilde{P}_2(p_1, ..., p_7)$ where A and B denotes by abuse of notation also the pullback of the class of a hyperplane divisor and a ruling on P_2 respectively.

We give an outline how to construct the local one paramter family with the desired properties:

1. In the first step we separate infinitely near double points, i.e. we give a local one paramter family of curves on the cone Y that have the same number of $g_5^1{}'s$ (respecting their multiplicities) as C', but exactly 7 *distinct* double points. This shows that we can assume for C' to have exactly 7 distinct double points.

2. Then we show that we can choose F_0 in such a way that $\Gamma = V(F_0)$ has multiplicity 2 in each of the points p_i, i.e. the singularities of C' in the points p_i come from singularities of Γ in p_i.

3. In the last step we get a deformation $C'_\lambda := \Gamma \cap Y_\lambda$ of C' by fixing the quintic Γ and deforming Y into a smooth quadric $Y_\lambda \cong \mathbb{P}^1 \times \mathbb{P}^1$ that passes through the points p_i. Then C'_λ becomes a divisor of class $(5,5)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with double points p_i. This separates the infinitesimal near $g_5^1{}'s$ of C' that are cut out by the class of a ruling on the cone Y.

We start with the following lemma:

Lemma 4.6.6 *Let $C' \in |5A|$ be an irreducible curve on P_2 that has exactly 7 double points $p_1, ..., p_7$ as only singularities. Further assume that p_7 lies infinitely near to p_6. Let $\Sigma \subset P_2$ be a rational curve that passes through p_7 and*

$(q_\lambda)_{\lambda \in \mathbb{A}^1}$ *be a local parametrisation of* Σ *with* $q_0 = p_7$. *Then there exists a local one parameter family* $(C'_\lambda)_{\lambda \in \mathbb{A}^1}$ *of curves in the divisor class* $5A$ *that have double points in* $p_1, ..., p_6, q_\lambda$ *as only singularities.*

Proof. We blow up P_2 in the points $p_1, ..., p_6, q_\lambda$:

$$\sigma_\lambda : S_\lambda = \tilde{P}_2(p_1, ..., p_{6}, q_\lambda) \to S = \tilde{P}_2(p_1, ..., p_6) \to P_2$$

E_i denotes the total transforms of p_i and $E \sim A - 2B$ the exceptional divisor on P_2. We consider the complete linear system $|L| = \left|5A - \sum_{i=1}^{6} 2E_i\right|$ of strict transforms of curves passing through the points $p_1, ..., p_6$ with multiplicity 2. Due to Corollary 2.3.1 we know that $|L|$ is very ample on $S \backslash (\bigcup_{i=1}^{5} E_i \cup E)$ and therefore, passing through an arbitrary point q_λ with multiplicity two gives three independent conditions for $|L|$. Thus the dimension $d_\lambda :=$ $\dim \left|5A - \sum_{i=1}^{6} 2E_i - 2E_{q_\lambda}\right|$ for the complete linear system of curves passing through $p_1, ..., p_6, q_\lambda$ with multiplicity 2 is independent from $\lambda \in \mathbb{A}^1$. Let $\tilde{S}$ be the blowup of $\mathbb{A}^1 \times S$ along the subscheme $\left\{(\lambda, q_\lambda) : \lambda \in \mathbb{A}^1\right\} \subset \mathbb{A}^1 \times S$, then we obtain the following picture:

$$\begin{array}{ccccc} S_\lambda & \subset & \tilde{S} & \to & \mathbb{A}_1 \times S \\ & \searrow & \downarrow \pi & \swarrow & \\ & & \mathbb{A}_1 & & \end{array}$$

The fibres of $\tilde{S}$ are given by the blowups S_λ with exceptional divisor E_λ. For the sheaf $\mathcal{F} := \mathcal{O}_{\tilde{S}}(5A - \sum_{i=1}^{6} 2E_i - 2E_\lambda)$ on $\tilde{S}$, we know that $\dim_{\mathbb{k}(\lambda)} H^0(S_\lambda, \mathcal{F}_\lambda)$ is constant on $\mathbb{A}^1$. It follows that $\pi_* \mathcal{F}$ is a vector bundle of dimension $d = d_\lambda$ on $\mathbb{A}^1$. Then we can choose a global section of $\pi_* \mathcal{F}$, that lifts to a one parameter family of elements $C'_\lambda \sim |5A - \sum_{i=1}^{6} 2E_i - 2q_\lambda|$ on S. ■

According to the above approach we can successively separate infinitely near double points. We remark that in the situation where all double points p_i are lying on a rational curve Σ (especially in the case where we have three infinitesimal near $g_5^1{}'s$) this can be done in such a way that all double points which have been separated lie again on Σ. This shows **1.**

To handle step **2.** we have to show that we can choose F_0 in such a way that p_i is a point of multiplicity 2 of $\Gamma = V(F_0)$. As p_i is a double point of the complete intersection $C' = \Gamma \cap Y$, we must have

$$F_0(p_i) = Q_0(p_i) = 0 \text{ and } dF_0(p_i) = \lambda_i dQ_0(p_i)$$

with elements $\lambda_i \in \mathbb{k}$. Now we replace F_0 by $F_G := F_0 + G \cdot Q_0 \in I_{C'}$ where G is a homogeneous, cubic polynomial. If it is possible to choose G in such a way that $F_G(p_i) = 0$ and $dF_G(p_i) = dF_0(p_i) + dG(p_i) \cdot Q_0(p_i) + dQ_0(p_i) \cdot G(p_i) = dF_0(p_i) + dQ_0(p_i) \cdot G(p_i) = 0$, then F_G has the desired property. This is fulfilled if $G(p_i) = -\lambda_i$ for $i = 1, ..., 7$. If three of the double points are colinear, projection from one of them would lead to a plane model for C of degree 8 that has a point with multiplicity of at least four, hence we would get a g_4^1 or a special linear series with even lower Clifford index, so that we can ommit this case. Then we can find hyperplanes, such that each of them contains exactly two of the points p_i and no two of them intersect in one of the points p_i. For the product G_j of their defining equations we obtain $G_j \in \Bbbk[x_0,...,x_3]_3$, $j = 1,...,7$, with $G_j(p_i) = \delta_{ij} \cdot G_j(p_j)$ where $G_j(p_j) \neq 0$. Then for $G := \sum_{j=1}^{7} - \frac{\lambda_j}{G_j(p_j)} G_j$ we get $G(p_i) := \sum_{j=1}^{7} - \frac{\lambda_j}{G_j(p_j)} G_j(p_i) = -\lambda_i$.

According to above considerations we can assume that C' is given by the complete intersection of the cone $Y = V(Q_0)$ and a hypersurface $\Gamma = V(F_0)$ of degree 5 that passes through the points p_i with multiplicity 2. As mentioned in **3.** we want to deform $C_0 = C$ by fixing Γ as well as the points p_i and varying the quadratic form Q_λ in such a way that Q_λ defines a smooth, irreducible hypersurface of degree 2 in $\mathbb{P}^3$ for $\lambda \neq 0$, thus $Y = V(Q_\lambda) \cong \mathbb{P}^1 \times \mathbb{P}^1$. The only condition on $Y_\lambda := V(Q_\lambda)$ is that it has to contain the points p_i. From $\dim \Bbbk[x_0,...,x_3]_2 = \binom{3+2}{2} = 10$, it follows the existence of three linear independent quadratic forms $Q', Q'', Q''' \in \Bbbk[x_0,...,x_3]_2$ that vanish at all points p_i. Further we can assume that $Q''' = Q_0$. We denote $Y_1 := V(Q')$ and $Y_2 := V(Q'')$. On the blowup $S := \tilde{P}_2(p_1,...,p_7)$ they give two effective divisors Γ_1 and Γ_2 of type $2A - \sum_{i=1}^{7} E_i$ and the pencil spanned by these two divisors cut out the linear system $|K_C - 2D| = |(3A - 2B - \sum_{i=1}^{7} E_i)|_C| = |(2A - \sum_{i=1}^{7} E_i)_C|$ as $E|_C \sim (A - 2B)|_C \sim 0$. Then it follows

Lemma 4.6.7 *The complete linear system $|2A - \sum_{i=1}^{7} E_i|$ has a rational curve $\Sigma \sim A + B - \sum_{i=1}^{7} E_i$ as base locus if and only if $m_{|D|} = 3$. For $m_{|D|} = 2$ this system has exactly one base point $q \in C$ if C admits a further g_5^1 otherwise its base locus is a point $q \in S \subset C$ or a divisor of type $A + B - \sum_{i \in \Delta} E_i$ with $|\Delta| = 5$.*

Proof. For $m_{|D|} = 3$ we get $1 = h^0(C, K_C - 3D) = h^0(C, \mathcal{O}_C((3A - 3B)|_C - \sum_{i=1}^{7} E_i|_C)) = h^0(C, \mathcal{O}_C((2A - B)|_C - \sum_{i=1}^{7} E_i|_C))$ since $(A - 2B)|_C \sim 0$. Hence, it follows the existence of an element $\gamma \in H^0(C, \mathcal{O}_C((2A - B)|_C - \sum_{i=1}^{7} E_i|_C))$. Denoting the global generators of $H^0(S, \mathcal{O}_S(B))$ by s and t, we can assume that the rulings R_s, $R_t \sim R$ given by s and t do not pass through any point p_i. The complete linear system

$$H^0(C, \mathcal{O}_C(2A|_C - \textstyle\sum_{i=1}^{7} E_i|_C)) = \langle s|_C \cdot \gamma, t|_C \cdot \gamma \rangle$$

is cut out by a pencil of effective divisors of type $2A - \sum_{i=1}^{7} E_i$, thus we have $q_1|_C = s|_C \cdot \gamma$ and $q_2|_C = t|_C \cdot \gamma$ with $\langle q_1, q_2 \rangle = H^0(S, \mathcal{O}_S(2A - \sum_{i=1}^{7} E_i))$. Since the rulings R_s and R_t cut out the same divisor on C as $\Gamma_1 = (q_1)$ and $\Gamma_2 = (q_2)$ respectively they must have 5 in common. We have $R_s.\Gamma_1 = R_t.\Gamma_2 = 2$, that is only possible if R_s and R_t are components of Γ_1 and Γ_2 respectively. Hence we get a common component $\Sigma \sim 2A - B - \sum_{i=1}^{7} E_i$ of Γ_1 and Γ_2 on S. The intersection product of Σ with the exceptional divisor $E \sim A - 2B$ is negative, hence E is a component of Σ. Substracting E once we get a rational curve $\Sigma \sim A + B - \sum_{i=1}^{7} E_i$. For the other direction let us assume the existence of such an effective divisor Σ on S. Then $h^0(C, \mathcal{O}_C(K_C - 3D)) = h^0(C, \mathcal{O}_C((3A - 3B)|_C - \sum_{i=1}^{7} E_i|_C)) = h^0(C, \mathcal{O}_C((A + B)|_C - \sum_{i=1}^{7} E_i|_C)) = 1$ from which $m_{|D|} = 3$ follows.

It remains to consider the case $m_{|D|} = 2$: If two divisors in $|2A - \sum_{i=1}^{7} E_i|$ have no common component they intersect in exactly $(2A - \sum_{i=1}^{7} E_i)^2 = 8 - 7 = 1$ point $q \in S$. In the case where C admits no further g^1_5 the point q cannot lie on C as otherwise we would get a $g^1_5 = |(2A - \sum_{i=1}^{7} E_i)|_C - q|$ that is different from $|D|$ because of $0 = h^0(C, \mathcal{O}_C(K_C - 3D)) = h^0(C, \mathcal{O}_C((2A - B)|_C - \sum_{i=1}^{7} E_i|_C))$. If C has a further g^1_5 then according to our results in Theorem 4.5.6 this linear series is given by $|K_C - 2D - q'| = |(2A - B)|_C - q'|$ with a $q' \in C$, but then $q' = q$.

We consider the situation where $|2A - \sum_{i=1}^{7} E_i|$ has a common divisor $\Sigma \sim aA + bB - \sum_{i=1}^{7} \lambda_i E_i$, $a, b, \lambda_i \in \mathbb{Z}$, in its base locus. Let $\Gamma_1, \Gamma_2 \in |2A - \sum_{i=1}^{7} E_i|$ be two different divisors. Then we can assume that Σ is maximal in the sense that $\Gamma_1 - \Sigma$ and $\Gamma_2 - \Sigma$ contain no further component, i.e.

(I) $0 \leq (\Gamma_1 - \Sigma).(\Gamma_2 - \Sigma) = 2(2 - a)(2 - a - b) - \sum_{i=1}^{7} (1 - \lambda_i)^2$
$\Leftrightarrow \sum_{i=1}^{7} 2\lambda_i \geq 7 - 2(2 - a)(2 - a - b) + \sum_{i=1}^{7} \lambda_i^2$

Moreover $|2A - \sum_{i=1}^{7} E_i|$ cuts out on C a linear system of dimension 2 and degree

(II) $d = C.((2-a)A - bB - \sum_{i=1}^{7} (1 - \lambda_i)E_i) = 10(2-a) - 5b - \sum_{i=1}^{7} 2(1 - \lambda_i) \geq 5$
$\Leftrightarrow \Sigma.C = 10a + 5b - \sum_{i=1}^{7} 2\lambda_i \leq 1$

and as C is irreducible

$$0 \leq \Sigma.C = 10a + 5b - \sum_{i=1}^{7} 2\lambda_i \leq 1$$

If some of the λ_i are negative then one of the exceptional divisors E_i is contained in the base locus of our complete linear series, hence we must have $\Sigma.C \geq E_i.C = 2$, a contradiction. From (I) it follows that at least $7 - 2(2 - a)(2 - a - b)$ coefficiants λ_i have to be equal to one. Therefore we get

$$0 \leq 10a + 5b - \sum_{i=1}^{7} 2\lambda_i \leq 10a + 5b - 14 + 4(2 - a)(2 - a - b)$$

We distinguish the cases $a = 0, 1$ and 2 :

$a = 0$: Here we have $2 - 3b \geq 0$, hence $b = 0$ and therefore $\Sigma \sim 0$.

$a = 1$: In this case it follows that $b \geq 0$ and from (I) that $b \leq 1$. Then either $\Sigma \sim A - \sum_{i\in\Delta} E_i$ for $\Delta \subset \{1,...,7\}$ and $|\Delta| = 5$ or $\Sigma \sim A + B - \sum_{i=1}^{7} E_i$. The last possibility can be excluded since then it follows that $m_{|D|} = 3$.

$a = 2$: From (I) we see that $\lambda_i = 1$ for all $i = 1,...,7$ and from (II): $0 \leq \Sigma.C = 6 + 5b \leq 1$, hence $b = -1$. Then we get the existence of an effective divisor Σ of type $2A - B - \sum_{i=1}^{7} E_i$ and thus $m_{|D|} = 3$ as above. In the case where $\Sigma \sim A - \sum_{i\in\Delta} E_i$, we assume $\Sigma \sim A - \sum_{i=1}^{5} E_i$, the complete linear system cut out on C by the pencil of divisors spanned by $\Gamma_1 - \Sigma$ and $\Gamma_2 - \Sigma \sim A - E_6 - E_7$ cannot be of type g_5^1 as the divisors $\Gamma_1 - \Sigma$ and $\Gamma_2 - \Sigma$ do not intersect and $(\Gamma_i - \Sigma).C = 6$ for $i = 1, 2$. If C admits a further g_5^1 then it is cut out by the pencil of divisors spanned by Γ_1 and Γ_2, hence in this situation there exists no common divisor in the base locus of $|2A - \sum_{i=1}^{7} E_i|$. In this case, i.e. if $\Sigma \sim 0$, Γ_1 and Γ_2 intersect in exactly one point $q \in S$ and we have $q \in C$ if and only if C has a further g_5^1. ■

We have collected enough information to formulate our first result:

Theorem 4.6.8 *Let C be an irreducible, smooth, canonical curve of genus 9 with* $\mathrm{Cliff}(C) = 3$ *that admits no g_7^2. If C admits exactly one $g_5^1 = |D|$, having multiplicity $m_{|D|} = 2$, then there exists a local one parameter family $(C_\lambda)_{\lambda\in\mathbb{A}^1}$ with $C_0 = C$ and $C_\lambda \in \mathcal{M}_9$ having exactly two different $g_5^1{}'s$ of ordinary multiplicity.*

Proof. We consider the image C' on the cone $Y \subset \mathbb{P}^3$ of C under the map obtained from the complete linear system $|2D|$. We have already shown that we can assume for C' to have exactly 7 distinct double points $p_1, ..., p_7$ as only singularities. Further C' is given as complete intersection of the singular quadric $Y = V(Q_0)$ and a quintic Γ that passes through the points p_i with multiplicity 2. From the above lemma we know that the quadrics $Y_1 = V(Q')$ and $Y_2 = V(Q'')$ either intersect in a one dimensional component Σ' or in the points p_i and one further point which we denote by q. In the first case we have shown that on the blowup S the strict transform Σ of Σ' is a divisor of type $A - \sum_{i\in\Delta} E_i$ with $|\Delta| = 5$. This situation occurs if five double points are lying on a hyperplane. In the second case C does not pass through p. Then setting $Q_\lambda := Q_0 + \lambda Q'$ and accordingly $Y_\lambda = V(Q_\lambda)$, this gives us an adequate one parameter family of smooth quadrics and $C_\lambda := Y_\lambda \cap \Gamma$ the corresponding deformation of $C' = C_0$: Y_λ is an irreducible, smooth quadric in $\mathbb{P}^3$ which can be considered as an image of $\mathbb{P}^1 \times \mathbb{P}^1$ via a Segre embedding in $\mathbb{P}^3$ (cf. [Ha92] page 285). The quintic Γ cuts out a divisor of class $(5, 5)$ on $Y_\lambda \cong \mathbb{P}^1 \times \mathbb{P}^1$, hence C_λ is a divisor of class $(5, 5)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ having double points p_i as only singularities. Now C_λ has two distinct special linear series of type g_5^1 obtained from projection from each factor of $\mathbb{P}^1 \times \mathbb{P}^1$.

From Theorem 4.5.6 we already know that C_λ has a third g_5^1 if and only if it is cut out by the pencil of quadrics passing through the points p_i. In the case where Y, Y_1 and Y_2 intersect in the common component Σ' the additional g_5^1 has to be cut out on $C_\lambda \subset \mathbb{P}^1 \times \mathbb{P}^1$ by the pencil of divisors $|(1,1) - p_6 - p_7|$ which is not possible as this systems cuts out a g^1_6 on C_λ and two divisors of this linear system do exactly intersect in the points p_1 and p_2. In the situation where Y, Y_1 and Y_2 intersect in exactly the points $p_1, ..., p_7$ and q, we must have $q \in C_\lambda$ if C_λ has a third g^1_5. Thus it follows that C_λ has exactly two different $g^1_5{}'s$. ■

Now let us consider the case where C has an additional $g^1_5 = |D^*|$, $D^* \nsim D$, $m_{|D^*|} = 1$. Then the three quadrics $Y = V(Q_0)$, $Y_1 := V(Q')$ and $Y_2 := V(Q'')$ intersect in the points p_i and one further point q, the base point of the complete linear system $|2H - p_1 - ... - p_7|$. According to Theorem 4.5.7 we have $q \in C'$ and on the corresponding $\mathbb{P}^1$-bundle P_2, the linear system $|D^*|$ is cut out by the pencil of divisors $|2A - p_1 - ... - p_7|$. With $Q_\lambda := Q_0 + \lambda Q'$ and accordingly $Y_\lambda = V(Q_\lambda)$ we get an adequate one parameter family of quadrics $Y_\lambda \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $C_\lambda := Y_\lambda \cap \Gamma$ a divisor of class $(5,5)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ the corresponding local deformation of $C' = C_0 : C_\lambda$ has two $g^1_5{}'s$ from projection along each factor of $\mathbb{P}^1 \times \mathbb{P}^1$ and as $q \in C'$, a third g^1_5 is cut out by the pencil of divisors of class $(2,2)$ passing through the points $p_1, ..., p_7$. We conclude:

Theorem 4.6.9 *Let C be an irreducible, smooth, canonical curve of genus 9 with* $\mathrm{Cliff}(C) = 3$ *that admits no g^2_7. If C admits exactly one $g^1_5 = |D|$ with multiplicity $m_{|D|} = 2$ and a further one with multiplicity one then there exists a local one parameter family $(C_\lambda)_{\lambda \in \mathbb{A}^1}$ with $C_0 = C$ and $C_\lambda \in \mathcal{M}_9$ having exactly three different g^1_5 of ordinary multiplicity.*

It remains to examine the most special case where C has a g^1_5 of multiplicity 3. We will see that the same deformation as above leads to a local one parameter family $(C_\lambda)_{\lambda \in \mathbb{A}^1}$ of curves that have a g^1_5 with multiplicity two and a distinct one of ordinary multiplicity. Together with the theorem above, where we have dicussed this case, it follows:

Theorem 4.6.10 *Let C be an irreducible, smooth, canonical curve of genus 9 with* $\mathrm{Cliff}(C) = 3$, *that has exactly one linear system $|D|$ of type g^1_5 and admits no g^2_7. If $m_{|D|} = 3$, then there exists a local one parameter family $(C_\lambda)_{\lambda \in \mathbb{A}^1}$ with $C_0 = C$ and $C_\lambda \in \mathcal{M}_9$ having exactly three different, ordinary g^1_5.*

Proof. In Lemma 4.6.7 we have shown that C has a g^1_5 of multiplicity 3 if and only if the three quadrics Y, Y_1 and Y_2 intersect in a rational curve $\Sigma' \subset Y$, such that its strict transform Σ is a divisor of type $A + B - \sum_{i=1}^7 E_i$ on the blowup S. As above we define $Q_\lambda := Q_0 + \lambda Q'$ and accordingly $Y_\lambda = V(Q_\lambda)$. Then, on $Y_\lambda \cong \mathbb{P}^1 \times \mathbb{P}^1$ the curve Σ' is a divisor of type $(2,1)$ that passes through the points $p_1, ..., p_7$. It follows from Theorem 4.5.7 that $C_\lambda := Y_\lambda \cap \Gamma$, a divisor of class $(5,5)$ on Y_λ, has exactly two different $g^1_5{}'s$ given by the divisors $D \sim (1,0)|_{C_\lambda}$ and $D^* \sim (0,1)|_{C_\lambda}$ with $m_{|D|} = 1$ and $m_{|D^*|} = 2$. Therefore C_λ gives an adequate one parameter family with the desired properties. ■

5

Summary

The result of this thesis is a complete description of irreducible, smooth, canonical curves of genus 9 concerning their Betti tables and the corresponding Clifford index of C. For $\mathrm{Cliff}(C) \leq 2$ a collection of all Betti tables that occur was already determined in [S86]. In the case for a hyperelliptic or a tetragonal curve there exists a unique Betti table.
For tetragonal curves the Betti table depends on the existence of a further special linear system of Clifford index 2 or 3. Here we can distinguish three cases. The most interesting situation is that of a pentagonal curve: From Theorem 0.0.2 we know that for odd genus $g = 2l - 1$ there exist extra syzygies if and only if C carries a pencil of degree l, i.e. especially for $g = 9$ this is exactly the case if C has a g^1_5. For curves C with Clifford index 3 there exist 4 different Betti tables that correspond to curves having a g^2_7 or k $g^1_5{}'s$ (counted with multiplicities) with $k = 1, 2$ or 3. We calculated the minimal free resolution for the homogenous coordinate ring S_C of $C \subset \mathbb{P}^8$ applying the structure theorem in codimension 3 to $C \subset X$ where X is the scroll constructed from a g^1_5. This calculation even shows that the given collection of Betti tables is correct for arbitrary fields $\Bbbk$ with $\mathrm{char}(\Bbbk) \neq 3$. For $\mathrm{char}(\Bbbk) = 3$ the following Betti tables are different:

general									$\exists\ g^1_5$									$\exists$ two g^1_5								
	0	1	2	3	4	5	6	7		0	1	2	3	4	5	6	7		0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-	0	1	-	-	-	-	-	-	-	0	1	-	-	-	-	-	-	-
1	-	21	64	70	4	-	-	-	1	-	21	64	70	6	-	-	-	1	-	21	64	70	10	-	-	-
2	-	-	-	4	70	64	21	-	2	-	-	-	6	70	64	21	-	2	-	-	-	10	70	64	21	-
3	-	-	-	-	-	-	-	1	3	-	-	-	-	-	-	-	1	3	-	-	-	-	-	-	-	1

In [S03], due to computational evidence, the author also gives conjectural Betti tables for canonical curve of genus $g = 10$ and $g = 11$. For $g = 10$ the methods of this thesis can be applied to check these Betti tables with the exception of that of a curve with Clifford index 4 since an analogue of Theorem 0.0.2 for even genus is not proven yet. In the case $g = 11$ the method to determine the minimal free resolution of S_C from a free resolution on a scroll fails for hexagonal curves as we do not have any structure theorem for varieties of codimension 4. The reader is encouraged to continue this work.

6 Appendix

The following calculations are done in Macaulay2:

1. Lemma 4.5.1: We denote by $f_1, ..., f_5$ the generators of $H^0(X, \mathcal{O}_X(H-R))$.

We have to distinguish four cases:

```
restart
kk=QQ
kk=ZZ/3
R=kk[f_1..f_5,f_6,SkewCommutative=>true]
Mpsi=matrix{{0,f_1,f_2,f_3},{-f_1,0,f_4,f_5},{-f_2,-f_4,0,f_6},{-f_3,-f_5,-f_6,0}}
```

For $\psi \sim A$ we have:

```
betti syz substitute(Mpsi,{f_1=>f_6})
```

$\psi \sim B$:

```
betti syz substitute(Mpsi,{f_6=>0})
```

$\psi \sim C$:

```
betti syz substitute(Mpsi,{f_1=>0,f_6=>0})
```

$\psi \sim D$

```
betti syz substitute(Mpsi,{f_1=>f_6,f_3=>f_4})
```

2. Theorem 4.5.2:

```
Momega=matrix{{f_1,f_2},{f_3,f_4}}
betti syz Momega
betti syz substitute(Momega,{f_2=>f_3})
```

3. Theorem 4.6.4: We denote $f_1 = s\varphi_0$, $f_2 = t\varphi_0$, $f_3 = s\varphi_1$, $f_4 = t\varphi_1$ and $f_5 = \varphi_2$:

```
restart
```

```
kk=QQ
kk=ZZ/3
R=kk[f_1..f_5,f_6,SkewCommutative=>true]
Dach2=mingens (ideal(f_1..f_5))^2
Dach3=mingens (ideal(f_1..f_5))^3
M1a=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M1b=f_5*Dach2
M1c=f_3*Dach2
M1d=f_4*Dach2
M1=diff(Dach3,transpose (M1a|M1b|M1c|M1d))
M2a=-f_5*Dach2
M2b=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M2c=f_1*Dach2
M2d=f_2*Dach2
M2=diff(Dach3,transpose (M2a|M2b|M2c|M2d))
M3a=-f_3*Dach2
M3b=-f_1*Dach2
M3c=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M3d=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M3=diff(Dach3,transpose (M3a|M3b|M3c|M3d))
M4a=-f_4*Dach2
M4b=-f_2*Dach2
M4c=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M4d=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M4=diff(Dach3,transpose (M4a|M4b|M4c|M4d))
betti ker (M1|M2|M3|M4)
```

4. Theorem 4.6.5: We denote $f_1 = s^2\varphi_0$, $f_2 = st\varphi_0$, $f_3 = t^2\varphi_0$, $f_4 = \varphi_1 = \varphi$ and $f_5 = \varphi_2$:

```
Dach2=mingens (ideal(f_1..f_5))^2
Dach3=mingens (ideal(f_1..f_5))^3
M1a=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M1b=f_4*Dach2
M1c=f_2*Dach2
M1d=f_3*Dach2
M1=diff(Dach3,transpose (M1a|M1b|M1c|M1d))
M2a=-f_4*Dach2
M2b=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M2c=f_1*Dach2
M2d=f_2*Dach2
M2=diff(Dach3,transpose (M2a|M2b|M2c|M2d))
M3a=-f_2*Dach2
M3b=-f_1*Dach2
M3c=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M3d=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
```

```
M3=diff(Dach3,transpose (M3a|M3b|M3c|M3d))
M4a=-f_3*Dach2
M4b=-f_2*Dach2
M4c=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M4d=substitute(matrix{{0,0,0,0,0,0,0,0,0,0}},R)
M4=diff(Dach3,transpose (M4a|M4b|M4c|M4d))
betti ker (M1|M2|M3|M4)
```

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