

Dissertation zur Erlangung des Grades des Doktors der Naturwissenschaften
der Naturwissenschaftlich-Technischen Fakultät I der
Universität des Saarlandes

Matrix Convex State Spaces of C^* - and W^* -algebras

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2004

Saarbrücken

Tag des Kolloquiums: 23. Februar 2005

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Einleitung

Sei \mathcal{A} eine unitale C^* -Algebra und $1 \in \mathcal{A}$ ihre Eins. Dann ist der reelle Unterraum \mathcal{A}_h der selbstadjungierten (oder hermiteschen) Elemente versehen mit dem Kegel der positiven Elemente von \mathcal{A} ein geordneter Vektorraum. Ferner gilt $-\|a\|1 \leq a \leq \|a\|1$ für alle $a \in \mathcal{A}_h$. Dies zeigt, daß (die algebraische) 1 eine sogenannte Ordnungseins ist, und daß die durch die Ordnungseins definierte Norm $\|a\|_e = \inf \{ r \geq 0 \mid -r1 \leq a \leq r1 \}$ mit der Norm von \mathcal{A} übereinstimmt. Mithin ist \mathcal{A}_h ein vollständiger Ordnungseinsraum, also insbesondere ein geordneter Banachraum. Sei \mathcal{B} eine weitere unitale C^* -Algebra. Kadison bemerkte, daß ein linearer Isomorphismus $\varphi: \mathcal{A}_h \rightarrow \mathcal{B}_h$ genau dann ein unitaler Ordnungsisomorphismus ist, wenn $\varphi(a^2) = \varphi(a)^2$ für alle $a \in \mathcal{A}_h$. Das heißt, es gibt eine Beziehung zwischen der Ordnungsstruktur auf \mathcal{A}_h und der Multiplikation auf \mathcal{A} , genauer dem von der assoziativen Multiplikation abgeleiteten Jordan Produkt $a \circ b = \frac{1}{2}(ab + ba)$ auf \mathcal{A}_h . Dies legt es nahe, zum besseren Verständnis von C^* -Algebren zunächst die Struktur der Ordnungseinsräume \mathcal{A}_h zu untersuchen. Der erste Schritt bestünde dann darin, diejenigen Ordnungseinsräume zu beschreiben, die als selbstadjungierte Teile von C^* -Algebren auftauchen können. Wegen der Korrespondenz zwischen Ordnungseinsräumen und deren Zustandsräumen ist dies äquivalent damit zu charakterisieren, welche kompakten und konvexen Mengen Zustandsräume von C^* -Algebren sind. Für kommutative (unitale) C^* -Algebren, die bekanntermaßen isomorph zum Raum $C(X)$ der stetigen Funktionen auf einem kompakten Hausdorffraum sind, sind dies genau die Bauer Simplexe, das heißt, die Choquet Simplexe, deren Extrempunktmenge abgeschlossen ist. Für beliebige C^* -Algebren ist die Situation weitaus komplizierter. Bei obigem Ordnungsansatz ist sofort klar, daß die Ordnungsstruktur der C^* -Algebra lediglich mit ihrer Jordanstruktur korrespondiert. Entsprechend wurden in [8] zunächst Zustandsräume von JB-Algebren (Jordan-Banach-Algebren) untersucht und abstrakt charakterisiert. Darauf aufbauend mußte in [7] ein neues Konzept, die sogenannte *Orientierung*, die nicht mit der Ordnungsstruktur zusammenhängt, eingeführt werden, um beschreiben zu können, welche Zustandsräume von JB-Algebren Zustandsräume von C^* -Algebren sind. Insgesamt wurde damit also beantwortet, welche kompakten und konvexen Mengen Zustandsräume von C^* -Algebren sind, nämlich diejenigen Zustandsräume von JB-Algebren, die orientierbar sind. Dieses an sich schöne Ergebnis hat bisher, obschon es bis in die Gegenwart hinein Interesse aus der Quantenphysik an den Arbeiten von Alfsen und Shultz gibt (vgl. z. b. [41, 42] und [43]) leider keine allzugroße mathematische Beachtung gefunden. Ein Grund dafür mag sein, daß die Originalarbeiten recht schwer lesbar sind. So erschienen inzwischen zwei Bücher [5, 6], in denen Alfsen und Shultz das Konzept der Orientierung und die darauf beruhende Charakterisierung von Zustandsräumen von C^* -Algebren und auch von normalen Zustandsräumen von W^* -Algebren ausführlich vorstellen. Andere Gründe für die geringe Beachtung könnten aber auch schlicht im Umfang der Arbeit liegen und darin, daß eine Charakterisierung mit Hilfe der Orientierung schlecht handhabbar scheint. Welche kompakten und konvexen Mengen sind denn orientierbar? Um diese Frage zu beantworten, müßte man zunächst wissen, daß jede von zwei Extrempunkten erzeugte Seite (abge-

sehen von den trivialen Fällen einpunktig oder die Verbindungsstrecke beider Punkte zu sein) affin-isomorph zur abgeschlossenen Einheitskugel des \mathbb{R}^3 ist. Zu diesen Seiten läßt sich dann eine von zwei möglichen Orientierungen wählen, abhängig von der Basiswahl des \mathbb{R}^3 . Alsdann müßte man zeigen, daß es eine Auswahl von Orientierungen zu all diesen Seiten gibt, die über eine gewisse Stetigkeitseigenschaft verfügt. Ein weiterer Nachteil ist, daß Orientierung in Abhängigkeit von der Existenz von Extrempunkten definiert ist. Daher ließ sich dieser Ansatz nicht ohne weiteres dazu benutzen, normale Zustandsräume von W^* -Algebren zu charakterisieren, siehe hierzu [34].

Meine These ist es, ausschließlich die Ordnungsstruktur, genauer die Matrixordnung, von C^* - und W^* -Algebren zu untersuchen. Hintergrund sind dabei folgende seit langem bekannte Feststellungen: Wenn \mathcal{A} und \mathcal{B} zwei C^* -Algebren sind, so sind auch $M_n(\mathcal{A}) = \mathcal{A} \otimes M_n$ und $M_n(\mathcal{B})$ C^* -Algebren für alle $n \in \mathbb{N}$, und zwar auf genau eine Weise, da M_n nuklear ist. Insbesondere sind $M_n(\mathcal{A})$ und $M_n(\mathcal{B})$ geordnete Vektorräume. Eine lineare Abbildung $f: \mathcal{A} \rightarrow \mathcal{B}$ heißt n -positiv, wenn ihre n -te Amplifikation $f^{(n)}([a_{ij}]) = [f(a_{ij})]$ eine positive Abbildung ist. Nun gilt, daß ein linearer Isomorphismus $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ genau dann ein $*$ -Isomorphismus ist, wenn er ein 2-bipositiver unitaler Ordnungsisomorphismus ist, vgl. [60, 24]. Dies bedeutet, daß die Multiplikation von C^* -Algebren – ohne Umweg über das Jordanprodukt – durch ihre Matrixordnung bestimmt ist. Ich werde beschreiben, welche Operatorsysteme C^* - bzw. W^* -Algebren sind, siehe Theorem 3.83 und Theorem 2.19. Wegen der Korrespondenz zwischen Operatorsystemen und deren matrix-konvexen Zustandsräumen ist dies äquivalent dazu, diejenigen kompakten und matrix-konvexen Mengen zu charakterisieren, die matrix-konvexe Zustandsräume von C^* -Algebren sind bzw. diejenigen matrix-konvexen Mengen, die normale matrix-konvexe Zustandsräume von W^* -Algebren sind. Das Voraussetzen einer Matrixordnung ist zwar eine stärkere Forderung als Ordnung plus Orientierung. Andererseits erklärt die Matrixordnung die Orientierung, und es ergibt sich ein einfacherer und durchsichtigerer Zugang. Am Ende liefern die hier vorgestellten Methoden sogar eine abstrakte Beschreibung der reinen Matrixzustände von C^* -Algebren, siehe Theorem 3.87. Ein solches Ergebnis wurde bisher nicht erreicht und folgt auch nicht aus [5, 6]. Es ist wohl möglich, eine Charakterisierung des reinen Zustandsraumes unter Verzicht von Matrixordnung zu erhalten (ähnlich der Arbeiten von Alfsen und Shultz über den ganzen Zustandsraum). Landsman schlägt hierzu in [43] (inspiriert von [56]) sogenannte „*uniform Poisson spaces with transition probability*“ vor, ohne aber ein volles Resultat zu liefern, siehe die auf Theorem 3.87 folgenden Bemerkungen.

Der Inhalt der Arbeit gliedert sich wie folgt: Im ersten Kapitel werden die matrix-geordneten Versionen von (approximativen) Ordnungseinsräumen und basis-normierten Räumen eingeführt. Dies sind die (approximativen) Operatorsysteme und die matrix-basis-normierten Räume. Es wird eine kurze Einführung in die Theorie matrix-konvexer Mengen aus [30] gegeben. Diese umfaßt die sogenannte matrix-affinen Abbildungen und die Korrespondenz zwischen Operatorsystemen und matrix-konvexen Mengen aus [62]. Während die Theorie der Operatorsysteme seit langem bekannt ist, vgl. [18], tauchen matrix-basis-normierte Räume als matrixgeordnete Operatorräume meines Wissens in der Literatur bisher nicht auf. Überhaupt hat die Theorie der Operatorräume, also die normierte Theorie, weit mehr Aufmerksamkeit erlangt, als die geordnete Theorie. Für die Anwendung auf C^* -Algebren ist jedoch die geordnete Theorie entscheidend. Daher wird die Dualitätstheorie von Ordnungseinsräumen und basis-normierten Räumen auf Operatorsysteme und matrix-basis-normierte Räume übertragen. Es folgt die Bemerkung, daß Präduale

von W^* -Algebren eine bis auf Isomorphie eindeutige Matrixbasismatrixnormstruktur besitzen. Wird der Prädual einer W^* -Algebra in diesem Sinne als matrix-geordneter Operatorraum verstanden, so ist die duale Matrixordnung die Matrixordnung der W^* -Algebra, die die komplette W^* -Struktur der Algebra bestimmt. (Wird der Prädual lediglich als Banachraum gesehen, so liefert er nur die Banachraumstruktur der W^* -Algebra.) Das Kapitel schließt mit der Einführung matrix-konvexer Seiten, die zur abstrakten Beschreibung von Zustandsräumen von C^* -Algebren im dritten Kapitel benötigt wird.

Im zweiten Kapitel werden diejenigen matrix-basis-normierten Räume charakterisiert, die Präduale von W^* -Algebren sind. Damit erhalten wir eine bijektive Korrespondenz zwischen W^* -Algebren und bestimmten matrix-basis-normierten Räumen, so daß sich die Theorie der W^* -Algebren mathematisch äquivalent als Theorie bestimmter matrix-basis-normierten Räume formulieren läßt. Dazu wird für ein vorgegebenes Operatorsystem dessen sogenannte Multiplialgebra direkt aus der Matrixordnung konstruiert. Diese Konstruktion wurde aus [54] übernommen und stellt im Vergleich zu den Arbeiten [6, 5] und auch zu [63, 64], wo eine Algebra über sogenannte P -Projektoren erzeugt wird, eine erhebliche Vereinfachung dar. Es wird sodann gezeigt, daß es zu einem dualen Paar, bestehend aus einem matrix-basis-normierten Raum und einem Operatorsystem, einen Hilbertraum H und eine gemeinsame Darstellung von dem Operatorsystem und dessen Multiplialgebra in $\mathcal{B}(H)$ gibt. In $\mathcal{B}(H)$ besteht die Multiplialgebra aus denjenigen Operatoren, die das Operatorsystem invariant lassen. Da das Operatorsystem die Eins von $\mathcal{B}(H)$ enthält, liegt die Multiplialgebra in dem Operatorsystem. Es wird inspiriert von [10] eine Seitenbedingung formuliert, die sicherstellt, daß die Multiplialgebra mit dem Operatorsystem übereinstimmt und folglich eine C^* -Algebra mit Prädual, also eine W^* -Algebra ist. Das Kapitel schließt mit einer ersten Beschreibung von matrix-konvexen Zustandsräumen von C^* -Algebren, die der Charakterisierung in [7, Cor. 8.6] in gewisser Weise ähnlich ist.

Das dritte Kapitel stellt den Hauptteil meiner Doktorarbeit dar. Es wird dort eine nicht-kommutative Version der Aussage, daß die Zustandsräume von kommutativen C^* -Algebren genau die Bauer Simplexe sind, bewiesen. Ferner werden (unitale) C^* -Algebren in Analogie zum kommutativen Fall als gleichmäßig stetige und equivariante Funktionen $\mathcal{C}_u^E(X)$ auf der Matrixmenge ihrer reinen Matrixzustände dargestellt. Dabei wird der Raum der Matrixzustände abstrakt als eine Art nicht-kommutativer topologischer Raum beschrieben, so daß sich im kommutativen Spezialfall gerade die kompakten Hausdorffräume ergeben. Um diese Ergebnisse zu erhalten, werden zunächst equivariante Matrixmengen eingeführt und deren Eigenschaften (aufbauend auf den Eigenschaften der reinen Matrixzustände einer C^* -Algebra) untersucht. Es werden dann equivariante Abbildungen auf equivarianten Matrixmengen eingeführt, das heißt solche Abbildungen die mit der (nicht-kommutativen) Matrixstruktur kompatibel sind. Für solche Abbildungen wird ein nicht-kommutatives Produkt erklärt, daß sich im kommutativen Spezialfall zum punktweisen Produkt von Funktionen vereinfacht. Es wird weiter gezeigt, daß der Raum $\mathcal{F}_b^E(X)$ der beschränkten equivarianten Funktionen auf einer Matrixmenge eine atomare W^* -Algebra ist, wobei sich die Matrixmenge X genau mit den normalen und reinen Matrixzuständen identifizieren läßt. (Die kommutative Version hiervon, daß die beschränkten Funktionen auf einer beliebigen Menge eine kommutative W^* -Algebra sind, so daß die Menge via Punktauswertung genau die normalen reinen Zustände sind, ist bekannt.) Alsdann werden die normalen Matrixzustände von atomaren W^* -Algebren abstrakt charakterisiert. Zwar haben atomare W^* -Algebren als Summe von Typ I Faktoren eine sehr einfache Struktur, so daß die Beschreibung ihrer normalen Matrixzustände nicht sehr

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spannend erscheint. Es wird sich jedoch zeigen, daß mit diesem Zwischenergebnis die Charakterisierung von Matrixzustandsräumen von C^* -Algebren als sogenannte matrix-konvexe (Bauer) Simplexe gelingt. Die Anlehnung an Bauer Simplexe rechtfertigt sich dadurch, daß Zustandsräume von C^* -Algebren tatsächlich eine leicht modifizierte Bauer Simplexeigenschaft besitzen. Zwar können nicht mehr alle gleichmäßig stetige Abbildungen auf den Extrempunkten stetig zu affinen Abbildungen auf dem Zustandsraum fortgesetzt werden, wohl aber eine Teilmenge dieser, nämlich die equivarianten Abbildungen, das heißt, genau diejenigen, die mit der Matrixstruktur der C^* -Algebra kompatibel sind. Man bemerke dabei, daß die Matrixstruktur kommutativer C^* -Algebren trivial ist, so daß alle (gleichmäßig) stetigen Abbildungen auf den reinen Zuständen mit dieser kompatibel sind. Diese Simplexeigenschaft von Zustandsräumen von C^* -Algebren ist zwar schon in anderer Formulierung bekannt, vgl. [3, 13, 31], reicht aber zur abstrakten Beschreibung der Matrixzustandsräume nicht aus. Die in meiner Doktorarbeit vorgestellten Methoden jedoch liefern neben einer Charakterisierung der Matrixzustandsräume in Theorem 3.83 auch eine Darstellung von (unitalen) C^* -Algebren als Raum $\mathcal{C}_u^E(X)$ von gleichmäßig und equivarianten Abbildungen auf dem reinen Matrixzustandsraum X versehen mit der w^* -Uniformität. Diese Darstellung ist zusammen mit einer Formel für das nicht-kommutative Produkt von Funktionen neu. Sie führte zur Frage, ob man auch direkt reine Matrixzustandsräume von C^* -Algebren abstrakt als gewisse nicht-kommutative topologische Räume beschreiben kann. Eine solche Beschreibung gelingt in Theorem 3.87, womit die Dissertation schließt.

Mein besonderer Dank gilt Herrn Prof. Gerd Wittstock. Ohne seine (schon seit meiner Diplomarbeit andauernde) geduldige Betreuung, die zahlreichen Diskussionen mit ihm und seine wertvollen Hinweise wäre diese Arbeit sicherlich nicht entstanden. So profitierte ich zum Beispiel – neben vielem anderem – im zweiten Kapitel von seiner gemeinsamen Arbeit mit L. Schmidt [54] und von einem unveröffentlichtem Preprint einer vereinfachten Version der Dissertation von K.-H. Werner [64].

Danken für die gemeinsame mathematische Zeit möchte ich auch den Teilnehmern der AG Operatorräume und allen Teilnehmern des Oberseminars Funktionalanalysis.

Preface

The title of my thesis resembles the title of [7]. In that paper Alfsen, Hanche-Olsen and Shultz describe state spaces of C^* -algebras—based upon their characterization of state spaces of Jordan algebras, cf. [8]—as compact convex sets that fulfill certain conditions. An essential feature of state spaces of C^* -algebras (and also W^* -algebras) that distinguishes them from state spaces of Jordan algebras is that they are orientable. More recently Alfsen and Shultz published the two books [5, 6], where they explain their work in detail, including the later characterization of normal state spaces of W^* -algebras that first appeared in [34].

In my dissertation I present a characterization of W^* - and C^* -algebras using purely the concept of orderings. The background of this idea already pursued in [63, 64] is the observation that C^* -algebras are completely determined by their matrix orderings. That is, if \mathcal{A} is a C^* -algebra, then $M_n(\mathcal{A}) = \mathcal{A} \otimes M_n$ is a C^* -algebra in a unique way for all $n \in \mathbb{N}$. In particular the matrix algebras $M_n(\mathcal{A})$ are ordered vector spaces, and these orderings determine the C^* -algebra. This follows from the long known fact that a unital complete order isomorphism between unital C^* -algebras must be a unital $*$ -isomorphism, cf. [60, 24]. Since each chapter of the dissertation contains an introduction and explanations of what will be done and why, I restrict myself here to give only a short description of the contents of each chapter:

In the first chapter I explain the basics about matrix ordered spaces, operator systems and their matrix state spaces, which are matrix convex sets. Compact matrix convex sets correspond to operator systems (that are the non-commutative versions of order unit spaces) in the same way as compact and convex sets correspond to order unit spaces. Then a matrix version of base norm spaces (that are the dual spaces of order unit spaces) is defined. After establishing a duality theory between (approximate) operator systems and matrix base norm spaces, the chapter ends with the introduction of matrix versions of faces, in particular of split faces, that will be useful in the third chapter. The second chapter is about characterizing the normal matrix state spaces of W^* -algebras. The main tool to achieve this goal is the so-called multiplier algebra of an operator system constructed directly using matrix orderings. The construction is borrowed from [54]. Another source of inspiration was [10] that, with the help of projective faces, led to a condition that ensures the presence of sufficiently many projections in the multiplier algebra. Finally, the main part of the dissertation is contained in the third chapter. I prove that matrix state spaces of C^* -algebras can be described abstractly as non-commutative (Bauer) simplexes. Moreover, I obtain characterization theorems for the normal matrix state spaces and the normal pure matrix state spaces of atomic W^* -algebras. The latter can be interpreted as non-commutative sets. Based on this observation the chapter ends with an abstract characterization of the pure matrix state spaces of C^* -algebras. Recall that commutative (unital) C^* -algebras correspond with compact Hausdorff space, which are their pure state spaces. In this sense the pure matrix state spaces of C^* -algebras can be interpreted as non-commutative topological spaces.

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1. Matrix Orderings

In classical analysis a partially ordered vector space is a real vector space with a distinguished cone. We do not like to give an account of the theory of ordered vector spaces. Instead, we are aiming at characterizing C^* -algebras by their order structure. Here the classical ordered vector spaces of interest are the real *order unit spaces* (or Kadison's function systems) and their duals, the *base norm spaces*. These ordered spaces don't carry only an order structure, but have also a norm that is related to their order. However, order unit spaces are applicable only to the self-adjoint parts of C^* -algebras. This is what makes things difficult. A C^* -algebra \mathcal{A} is a complex vector space with an involution and all matrix algebras $M_n(\mathcal{A})$ are ordered, too. So, what we need are complex vector spaces with an involution and a matrix order structure. These spaces should also be normed and the norm should be related to the ordering. The reader should keep in mind that forgetting matrices and taking only the self-adjoint part of the spaces, that we will introduce soon, these spaces are nothing but the classical order unit and base norm spaces. The additional matrix structure is required, though, to make them applicable to C^* -algebras and their duals. Matrix ordered spaces and operator systems were introduced in [18]. In this chapter we will recall their basic definitions. First we need some matrix conventions. Let V be a vector space. Our vector spaces and linear maps will be *complex* vector spaces and complex linear maps, unless stated otherwise. For $n \in \mathbb{N}$ we let $M_{n,m}(V)$ denote the vector space of n by m matrices $v = [v_{ij}]$ with $v_{ij} \in V$. We write v^{tr} for the transpose matrix $[v_{ji}] \in M_{m,n}(V)$. Notice the following abbreviations: $M_n(V) = M_{n,n}(V)$, $M_{n,m} = M_{n,m}(\mathbb{C})$ and $M_n = M_{n,n}$. We denote the unit of M_n as $\mathbb{1}_n$ and for $l < n$ we let $\mathbb{1}_{n,l} = \begin{pmatrix} \mathbb{1}_l \\ 0 \end{pmatrix} \in M_{n,l}$.

Given $v \in M_n(V)$, $w \in M_k(V)$, $\alpha \in M_{m,n}$ and $\beta \in M_{n,m}$ we have the *matrix product*

$$\alpha v \beta = \left(\sum_{l,k=1}^n \alpha_{il} v_{lk} \beta_{kj} \right) \in M_m(V) \quad (1.1)$$

and the *direct sum*

$$v \oplus w = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in M_{n+k}(V). \quad (1.2)$$

For $v_1, \dots, v_n \in V$ we let

$$\text{diag}(v_1, \dots, v_n) = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & v_n \end{pmatrix}.$$

Since $M_n(V) = V \otimes M_n$ we will write also $v \otimes \mathbb{1}_n$ for $\text{diag}(v, \dots, v) \in M_n(V)$.

Let V be a vector space with an involution, i.e., a conjugate linear map $*$: $V \rightarrow V$ such that $v = v^{**}$. We call such a V a $*$ -vector space. An element $v \in V$ is called self-adjoint

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or hermitian if $v = v^*$. We write V_h for the set of all hermitian elements of V . This is a real vector space and we have $v = \operatorname{Re} v + i \operatorname{Im} v$, where $\operatorname{Re} v = \frac{v+v^*}{2}$ and $\operatorname{Im} v = \frac{v-v^*}{2i}$. Thus $V = V_h + iV_h$ and, since $V_h \cap iV_h = \{0\}$, we get $x = \operatorname{Re} v$ and $y = \operatorname{Im} v$, whenever $v = x+iy$ for $x, y \in V_h$. Notice that for all $n \in \mathbb{N}$ an involution on V induces an involution on $M_n(V)$ in the usual way, that is, by letting $v^* = [v_{ji}^*]$ for $v = [v_{ij}] \in M_n(V)$. For a $*$ -vector space V we always give $M_n(V)$ this induced involution.

We say that a vector space V is (partially) ordered if V is a $*$ -vector space having a distinguished cone $V_+ \subset V_h$ (i.e., $V_+ + V_+ \subset V_+$ and $\mathbb{R}_+ V_+ \subset V_+$). The cone V_+ is called *generating*, if $V_h = V_+ - V_+$. In this case V_h is called *directed*. For $w - v \in V_+$ we write $v \leq w$ or $w \geq v$. The cone V_+ is called *proper* if $V_+ \cap -V_+ = \{0\}$. Now we can already state the definition of a matrix ordered vector space.

Definition 1.1. A complex vector space V is a *matrix ordered vector space* if V is a $*$ -vector space and $M_n(V)$ is partially ordered for all $n \in \mathbb{N}$ such that the distinguished cone V_+ is proper and generating, and such that $\alpha^* M_l(V)_+ \alpha \subset M_n(V)_+$ for all $\alpha \in M_{l,n}$ and $l, n \in \mathbb{N}$.

Apart from the matrix ordering, the spaces that we are going to study are also operator spaces. For the general theory of operator spaces we refer to [26]. Notice that we do not require operator spaces to be complete in the norm, i.e., to be Banach spaces, in general. If the norm of an operator space must be complete, we will write ‘complete operator space’ to stress it. For the notion of matrix ordered operator spaces we refer to [55], where we find the following definition:

Definition 1.2. A matrix ordered operator space V is an operator space that is a matrix ordered vector space in such a way that the involution on $M_n(V)$ is an isometry and the cone $M_n(V)_+$ is closed in the norm topology for all $n \in \mathbb{N}$.

Let $\varphi: V \rightarrow W$ be a linear map between vector spaces. We let $L(V, W)$ be the vector space of all linear maps from V to W . For $n \in \mathbb{N}$ we denote the n -th amplification of φ as $\varphi^{(n)}: M_n(V) \rightarrow M_n(W)$, where $\varphi^{(n)}(v) = [\varphi(v_{ij})]$ for $v = [v_{ij}] \in M_n(V)$. If V and W are $*$ -vector spaces, we define an involution on the vector space $L(V, W)$ by $\varphi^*(v) = \varphi(v^*)^*$. If V and W are ordered vector spaces, then the linear map $\varphi: V \rightarrow W$ is positive if $\varphi = \varphi^*$ and $\varphi(V_+) \subset W_+$. If V and W are matrix ordered and $n \in \mathbb{N}$, then φ is called n -positive, if $\varphi^{(n)}$ is positive. If $\varphi^{(n)}$ is positive for all $n \in \mathbb{N}$, then φ is called *completely positive*. We write $CP(V, W)$ for the set of all completely positive maps from V to W . Notice that we can define a matrix order on $L(V, W)$ by the identification $M_n(L(V, W))_+ = CP(V, M_n(W))$. We use the symbol $\psi \leq_{cp} \phi$ or $\phi \geq_{cp} \psi$ to indicate that $\phi - \psi$ is completely positive. For operator spaces V and W we let $CB(V, W)$ denote the operator space of all completely bounded linear maps from V to W . We will denote the completely bounded norm of $f \in CB(V, W)$ by $\|f\|_{cb}$, i.e., $\|f\|_{cb} = \sup \{ \|f^{(n)}\| \mid n \in \mathbb{N} \}$.

If V and W are matrix ordered operator spaces, then $CB(V, W)$ is also a matrix ordered operator space, where we define $M_n(CB(V, W))_+ = CP(V, M_n(W)) \cap CB(V, M_n(W))$, cf. [55, Theorem 3.1]. In particular, setting $W = \mathbb{C}$ we define:

Definition 1.3. Let V be a matrix ordered operator space. Then the matrix ordered operator space $V^* = CB(V, \mathbb{C})$ is called the (*operator*) *dual* of V .

Remark 1.4. To distinguish between operator duals and the dual of a normed space where necessary, we denote the dual of a normed space E as E' . Of course, seeing an

operator space V as normed space, we have $V^* = {}_1 V'$ on the first level. But, recalling the duality between $M_n(V)$ and $M_n(V')$ (see [26, p. 6ff]), we may distinguish between $M_n(V^*) = CB(V, M_n)$ and $M_n(V)'$, where the latter is only dual space of the normed space $M_n(V)$. Moreover, V^* carries the w^* -topology induced from V , and we endow $M_n(V^*)$ with the product topology that we call the w^* -topology on $M_n(V^*)$. That is, a net $(f^\nu)_\nu = ([f_{ij}^\nu])_\nu$ in $M_n(V^*)$ converges to $f = [f_{ij}] \in M_n(V^*)$ if and only if $f_{ij}^\nu \rightarrow f_{ij}$ for all $1 \leq i, j \leq n$ in the w^* -topology, cf. [26, p. 43]. See also Lemma A.5.

The relation between the ordering and the operator space structure required in the definition of a matrix ordered operator space is rather weak. A somewhat stronger relation is the notion of regularity. From [55] we have the following definition:

Definition 1.5. A matrix ordered operator space V is called *matrix regular* if for all $v \in M_n(V)$ the following is equivalent:

- (i) $\|v\| < 1$.
- (ii) There are $v_1, v_2 \in M_n(V)_+$ such that $\begin{pmatrix} v_1 & v \\ v^* & v_2 \end{pmatrix} \geq 0$ and $\|v_1\|, \|v_2\| < 1$.

Remark 1.6. Let V be a matrix ordered operator space. Then V is matrix regular if and only if for each $n \in \mathbb{N}$ and for all $v \in M_n(V)_h$ the following holds:

- (i) $w \in M_n(V)_h$ and $-w \leq v \leq w$ implies that $\|v\| \leq \|w\|$, and
- (ii) $\|v\| < 1$ implies that there is $w \in M_n(V)_h$ such that $\|w\| < 1$ and $-w \leq v \leq w$.

These two conditions mean that the real spaces $M_n(V)_h$ are regularly normed (in the classical sense) for all $n \in \mathbb{N}$.

Sometimes we will need results from the literature that are stated only for real vector spaces. Being concerned with complex spaces the following remarks are useful in such cases.

Remark 1.7. Let X be an ordered vector space that is also a Banach space in such a way that the involution is an isometry. Then $(X')_h = (X_h)'$ isometrically.

Proof. Obviously $\|f\| \geq \sup \{ |f(x)| \mid \|x\| = 1, x = x^* \}$. If $f = f^*$, then we have

$$\|f\| = \sup \{ \operatorname{Re} f(x) \mid \|x\| = 1 \} = \sup \{ f(x) \mid \|x\| = 1, x = x^* \},$$

because the involution is an isometry. □

Remark 1.8. Let V be a $*$ -vector space. Then any real linear map $f: V_h \rightarrow \mathbb{R}$ extends uniquely to a linear map $\tilde{f}: V \rightarrow \mathbb{C}$ given by $\tilde{f}(v) = f(\operatorname{Re} v) + if(\operatorname{Im} v)$. Moreover, \tilde{f} is self-adjoint.

Operator systems and m -convex sets

Since our matrix ordered spaces shall carry a norm structure that is related to the ordering, we need some additional properties of the order. Let V be an ordered vector space with distinguished cone V_+ . Recall that V is called *archimedian* ordered if $rv \leq w$ for all $r \geq 0$ implies $v \leq 0$, where $v, w \in V_h$. In this case we also say the cone V_+ is archimedian. Furthermore, a net $(e_\lambda)_{\lambda \in \Lambda}$ in V_+ is called an *approximate order unit* if

1. Matrix Orderings

$(e_\lambda)_{\lambda \in \Lambda}$ is monotone increasing and for any $v \in V_h$ there is $\lambda = \lambda(v) \in \Lambda$ and a number $r = r(v) > 0$ such that $-re_\lambda \leq v \leq re_\lambda$, cf. [47]. A constant approximate order unit, i.e., $e_\lambda = e$ for some $e \in V_+$, is called an order unit.

In [18, Theorem 4.4] an operator system is defined to be a matrix ordered space X such that X_+ is a proper cone with a distinguished order unit e , and the cones $M_n(X)_+$ are archimedean for all $n \in \mathbb{N}$. It is shown that there exist a Hilbert space H and a unital complete order isomorphism $\pi: X \rightarrow \mathcal{B}(H)$ into $\mathcal{B}(H)$. Hence the operator systems are nothing but the self-adjoint subspaces of unital C^* -algebras containing the unit. If (X, e) is an operator system, we let $e_n = e \otimes \mathbb{1}_n$ and define the so-called matrix order unit norm by

$$\|x\|_e = \inf \left\{ r \geq 0 \mid \begin{pmatrix} re_n & x \\ x^* & re_n \end{pmatrix} \geq 0 \right\}$$

for all $x \in M_n(X)$ and $n \in \mathbb{N}$. If $\|x\|_e = 0$, the archimedean property ensures that $x = 0$. Moreover, X together with the matrix order unit norm is obviously a matrix ordered operator space that is matrix regular. Usually in the literature, for instance [48] or [26], only operator systems are considered. However, we would like to handle non-unital C^* -algebras, too, without adjoining a unit. For this reason we gave the definition of an approximate order unit, and we want to discuss shortly how to extend the concept of operator systems to the non-unital case.

Remark 1.9. Recall from the proof of [18, Theorem 4.4] that if V is a matrix ordered vector space and the cone V_+ is proper, then $M_n(V)_+$ is proper for all $n \in \mathbb{N}$. It is shown there also that if $e \in V_+$ is an order unit, then $e \otimes \mathbb{1}_n \in M_n(V)_+$ is an order unit for all $n \in \mathbb{N}$. This proof translates verbatim to the case of an approximate order unit (e_λ) (noting that an approximate order unit is directed and monotone increasing), i.e., if (e_λ) is an approximate order unit in V_+ then $(e_\lambda \otimes \mathbb{1}_n)$ will be an approximate order unit in $M_n(V)_+$ for all $n \in \mathbb{N}$.

Remark 1.10. Let X be a matrix ordered space. Then the cones $CP(X, M_n)$ are archimedean for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. Given $\varphi, \psi \in CP(X, M_n)$ such that $r\varphi \leq_{cp} \psi$ for all $r \geq 0$ we have to show that $\varphi \leq_{cp} 0$. But for $x \in M_n(X)_+$ it follows from $r\varphi^{(n)}(x) \leq \psi^{(n)}(x)$ for all $r \geq 0$ that $\varphi^{(n)}(x) \leq 0$, because $M_n(M_n) = M_{n^2}$ is archimedean ordered. Hence $\varphi \leq_{cp} 0$. \square

Definition 1.11. Let X be a matrix ordered space with an approximate order unit $(e_\lambda)_{\lambda \in \Lambda}$. Let $e_\lambda^n = e_\lambda \otimes \mathbb{1}_n$. Then (X, e_λ) is an *approximate operator system*, if X_+ is proper, the seminorms defined by

$$\|x\|_e = \inf \left\{ r \geq 0 \mid \exists \lambda \in \Lambda \begin{pmatrix} re_\lambda^n & x \\ x^* & re_\lambda^n \end{pmatrix} \geq 0 \right\}$$

for all $x \in M_n(X)$ are norms, and $M_n(X)_+$ is a closed set in this norm for all $n \in \mathbb{N}$.

It is enough that $\|x\| = \inf \{ r \geq 0 \mid \exists \lambda -re_\lambda \leq x \leq re_\lambda \}$ is a norm to ensure that the (operator) seminorms on $M_n(X)$ above are norms.

Proposition 1.12. Let (X, e_λ) be an approximate operator system. Then the bi-dual X^{**} is an operator system with order unit $e = w^*\text{-}\lim_\lambda \hat{e}_\lambda$, where \hat{e}_λ denotes the canonical image of e_λ in X^{**} . Furthermore, there is a complete isometric order isomorphism from (X, e_λ) into (X^{**}, e) .

Proof. From Remark 1.10 we know that the cones $M_n(X^{**})_+$ are archimedean for all $n \in \mathbb{N}$. We need to show that $e = w^*\text{-}\lim_{\lambda} \hat{e}_{\lambda}$ exists and is an order unit for X^{**} . Let $f \in X^*$ be positive. Then from $\hat{e}_{\lambda}(f) = f(e_{\lambda})$ we see that $(\hat{e}_{\lambda}(f))$ is a monotone increasing net of positive numbers that is bounded by $\|f\|$, since $\|\hat{e}_{\lambda}\| \leq 1$. Hence the net converges and we set $e(f) = \lim_{\lambda} \hat{e}_{\lambda}(f)$. In this way we get a map $e: X^*_+ \rightarrow \mathbb{R}_+$ that is obviously additive, positive homogeneous and bounded. Notice that X^* is directed. To see this let $g \in X^*$ be self-adjoint. Then the restriction $g' = g|_{X_h}$ is a bounded real linear map. Since (X_h, e_{λ}) is a (real) approximate order unit space, it is 1-normal and we can apply the Grotsberg-Krein theorem (e.g., [47, Thm. 1]) to find a decomposition $g' = g_1 - g_2$ such that $\|g_1\| + \|g_2\| = \|g'\|$ and $g_i: X_h \rightarrow \mathbb{R}$ real linear, bounded and positive for $i = 1, 2$. By Remark 1.8 we can uniquely extend g_1 and g_2 to linear maps \tilde{g}_1 and \tilde{g}_2 on X , that are bounded and stay positive. So, we obviously obtain $g = \tilde{g}_1 - \tilde{g}_2$ for $\tilde{g}_1, \tilde{g}_2 \in X^*_+$. Hence X^* is linearly generated by X^*_+ , and since the limit process is linear, we see that $e = w^*\text{-}\lim_{\lambda} \hat{e}_{\lambda}$ exists on all of X^* . It is obvious that $e \in X^{**}_+$. Also $\|e\| \leq 1$, because $\|f\| \geq |f(e_{\lambda})| = |\hat{e}_{\lambda}(f)|$, so that $|e(f)| \leq \|f\|$ for all $f \in X^*$. We have to show that e is an order unit. Let $\varphi \in X^{**}_+$. We can assume that $\|\varphi\| = 1$. The image of the unit ball of X is w^* -dense in the unit ball of X^{**} . Hence there is a net (\hat{x}_{ν}) such that $\hat{x}_{\nu}(f) \rightarrow \varphi(f)$ for all $f \in X^*$. Passing to the real part we can assume that \hat{x}_{ν} is self-adjoint. For positive f we find for all ν some $\lambda(\nu)$ such that

$$-e(f) \leq -\hat{e}_{\lambda(\nu)}(f) \leq \hat{x}_{\nu}(f) \leq \hat{e}_{\lambda(\nu)}(f) \leq e(f).$$

Thus $-e(f) \leq \varphi(f) \leq e(f)$ for all positive f .

It is known that the canonical embedding of the operator space X into X^{**} is a complete isometry. Since the cones $M_n(X)_+$ are norm closed by assumption, the canonical embedding is also a complete order isomorphism. We have still to show that the matrix order unit norm defined by e coincides with the cb -norm of X^{**} . Let $\varphi \in M_n(X^{**})$. Letting $r = \|\varphi\|_{cb}$ we see that the map

$$\Psi: X^* \rightarrow M_{2n} \text{ defined by } f \mapsto \begin{pmatrix} re_n(f) & \varphi(f) \\ \varphi(f)^* & re_n(f) \end{pmatrix},$$

where $e_n = e \otimes 1_n$, is completely positive. Indeed there is a unitary $u \in M_{2n}$ such that

$$\Psi^{(l)}(f) = \begin{pmatrix} re_n(f_{ij}) & \varphi(f_{ij}) \\ \varphi(f_{ij})^* & re_n(f_{ij}) \end{pmatrix} = u^* \begin{pmatrix} re_n^{(l)}(f) & \varphi^{(l)}(f) \\ \varphi^{(l)}(f)^* & re_n^{(l)}(f) \end{pmatrix} u \geq 0$$

for all positive $f \in M_l(X^*)$, because $\|\varphi^{(l)}\| \leq r$. This shows that $\|\varphi\|_e \leq r = \|\varphi\|_{cb}$, where $\|\varphi\|_e = \inf \left\{ t \geq 0 \mid \begin{pmatrix} te_n & \varphi \\ \varphi^* & te_n \end{pmatrix} \geq_{cp} 0 \right\}$. On the other hand, given $\varphi \in M_n(X^{**})$ such that $\|\varphi\|_e < s < 1$, we have $\begin{pmatrix} se & \varphi \\ \varphi^* & se \end{pmatrix} \geq_{cp} 0$. From [55] we know that the dual of a matrix regular space is again matrix regular. Since the approximate operator system (X, e_{λ}) is matrix regular, its dual is matrix regular and hence the bidual X^{**} is matrix regular, too. Obviously $se \geq_{cp} 0$ and $\|se\|_{cb} = s < 1$, so that by matrix regularity we get $\|\varphi\|_{cb} < 1$ and the proof is complete. \square

Remark 1.13. Let (X, e) be a dual operator system. Then we see from Proposition 2.10 that there is a Hilbert space H and a unital completely order isomorphism into $\mathcal{B}(H)$ that is w^* - w^* -continuous. Hence the approximate operator systems are nothing but the self-adjoint subspaces of some $\mathcal{B}(H)$ that have an approximate order unit (e_{λ}) which converges to the identity of $\mathcal{B}(H)$.

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Definition 1.14 (Matrix States). For an operator system (X, e) and an approximate operator system (Y, e_λ) we define $CS_n(X) = \{ f \in CP(X, M_n) \mid f(e) = \mathbb{1}_n \}$ and $CQ_n(Y) = \{ g \in CP(Y, M_n) \mid \|g\|_{cb} \leq 1 \}$ for all $n \in \mathbb{N}$. For $n \geq 2$ the elements of $CS_n(X)$ are called *matrix states* (*m-states*) of X and the elements of $CQ_n(Y)$ are called *quasi matrix states* (*quasi m-states*) of Y . The collections $CS(X) = (CS_n(X))_n$ and $CQ(Y) = (CQ_n(Y))_n$ are called the *matrix convex state space* of X and the *matrix convex quasi state space* of Y , respectively. Moreover, we define the m-states of the approximate operator system Y to be the subset $CS_n(Y) = \{ f \in CP(X, M_n) \mid \lim_\lambda f(e_\lambda) = \mathbb{1}_n \}$ of $CQ_n(Y)$. Maps from (Y, e_λ) to M_n such that $\lim_\lambda f(e_\lambda) = \mathbb{1}_n$ are called *approximately unital maps*.

The sets introduced in the preceding definition are the replacements for the (usual) state and quasi state spaces. All these sets are convex, but much more is true. We need to give a short introduction into the theory of matrix convex sets now.

Definition 1.15 (Matrix Convex Set). A *matrix convex* (or *m-convex*) set in a vector space V is a sequence of subsets $K = (K_l)_l$ such that $K_l \subset M_l(V)$ for all $l \in \mathbb{N}$ and

$$\sum_{i=1}^m \alpha_i^* v_i \alpha_i \in K_n \quad (1.3)$$

for all $n_i, n, m \in \mathbb{N}$, $v_i \in K_{n_i}$ and $\alpha_i \in M_{n_i, n}$ such that $\sum_{i=1}^m \alpha_i^* \alpha_i = \mathbb{1}_n$. Occasionally, given a fixed $l \in \mathbb{N}$, we will consider also subsets $C \subset M_l(V)$ such that $\sum_{i=1}^m \alpha_i^* v_i \alpha_i \in C$ for all $m \in \mathbb{N}$, $v_i \in C$ and $\alpha_i \in M_l$ for which $\sum_{i=1}^m \alpha_i^* \alpha_i = \mathbb{1}_l$. We call such sets *M_l -convex*. So given an m-convex set $K = (K_l)_l$, the sets K_l are in particular M_l -convex for all $l \in \mathbb{N}$. Furthermore, in case V has a locally convex topology, we call a matrix convex set K compact if K_n is compact with respect to the product topology on $M_n(V)$ for all $n \in \mathbb{N}$.

Naturally because of the topic of this thesis we will be regularly concerned with matrix versions of classical concepts like, for instance, state and convexity in the preceding definitions. Matrix regularity is another example, but there are some more to come like matrix affine and matrix base. In addition there will be some new concepts like ‘matricial relation’ or ‘matrix related’. Since we don’t like to write constantly ‘matrix’ or ‘matricial’, the reader should be aware that we simply use the abbreviation ‘m-’ for ‘matrix’ (or sometimes also ‘matricial’), that is, we write m-convex, m-affine, m-relation and so on, possibly even without further notice. Moreover, we pronounce the ‘m-’ like ‘em’, so we write for instance ‘an m-convex set’.

The following observation is easy to prove:

Remark 1.16. Let V be a vector space. A sequence of subsets $K = (K_l)_l$ such that $K_l \subset M_l(V)$ for all $l \in \mathbb{N}$ is m-convex if and only if

- (i) $K_n \oplus K_m \subset K_{n+m}$, and
- (ii) $\alpha^* K_n \alpha \subset K_m$,

for all $n, m \in \mathbb{N}$ and $\alpha \in M_{n, m}$ such that $\alpha^* \alpha = \mathbb{1}_m$.

We will say that $v = \sum_{i=1}^m \alpha_i^* v_i \alpha_i$ as in (1.3) is a *matrix convex combination* of v_1, \dots, v_m . An m-convex combination is called *proper* if $\alpha_i \neq 0$ for $i = 1, \dots, m$. The

intersection of m -convex sets is again m -convex. Hence, given a sequence $Y = (Y_n)_{n \in \mathbb{N}}$ of subsets $Y_n \subset M_n(V)$, the m -convex hull of Y is the smallest m -convex subset of V containing Y . We denote the m -convex hull of Y as $\text{mco}(Y) = (\text{mco}_n(Y))_{n \in \mathbb{N}}$. Notice that calculations with m -convex combinations is close to calculations with convex combinations. Indeed, from [30] we have

$$\sum_{i=1}^m \alpha_i^* v_i \alpha_i = \alpha_1^* v_1 \alpha_1 + \beta^* w \beta, \quad (1.4)$$

where $w \in \text{mco}(\{v_2, \dots, v_m\})$ and $\beta^* \beta = \sum_{i=2}^m \alpha_i^* \alpha_i$.

Definition 1.17 (Structural Elements). Let $K = (K_n)_{n \in \mathbb{N}}$ be an m -convex set in V . Then $v \in K_n$ is a *structural element* of K_n if whenever $v = \sum_{i=1}^m \alpha_i^* v_i \alpha_i$ is a proper m -convex combination of $v_1, \dots, v_m \in K_n$ then there are unitary $u_i \in M_n$ and numbers $\lambda_i \in \mathbb{C}$ such that $v = u_i^* v_i u_i$ and $\alpha_i = \lambda_i u_i$ for $i = 1, \dots, m$. We write $\text{str}(K_n)$ for the set of the structural elements of K_n and we let $\text{str}(K) = (\text{str}(K_n))_n$. Obviously, structural elements are in particular extreme points, and $\text{str}(K_1) = \text{ex}(K_1)$, where $\text{ex}(K_1)$ denotes the set of the extreme points of K_1 .

Notice that if $x \in \text{str}(K_n)$ then $u^* x u \in \text{str}(K_n)$ for all unitaries $u \in M_n$. We call $x, y \in M_n(V)$ unitarily equivalent if there is a unitary $u \in M_n$ such that $y = u^* x u$. We write $\mathcal{U}(x)$ for the unitary equivalence class of x , i.e., for the set $\{u^* x u \mid u \in M_n\}$. Notice also from [30] that in the definition of structural elements it is actually enough to require that there exists $u_i \in M_n$ such that $v = u_i^* v_i u_i$ for $i = 1, \dots, m$. This implies that $\alpha_i = \lambda_i u_i$ for $\lambda_i \in \mathbb{C}$. Notice that the definition is equivalent with the definition of *matrix extreme points* in [62, Def. 2.1]. However, we have reserved the word matrix extreme points for special structural elements.

Remark 1.18. From [30] structural elements are m -irreducible. Recall that a matrix $v = [v_{ij}] \in M_n(V)$, where $n \in \mathbb{N}$ and V is a vector space, is m -reducible, if there are $1 \leq l < n$ and a unitary $u \in M_n$ such that $v = u^*(w_1 \oplus w_2)u$ for $w_1 \in M_l(V)$ and $w_2 \in M_{n-l}(V)$. Of course, v is called m -irreducible, if v is not m -reducible.

It is known that the extreme points of the convex quasi state space are the pure maps with norm one and the zero map. A special case of the next proposition has appeared in [29, Thm. B (1)] (Part (2) of that theorem is contained in more general form with simple proof in [30].) Recall that a completely positive map φ is called pure, if whenever ψ is a completely positive map such that $\psi \leq_{cp} \varphi$ then $\psi = r\varphi$ for some $0 \leq r \leq 1$.

Proposition 1.19. *Let (X, e_λ) be an approximate operator system. Let $n > 1$. The structural elements of $CQ_n(X)$ are exactly the pure maps of $CS_n(X)$, that is, those maps of $CQ_n(X)$ that are pure and approximately unital.*

Proof. Let $K_m = \{f \in CQ_m(X) \mid \lim f(e_\lambda) = \mathbb{1}_m\}$ for all $m \in \mathbb{N}$. Let $\phi \in K_n$ be a pure map and let $\phi = \sum_{i=1}^l \alpha_i^* \phi_i \alpha_i$ be a proper m -convex combination of $\phi_i \in CQ_n(X)$. Then $\phi - \alpha_i^* \phi_i \alpha_i$ is completely positive for all $i = 1, \dots, l$. Since ϕ is pure, there is $t_i^2 \in (0, 1)$ such that $\alpha_i^* \phi_i \alpha_i = t_i^2 \phi$. This implies $\alpha_i^* \beta_i^* \beta_i \alpha_i = t_i^2 \mathbb{1}_n$, where $\beta_i^* \beta_i = \lim \phi_i(e_\lambda) \leq \mathbb{1}_n$. Hence α_i and β_i must have full rank for $i = 1, \dots, l$. From

$$\mathbb{1}_n = \sum_{i=1}^l \alpha_i^* \beta_i^* \beta_i \alpha_i \quad (1.5)$$

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we obtain $\beta_i^* \beta_i = \mathbb{1}_n$ for $i = 1, \dots, l$. Indeed, assume without loss of generality that $\gamma = \beta_1^* \beta_1 \neq \mathbb{1}_n$. Since γ is positive, we can assume that this matrix is diagonal (if not there is a unitary $u \in M_n$ such that $u^* \gamma u$ is diagonal). By assumption we have an index j such that $\gamma_{jj} < 1$, where we can take $j = 1$. Now, evaluating the $(1, 1)$ -entry of equation (1.5) we obtain

$$\begin{aligned} 1 &= [\alpha_1^* \gamma \alpha_1]_{11} + \left[\sum_{i=2}^l \alpha_i^* \beta_i^* \beta_i \alpha_i \right]_{11} \leq \gamma_{11} |\alpha_{11}|^2 + \sum_{j=2}^n \gamma_{jj} |\alpha_{j1}|^2 + \left[\sum_{i=2}^l \alpha_i^* \alpha_i \right]_{11} \\ &\leq \gamma_{11} |\alpha_{11}|^2 + \sum_{j=2}^n |\alpha_{j1}|^2 + \left[\sum_{i=2}^l \alpha_i^* \alpha_i \right]_{11} \\ &< [\alpha_1^* \alpha_1]_{11} + \left[\sum_{i=2}^l \alpha_i^* \alpha_i \right]_{11} \leq 1, \end{aligned}$$

which is a contradiction. So, it is proved that $\beta_i^* \beta_i = \mathbb{1}_n$ for $i = 1, \dots, l$. This means $\phi_i \in K_n$ and we obtain $\alpha_i^* \alpha_i = t_i^2 \mathbb{1}_n$ for all $i = 1, \dots, l$, so that $u_i = \alpha_i / t_i$ is a unitary matrix and $\phi = u_i^* \phi_i u_i$ for each i . It follows that $\phi \in \text{str}(CQ_n(X))$.

For the converse, let $\phi \in \text{str}(CQ_n(X))$ and let ψ be a completely positive map from X to M_n such that $\phi - \psi$ completely positive. By Corollary 1.45 X^* is an m-base norm space with m-base $K = (K_m)_m$. So by definition of an m-base there exist $f, g \in K_n$ such that $\psi = \alpha^* f \alpha$ and $(\phi - \psi) = \beta^* g \beta$, where $\alpha, \beta \neq 0$ (since we can assume $\psi \neq 0$ and $\psi \neq \phi$). Hence we can write $\phi = \psi + (\phi - \psi) = \alpha^* f \alpha + \beta^* g \beta$, which applying Lemma 1.37 yields

$$0 \leq \alpha^* \alpha + \beta^* \beta \leq \|\alpha^* \alpha + \beta^* \beta\| \mathbb{1}_n = \|\phi\|_{cb} \mathbb{1}_n \leq \mathbb{1}_n.$$

So we see that $\phi = \alpha^* f \alpha + \beta^* g \beta$ is an m-convex combination (otherwise there would be a proper m-convex combination $\phi = \alpha^* f \alpha + \beta^* g \beta + \gamma^* \gamma$, where $\gamma^2 = \mathbb{1}_n - \alpha^* \alpha - \beta^* \beta$, which is impossible because structural elements are m-irreducible). Therefore there are $\lambda \in \mathbb{C}$ and a unitary $u \in M_n$ such that $\alpha = \lambda u$ and $f = u \phi u^*$, so in particular $\phi \in K_n$. Moreover, $\psi = \alpha^* f \alpha = |\lambda|^2 \phi$, which shows that ϕ is pure, and the proof is complete. \square

Corollary 1.20. *Let (X, e) be an operator system. The structural elements of $CS_n(X)$ are exactly the pure maps of $CS_n(X)$ for all $n \in \mathbb{N}$.*

The following definition is from [30] inspired by [44].

Definition 1.21. Let K be an m-convex set. For $n \in \mathbb{N}$ we let

$$\text{mext}(K_n) = \left\{ x \in \text{str}(K_n) \mid x \notin \bigcup_{l>n} \mathbb{1}_{l,n}^* \text{str}(K_l) \mathbb{1}_{l,n} \right\}.$$

The sequence $\text{mext}(K) = (\text{mext}(K_n))_n$ is called the set of *matrix extreme points* of K .

The next proposition characterizes the matrix extreme points among the structural elements in the special case where we consider the quasi state space of C^* -algebras. Notice that we regard a C^* -algebra as an approximate operator system by choosing the positive part of the open unit ball as approximate order unit, see Remark A.8. If the C^* -algebra should be unital, the unit is an upper bound for the open unit ball, cf. Remark 1.43.

Proposition 1.22. *Let $n \in \mathbb{N}$ and let \mathcal{A} be a C^* -algebra. Then $\varphi: \mathcal{A} \rightarrow M_n$ is an approximately unital irreducible representation if and only if φ is a non-zero matrix extreme point of $CQ(\mathcal{A})$.*

Proof. Let $\varphi: \mathcal{A} \rightarrow M_n$ be a non-zero matrix extreme point. In particular φ is a structural element of $CQ_n(\mathcal{A})$, so by Proposition 1.19 φ is pure and approximately unital. Let $\varphi = \mathcal{V}^* \pi \mathcal{V}$ be the essentially unique minimal Stinespring representation of φ , where $\pi: \mathcal{A} \rightarrow \mathcal{B}(H_\pi)$ is an approximately unital $*$ -representation and $\mathcal{V}: \mathbb{C}^n \rightarrow H_\pi$ is an isometry, cf. Theorem A.9. Since φ is pure, π is an irreducible representation, cf. Theorem A.11. Assume that the dimension of H_π would be greater than n . Then $\mathcal{V}(\mathbb{C}^n)$ is a subspace of H_π of dimension n , so there exists $\eta \in \mathcal{V}(\mathbb{C}^n)^\perp \subset H_\pi$ such that $\|\eta\| = 1$. Define $\mathcal{W}: \mathbb{C}^{n+1} \rightarrow H_\pi$ by $\mathcal{W}(\xi_1, \dots, \xi_{n+1}) = \mathcal{V}(\xi_1, \dots, \xi_n) + \xi_{n+1}\eta$. Notice that \mathcal{W} is an isometry. Therefore the completely positive map $\psi = \mathcal{W}^* \pi \mathcal{W}$ is in $CQ_{n+1}(\mathcal{A})$ and, since π is irreducible, ψ is pure, so that $\psi \in \text{str}(CQ_{n+1}(\mathcal{A}))$ by Proposition 1.19. This leads immediately to a contradiction, since φ is a matrix extreme point, i.e., a structural element that is not a compression of another structural element. However we have $\varphi = (\mathbb{1}_n \ 0) \psi \begin{pmatrix} \mathbb{1}_n \\ 0 \end{pmatrix}$. Consequently, the dimension of H_π must be n , so that we can identify H_π with \mathbb{C}^n . Then \mathcal{V} is a unitary matrix and φ is unitarily equivalent to the irreducible representation π from \mathcal{A} onto M_n , so φ is itself an irreducible representation onto M_n .

Conversely, let $\varphi: \mathcal{A} \rightarrow M_n$ be an approximately unital and irreducible representation. Then φ is obviously completely positive and also pure by Theorem A.11. Hence φ is a structural element of $CQ_n(\mathcal{A})$. Suppose for contradiction that there would be $l > n$ and $\psi \in \text{str}(CQ_l(\mathcal{A}))$ such that $\varphi = \mathbb{1}_{l,n}^* \psi \mathbb{1}_{l,n}$. Let $\psi = \mathcal{V}^* \pi \mathcal{V}$ be the minimal Stinespring representation of ψ , where $\pi: \mathcal{A} \rightarrow \mathcal{B}(H_\pi)$ is irreducible, because ψ is pure, and $\mathcal{V}: \mathbb{C}^l \rightarrow H_\pi$ is an isometry, so $\dim(H_\pi) \geq l$. Notice that $\mathcal{W} = \mathcal{V} \mathbb{1}_{l,n}$ is an isometry and that $\varphi = \mathcal{W}^* \pi \mathcal{W}$. Since π is irreducible, we get $\text{span}(\pi(\mathcal{A}) \mathcal{W} \mathbb{C}^n) = H_\pi$. This means that $\varphi = \mathcal{W}^* \pi \mathcal{W}$ is the essentially unique minimal Stinespring representation of φ . Since φ , being an irreducible (and approximately unital) representation, is already its own minimal Stinespring representation, φ and π must be unitarily equivalent. This leads to the contradiction $\dim(H_\pi) \geq l > n = \dim(H_\pi)$. \square

The previous proposition shows directly that there are compact m -convex sets K such that the set of matrix extreme points of K is empty—just take for K the (quasi) state space of a C^* -algebra that has no irreducible finite dimensional representations.

Matrix affine mappings

So far we have seen that (approximate) operator systems give rise to m -convex sets. The m -convex (quasi) state spaces of (approximate) operator systems are compact m -convex sets and we characterized their structural elements. From the scalar theory it is known that compact and convex sets are a dual object for order unit spaces. Let C be a compact convex set. Then the space $A(C)$ of all continuous affine functions is a complete order unit space. Its state space is affinely homeomorphic to C . Moreover, any order unit space that is complete in the order unit norm is unitaly order isomorphic to the order unit space of the continuous affine functions on its state space. It is shown in [62, Prop. 3.5] that this can be generalized to operator systems. Since this observation is basic for what follows, we repeat it here. It starts with defining so-called matrix affine maps.

Definition 1.23. Let $K = (K_n)_n$ be a matrix convex set and W a complex vector space. A *matrix affine map* $\phi: K \rightarrow W$ is a sequence $\phi = (\phi_n)_n$ of maps $\phi_n: K_n \rightarrow M_n(W)$

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such that

$$\phi_n \left(\sum_{i=1}^m \alpha_i^* x_i \alpha_i \right) = \sum_{i=1}^m \alpha_i^* \phi_{n_i}(x_i) \alpha_i$$

for all $n_i, n, m \in \mathbb{N}$, $x_i \in K_{n_i}$ and $\alpha_i \in M_{n_i, n}$ such that $\sum_{i=1}^m \alpha_i^* \alpha_i = \mathbb{1}_n$.

Let $\mathcal{A}(K, W)$ denote the complex vector space of all matrix affine maps from K to W with point-wise operations induced by $M_n(W)$, $n \in \mathbb{N}$. If W is a $*$ -vector space then $\theta^* = (\theta_n^*)$ where $\theta_n^*(x) = \theta_n(x)^*$ defines an involution on $\mathcal{A}(K, W)$. If W is a matrix ordered space, $\mathcal{A}(K, W)$ is ordered by point-wise evaluation, i.e., $\psi = (\psi_n) \geq 0$ if $\psi_n(x) \geq 0$ for all $n \in \mathbb{N}$ and $x \in K_n$. Especially for $W = \mathbb{C}$ there is a matrix order structure on $\mathcal{A}(K) = \mathcal{A}(K, \mathbb{C})$ by identifying $M_n(\mathcal{A}(K))$ with $\mathcal{A}(K, M_n)$ and letting $M_n(\mathcal{A}(K))_+ = \mathcal{A}(K, M_n)_+$. We let $A_b(K, M_l) \subset \mathcal{A}(K, M_l)$ denote the subspace of all bounded matrix affine maps from K to M_l , i.e., all matrix affine maps $f = (f_n)$ such that f_1 is bounded. This means that there is $r \geq 0$ such that $\|f_1(x)\| \leq r$ for all $x \in K_1$. If f is self-adjoint and bounded by r , we see from (1.7) that

$$\|f_n(x)\| = \sup \{ |\langle f_n(x)\xi | \xi \rangle| \mid \|\xi\| = 1 \} \leq r \quad (1.6)$$

for all $x \in K_n$. For an arbitrary $f \in A_b(K, M_l)$ we get $\|f_n(x)\| \leq 2r$ for all $x \in K_n$ and $n \in \mathbb{N}$, where r is a bound of f_1 . So for $f = (f_n) \in A_b(K, M_l)$ we define the norm

$$\|f\| = \sup \{ \|f_n(x)\| \mid x \in K_n, n \in \mathbb{N} \}.$$

We let again $A_b(K) = A_b(K, \mathbb{C})$.

Remark 1.24. Notice that there is an order isomorphism between $\mathcal{A}(K)$ ($A_b(K)$) and the space $\mathcal{A}(K_1)$ ($A_b(K_1)$) of (bounded) affine complex-valued functions on K_1 , given by $f = (f_n) \mapsto f_1$ for $f \in A_b(K)$. This follows easily from the identity

$$\langle f_n(x)\xi | \xi \rangle = \xi^* f_n(x) \xi = f_1(\xi^* x \xi) \quad (1.7)$$

for all unit vectors $\xi \in \mathbb{C}^n$. Indeed, let K be a matrix convex set and $f_1: K_1 \rightarrow \mathbb{C}$ an affine map. If we can define maps $f_n: K_n \rightarrow M_n$ by the rule $\langle f_n(x)\xi | \xi \rangle = f_1(\xi^* x \xi)$ for all $n > 1$, $x \in K_n$ and $\xi \in \mathbb{C}^n$ such that $\xi^* \xi = 1$, then $f = (f_n)_{n \in \mathbb{N}}$ will be a m-affine map. To show that f_n is a well-defined map for all $n \in \mathbb{N}$ we need to prove that the map $h(\xi) = \|\xi\|^2 f_1(\xi_1^* x \xi_1)$ is a quadratic form on \mathbb{C}^n , where $\xi_1 = \xi / \|\xi\|$. So, let $\xi, \eta \in \mathbb{C}^n$. By the parallelogram identity, we have $\|\xi + \eta\|^2 + \|\xi - \eta\|^2 = 2(\|\xi\|^2 + \|\eta\|^2)$. With $d = 2(\|\xi\|^2 + \|\eta\|^2)$ we obtain

$$\begin{aligned} h(\xi + \eta) + h(\xi - \eta) &= \|\xi + \eta\|^2 f_1((\xi + \eta)_1^* x (\xi + \eta)_1) + \|\xi - \eta\|^2 f_1((\xi - \eta)_1^* x (\xi - \eta)_1) \\ &= df_1 \left(\frac{1}{d} ((\xi + \eta)^* x (\xi + \eta) + (\xi - \eta)^* x (\xi - \eta)) \right) \\ &= df_1 \left(\frac{1}{d} (2\xi^* x \xi + 2\eta^* x \eta) \right) \\ &= 2(\|\xi\|^2 f_1(\xi_1^* x \xi_1) + \|\eta\|^2 f_1(\eta_1^* x \eta_1)) \\ &= 2(h(\xi) + h(\eta)), \end{aligned}$$

which shows that h is a quadratic form. Then there is a unique matrix $f_n(x) \in M_n$ such that $\langle f_n(x)\xi | \xi \rangle = h(\xi)$. Obviously, if f_1 is bounded, then $(f_n)_n$ is bounded, cf. (1.6).

Looking at the preceding remark the reader should keep in mind that the rather technical definition of matrix affine maps comes down to nothing but to supply the space of the affine functions with a matrix order structure in such a way that it becomes an operator system with m -convex state space m -affinely homeomorphic to the given m -convex set.

Lemma 1.25. *Let K be a matrix convex set. Then $A_b(K)$ together with the induced matrix order structure as subspace of $\mathcal{A}(K)$ and with the distinguished unit e , where $e_n(x) = \mathbb{1}_n$ for all $n \in \mathbb{N}$ and $x \in K_n$, is an operator system. The matrix order unit norm is identical with the supremum norm, i.e., $\|f\| = \|f\|_e$ for all $f \in M_n(A_b(K)) = A_b(K, M_n)$ and $n \in \mathbb{N}$.*

Proof. We need to show first that the cones $M_n(A_b(K))$ are archimedean for all $n \in \mathbb{N}$ and that e is an order unit. The archimedean property of the cones follows immediately from the fact that the cones $M_n(M_r)_+ = M_{nr}^+$ are archimedean for all $r, n \in \mathbb{N}$. Let $f = (f_n) \in A_b(K)_h$. Then

$$-\|f\|e_n(x) \leq -\|f_n(x)\|e_n(x) \leq f_n(x) \leq \|f_n(x)\|e_n(x) \leq \|f\|e_n(x)$$

for all $x \in K_n$ and $n \in \mathbb{N}$ shows directly that e is an order unit. To show the norm equality, let $f \in A_b(K, M_l)$. Since $\|f_n(x)\| \leq \|f\|$ for all $x \in K_n$ and $n \in \mathbb{N}$, we get $\begin{pmatrix} \|f\|e & f \\ f^* & \|f\|e \end{pmatrix} \geq 0$. Hence by definition of the matrix order unit norm $\|f\|_e \leq \|f\|$. On the other hand, if there would be $r > 0$ such that $\|f\|_e < r < \|f\|$ then the matrix $\begin{pmatrix} re & f \\ f^* & re \end{pmatrix}$ would be positive, which means point-wise positive. Thus $\|f_n(x)\| \leq r$ for all $x \in K_n$ and $n \in \mathbb{N}$, so that $\|f\| \leq r$. This is a contradiction. So $\|f\|_e = \|f\|$. \square

When K is a matrix convex subset of a topological vector space, $A(K) \subset A_b(K)$ denotes the operator system of all continuous matrix affine maps. The basis for our further studies is the following proposition from [62, Prop. 3.5]:

Proposition 1.26. *Let X be an operator system. Then X is unitaly completely order isomorphic to $A(CS(X))$. Furthermore, if K is a compact m -convex subset of a locally convex vector space V , then K is matrix affinely homeomorphic to $CS(A(K))$.*

Duals of operator systems

In this section we will define the matrix ordered version of base norm spaces that will be the dual of approximate operator systems in the operator space sense. Hence our matrix base norm spaces will be matrix ordered operator spaces, such that there is a matrix convex base of the matrix cones. Furthermore we establish the duality theory between approximate operator systems and matrix base norm spaces. First we need some useful lemmas:

Lemma 1.27. *Let V be a matrix ordered vector space and K a matrix convex subset. If $K_1 \subset V_h$ then $K_n \subset M_n(V)_h$ for all $n \in \mathbb{N}$.*

Proof. Let

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_2.$$

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We have to show that $b = c^*$. Because K is matrix convex, it follows that

$$\frac{1}{2}(1 \ 1) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}(a + b + c + d) \in K_1 \subset V_h.$$

This implies that $b + c$ is self-adjoint. It follows $b - b^* = -(c - c^*)$, i.e., $\text{Im } b = -\text{Im } c$. On the other hand

$$\frac{1}{2}(1 \ i) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2}(a - ib + ic + d) \in K_1 \subset V_h,$$

which implies $\text{Re } b = \text{Re } c$. Thus $b = \text{Re } b + i \text{Im } b = \text{Re } c - i \text{Im } c = c^*$. It is obvious how to show $x_{ij} = x_{ji}^*$ for $x = (x_{ij}) \in K_n$. \square

Let V be an ordered vector space. Recall that a convex subset $C \subset V_+$ is called a base of V_+ if every non-zero $v \in V_+$ has a unique representation $v = rx$, where $x \in C$ and $r > 0$.

Lemma 1.28. *Let V be an ordered vector space with generating cone V_+ , and let C be a base of V_+ . Then for each affine map from C to some complex vector space W there is a unique extension to a complex linear map from V to W .*

Proof. Let $\psi: C \rightarrow W$ be an affine map. For each $x \in V_+$ there are $r \in \mathbb{R}_+$ and $z \in C$ such that $x = rz$ uniquely, since C is a base of V_+ . So we can define a map $\phi: V_+ \rightarrow W$ by $\phi(x) = r\psi(z)$. This is obviously a positive homogeneous map. We show that ϕ is additive. Let $u, v \in V_+$. Then $u = rx$ and $v = sy$ where $r, s \in \mathbb{R}_+$ and $x, y \in C$. It follows

$$\phi(u + v) = (r + s)\psi\left(\frac{r}{r + s}x + \frac{s}{r + s}y\right) = r\psi(x) + s\psi(y) = \phi(u) + \phi(v).$$

Since $V_h = V_+ - V_+$ it is now clear that ϕ can be extended to V_h . Moreover, given $x \in V$ we have $x = \text{Re } x + i \text{Im } x$ uniquely, so that it is straightforward that ϕ can be uniquely extended to a complex linear map $V \rightarrow W$. \square

Lemma 1.29. *Let V be an ordered vector space with generating cone V_+ , and let K be a matrix convex subset such that K_1 is a base of V_+ . Let $\psi = (\psi_n)$ be a matrix affine map from K to an involutive complex vector space W . Let $\phi: W \rightarrow V$ be the unique extension of the affine map ψ_1 that exists by Lemma 1.28. Then $\psi_n = \phi^{(n)}|_{K_n}$ for all $n \in \mathbb{N}$.*

Proof. For $\psi \in \mathcal{A}(K, W)_h$ let ϕ be the linear extension of ψ_1 which is self-adjoint because ψ_1 is self-adjoint. By Lemma 1.27 $K \subset V_h$. For $x = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \in K_2$ let $\begin{pmatrix} a' & w \\ w^* & d' \end{pmatrix} = \psi_2(x) \in M_2(W)_h$. By matrix affinity we have $\psi_1(a) = a'$ and $\psi_1(d) = d'$. Moreover

$$\frac{1}{2}(1 \ -i)\psi_2\left(\begin{pmatrix} a & b \\ b^* & d \end{pmatrix}\right)\begin{pmatrix} 1 \\ i \end{pmatrix} = \psi_1\left(\frac{1}{2}(a + ib - ib^* + d)\right)$$

implies

$$a' + iw - iw^* + d' = \psi_1(a) + i\phi(b) - i\phi(b^*) + \psi_1(d).$$

This means that $\text{Im } w = \text{Im } \phi(b)$ since $\phi(b^*) = \phi(b)^*$. Similarly one gets $\text{Re } w = \text{Re } \phi(b)$. Thus $w = \phi(b)$ and accordingly $\psi_2(x) = \phi^{(2)}(x)$ for $x \in K_2$ and self-adjoint ψ .

It is obvious that this holds also for any $x = [x_{ij}] \in K_n$ because $x_{ij} = x_{ji}^*$. If ψ is not self-adjoint, it can be uniquely decomposed into its real and imaginary parts. Since the process of extension of an affine map to a linear map is complex linear the claim follows immediately. \square

Let E be a real ordered vector space with positive cone E_+ . Recall that a non-empty convex subset B of E_+ is called a base for E_+ , if every non-zero $x \in E_+$ has a unique representation $x = rb$, where $b \in B$ and $r > 0$. This is equivalent to the existence of a strictly positive real linear functional on E . The right approach to translate this concept to matrix ordered vector spaces is by replacing convex sets with matrix convex sets. Hence we define:

Definition 1.30 (Matrix Convex Base). Let K be a matrix convex subset of a matrix ordered vector space V . Then K is a *matrix convex base* (or simply, an *m-base*) of V , if

- (i) $M_n(V)_+ = \{ \alpha^* K_m \alpha \mid m \leq n, \alpha \in M_{m,n} \}$ for all $n \in \mathbb{N}$, and
- (ii) $\alpha^* x \alpha = \beta^* y \beta$ implies $\alpha^* \alpha = \beta^* \beta$ for all $x \in K_l, y \in K_m, \alpha \in M_{l,n}, \beta \in M_{m,n}$ and $l, m, n \in \mathbb{N}$.

Notice that the definition implies that $0 \notin K_1$.

Proposition 1.31. Let K be a matrix convex subset of a matrix ordered vector space V . Then K is a matrix convex base of V if and only if there is a strictly positive linear map $\phi: V \rightarrow \mathbb{C}$ such that $K_n = \{ x \in M_n(V)_+ \mid \phi^{(n)}(x) = \mathbb{1}_n \}$ for all $n \in \mathbb{N}$.

Proof. Suppose that there is a strictly positive map $\phi: V \rightarrow \mathbb{C}$ with the stated property. Then the second condition of the definition of an m-base obviously holds. So all we have to show is that $M_n(V)_+ \subset \{ \alpha^* K_m \alpha \mid m \leq n, \alpha \in M_{m,n} \}$ for all $n \in \mathbb{N}$. To this end let $v = [v_{ij}] \in M_n(V)_+$ and assume $v \neq 0$. Then $v_{ii} \neq 0$ for some $0 \leq i \leq n$. If $\beta = \phi^{(n)}(v) \in M_n^+$ would be invertible, then $\phi^{(n)}(\beta^{-\frac{1}{2}} v \beta^{-\frac{1}{2}}) = \mathbb{1}_n$ and we would be through. Now, if $\beta \geq 0$ does not have full rank, there is a unitary matrix $u \in M_n$ such that $0 \leq u^* \beta u = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$. Thus there are $0 < m < n$ and an invertible $\gamma \in M_m$ such that $u^* \beta u = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$. A short calculation shows

$$\phi^{(m)} \left(\underbrace{\mathbb{1}_{n,m}^* \begin{pmatrix} \gamma^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} u^* v u \begin{pmatrix} \gamma^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \mathbb{1}_{n,m}}_{=x} \right) = \mathbb{1}_m.$$

Therefore $x \in K_m$ and $\phi^{(n)}(w) = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$, where $w = u^* v u$. We conclude that $\phi(w_{ii}) = 0$ for $m < i \leq n$, which implies $w_{ii} = 0$, because ϕ is strictly positive. Thus $u^* v u = \begin{pmatrix} v' & 0 \\ 0 & 0 \end{pmatrix}$. With $\alpha = \gamma^{\frac{1}{2}} (\mathbb{1}_m \ 0) u^* \in M_{m,n}$ we obtain $v = \alpha^* x \alpha$, which proves the claim.

Suppose now that K is an m-base of V . Let $v \in V$ be positive and non-zero. By definition of an m-base there are $r > 0$ and $x \in K_1$ such that $v = rx$, and if there is another pair $s \geq 0$ and $y \in K_1$ such that $v = sy$ then $\sqrt{r}x\sqrt{r} = \sqrt{s}y\sqrt{s}$ implies $r = s$, so that $x = y$. This means that each non-zero $v \in V_+$ has a unique representation $v = rx$ such that $r > 0$ and $x \in K_1$. Consequently, K_1 is a convex base (in the usual sense) of the cone $V_+ \subset V_h$, so there is a strictly positive real linear map $\phi: V_h \rightarrow \mathbb{R}$ such that $K_1 = \{ x \in V_+ \mid \phi(x) = 1 \}$. We can extend ϕ to a complex linear map on V that we still denote ϕ . Obviously the map stays strictly positive. Letting

$$C_n = \left\{ x \in M_n(V)_+ \mid \phi^{(n)}(x) = \mathbb{1}_n \right\},$$

we have to show $K_n = C_n$ for all $n \in \mathbb{N}$. We have $\langle \phi^{(n)}(x) \xi \mid \xi \rangle = \phi(\xi^* x \xi) = 1$ for $n \in \mathbb{N}$, $x \in K_n$ and for all unit vectors $\xi \in \mathbb{C}^n$. It follows that $\phi^{(n)}(x) = \mathbb{1}_n$ and hence $K_n \subset C_n$

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for all $n \in \mathbb{N}$. Conversely, let $v \in C_n$. Then v is positive and since by assumption K is an m -base, there are $m \leq n$, $\alpha \in M_{m,n}$ and $x \in K_m \subset C_m$ such that $v = \alpha^* x \alpha$. Then $\mathbb{1}_n = \phi^{(n)}(v) = \alpha^* \phi^{(m)}(x) \alpha = \alpha^* \alpha$ implies $v \in K_n$, because K is an m -convex set. This shows that $K_n = C_n$ for all $n \in \mathbb{N}$, and the proof is complete. \square

Recall that for a real base norm space E with base C , the unit ball of E is given by $\text{conv}(C \cup -C)$. Since E is a real vector space this means that the unit ball of E is the absolute convex hull of the base. It is well-known that there is a correspondence between norms and absolute convex sets, i.e., the unit balls of normed spaces are absolute convex sets. Conversely, one can use an absolute convex set to define a (semi-)norm, such that the given absolute convex set will be the unit ball in this norm. The analogous concept of an absolute convex set for operator spaces is a so-called absolutely matrix convex set. Recall the following definition from [28] or [26]:

Definition 1.32. Let V be a vector space. Let $K = (K_n)$ be a sequence of sets such that $K_n \subset M_n(V)$ for all $n \in \mathbb{N}$. Then K is an absolutely matrix convex set if

$$\sum_i \alpha_i x_i \beta_i \in K_n$$

whenever $x_i \in K_{n_i}$ and $\alpha_i \in M_{n,n_i}$, $\beta_i \in M_{n_i,n}$ such that $\sum_i \alpha_i \alpha_i^* \leq \mathbb{1}_n$, $\sum_i \beta_i^* \beta_i \leq \mathbb{1}_n$. Moreover, the intersection of absolutely m -convex sets is again absolutely m -convex, so given a sequence $Y = (Y_n)_{n \in \mathbb{N}}$ of subsets $Y_n \subset M_n(V)$, the *absolutely m -convex hull* of Y is the smallest absolutely m -convex subset of V containing Y . We denote the absolutely m -convex hull of Y as $\text{amco}(Y) = (\text{amco}_n(Y))_{n \in \mathbb{N}}$.

Since we can rewrite an absolutely m -convex combination as $\sum_i \alpha_i x_i \beta_i = \alpha x \beta$, where $x = \oplus_i x_i$ and $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)^{\text{tr}}$ are contractions, we conclude:

Remark 1.33. K is absolutely matrix convex if and only if for all $n, m \in \mathbb{N}$

- (i) $K_n \oplus K_m \subset K_{n+m}$, and
- (ii) $\alpha K_n \beta \subset K_m$ for contractions $\alpha \in M_{m,n}$ and $\beta \in M_{n,m}$.

Lemma 1.34. Let V be a $*$ -vector space and let $K = (K_n)$ be a sequence of subsets $K_n \subset M_n(V)_h$ for all $n \in \mathbb{N}$. Then $\text{amco}(K)_h = \text{mco}(K \cup -K)$. If K is m -convex, then $\text{amco}_n(K)_h = \text{mco}_n(K_n \cup -K_n)$.

Proof. Let $v \in \text{amco}_n(K)_h$. There are $l \in \mathbb{N}$ and $x \in K_l$ and contractions $\alpha, \beta \in M_{l,n}$ such that $v = \alpha^* x \beta$. Since $v = v^* = \beta^* x \alpha$, we find

$$v = \frac{1}{4}((\alpha + \beta)^* x (\alpha + \beta) - (\alpha - \beta)^* x (\alpha - \beta)). \quad (1.8)$$

In addition

$$\frac{1}{4}((\alpha + \beta)^* (\alpha + \beta) + (\alpha - \beta)^* (\alpha - \beta)) = \frac{1}{2}(\alpha^* \alpha + \beta^* \beta) \leq \mathbb{1}_n,$$

since α and β are contractions. So we see that $v \in \text{mco}_n(K \cup -K)$, because 0 is contained in $\text{mco}_n(K \cup -K)$. If K is assumed to be m -convex, and hence $-K$ is also m -convex, then we can rewrite the m -convex combination of equation (1.8) (possibly adding 0) into an M_n -convex combination such that $v \in \text{mco}_n(K_n \cup -K_n)$, cf. [30]. \square

Let V be an operator space. For the following definition, recall that there is a correspondence between absolute matrix convex sets and operator space (semi)norms, cf. [28, p. 171ff]. This correspondence is given by the Minkowsky functionals on each matrix level, i.e., by

$$\|v\|_n = \inf \{ \lambda \geq 0 \mid v \in \lambda B_n \},$$

where $v \in M_n(V)$ and $(\|\cdot\|_n)_n$ is the family of operator space norms and $B = (B_n)_n$, where $B_n = \text{Ball}(M_n(V))$ for all $n \in \mathbb{N}$, is the absolutely matrix convex set of the unit balls of $M_n(V)$.

Definition 1.35. Let V be a matrix ordered vector space such that $V_h = V_+ - V_+$. Then V is a *matrix base norm space* (or *m-base norm space*), if V has an m-base K such that its absolute matrix convex hull $B = \text{amco}(K)$ determines an operator space norm by $\|v\|_n = \inf \{ \lambda \geq 0 \mid v \in \lambda B_n \}$ for all $n \in \mathbb{N}$. Note that it is sufficient, if $\|\cdot\|_1$ is a norm. This will be the case if, for instance, B_1 is linearly bounded.

Remark 1.36. We should mention that the term ‘matrix base norm space’ appears in [38]. However, the spaces considered there are neither operator spaces nor do they have an m-base. What Karn and Vasudevan use is actually the old Choi-Effros dual of an operator space. This means given an operator space V the dual norms are defined by identifying $M_n(V^*) = M_n(V)'$. Hence their matrix base norm spaces are no operator spaces, but are L^1 -normed, i.e., they satisfy $\|v \oplus w\| = \|v\| + \|w\|$. Moreover, let W be a matrix base norm space in the sense of [38], then $M_n(W)_h$ is a real base norm space for all $n \in \mathbb{N}$ in the usual sense. This means given a dual pair X and W , where W is such a matrix base norm space and X is an approximate operator system, $M_n(W)_h$ is just the Banach space dual of the real approximate order unit space $M_n(X)_h$. In addition, while $M_n(W)_h$ has a convex base C_n for all $n \in \mathbb{N}$, the collection $(C_n)_n$ of these bases does not define an m-convex set. So, these matrix base norm spaces are quite different from ours.

Lemma 1.37. Let (V, K) be an m-base norm space. Suppose that $v = \alpha^* x \alpha$, where $x \in K_m$, $\alpha \in M_{m,n}$ and $m, n \in \mathbb{N}$. Then $\|v\| = \|\alpha^* \alpha\|$.

Proof. Obviously $\|v\| \leq \|\alpha\|^2$, since we have an operator space norm and $\|x\| \leq 1$. Given an arbitrary $\varepsilon > 0$ we have by definition of the norm $v \in (\|v\| + \varepsilon)B_n$. So we find some $m \in \mathbb{N}$, $y \in K_m$, $\beta \in M_{n,m}$ and $\gamma \in M_{m,n}$ such that $\|\beta\|, \|\gamma\| \leq 1$ and

$$\alpha^* x \alpha = v = (\|v\| + \varepsilon) \beta y \gamma.$$

From Proposition 1.31 there is a strictly positive functional ϕ determined by the m-base. Applying ϕ to the above equation yields $\alpha^* \alpha = (\|v\| + \varepsilon) \beta \gamma$. Hence we find $\|\alpha\|^2 \leq \|v\| + \varepsilon$. Since this holds for all $\varepsilon > 0$ the claim follows. \square

Proposition 1.38. Let (V, K) and (W, C) be m-base norm spaces. Let $\psi = (\psi_n)_n$ be an m-affine isomorphism between K and C , then ψ_1 extends to a complete order isomorphism that is also a complete isometry. Conversely, if $f: V \rightarrow W$ is a complete order isomorphism and a complete isometry, then the restrictions $(f^{(n)}|_{K_n})_n$ form an m-affine isomorphism between K and C . For this reason we call an m-affine isomorphism between the m-bases an isomorphism of m-base norm spaces.

Proof. Let $x \in K_n$. Given a complete isometry $f: V \rightarrow W$ that is a complete order isomorphism and setting $g = f^{-1}$, we have $0 \leq f^{(n)}(x) = \alpha^* y \alpha$ for some $\alpha \in M_{l,n}$,

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$y \in C_l$ and $l \leq n$. Moreover, $1 = \|f^{(n)}(x)\| = \|\alpha\|^2$ and thus $\alpha^*\alpha \leq \mathbb{1}_n$. Similarly $0 \leq g^{(l)}(y) = \beta^*\tilde{x}\beta$ for some $\beta \in M_{m,l}$ with $\beta^*\beta \leq \mathbb{1}_l$, $\tilde{x} \in K_m$ and $m \leq l$. Since K is an m-base, the identity $x = \alpha^*\beta^*\tilde{x}\beta\alpha$ implies $\alpha^*\beta^*\beta\alpha = \mathbb{1}_n$. Hence we see $l = n = m$ and $\beta^*\beta = \mathbb{1}_n$ and $\alpha^*\alpha = \mathbb{1}_n$. This means that $f^{(n)}(x) \in C_n$ for all $x \in K_n$. So we have shown that $f^{(n)}(K_n) \subset C_n$ and $g^{(n)}(C_n) \subset K_n$ for all $n \in \mathbb{N}$. It is clear that the maps are m-affine.

Conversely, suppose that we have an m-affine isomorphism $\psi = (\psi_n)$ between K and C . Let f and g be the unique linear extensions of ψ_1 and ψ_1^{-1} , respectively, that exist by Lemma 1.28. Then f and g are inverse to each other so that f is a linear isomorphism between V and W . Furthermore from Lemma 1.29 we have $f^{(n)}|_{K_n} = \psi_n$ and $g^{(n)}|_{C_n} = \psi_n^{-1}$. Let $v \in M_n(V)_+$. Then $v = \alpha^*x\alpha$ for a suitable matrix α and x in K and so $f^{(n)}(v) = \alpha^*\psi_n(x)\alpha \geq 0$. Conversely suppose $f^{(n)}(v) \geq 0$ for $v \in M_n(V)$. Then $f^{(n)}(v) = \beta^*y\beta$ for a suitable matrix β and y in C . Applying the inverse map yields $v = \beta^*g^{(n)}(y)\beta = \beta^*\psi_n^{-1}(y)\beta \geq 0$. This shows that f is a complete order isomorphism. To show that f is an isometry let $v \in M_n(V)$ and $\varepsilon > 0$. Obviously $v \in (\|v\| + \varepsilon)B_n$. Thus we may write $v = (\|v\| + \varepsilon)\alpha x\beta$ for x in K . Then $f^{(n)}(v) = (\|v\| + \varepsilon)\alpha\psi_1(x)\beta$, which implies $\|f^{(n)}(v)\| \leq \|v\| + \varepsilon$ for all $\varepsilon > 0$ by definition of the base norm. Thus $\|f^{(n)}(v)\| \leq \|v\|$. On the other hand $f^{(n)}(v) = (\|f^{(n)}(v)\| + \varepsilon)\alpha y\beta$ and hence applying the inverse map $v = (\|f^{(n)}(v)\| + \varepsilon)\alpha\psi_1^{-1}(y)\beta$ which yields $\|v\| \leq \|f^{(n)}(v)\|$. This proves that f is a complete isometry as claimed. \square

It is known that approximate operator systems are in particular matrix regular spaces as defined in Definition 1.5. We will show next that our matrix base norm spaces are also matrix regular. This will be of help when proving the duality relations between approximate operator systems and matrix base norm spaces.

Remark 1.39. Let (V, K) be an m-base norm space. Let $n \in \mathbb{N}$. Given $v \in M_n(V)$ such that $v \geq 0$ and $\|v\| \leq 1$ there is $x \in K_n$ such that $x \geq v$.

Proof. Let ϕ be the strictly positive functional determined by K , see Proposition 1.31. Since $v \geq 0$ there is $\alpha \in M_n$ and $y \in K_n$ such that $v = \alpha^*y\alpha$. Hence $\phi^{(n)}(v) = \alpha^*\alpha$ and from Lemma 1.37 $\alpha^*\alpha \leq \|\alpha^*\alpha\|\mathbb{1}_n = \|v\|\mathbb{1}_n \leq \mathbb{1}_n$. Thus we find $\beta \in M_n$ such that $\beta^*\beta = \mathbb{1}_n - \alpha^*\alpha$. We let $x = v + \beta^*y\beta \in K_n$. \square

Lemma 1.40. *Let K be an m-base of a matrix ordered vector space V . Let $n \in \mathbb{N}$. Then $v \in \text{amco}_n(K)$ if and only if there are $x_1, x_2 \in K_n$ such that $\begin{pmatrix} x_1 & v \\ v^* & x_2 \end{pmatrix} \geq 0$.*

Proof. If $v \in \text{amco}_n(K)$ there are $l \in \mathbb{N}$, $y \in K_l$ and contractions $\alpha, \beta \in M_{l,n}$ such that $v = \alpha^*y\beta$. Then we see

$$0 \leq \begin{pmatrix} \alpha^* & 0 \\ 0 & \beta^* \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \alpha^*y\alpha & \alpha^*y\beta \\ \beta^*y\alpha & \beta^*y\beta \end{pmatrix}.$$

For $y_1 = \alpha^*y\alpha$ and $y_2 = \beta^*y\beta$ there are $x_1, x_2 \in K_n$ such that $y_i \leq x_i$ for $i = 1, 2$, cf. Remark 1.39. Thus the claim follows at once. For the converse let $v \in M_n(V)$ such that $w = \begin{pmatrix} x_1 & v \\ v^* & x_2 \end{pmatrix} \geq 0$ for some $x_1, x_2 \in K_n$. Since the m-base K generates the matrix cones, there is $y \in K_{2n}$ and $\alpha \in M_{2n}$ such that $w = \alpha^*y\alpha$. Then $v = (1 \ 0)\alpha^*y\alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let ϕ be the strictly positive functional determined by K , cf. Proposition 1.31. Letting

$\beta^* = (1 \ 0)\alpha^*$ and $\gamma = \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we find

$$\begin{aligned} \beta^*\beta &= \beta^*\phi^{(2n)}(y)\beta = (1 \ 0)\alpha^*\phi^{(2n)}(y)\alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (1 \ 0)\begin{pmatrix} \phi^{(n)}(x_1) & \phi^{(n)}(v) \\ \phi^{(n)}(v^*) & \phi^{(n)}(x_1) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \phi^{(n)}(x_1) = \mathbb{1}_n, \end{aligned}$$

and similarly $\gamma^*\gamma = \mathbb{1}_n$. This shows that $v \in \text{amco}_n(K_{2n})$. \square

Proposition 1.41. *Let (V, K) be an m -base norm space. Then V is matrix regular.*

Proof. Let $n \in \mathbb{N}$ and $v \in M_n(V)$ such that $\|v\| < r < 1$. Then there is $b \in \text{amco}_n(K)$ such that $v = rb$. We know that $\begin{pmatrix} x_1 & b \\ b^* & x_2 \end{pmatrix} \geq 0$ for some $x_1, x_2 \in K_n$. Then $r\begin{pmatrix} x_1 & b \\ b^* & x_2 \end{pmatrix} \geq 0$, $rx_1, rx_2 \geq 0$ and $\|rx_1\| = \|rx_2\| = r < 1$. On the other hand, if $v \in M_n(V)$ such that $\begin{pmatrix} v_1 & v \\ v^* & v_2 \end{pmatrix} \geq 0$ for some positive $v_1, v_2 \in M_n(V)$ with $\|v_1\|, \|v_2\| < 1$, there are $x_1, x_2 \in K_n$ such that $x_1 \geq v_1$ and $x_2 \geq v_2$, cf. Remark 1.39. Hence $\begin{pmatrix} x_1 & v \\ v^* & x_2 \end{pmatrix} \geq 0$, which implies $v \in \text{amco}_n(K)$ by Lemma 1.40. \square

For a real ordered vector space E it is known that E is an approximate order unit space if and only if the dual E' is a base norm space. Furthermore, E is a base norm space if and only if E' is an approximate order unit space. We start now with establishing these duality relations between approximate operator systems and matrix base norm spaces.

Proposition 1.42. *Let (V, K) be a matrix base norm space with base K . Then there is a complete isometrically order isomorphism from the dual space $(V, K)^*$ onto the space $A_b(K)$ of all bounded matrix affine maps on the m -base K .*

Proof. Let $\psi = (\psi_n)$ be a matrix affine map from $K \rightarrow \mathbb{C}$. By Lemma 1.28 there is a unique linear extension f of ψ_1 to V . Notice from Lemma 1.29 that $\psi_n = f^{(n)}|_{K_n}$ for all $n \in \mathbb{N}$. Hence the linear map $\Phi: V^* \rightarrow A_b(K)$ defined by $f \mapsto \Phi(f) = (f^{(n)}|_{K_n})$ will be bijective, if we can show that the linear extension of a bounded affine map from $K_1 \rightarrow \mathbb{C}$ is still bounded. But this is clear from the definition of an m -base norm space. Indeed for $v \in B_n$, the unit ball of $M_n(V)$, we have $v = \alpha x \beta$ for some $m \in \mathbb{N}$ and $x \in K_m$ and $\alpha \in M_{n,m}, \beta \in M_{m,n}$ such that $\|\alpha\|, \|\beta\| \leq 1$. Thus

$$\|f^{(n)}(v)\| = \|\alpha f^{(m)}(x)\beta\| \leq \|\psi_m(x)\| \leq 2\|\psi_1\|$$

for all $v \in B_n$ and $n \in \mathbb{N}$. This means that f is completely bounded. So we only need to prove that Φ is completely bi-positive. Let $f = [f_{ij}] \in M_r(V^*)_+$. This means that f read as map from $V \rightarrow M_r$ is completely positive. Then $\Phi^{(r)}(f) = [f_{ij}^{(n)}|_{K_n}]$ is positive because $[f_{ij}^{(n)}(x_{lk})] \geq 0$ for all $x = [x_{lk}] \in M_m(V)_+$ and all $m \in \mathbb{N}$, especially so for all x in K . On the other hand, if $[f_{ij}^{(n)}(x_{lk})] \geq 0$ for all $x = [x_{lk}] \in K_m$ and all $m \in \mathbb{N}$, i.e., $\Phi^{(r)}(f) \geq 0$, then since the m -base K generates the matrix order of V , it follows immediately that f is completely positive. Hence Φ is a complete order isomorphism between V^* and $A_b(K)$. It is left to show that Φ is a complete isometry. Let $f \in M_n(V^*) = CB(V, M_n)$. Let $n \in \mathbb{N}$ and $v \in M_n(V)$ such that $\|v\| < 1$. Then there are $m \in \mathbb{N}$ and $\alpha \in M_{n,m}, \beta \in M_{m,n}$ and $x \in K_m$ such that $v = \alpha x \beta$ and $\|\alpha\|, \|\beta\| \leq 1$. Hence for all $n \in \mathbb{N}$ and $\|v\| < 1$ we find some $m \in \mathbb{N}$ such that $\|f^{(n)}(v)\| \leq \|f^{(m)}(x)\| \leq \|\Phi(f)\|$. This implies $\|f\|_{cb} \leq \|\Phi(f)\|$. Obviously $\|\Phi(f)\| \leq \|f\|_{cb}$, because the m -base K lies in the unit ball of V . Recall from Lemma 1.25 that $\|\Phi(f)\| = \|\Phi(f)\|_e$. \square

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Remark 1.43. Let (X, e_λ) be an approximate operator system. If there is an element $e \in X_+$ such that $\|e\| \leq 1$ and $e_\lambda \leq e$ for all λ , then (X, e) is an operator system under the same matrix ordering and the approximate order unit and the order unit norms coincide.

Proof. We have $e_\lambda^n = e_\lambda \otimes \mathbb{1}_n \leq e \otimes \mathbb{1}_n = e^n$ for all $n \in \mathbb{N}$. Hence $(M_n(X)_h, e_\lambda^n)$ and $(M_n(X)_h, e^n)$ coincide as (real) approximate order unit and order unit spaces for all $n \in \mathbb{N}$ by [47, Lemma 4]. By the proof of Lemma 1.46 the assertion follows easily. \square

Theorem 1.44. *Let V be a matrix ordered complete operator space. If its dual space V^* is an approximate operator system, then it is an operator system and V is a matrix base norm space.*

Proof. Since the unit ball of V^* is w^* -compact, we see from the prove of [47, Lemma 4] that there is a least upper bound $e \in V_+^*$ for the net $(e_\lambda)_\lambda$, where $\|e\| \leq 1$. Thus (V^*, e) is an operator system by Remark 1.43. We will show that $e: V \rightarrow \mathbb{C}$ is strictly positive. It is clear that e is positive. Let $v \in V_+$ such that $v \neq 0$. By the Hahn-Banach theorem there is a bounded linear $x: V_h \rightarrow \mathbb{R}$ such that $x(v) \neq 0$. By Remark 1.7 we can consider x as element of $(V^*)_h = (V')_h = (V_h)'$. Since e is an order unit, there is $r \geq 0$ such that $-re \leq x \leq re$ and in particular $-re(v) \leq x(v) \leq re(v)$. Hence $e(v)$ cannot vanish. Since e is a strictly positive functional, we see that $K = (K_n)_n$, where $K_n = \{v \in M_n(V)_+ \mid e^{(n)}(v) = \mathbb{1}_n\}$ for all $n \in \mathbb{N}$, is an m-base of V .

Let $B = (B_n)_n$, where $B_n = \text{Ball}(M_n(V))$ for all $n \in \mathbb{N}$, be the absolutely matrix convex set of the unit balls of $M_n(V)$. It is clear that $\text{amco}(K) \subset B$. It follows that the sequence of semi-norms generated by $\text{amco}(K)$ is an operator space norm, which we denote by $\|\cdot\|_K$. In order to see that the m-base K generates the given norm of V , we have to show that the Minkowsky functionals of B_n and $\text{amco}_n(K)$ coincide for all $n \in \mathbb{N}$. This will be the case if B_n is contained in the $\|\cdot\|_K$ -norm closure of $\text{amco}_n(K)$. So, let $n \in \mathbb{N}$ and $v \in B_n$. We can interpret v as map from $X = V^*$ to M_n by $v(x) = x^{(n)}(v)$ for all $x \in X$. Since the operator system X has a complete predual, there is a Hilbert space H and a unital complete order isomorphism $\pi: X \rightarrow \mathcal{B}(H)$ that is a homeomorphism with respect to the $\sigma(X, V)$ and $\sigma(\mathcal{B}(H), \mathcal{B}(H)_*)$ topologies, cf. Proposition A.4. Hence we can assume that X is a w^* -closed, self-adjoint subspace of $\mathcal{B}(H)$. By [26, 4.1.5] there is an extension $\phi: \mathcal{B}(H) \rightarrow M_n$ of v such that $\|\phi\|_{cb} \leq 1$. Then from Proposition A.3 the map $\phi: \mathcal{B}(H) \rightarrow M_n$ can be approximated pointwise by a net (ϕ_λ) of normal maps such that $\|\phi_\lambda\|_{cb} \leq 1$. By [26, Thm. 5.3.2] for each λ we find two completely positive and unital maps ψ_λ and ρ_λ from $\mathcal{B}(H)$ to M_n such that the maps

$$M_2(\mathcal{B}(H)) \rightarrow M_2(M_n); \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \psi_\lambda(a) & \phi_\lambda(b) \\ \phi_\lambda^*(c) & \rho_\lambda(d) \end{pmatrix}$$

are completely positive. Then the maps

$$\vartheta_\lambda: \mathcal{B}(H) \rightarrow M_2(M_n); \vartheta_\lambda(a) = \begin{pmatrix} \psi_\lambda(a) & \phi_\lambda(a) \\ \phi_\lambda^*(a) & \rho_\lambda(a) \end{pmatrix}$$

are completely positive, cf. [26, Prop. 5.4.2]. Recall that we have unique decompositions $\psi_\lambda = \psi_\lambda^\sigma + \psi_\lambda^s$ of ψ_λ into its normal and singular parts. Moreover, since ψ_λ is completely positive, ψ_λ^σ and ψ_λ^s are completely positive, too, cf. [53, Lemma 3.4]. We also decompose ρ_λ into its normal and singular parts. Since all these decompositions are unique, we

find that the normal part of ϑ_λ is given by $\vartheta_\lambda^\sigma = \begin{pmatrix} \psi_\lambda^\sigma & \phi_\lambda^\sigma \\ \phi_\lambda^\sigma & \rho_\lambda^\sigma \end{pmatrix}$. Since ϑ_λ is completely positive, ϑ_λ^σ is completely positive. Hence its restriction to X is completely positive and belongs to $M_{2n}(V)$ since $\psi_\lambda^\sigma|_X, \rho_\lambda^\sigma|_X \in M_n(V)_+$ (recall that π is a homeomorphism). By Remark 1.39 there are $w_\lambda^1, w_\lambda^2 \in K_n$ such that $w_\lambda^1 \geq \psi_\lambda^\sigma|_X$ and $w_\lambda^2 \geq \rho_\lambda^\sigma|_X$. Letting $v_\lambda = \phi_\lambda|_X \in M_n(V)$, it follows that $\begin{pmatrix} w_\lambda^1 & v_\lambda \\ v_\lambda^* & w_\lambda^2 \end{pmatrix} \geq 0$, which implies $v_\lambda \in \text{amco}_n(K_{2n})$ by Lemma 1.40. Since (ϕ_λ) converges pointwise to ϕ , the restrictions (v_λ) converges pointwise to v on X . Hence v is in the weak closure (that is, the $\sigma(V, X)$ -closure) of $\text{amco}_n(K)$. Since the latter is a convex set, the weak closure and the norm closure of $\text{amco}_n(K)$ are equal (e.g., [50, 2.4.8]). Thus v is in the norm closure of $\text{amco}_n(K)$. Now, it follows from [12, Cor. 3.9] that the norm of V and $\|\cdot\|_K$ coincide on V_h . Since $\|\cdot\|_K$ is an operator space norm, $\|\cdot\|_K$ is (topologically) equivalent to the operator space norm of V on all matrix levels. Therefore v is in the $\|\cdot\|_K$ -closure of $\text{amco}_n(K)$, too. It follows immediately that $\|\cdot\|_K$ is equal to the given norm on V . Indeed, suppose for contradiction that there is $v \in M_n(V)$ such that $\|v\|_K > r > \|v\|$. Then there is $r > \lambda > \|v\|$ and $b \in B_n$ such that $v = \lambda b$. Since b is in the $\|\cdot\|_K$ -closure of $\text{amco}_n(K)$ there is a sequence $(d_l)_l$ in $\text{amco}_n(K)$ such that $\|d_l - b\|_K \rightarrow 0$. Therefore $\|v\|_K = \lambda$, which contradicts $\lambda < r < \|v\|_K$, and the proof is complete. \square

Corollary 1.45. *Let (X, e_λ) be an approximate operator system. Then X^* is a matrix base norm space with m -base $K = (K_n)_n$, where*

$$K_n = \left\{ f \in CP(X, M_n) \mid \lim_\lambda f(e_\lambda) = \mathbb{1}_n \right\},$$

for all $n \in \mathbb{N}$, i.e., K_n consists of the m -states of X . Furthermore,

$$\|x\| = \sup \left\{ \|f^{(n)}(x)\| \mid f \in K_{2n} \right\}$$

holds for all $x \in M_n(X)$ and $n \in \mathbb{N}$.

Proof. If (X, e_λ) is an approximate operator system then from Proposition 1.12 the bidual (X^{**}, e) , where $e = w^*\text{-}\lim_\lambda \hat{e}_\lambda$, is an operator system. Hence by Theorem 1.44 the predual X^* is a matrix base norm space. Furthermore its m -base is given by the collection $K_n = \{ f \in M_n(X^*)_+ \mid e^{(n)}(f) = \mathbb{1}_n \}$. Obviously, $\lim_\lambda f(e_\lambda) = \lim_\lambda e_\lambda^{(n)}(f) = e^{(n)}(f)$ for all $f \in M_n(X^*)$. This shows that $K_n = \{ f \in M_n(X^*)_+ \mid \lim_\lambda f(e_\lambda) = \mathbb{1}_n \}$.

It is only left to verify the norm equation. Let $n \in \mathbb{N}$ and $x \in M_n(X)$. Then canonically embedding X into its operator bidual $X^{**} = V^*$ yields

$$\|x\| = \|x\|_{cb} = \|x^{(n)}\| = \sup \left\{ \|\phi^{(n)}(x)\| \mid \phi \in B_n \right\},$$

because $\phi^{(n)}(x) = x^{(n)}(\phi)$. But then it is obvious that $\|x\| \geq \sup \{ \|f^{(n)}(x)\| \mid f \in K_{2n} \}$. To see equality, let $\phi = [\phi_{\mu\nu}] \in B_n$. Then from the proofs of Theorem 1.44 and Lemma 1.40 we have $\phi = \alpha f \beta$ for some $f \in K_{2n}$, $\alpha \in M_{n,2n}$ and $\beta \in M_{2n,n}$, where $\|\alpha\|, \|\beta\| \leq 1$. This yields

$$\begin{aligned} \|\phi^{(n)}(x)\| &= \|[\phi(x_{\mu\nu})]\| = \|[\alpha f(x_{\mu\nu})\beta]\| \\ &= \|(\alpha \otimes \mathbb{1}_n) f^{(n)}(x) (\beta \otimes \mathbb{1}_n)\| \\ &\leq \|\alpha\| \|f^{(n)}(x)\| \|\beta\| \\ &\leq \|f^{(n)}(x)\|, \end{aligned}$$

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and so $\|x\| = \sup \{ \|f^{(n)}(x)\| \mid f \in K_{2n} \}$. \square

Lemma 1.46. *Let X be a matrix ordered operator space such that $M_n(X)_h$ is a real approximate order unit space for all $n \in \mathbb{N}$. Then X is an approximate operator system under the given norm and order structure.*

Proof. By assumption there is for all $n \in \mathbb{N}$ an approximate order unit $(u_\lambda^n)_{\lambda \in \Lambda}$ in $M_n(X)_+$ such that

$$\|x\| = \inf \{ r \geq 0 \mid \exists \lambda \in \Lambda \quad -ru_\lambda^n \leq x \leq ru_\lambda^n \},$$

for all $x \in M_n(X)_h$. Letting $e_\lambda = u_\lambda^1$, we have to show that $e_\lambda^n = e_\lambda \otimes \mathbb{1}_n$ is an approximate order unit for $M_n(X)_h$ that generates the given norm on $M_n(X)_h$. It is obvious that $(e_\lambda^n)_\lambda$ is an increasing net in $M_n(X)_+$. Let G denote the finite and commutative group of n by n matrices $[\pm \delta_{ij}]$ with unit $\mathbb{1}_n$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. Notice that $g = g^{-1}$ for all $g \in G$. Given $x = [x_{ij}] \in M_n(X)_h$, where we may assume $\|x\| \leq 1$, we see that $\|g^{-1}xg\| \leq \|x\| \leq 1$ for all $g \in G$, since X is an operator space. Since $(u_\lambda^n)_\lambda$ generates the norm and Λ is directed, there is $\mu \in \Lambda$ such that $-u_\mu^n \leq g^{-1}xg \leq u_\mu^n$ for all $g \in G$. This is of course equivalent with $-g^{-1}u_\mu^n g \leq x \leq g^{-1}u_\mu^n g$ for all $g \in G$. Hence letting

$$u = \frac{1}{|G|} \sum_{g \in G} g^{-1} u_\mu^n g,$$

we see that $-u \leq x \leq u$. Moreover, u must be a diagonal matrix, because u is invariant under G . Indeed for any $h \in G$ we obtain

$$h^{-1}uh = \frac{1}{|G|} \sum_{g \in G} h^{-1}g^{-1}u_\mu^n gh = u,$$

since the multiplication with a group element is a group isomorphism. But then it follows easily that u is diagonal. This means that $u = \oplus_i [u_\mu^n]_{ii}$, because simultaneously multiplying from left and right with $g \in G$ does not change the entries on the diagonal. Furthermore, $1 \geq \|u\| = \max \{ [u_\mu^n]_{ii} \mid 1 \leq i \leq n \}$, where the $[u_\mu^n]_{ii}$ are the diagonal entries of u_μ^n . Consequently, there is $\nu \in \Lambda$ such that $-e_\nu^n \leq [u_\mu^n]_{ii} \leq e_\nu^n$ for $1 \leq i \leq n$, so we obtain $-e_\nu^n \leq -u \leq x \leq u \leq e_\nu^n$. It follows that $(e_\lambda^n)_\lambda$ is an approximate order unit and $\|x\|_e \leq 1$ for the seminorm it generates. On the other hand we have $\|e_\lambda^n\| = \|e_\lambda\| \leq 1$ for all $\lambda \in \Lambda$, since $(e_\lambda)_\lambda$ is an approximate order unit of X_h . Given $x \in M_n(X)_h$ such that $\|x\|_e \leq 1$, there exists by definition of $\|\cdot\|_e$ a $\mu \in \Lambda$ such that $-e_\mu^n \leq x \leq e_\mu^n$. Since $\|e_\mu^n\| \leq 1$ there is by assumption a $\nu \in \Lambda$ such that $-u_\nu^n \leq e_\mu^n \leq u_\nu^n$. Thus $-u_\nu^n \leq x \leq u_\nu^n$ which implies $\|x\| \leq 1$. So, we have proved that $(e_\lambda^n)_\lambda$ is an approximate order unit for $M_n(X)_h$ and generates the given norm on $M_n(X)_h$ for all $n \in \mathbb{N}$. But then it follows easily from

$$\|x\| = \left\| \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \right\|_e = \|x\|_e,$$

where $\|x\|_e = \inf \{ r \geq 0 \mid \exists \lambda \left(\begin{pmatrix} re_\lambda^n & x \\ x^* & re_\lambda^n \end{pmatrix} \geq 0 \right) \}$ for $x \in M_n(X)$ is the approximate order unit operator space norm, that (X, e_λ) is an approximate operator system under the given norm and order. \square

Theorem 1.47. *Let X be a matrix ordered complete operator space. If its dual space $V = X^*$ is a matrix base norm space, then X is an approximate operator system.*

Proof. If V is a matrix base norm space then $V^* = X^{**}$ is an operator system. Notice that $M_n(X)'' = M_n(X^{**})$ for all $n \in \mathbb{N}$. Since $M_n(X^{**})_h$ is a real order unit space it follows that $(M_n(X)_h)'$ is a real base norm space for all $n \in \mathbb{N}$. Then, because $M_n(X)_h$ is an ordered Banach space whose dual is a base norm space, $M_n(X)_h$ is an approximate order unit space for all $n \in \mathbb{N}$. Now we see from Lemma 1.46 that X is an approximate operator system. \square

To summarize, Corollary 1.45 and Theorem 1.47 show that a matrix ordered complete operator space X is an approximate operator system if and only if its dual X^* is a matrix base norm space (cp. [12, Thm. 2.3]). Furthermore, Proposition 1.42 and Theorem 1.44 show that a matrix ordered complete operator space V is a matrix base norm space if and only if V^* is an approximate operator system (cp. [12, Cor. 3.9]).

The step from real order unit spaces and base norm spaces to the matrix ordered versions of these spaces, i.e., the operator systems and the matrix base norm spaces, was done for applying these spaces more easily to C^* -algebras. As a first example of this we are going to verify now that preduals of W^* -algebras are not only Banach spaces. They are m-base norm spaces, where the m-base is the m-convex normal state space of the W^* -algebra.

Proposition 1.48. *Let \mathcal{M} be a W^* -algebra. Then there is an m-base norm space (V, K) such that $(V, K)^* =_{cb} \mathcal{M}$. Moreover, V is complete in the m-base norm and uniquely determined up to isomorphism. The m-base K is m-affinely isomorph to the normal m-convex state space $CS^\sigma(\mathcal{M})$ and hence is norm-closed.*

Proof. Let V be the predual of \mathcal{M} . We can embed V isometrically into the dual \mathcal{M}^* of \mathcal{M} in the usual way. Since \mathcal{M} is a unital C^* -algebra, it is especially an operator system, i.e., \mathcal{M} carries a matrix order and an operator space structure. Hence \mathcal{M}^* is a matrix ordered operator space, actually an m-base norm space, since it is the dual of an operator system. We can identify V with the normal or $\sigma(\mathcal{M}, V)$ -continuous functionals on \mathcal{M} . Now we will give V the m-base structure inherited from \mathcal{M}^* . This means, identifying $M_n(V) = \{ f \in M_n(\mathcal{M}^*) \mid f \text{ } w^*\text{-continuous} \}$ for all $n \in \mathbb{N}$, we set

$$M_n(V)_+ = \{ f: \mathcal{M} \rightarrow M_n \mid f \text{ completely positive and } \sigma\text{-continuous} \}$$

and

$$K_n = \{ f \in M_n(V)_+ \mid f(e) = \mathbb{1}_n \},$$

where e denotes the unit of \mathcal{M} . Notice from [52] that $V_+ \neq \emptyset$ and that $V_h = V_+ - V_+$. Hence we have a matrix ordered space and we will show next that $K = (K_n)_n$ is an m-base of V . Defining $\hat{e}: V \rightarrow \mathbb{C}$ by $\hat{e}(f) = f(e)$, it is obvious that \hat{e} is a positive linear map. Assume $f \in V_+$ such that $\hat{e}(f) = f(e) = 0$. Then, since e is an order unit, we find $-\|x\|f(e) \leq f(x) \leq \|x\|f(e)$ for all $x \in \mathcal{M}_h$. This implies $f = 0$ on \mathcal{M}_h and hence on \mathcal{M} . Thus \hat{e} is strictly positive and by definition of K_n we have $K_n = \{ f \in M_n(V)_+ \mid \hat{e}^{(n)}(f) = f(e) = \mathbb{1}_n \}$ for all $n \in \mathbb{N}$. So, K is an m-base of V by Proposition 1.31. Notice that it is also shown that \hat{e} is an order unit for V^* . To see that (V, K) is an m-base norm space it is left to show that $\text{amco}(K) = B$, where $B = (B_n)_{n \in \mathbb{N}}$ is the absolutely m-convex set of the unit balls of $M_n(V)$, i.e., $B_n = \{ f \in M_n(V) \mid \|f\|_{cb} \leq 1 \}$. Notice that $f \in K_n$ is completely positive and unital. Hence $\|f\|_{cb} = \|f(e)\| = 1$. Since we have an operator space norm (induced from \mathcal{M}^*), it is

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obvious that $\text{amco}(K) \subset B$. On the other hand, given $f \in B_n$ there are two unital and completely positive maps $\psi_1, \psi_2: \mathcal{M} \rightarrow M_n$ such that the map

$$\phi: \mathcal{M} \rightarrow M_{2n}; \quad \phi(x) = \begin{pmatrix} \psi_1(x) & f(x) \\ f^*(x) & \psi_2(x) \end{pmatrix}$$

is completely positive, cf. [26, Thm. 5.3.2, Prop. 5.4.2]. Recall that we have for $j = 1, 2$ a unique decomposition $\psi_j = \psi_j^\sigma + \psi_j^s$ of ψ_j into its normal and singular parts. Moreover, since ψ_j is completely positive, ψ_j^σ and ψ_j^s are completely positive, too, cf. [53, Lemma 3.4]. Since these decompositions are unique, we find that the normal part of ϕ is given by $\phi^\sigma = \begin{pmatrix} \psi_1^\sigma & f \\ f^* & \psi_2^\sigma \end{pmatrix}$. Since ϕ is completely positive, ϕ^σ is completely positive. It follows from Lemma 1.40 in combination with Remark 1.39 that $f \in \text{amco}(K)$, because K is an m-base. Thus we have shown so far that (V, K) is an m-base norm space. Next we will show that the dual matrix order defined by V corresponds with the given matrix order of \mathcal{M} . The W^* -algebra \mathcal{M} carries the w^* -topology (i.e., $\sigma(\mathcal{M}, V)$ -topology) and the w^* -topology of the W^* -algebra $M_n(\mathcal{M})$ coincides with the product topology, cf. Lemma A.5. Recall that $M_n(\mathcal{M})_+$ is w^* -closed, cf. [52, Lem. 1.7.1]. With the identification $M_n(V^*) = L(V, M_n)$ let the dual order be

$$M_n(\mathcal{M})^+ = \left\{ x \in M_n(V^*) \mid x^{(n)}(f) \geq 0 \text{ for all } f \in M_n(V)_+ \right\}.$$

Obviously, if $x \in M_n(\mathcal{M})_+$ then $x^{(n)}(f) = f^{(n)}(x) \geq 0$ for all $f \in M_n(V)_+$ so that $M_n(\mathcal{M})_+ \subset M_n(\mathcal{M})^+$. For the converse let $x \notin M_n(\mathcal{M})_+$. Since $M_n(\mathcal{M})_+$ is w^* -closed, there is a w^* -continuous map $\varphi \in CP(\mathcal{M}, M_n)$ such that $\varphi^{(n)}(x) \not\geq 0$ by Remark A.2. Therefore, $x \notin M_n(\mathcal{M})^+$. So, the dual matrix order induced from (V, K) is the given matrix order of \mathcal{M} . Since \mathcal{M} is a unital C^* -algebra, its norm is the matrix order unit norm. From Proposition 1.42 we have $A_b(K) =_{cp} (V, K)^* =_{cp} \mathcal{M}$. This implies $A_b(K) =_{cb} \mathcal{M}$ and again by Proposition 1.42 the matrix order unit norm is the cb -norm of V^* . Hence we see that $M_n(\mathcal{M}) = CB(V, M_n)$ for all $n \in \mathbb{N}$.

To see uniqueness, assume there is another complete m-base norm space (W, C) such that $(W, C)^* = \mathcal{M}$. Then $V =_1 W$ isometrically, since V and W are usual Banach preduals of \mathcal{M} , [52, Corollary 1.13.3]. But then a net in \mathcal{M} will converge in the $\sigma(\mathcal{M}, V)$ -topology exactly if it converges in the $\sigma(\mathcal{M}, W)$ -topology. Thus both topologies are equal. As shown there are complete isometric order isomorphism of (V, K) and (W, C) into \mathcal{M}^* , which is the operator bidual of both V and W . Under these embeddings the m-bases will map to the same m-convex set, namely those completely positive and unital maps from \mathcal{M} to the matrices, that are w^* -continuous. Hence K and C are m-affinely isomorphic. Therefore, V and W are isomorphic as m-base norm spaces by Remark 1.38. \square

Notice that it makes sense to consider preduals of W^* -algebras as m-base norm spaces. While the predual of a W^* -algebra is uniquely determined, the predual seen as Banach space does not determine the W^* -algebra in general. The algebraic structure of the W^* -algebra cannot be stored in the *Banach space structure* of its predual, since the predual is already determined by the order structure of the W^* -algebra. Given some W^* -algebra \mathcal{M} the opposite algebra \mathcal{M}^{op} has the same order structure as \mathcal{M} , so that \mathcal{M} and \mathcal{M}^{op} have the same predual \mathcal{M}_* . Therefore, if the algebraic structure of \mathcal{M} and \mathcal{M}^{op} would be determined by their common predual, \mathcal{M} and \mathcal{M}^{op} had to be isomorphic W^* -algebras, which is wrong in general, cf. [20]. However, considering the predual as m-base

norm space, which is a *matrix ordered* operator space, the predual generates the matrix ordering of its operator dual, that is, the matrix order structure of the W^* -algebra, and thus determines the multiplication of the algebra. Hence the preceding proposition establishes a bijective correspondence between W^* -algebras and their m-base norm preduals. The main result of the next chapter will characterize those m-base norm spaces that are (m-base norm) preduals of W^* -algebras among all m-base norm spaces, see Theorem 2.19. Consequently, the theory of W^* -algebras is equivalent to the theory of a certain class of m-base norm spaces. Moreover, interpreting the results of [19] in this context, there seems to be already an equivalent formulation of the type theory of W^* -algebras in terms of m-base norm spaces.

Matrix convex faces

In chapter 3 we will be confronted with the need for a matrix version of split faces. A split face F of a convex set C is a face such that there is another face F' of C such that every point of $x \in C$ can be written in a unique way as convex combination $x = ty + (1 - t)y'$, where $y \in F$ and $y' \in F'$. Now it is not obvious what a matrix convex face should be. In addition, there are always distinct matrix convex combinations expressing the same point, since $\alpha^*y\alpha = \alpha^*u^*uyu^*u\alpha$, where u is a unitary matrix. However, recalling the preceding section about the duality of operator systems and m-base norm spaces, it becomes clear that there should be something like ‘matrix convex split faces’. To have a more concrete example, consider a W^* -algebra \mathcal{L} . It is known, for instance, that \mathcal{L} splits uniquely into an atomic and a purely non-atomic part, say $\mathcal{L} = \mathcal{M} \oplus_\infty \mathcal{N}$. W^* -algebras have essentially unique preduals that are m-base norm spaces by Proposition 1.48. So we can identify $\mathcal{L}_* = \mathcal{M}_* \oplus_1 \mathcal{N}_*$. Now, the m-bases of \mathcal{M}_* and \mathcal{N}_* should generate the m-base of \mathcal{L}_* and should be—in a sense that we will make precise in the current section—a pair of m-convex split faces in the m-base of \mathcal{L}_* . It should be clear from the preceding sections that we cannot rely only on bases in the usual sense, since they determine only the order on the first level, but not the orderings on the higher matrix levels. We begin with defining matrix convex faces.

Definition 1.49. Let $K = (K_n)_n$ be an m-convex set in a vector space V . An *m-convex face* of K is an m-convex subset¹ $S \subset K$ such that S_n is a convex face of K_n in the usual sense for all $n \in \mathbb{N}$.

The next proposition is essentially taken from [57, Section 3].

Proposition 1.50. Let K be an m-convex set in V . Let $n \in \mathbb{N}$ and let $F \subset K_n$ be a non-empty face of K_n that is M_n -convex. Then there is a unique m-convex face S of K such that $S_n = F$.

Proof. We show the existence first. Let $F_1 = \mathbb{1}_{n,1}^* F \mathbb{1}_{n,1}$. Obviously F_1 is a convex subset of K_1 . Let $y_1, z_1 \in K_1$ such that $x_1 = ry_1 + (1 - r)z_1 \in F_1$. By definition of F_1 there is $x \in F$ such that $x_1 = \mathbb{1}_{n,1}^* x \mathbb{1}_{n,1}$. Let $(e_{ij})_{i,j=1}^n$ be the standard basis of M_n . Then $x_1 \otimes \mathbb{1}_n = \text{diag}(x_1, \dots, x_1) = \sum_{j=1}^n e_{1j}^* x e_{1j} \in F$, because F is M_n -convex. It follows immediately that F_1 is a face of K_1 , because F is a face of K_n .

For each $m \in \mathbb{N}$ let S_m be the set of matrices $[x_{ij}] \in K_m$ such that $x_{ii} \in F_1$ for $i = 1, \dots, m$. Obviously S_m is a convex face of K_m . Letting $\text{Sp}_m x = m^{-1} \sum_{i=1}^m x_{ii}$ for

¹The notation $S \subset K$ for matrix sets is an abbreviation for $S_n \subset K_n$ for all $n \in \mathbb{N}$.

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$x = [x_{ij}] \in K_m$ we see that $S_m = \{x \in K_m \mid \text{Sp}_m x \in F_1\}$ for all $m \in \mathbb{N}$, because F_1 is a face. We easily verify that $\text{Sp}_m(u^*xu) = \text{Sp}_m x$ and $\text{Sp}_m(\alpha x \alpha) = \text{Sp}_m(x \alpha^2)$ hold for unitary u and positive α in M_m . Thus in order to prove that S_m is M_m -convex, it is sufficient to show that $\sum_{i=1}^2 \alpha_i x_i \alpha_i \in S_m$, where $x_i \in S_m$ and $\alpha_i \in M_m^+$ for which $\sum_{i=1}^2 \alpha_i^2 = \mathbb{1}_m$, recall also (1.4). By assumption we have

$$\begin{aligned} F_1 \ni \frac{1}{2} \sum_{i=1}^2 \text{Sp}_m x_i &= \frac{1}{2} \sum_{i=1}^2 \text{Sp}_m \left(x_i \sum_{j=1}^2 \alpha_j^2 \right) \\ &= \frac{1}{2} \left(\text{Sp}_m \left(\sum_{i=1}^2 \alpha_i x_i \alpha_i \right) + \text{Sp}_m(\alpha_1 x_2 \alpha_1 + \alpha_2 x_1 \alpha_2) \right). \end{aligned}$$

Since the summands lie in K_1 and F_1 is a face of K_1 , it follows $\sum_{i=1}^2 \alpha_i x_i \alpha_i \in S_m$. So S_m is M_m -convex for each $m \in \mathbb{N}$. Then $S = (S_m)_{m \in \mathbb{N}}$ is an m-convex set, since S is by definition closed under compressions and direct sums.

It is left to show $S_n = F$ and that the construction is unique. Fix $m \in \mathbb{N}$. Let $B_m \subset K_m$ be a non-empty face of K_m that is M_m -convex. Assume that $\mathbb{1}_{m,1}^* B_m \mathbb{1}_{m,1} = F_1$. We will prove that in this case we must have $B_m = S_m$. By definition of S_m it is obvious that $B_m \subset S_m$. We will show $S_m \subset B_m$ by induction. Let $\text{diag}(x_1, \dots, x_m) \in S_m$. Then $x_1, \dots, x_m \in F_1 = \mathbb{1}_{m,1}^* B_m \mathbb{1}_{m,1}$, and since B_m is M_m -convex, it follows $x_i \otimes \mathbb{1}_m \in B_m$ for $i = 1, \dots, m$. Hence we obtain

$$\text{diag}(x_1, \dots, x_m) = \sum_{i=1}^m e_{ii}(x_i \otimes \mathbb{1}_m) e_{ii} \in B_m.$$

Now, let $x \in S_m$ be a matrix that has at most μ non-zero entries off its diagonal. We choose one of these entries, so let $x_{ij} \neq 0$. Let $\alpha = \text{diag}(r_1, \dots, r_m)$ such that $r_j = -1$ and $r_\nu = 1$ for $\nu \neq j$. Then $y = \alpha x \alpha \in S_m$ and we obtain $x_{\nu\nu} = y_{\nu\nu}$ for $\nu = 1, \dots, m$, $y_{ij} = -x_{ij}$ and $y_{\nu\mu} = 0$ if $x_{\nu\mu} = 0$. By inductive hypothesis $z = \frac{1}{2}(x + y) \in B_m$, since z has at most $\mu - 1$ non-zero entries off its diagonal. Consequently $x \in B_m$, because B_m is a face. Now, applying the last result in particular to the face F , which is M_n -convex, it follows that $S_n = F$, since we have $F_1 = \mathbb{1}_{n,1}^* F \mathbb{1}_{n,1}$ by definition. Moreover, if D is another m-convex face of K such that $D_n = F$, then $F_1 = \mathbb{1}_{m,1}^* D_m \mathbb{1}_{m,1}$ for all $m \in \mathbb{N}$, and consequently $S_m = D_m$ for all $m \in \mathbb{N}$, completing the proof. \square

Definition 1.51 (Matrix convex split face). Let $K = (K_n)_n$ be an m-convex set in a vector space V . An m-convex face F of K is called an *m-convex split face* of K , if there is an m-convex face F' of K such that F_1 and F'_1 are complementary split faces of K_1 and $K = \text{mco}(F \cup F')$. Obviously then F' is also an m-convex split face of K , and F and F' are called *complementary m-convex split faces* of K .

Proposition 1.52. Given two m -base norm spaces (W, F) and (W', F') , we define a matrix ordering on the algebraic direct sum² $V = W \oplus W'$ by setting

$$M_n(V)_+ = M_n(W)_+ \oplus M_n(W')_+,$$

for all $n \in \mathbb{N}$. Then V is an m -base norm space with m -base $K = \text{mco}(F \cup F')$ (reading F, F' as subsets of V), and F and F' are complementary m -convex split faces of K .

²Note that V is a $*$ -vector space under the involution $v^* = w^* \oplus w'^*$, where $v = w \oplus w'$.

Proof. Let $\phi: W \rightarrow \mathbb{C}$ and $\phi': W' \rightarrow \mathbb{C}$ be the m-base functionals for F and F' , respectively, cf. Proposition 1.31. We define a function $\psi: V \rightarrow \mathbb{C}$ by setting $\psi(v) = \phi(w) + \phi'(w')$ for $v = w \oplus w'$, where $w \in W$ and $w' \in W'$. Obviously, ψ is well-defined, and it is a simple exercise to check that ψ is linear. Moreover, by definition of the ordering on V , it is easily seen that ψ is strictly positive. Our claim is that $K_n = \{v \in M_n(V)_+ \mid \psi^{(n)}(v) = \mathbb{1}_n\}$ for all $n \in \mathbb{N}$. Now, if $x \in K_n$ for some $n \in \mathbb{N}$, then there is an m-convex combination $x = \alpha^* y \alpha + \beta^* y' \beta$, where $y \in F_n$, $y' \in F'_n$ and $\alpha^* \alpha + \beta^* \beta = \mathbb{1}_n$. Hence $x \in M_n(V)_+$, and $\psi^{(n)}(x) = \alpha^* \phi^{(n)}(y) \alpha + \beta^* \phi'^{(n)}(y') \beta = \mathbb{1}_n$. Conversely, if $v \in M_n(V)_+$ and $\psi^{(n)}(v) = \mathbb{1}_n$, then by definition of the ordering on V there are $w \in M_n(W)_+$ and $w' \in M_n(W')$ such that $v = w \oplus w'$. Since W and W' are m-base norm spaces, $w = \alpha^* y \alpha$ and $w' = \beta^* y' \beta$ for some $y \in F_n$, $y' \in F'_n$, and matrices $\alpha, \beta \in M_n$. It follows that

$$\mathbb{1}_n = \psi^{(n)}(v) = \alpha^* \phi^{(n)}(y) \alpha + \beta^* \phi'^{(n)}(y') \beta = \alpha^* \alpha + \beta^* \beta.$$

Thus $v = \alpha^* y \alpha + \beta^* y' \beta \in \text{mco}_n(F \cup F') = K_n$. This shows that K is an m-base of the matrix ordered vector space V . To see that (V, K) is an m-base norm space, cf. Definition 1.35, we have still to show that $V_h = V_+ - V_+$, (but this is trivial by definition of the involution of the direct sum), and that the seminorms

$$\|v\|_n = \inf \{ \lambda \geq 0 \mid v \in \lambda \text{amco}_n(K) \}$$

on $M_n(V)$ are norms for all $n \in \mathbb{N}$. If $\|v\|_n = 0$, then there is a monotone decreasing sequence $(\lambda_\nu)_{\nu \in \mathbb{N}}$ of positive numbers converging to 0, such that $v = \lambda_\nu b_\nu$, where $b_\nu \in \text{amco}_n(K)$. We find $b_\nu = \alpha_\nu x_\nu \beta_\nu$, where $x_\nu \in K$. Then $x_\nu = \gamma_{1,\nu}^* y_\nu \gamma_{1,\nu} + \gamma_{2,\nu}^* y'_\nu \gamma_{2,\nu}$ with $y_\nu \in F$ and $y'_\nu \in F'$. It follows that $v = \lambda_\nu \alpha_\nu \gamma_{1,\nu}^* y_\nu \gamma_{1,\nu} \beta_\nu + \lambda_\nu \alpha_\nu \gamma_{2,\nu}^* y'_\nu \gamma_{2,\nu} \beta_\nu$. Since $v = w \oplus w'$ uniquely, we find that $w = \lambda_\nu \alpha_\nu \gamma_{1,\nu}^* y_\nu \gamma_{1,\nu} \beta_\nu$ and $w' = \lambda_\nu \alpha_\nu \gamma_{2,\nu}^* y'_\nu \gamma_{2,\nu} \beta_\nu$ for all $n \in \mathbb{N}$. Hence $\|w\|_n, \|w'\|_n \leq \lambda_\nu$ for all $\nu \in \mathbb{N}$, so that $\|w\|_n = \|w'\|_n = 0$. Then $w = 0$ and $w' = 0$ and consequently $v = 0$, which shows that $\|\cdot\|_n$ is a norm.

So far, we have shown that (V, K) is an m-base norm space. Next we will show that F and F' are complementary m-convex split faces of K . Let $n \in \mathbb{N}$. If $z = r x_1 + (1-r) x_2 \in F_n$ for $r \in (0, 1)$ and $x_1, x_2 \in K_n \subset M_n(V)_+$, then, since F and F' are m-bases of W and W' , respectively, there are $\alpha_i, \beta_i \in M_n$, and $y_i \in F_n$ and $y'_i \in F'_n$, such that $x_i = \alpha_i^* y_i \alpha_i + \beta_i^* y'_i \beta_i$ for $i = 1, 2$. We find

$$\mathbb{1}_n = \psi^{(n)}(x_i) = \alpha_i^* \phi^{(n)}(y_i) \alpha_i + \beta_i^* \phi'^{(n)}(y'_i) \beta_i = \alpha_i^* \alpha_i + \beta_i^* \beta_i$$

for $i = 1, 2$. Moreover, because $z \in F_n \subset M_n(W)_+$, it follows from

$$z = r x_1 + (1-r) x_2 = r \alpha_1^* y_1 \alpha_1 + (1-r) \alpha_2^* y_2 \alpha_2 \oplus r \beta_1^* y'_1 \beta_1 + (1-r) \beta_2^* y'_2 \beta_2$$

that $r \beta_1^* y'_1 \beta_1 + (1-r) \beta_2^* y'_2 \beta_2 = 0$. Since $y'_1, y'_2 \in F'_n$, it follows that $r \beta_1^* \beta_1 + (1-r) \beta_2^* \beta_2 = 0$, and hence $\beta_1^* \beta_1, \beta_2^* \beta_2 = 0$. Consequently, $\alpha_1^* \alpha_1, \alpha_2^* \alpha_2 = \mathbb{1}_n$, which shows $x_i = \alpha_i^* y_i \alpha_i \in F_n$ for $i = 1, 2$. This proves that F_n and, by symmetry, F'_n are faces of K_n for all $n \in \mathbb{N}$. Since F_1 and F'_1 are bases, it is immediate that each $x \in K_1$ can be expressed as a unique convex combination $x = r y + (1-r) y'$, where $y \in F_1$, $y' \in F'_1$ and $r \in [0, 1]$. So, we have shown that F and F' are complementary m-convex split faces of K and the proof is complete. \square

1. Matrix Orderings

Proposition 1.53. *Let (V, K) be an m -base norm space. Given an m -convex split face F of K with complementary split face F' , we let $W = \text{lin } F_1$ and $W' = \text{lin } F'_1$. Then (W, F) and (W', F') are m -base norm spaces under the induced matrix ordering, and $(V, K) =_{cp} (W, F) \oplus_1 (W', F')$.*

Proof. It is easy to verify that $W_h = \text{lin}_{\mathbb{R}} F_1$ and that $W = W_h + iW_h$. Since $K_1 = F_1 \oplus_c F'_1$, we know from [4, Prop. II.6.1] that $\text{lin}_{\mathbb{R}} F_1 \cap \text{lin}_{\mathbb{R}} F'_1 = \{0\}$, which means that $W_h \cap W'_h = \{0\}$. If $x \in W \cap W'$, then obviously $\text{Re } x, \text{Im } x \in W_h \cap W'_h$, so that $W \cap W' = \{0\}$. Since $K_1 = F_1 \oplus_c F'_1$, it is also obvious that $V \subset W + W'$. Thus we have shown so far, that $V = W \oplus W'$ (which immediately implies $M_n(V) = M_n(W) \oplus M_n(W')$ for all $n \in \mathbb{N}$). Next we will verify that (W, F) (and hence by symmetry also (W', F')) is an m -base norm space under the matrix ordering induced by V . We have to show that F_n is a subset of $M_n(W)$ for all $n \in \mathbb{N}$, i.e., given $x = [x_{ij}] \in F_n$ we have to show $x_{ij} \in W$. Notice that $x_{ii} \in F_1$ for $i = 1, \dots, n$, $x_{ij} = x_{ji}^*$ and that it is sufficient to consider the case of 2×2 matrices only. So, let $x \in F_2$. Then it follows easily that $\text{Re } x, \text{Im } x \in W$, and thus $x \in W$. Hence we have verified $F_n \subset M_n(W)$ for all $n \in \mathbb{N}$. It is obvious that $W_+ = W \cap V_+$ is a proper and generating cone, and evidently $M_n(W)_+ = M_n(W) \cap M_n(V)_+$ determines a matrix ordering on W . Moreover, the restriction of the m -base functional ϕ of (V, K) , cf. Proposition 1.31, to W is easily seen to be an m -base functional for (W, F) , so that F is an m -base. Indeed, let $y \in M_n(W)_+$ such that $\phi^{(n)}(y) = 1_n$. Then $y \in M_n(V)_+$, which shows directly $y \in K_n$. Since F and F' are complementary split faces of K , there exists an m -convex combination $y = \alpha^* z \alpha + \beta^* z' \beta$, where $z \in F_n$ and $z' \in F'_n$. Since $M_n(V) = M_n(W) \oplus M_n(W')$, we obtain $\beta^* z' \beta = 0$. Thus $\beta^* \beta = 0$, so that $\alpha^* \alpha = 1_n$ and $y = \alpha^* z \alpha \in F_n$. We have verified that (W, F) and (W', F') are m -base norm spaces. To complete the proof we have to show that $M_n(V)_+ = M_n(W)_+ \oplus M_n(W')_+$. One direction is clear, for the other direction let $v \in M_n(V)_+$. Then $v = w + w'$ for uniquely determined elements $w \in M_n(W)$ and $w' \in M_n(W')$. Since (V, K) is an m -base norm space, $v = \gamma^* x \gamma$ for $x \in K_n$ and $\gamma \in M_n$. Since F and F' are complementary split faces, there is an m -convex combination $x = \alpha^* y \alpha + \beta^* y' \beta$, where $y \in F_n$ and $y' \in F'_n$. Hence $v = \gamma^* \alpha^* y \alpha \gamma + \gamma^* \beta^* y' \beta \gamma$, which by uniqueness implies $w = \gamma^* \alpha^* y \alpha \gamma$ and $w' = \gamma^* \beta^* y' \beta \gamma$. But now it is clear that w and w' are positive and the proof is complete. \square

Corollary 1.54. *Let (W, F) and (W', F') be m -base norm spaces. Then*

$$((W, F) \oplus_1 (W', F'))^* =_{cp} A_b(F) \oplus_{\infty} A_b(F').$$

Proof. We know that $(V, K) = (W, F) \oplus_1 (W', F')$ is an m -base norm space, and that $(V, K)^* =_{cp} A_b(K)$. Since F and F' are complementary split faces of K , it is easy to verify that $A_b(K) =_{cp} A_b(F) \oplus_{\infty} A_b(F')$. Hence the claim follows. \square

It is obvious that an extreme point of a face of a convex set is also an extreme point of the convex set. We end this chapter noting that a similar result, though less obvious, holds for structural elements of m -convex split faces.

Proposition 1.55. *Let F be an m -convex split face of an m -convex set K . If $n \in \mathbb{N}$ and $x \in \text{str}(F_n)$, then $x \in \text{str}(K_n)$.*

Proof. Suppose that $x \in \text{str}(F_n)$ and $x = \alpha_1^* x_1 \alpha_1 + \alpha_2^* x_2 \alpha_2$ is an M_n -convex combination, where $x_1, x_2 \in K_n$. Let F' be the m -convex split face that is complementary to F . Then

there are M_n -convex combinations $x_1 = \beta^*y\beta + \beta'^*y'\beta'$ and $x_2 = \gamma^*z\gamma + \gamma'^*z'\gamma'$ such that $y, z \in F_n$ and $y', z' \in F'_n$. Then we conclude from

$$x = \alpha_1^*\beta^*y\beta\alpha_1 + \alpha_2^*\gamma^*z\gamma\alpha_2 + \alpha_1^*\beta'^*y'\beta'\alpha_1 + \alpha_2^*\gamma'^*z'\gamma'\alpha_2$$

that the last two summands vanish, because $x \in F_n$. Since F' is an m-base, it follows that $\alpha_1^*\beta'^*\beta'\alpha_1, \alpha_2^*\gamma'^*\gamma'\alpha_2 = 0$, so we can assume that $x = \alpha_1^*\beta^*y\beta\alpha_1 + \alpha_2^*\gamma^*z\gamma\alpha_2$ is a proper m-convex combination. Therefore there are unitaries $u, v \in M_n$ and $r, s \in (0, 1)$ such that $\beta\alpha_1 = ru, \gamma\alpha_2 = sv, y = uxu^*$ and $z = vxv^*$. Thus in particular α_1 and α_2 are invertible, which yields that $\beta', \gamma' = 0$. Consequently β and γ are unitaries and $x_1 = \beta^*y\beta$ and $x_2 = \gamma^*z\gamma$. Now, obviously x_1 and x_2 are unitarily equivalent to x , which proves $x \in \text{str}(K_n)$. \square

2. Multiplier Algebra of Operator Systems

Following up Proposition 1.48, the aim of the current chapter is to characterize normal matrix convex state spaces of W^* -algebras among matrix convex sets. Instead of starting with an m -convex set K contained in some vector space, we assume for convenience that we have an m -base norm space (V, K) such that K is its m -base. (There is no essential loss of generality doing so.) If K should be the normal m -convex state space of a W^* -algebra, or in other words if (V, K) should be a predual of a W^* -algebra, then this W^* -algebra must be $(V, K)^* = A_b(K)$ up to isomorphism. So an intermediate step to achieve our aim is to provide an associative multiplication on the operator system $A_b(K)$. The essential observation is that given an operator system we can define a (multiplier) algebra by using purely the matrix order of the operator system. Then, having an order unit, we can embed the algebra into the operator system. However, the embedding does not need to be surjective in general, that means the constructed algebra can be small. To prove that the embedding is surjective, in which case $A_b(K)$ will be turned into a C^* -algebra with predual, and thus into a W^* -algebra, we will need to pose only a single additional condition on K that ensures the existence of sufficiently many projections in the multiplier algebra. The difference of our approach compared with [9, 10, 8, 7] or [6] is that we can define an associative algebra directly using matrix orderings. In the papers of Alfsen and Shultz so-called P -projections were introduced for a given dual pair consisting of a (real) base norm space V and an order unit space A . These P -projections serve as a kind of generalization of projections, i.e., in the special case that A is the self-adjoint part of a W^* -algebra the P -projections on A are the maps $a \mapsto pap$ with p a projection in A . Using P -projections, Alfsen and Shultz built a spectral theory that could be used to define a Jordan product. So they first defined projections and afterwards constructed a Jordan algebra from these. In [64] Werner defined P -projections for matrix ordered spaces and used them to construct a self-adjoint algebra. Later on in a simplification of his thesis Werner replaced P -projections by the notion of so-called nh -projections (neutral and hereditary projections), but he still built an algebra using these projections.

We will construct the multiplier algebra of an operator system directly by using its matrix order structure without the use of projections. The construction is borrowed from [54], where the multiplier algebra is constructed for matrix ordered Hilbert spaces. We repeat the construction here for convenience of the reader. For the parts that we need, the assumption of Hilbert spaces is not required.

So, let (X, e) be an operator system supplied with the matrix order norm. Let $L(X)$ denote the linear mappings from X to X . For $x \in X$ and $T \in L(X)$ we define a right multiplication by $xT^* = (Tx^*)^*$. Let $n \in \mathbb{N}$. For $x = [x_{ij}] \in M_n(X)$ and $T = [T_{ij}] \in M_n(L(X))$ we define left and right multiplication with the matrix T by standard matrix multiplication, i.e., $Tx = [\sum_j T_{ij}x_{jk}]$ and

$$xT^* = (Tx^*)^* = \left[\sum_j T_{ij}x_{kj}^* \right]^* = \left[\sum_j x_{ij}T_{kj}^* \right],$$

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that means $[T_{ij}]^* = [T_{ji}^*]$. We then define a Jordan product by setting

$$\llbracket TxS^* \rrbracket = \frac{1}{2} \left((Tx)S^* + T(xS^*) \right)$$

for $T, S \in M_n(L(X))$ and $x \in M_n(X)$. Let $\mathbb{1}$ denote the identity mapping on X . For the following calculation notice that for example

$$\llbracket \text{diag}(\mathbb{1}, T, \mathbb{1})[x_{ij}] \text{diag}(\mathbb{1}, T^*, \mathbb{1}) \rrbracket = \begin{pmatrix} x_{11} & x_{12}T^* & x_{13} \\ Tx_{21} & \llbracket Tx_{22}S^* \rrbracket & Tx_{23} \\ x_{31} & x_{32}T^* & x_{33} \end{pmatrix}. \quad (2.1)$$

Definition 2.1. Let (X, e) be an operator system. We define the *matrix multiplier algebra* $\mathcal{M}(X) = \mathcal{M}$ of X to be the set

$$\mathcal{M} = \left\{ T \in L(X) \mid \llbracket \text{diag}(T, \mathbb{1}, \dots, \mathbb{1})x \text{diag}(T^*, \mathbb{1}, \dots, \mathbb{1}) \rrbracket \geq 0, \forall x \in M_n(X)_+, n \in \mathbb{N} \right\}.$$

Lemma 2.2. Let (X, e) be an operator system and \mathcal{M} its multiplier algebra. Then $(Sx)T^* = \llbracket SxT^* \rrbracket = S(xT^*)$ for all $S, T \in \mathcal{M}$ and $x \in X$.

Proof. Notice that

$$\llbracket \text{diag}(\mathbb{1}, \dots, \mathbb{1}, T, \mathbb{1}, \dots, \mathbb{1})x \text{diag}(\mathbb{1}, \dots, \mathbb{1}, T, \mathbb{1}, \dots, \mathbb{1})^* \rrbracket \geq 0$$

for all $T \in \mathcal{M}$ and positive $x \in M_n(X)$, since $\alpha^* M_n(X)_+ \alpha \subset M_n(X)_+$ for $\alpha \in M_n$.

Letting $S, T \in \mathcal{M}$ and $x \in X_+$, we define the diagonal matrices $d_1 = \text{diag}(T, \mathbb{1}, \mathbb{1})$, $d_2 = \text{diag}(\mathbb{1}, \mathbb{1}, S)$, $d_3 = \text{diag}(\mathbb{1}, T, \mathbb{1})$. Let $\alpha = (1, -1, 1)$, $\beta = (\frac{1}{\lambda}, \frac{1}{\lambda}, \lambda\varepsilon)$ for $\lambda > 0$ and $\varepsilon = \pm 1, \pm i$. Then, recalling (2.1), we obtain

$$\begin{aligned} 0 &\leq \beta \llbracket d_3 \llbracket d_2 \llbracket d_1 (\alpha^* x \alpha) d_1^* \rrbracket d_2^* \rrbracket d_3^* \rrbracket \beta^* \\ &= \beta \llbracket d_3 \llbracket d_2 \begin{pmatrix} \llbracket TxT^* \rrbracket & -Tx & Tx \\ -xT^* & x & -x \\ xT^* & -x & x \end{pmatrix} d_2^* \rrbracket d_3^* \rrbracket \beta^* \\ &= \beta \begin{pmatrix} \llbracket TxT^* \rrbracket & -(Tx)T^* & (Tx)S^* \\ -T(xT^*) & \llbracket TxT^* \rrbracket & -T(xS^*) \\ S(xT^*) & -(Sx)T^* & \llbracket SxS^* \rrbracket \end{pmatrix} \beta^* \\ &= \lambda^2 |\varepsilon|^2 \llbracket SxS^* \rrbracket + \varepsilon (S(xT^*) - (Sx)T^*) + \bar{\varepsilon} ((Tx)S^* - T(xS^*)). \end{aligned}$$

Since the positive cone in an operator system is closed, this implies for $\lambda \rightarrow 0$ that $\varepsilon y + \bar{\varepsilon} y^* \geq 0$, where $y = S(xT^*) - (Sx)T^*$. The positive cone is also proper, thus $\text{Re } y = \text{Im } y = 0$. We obtain $S(xT^*) = (Sx)T^*$ for positive x and it is immediate that this holds for all $x \in X$, then. \square

Proposition 2.3. Let (X, e) be an operator system and $\mathcal{M} \subset L(X)$ its multiplier algebra. Then \mathcal{M} is an algebra.

Proof. Let $S, T \in \mathcal{M}$. It is obvious from Lemma 2.2 and equation (2.1) that $ST \in \mathcal{M}$. It is also clear that $\lambda T \in \mathcal{M}$ for all $\lambda \in \mathbb{C}$. To show that \mathcal{M} is closed under addition, we let $d_1 = \text{diag}(T, \mathbb{1}, \dots, \mathbb{1})$ and $d_2 = \text{diag}(\mathbb{1}, S, \mathbb{1}, \dots, \mathbb{1})$ in $M_{n+1}(\mathcal{M})$. With $\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1_{n-1} \end{pmatrix}$ we find

$$\text{diag}(T + S, \mathbb{1}, \dots, \mathbb{1})x \text{diag}(T + S, \mathbb{1}, \dots, \mathbb{1})^* = \alpha d_2 d_1 \alpha^* x \alpha d_1^* d_2^* \alpha^* \geq 0$$

for $x \in M_n(X)_+$. Hence $S + T \in \mathcal{M}$. \square

Proposition 2.4. *Let (X, e) be an operator system and \mathcal{M} its multiplier algebra. Then $TM_m(X)_+T^* \subset M_n(X)_+$ for all $T \in M_{n,m}(\mathcal{M})$ and $n, m \in \mathbb{N}$.*

Proof. The assertion obviously holds for $n = m$ and $T = \text{diag}(T_1, \dots, T_n)$. So, we will reduce the general case to this special case in the following way: For $T = [T_{ij}] \in M_{n,m}(\mathcal{M})$ and $x \in M_m(X)_+$ we define the block diagonal matrix

$$\alpha = \begin{pmatrix} 1 \dots 1 & 0 \dots 0 & 0 \dots 0 & \dots & \dots & 0 \dots 0 \\ 0 \dots 0 & 1 \dots 1 & 0 \dots 0 & \dots & \dots & 0 \dots 0 \\ 0 \dots 0 & 0 \dots 0 & \ddots & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \dots 0 \\ 0 \dots 0 & 0 \dots 0 & \dots & \dots & 0 \dots 0 & 1 \dots 1 \end{pmatrix} \in M_{m,nm}$$

and we let $\gamma = (\mathbb{1}_n, \dots, \mathbb{1}_n) \in M_{n,nm}$ and

$$d = \text{diag}(T_{11}, T_{21}, \dots, T_{n1}, T_{12}, T_{22}, \dots, T_{n2}, \dots, T_{1m}, \dots, T_{nm}).$$

Then we obtain $T = \gamma d \alpha^*$ and thus $TxT^* = \gamma d \alpha^* x \alpha d^* \gamma^* \geq 0$ and the proof is complete. \square

After the definition of the multiplier algebra of an operator system the next step is to show that there is an embedding of the multiplier algebra into the operator system.

Remark 2.5. So far we interpreted mappings $T = [T_{ij}] \in M_n(L(X))$ as mappings from $M_n(X)$ to $M_n(X)$ by matrix multiplication, i.e., $Tx = [\sum_j T_{ij}x_{jk}]$. But one can also identify $M_n(L(X))$ with $L(X, M_n(X))$ by setting $Tx = [T_{ij}x]$ for $x \in X$, which is actually done when supplying $CB(X)$ with the cb -norms to get the dual operator space of the operator space X . This means that the operator space structure of the dual is defined by $M_n(CB(X)) = CB(X, M_n(X))$ for all $n \in \mathbb{N}$. Note that for $T = [T_{ij}] \in M_n(L(X))$ and $x \in X$ we have $T(x \otimes \mathbb{1}_n) = [T_{ij}x] = Tx$.

Proposition 2.6. *Let (X, e) be an operator system. Then the linear map $\Omega: \mathcal{M}(X) \rightarrow X$ defined by $\Omega(T) = Te$ is a complete isometry from $\mathcal{M}(X)$ supplied with the cb -norms into X . This means the equations*

$$\|\Omega^{(k)}(T)\| = \|Te_k\| = \|[T_{ij}e]\| = \|T\| = \|T\|_{cb}$$

hold for all $T = [T_{ij}] \in M_k(\mathcal{M}(X))$ and $k \in \mathbb{N}$. In particular we have $\mathcal{M}(X) \subset CB(X)$.

Proof. Fix $k \in \mathbb{N}$ and let $T = [T_{ij}] \in M_k(\mathcal{M})$. Notice that by definition of the norms the inequalities

$$\|Te_k\| = \|[T_{ij}e]\| \leq \|T\| \leq \|T^{(n)}\| \leq \|T\|_{cb}$$

hold for all $n \in \mathbb{N}$. So it suffices to prove that $\|T\|_{cb} \leq \|Te_k\|$ is true. For this let $n \in \mathbb{N}$ and $x = [x_{ij}] \in M_n(M_k(X))$ such that $x_{ij} \in M_k(X)$ and $\|x\| \leq 1$. Then the matrix $\begin{pmatrix} e_{nk} & x \\ x^* & e_{nk} \end{pmatrix} \in M_{2nk}(X)$, where $e_{nk} = e \otimes \mathbb{1}_{nk}$ is positive. Hence

$$0 \leq \text{diag}(\underbrace{T, \dots, T}_{n\text{-times}}, \underbrace{\mathbb{1}_k, \dots, \mathbb{1}_k}_{n\text{-times}}) \begin{pmatrix} e_{nk} & x \\ x^* & e_{nk} \end{pmatrix} \text{diag}(T^*, \dots, T^*, \mathbb{1}_k, \dots, \mathbb{1}_k).$$

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Carrying out the matrix multiplication and using $Te_k T^* \leq \|Te_k T^*\| e_k$, it follows that

$$0 \leq \begin{pmatrix} \|Te_k T^*\| e_{nk} & [Tx_{ij}] \\ [x_{ij}^* T^*] & e_{nk} \end{pmatrix}.$$

This implies

$$\|T^{(n)}x\|^2 = \|[Tx_{ij}]\|^2 \leq \|(Te_k)T^*\| = \|T(Te_k)^*\| \leq \|T\|_k \|Te_k\| \leq \|T\|_{cb} \|Te_k\|. \quad (2.2)$$

Since these inequalities hold for all $x \in M_n(M_k(X))$ with $\|x\| \leq 1$ and $n \in \mathbb{N}$, we obtain $\|T\|_{cb} \leq \|Te_k\|$ and the proof is complete. \square

Corollary 2.7. *Let X be an operator system. Then $\mathcal{M}(X)$ is an operator subalgebra of $CB(X)$.*

Remark 2.8. Notice that if the operator system (X, e) is a C^* -algebra, then $\Omega(\mathcal{M}(X)) = X$ and Ω is multiplicative, so defining an involution on $\mathcal{M}(X)$ by $T^*e = (Te)^*$ for $T \in \mathcal{M}(X)$ the map Ω is a unital $*$ -isomorphism from $\mathcal{M}(X)$ onto X . In fact, for each $t \in X$ the map $T: X \rightarrow X$ defined by $Tx = tx$ is in $\mathcal{M}(X)$ simply by definition of the multiplier algebra. Obviously, $\Omega(T) = Te = te = t$, so Ω is surjective. Moreover, for $S, T \in \mathcal{M}(X)$ we let $s = Se, t = Te \in X$ and see that $\Omega(ST) = S(Te) = st = \Omega(S)\Omega(T)$, because Ω is injective by Proposition 2.6. It follows that Ω is a $*$ -isomorphism. Notice in particular that in case (X, e) is a W^* -algebra, all elements of $\mathcal{M}(X)$ are w^* - w^* -continuous, because they come from right multiplications of elements of X .

In the sequel we will assume always the special case that our operator system (X, e) is the dual of a matrix base norm space (V, K) that is complete in the matrix base norm. It is known that if an operator space, say W , is the dual of a complete operator space, then there is a Hilbert space H and a completely isometric injection $\varphi: W \rightarrow \mathcal{B}(H)$ that is a w^* - w^* -homeomorphism onto its image, cf. [26, Proposition 3.2.4].

In order to have Hilbert space theory at hand, we would like to represent X (together with its multiplier algebra) concretely as subspace of some $\mathcal{B}(H)$. To obtain w^* -continuity of the representation (since X is a dual space), we consider the subalgebra of weakly continuous multipliers, i.e.,

$$\mathcal{M}_\sigma(X) = \{ T \in \mathcal{M}(X) \mid T: X \rightarrow X \text{ } w^*-w^*\text{-continuous} \},$$

that we call the *weak multiplier algebra* of X . Again we will write simply \mathcal{M}_σ for the weak multiplier algebra of X .

Our aim is to find a Hilbert space H and representations $\tilde{\pi}: \mathcal{M}_\sigma \rightarrow \mathcal{B}(H)$ and $\pi: X \rightarrow \mathcal{B}(H)$ such that $\pi(TxS^*) = \tilde{\pi}(T)\pi(x)\tilde{\pi}(S)^*$ for all $T, S \in \mathcal{M}_\sigma$ and $x \in X$ and such that π is a w^* - w^* -homeomorphism onto its image. For this let $n \in \mathbb{N}$, $\varphi \in K_n$ and $x \in X$. We define the sesquilinear forms

$$[\cdot, \cdot]_\varphi^x: \mathcal{M}_\sigma^* \otimes \mathbb{C}^n \times \mathcal{M}_\sigma^* \otimes \mathbb{C}^n \rightarrow \mathbb{C}$$

by

$$\left[\sum_i T_i^* \otimes \xi_i, \sum_j S_j^* \otimes \eta_j \right]_\varphi^x = \sum_{i,j} \langle \varphi(S_j x T_i^*) \xi_i \mid \eta_j \rangle. \quad (2.3)$$

For $x \geq 0$ it is obvious that $[\cdot, \cdot]_\varphi^x$ is an inner product on $\mathcal{M}_\sigma^* \otimes \mathbb{C}^n$, since $\varphi: X \rightarrow M_n$ is completely positive.

Lemma 2.9. *For the sesquilinear forms defined above we obtain the following Cauchy-Schwarz like inequality:*

$$|[a, b]_\varphi^x| \leq \|x\|^2 [a, a]_\varphi^e [b, b]_\varphi^e \quad (2.4)$$

for all $x \in X$, $\varphi \in K_n$, $n \in \mathbb{N}$ and $a, b \in \mathcal{M}_\sigma^* \otimes \mathbb{C}^n$.

Proof. Let $n \in \mathbb{N}$, $\varphi \in K_n$ and $x \in X$. Without loss of generality we may assume $\|x\| \leq 1$ so that $\begin{pmatrix} e & x \\ x^* & e \end{pmatrix} \geq 0$. So for $a = \sum_{i=1}^l T_i^* \otimes \xi_i$ and $b = \sum_{j=1}^k S_j^* \otimes \eta_j$, where $T_i, S_j \in \mathcal{M}$, we let

$$\gamma^* = \begin{pmatrix} S_1^* & \cdots & S_k^* & 0 & \cdots & 0 \\ 0 & \cdots & 0 & T_1^* & \cdots & T_l^* \end{pmatrix}.$$

Then we see from Proposition 2.4 that $\gamma \begin{pmatrix} e & x \\ x^* & e \end{pmatrix} \gamma^* \geq 0$. Applying $\varphi^{(lk)}$ on this positive matrix and evaluating the scalar product on $M_{lk}(M_n)$ with the vector

$$\zeta = (\eta_1, \dots, \eta_k, \lambda \xi_1, \dots, \lambda \xi_l)^{\text{tr}} \in (\mathbb{C}^n)^{lk},$$

where $\lambda \in \mathbb{C}$ can be arbitrarily chosen, we obtain

$$0 \leq \zeta^* \varphi^{(lk)} \left(\gamma \begin{pmatrix} e & x \\ x^* & e \end{pmatrix} \gamma^* \right) \zeta = |\lambda|^2 [a, a]_\varphi^e + \lambda [a, b]_\varphi^x + \bar{\lambda} [b, a]_\varphi^{x^*} + [b, b]_\varphi^e.$$

Notice that $[b, a]_\varphi^{x^*} = \overline{[a, b]_\varphi^x}$. Hence choosing $\lambda = t \overline{[a, b]_\varphi^x}$ for some real t , we get

$$t^2 |[a, b]_\varphi^x|^2 [a, a]_\varphi^e + 2t |[a, b]_\varphi^x|^2 + [b, b]_\varphi^e \geq 0.$$

This implies $|[a, b]_\varphi^x|^2 \leq [a, a]_\varphi^e [b, b]_\varphi^e$ in the known way, and the claim is proved. \square

Proposition 2.10. *Let $(X, e) = (V, K)^*$, where (V, K) is a norm complete matrix base norm space. Then there are a Hilbert space H , a representation $\tilde{\pi}: \mathcal{M}_\sigma \rightarrow \mathcal{B}(H)$ and a unital completely positive order isomorphism onto its image $\pi: X \rightarrow \mathcal{B}(H)$ such that*

$$\pi(TxS^*) = \tilde{\pi}(T)\pi(x)\tilde{\pi}(S)^*$$

for all $x \in X$ and $S, T \in \mathcal{M}_\sigma$.

Proof. Let $n \in \mathbb{N}$ and $\varphi \in K_n$. Notice that $[\cdot, \cdot]_\varphi^e$ is a semidefinite inner product on the space $\mathcal{M}_\sigma^* \otimes \mathbb{C}^n$. Hence we see from the Cauchy-Schwarz inequality

$$|[a, b]_\varphi^e|^2 \leq [a, a]_\varphi^e [b, b]_\varphi^e$$

that the null space

$$N_\varphi = \{ a \in \mathcal{M}_\sigma^* \otimes \mathbb{C}^n \mid [a, a]_\varphi^e = 0 \} = \{ a \in \mathcal{M}_\sigma^* \otimes \mathbb{C}^n \mid [a, b]_\varphi^e = 0 \text{ for all } b \in \mathcal{M}_\sigma^* \otimes \mathbb{C}^n \}$$

is a subspace of $\mathcal{M}_\sigma^* \otimes \mathbb{C}^n$. It follows that the induced sesquilinear form on the quotient space $(\mathcal{M}_\sigma^* \otimes \mathbb{C}^n)/N$ defined by $[a+N, b+N]_\varphi^e = [a, b]_\varphi^e$ is a definite inner product. Let H_φ be the completion of the pre-Hilbert space $(\mathcal{M}_\sigma^* \otimes \mathbb{C}^n)/N$. We denote the scalar product of H_φ as $\langle \cdot | \cdot \rangle_\varphi$. Given $R \in \mathcal{M}_\sigma$ we define an antilinear and antimultiplicative mapping

$$\Lambda_\varphi(R): \mathcal{M}_\sigma^* \otimes \mathbb{C}^n \rightarrow \mathcal{M}_\sigma^* \otimes \mathbb{C}^n \text{ by } \Lambda_\varphi(R) \left(\sum_{i=1}^l T_i^* \otimes \xi_i \right) = \sum_{i=1}^l (R^* T_i^*) \otimes \xi_i.$$

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Notice that $\|ReR^*\|e - ReR^* \geq 0$. Hence from Proposition 2.4 we see that

$$\begin{aligned} 0 &\leq \text{diag}(T_1, \dots, T_l) \begin{pmatrix} 1 \\ 1 \end{pmatrix} (\|ReR^*\|e - ReR^*) \begin{pmatrix} 1 & 1 \end{pmatrix} \text{diag}(T_1, \dots, T_l)^* \\ &= \|ReR^*\| (T_i e T_j^* - (T_i ReR^* T_j^*)). \end{aligned}$$

Consequently, letting $a = \sum_{i=1}^l T_i^* \otimes \xi_i$, we find

$$\begin{aligned} [\Lambda_\varphi(R)a, \Lambda_\varphi(R)a]_\varphi^e &= \sum_{i,j} \langle \varphi(T_j ReR^* T_i^*) \xi_i \mid \xi_j \rangle \\ &\leq \|ReR^*\| \langle \varphi(T_j e T_i^*) \xi_i \mid \xi_j \rangle \\ &= \|ReR^*\| [a, a]_\varphi^e \end{aligned}$$

This shows that $\Lambda_\varphi(R)$ leaves N_φ invariant and thus defines an antilinear transformation on $(\mathcal{M}_\sigma^* \otimes \mathbb{C}^n)/N_\varphi$ that we still denote as $\Lambda_\varphi(R)$. We also see that $\|\Lambda_\varphi(R)\|^2 \leq \|ReR^*\|$. So $\Lambda_\varphi(R)$ extends to a bounded antilinear mapping on the completion H_φ , which we again denote as $\Lambda_\varphi(R)$. Then we define $\tilde{\pi}_\varphi(R) = \Lambda_\varphi(R)^*$ and get a bounded linear representation $\tilde{\pi}_\varphi: \mathcal{M}_\sigma \rightarrow \mathcal{B}(H_\varphi)$.

We still need to find a completely bipositive mapping π_φ from X into $\mathcal{B}(H_\varphi)$. Let $x \in X$. From equation (2.4) we see that $[\cdot, \cdot]_\varphi^x$ extends to a bounded sesquilinear form on H_φ . Hence there is an operator in $\mathcal{B}(H_\varphi)$, that we call $\pi_\varphi(x)$, such that $\langle \pi_\varphi(x)a \mid b \rangle_\varphi = [a, b]_\varphi^x$ for all $a, b \in H_\varphi$. Now we define the Hilbert space $H = \bigoplus H_\varphi$ and the mappings $\pi = \bigoplus \pi_\varphi$ and $\tilde{\pi} = \bigoplus \tilde{\pi}_\varphi$, where the sum runs over all $\varphi \in K_n$ and all $n \in \mathbb{N}$.

We have to show that $\pi_\varphi(Rx) = \tilde{\pi}_\varphi(R)\pi_\varphi(x)$ and $\pi_\varphi(xS^*) = \pi_\varphi(x)\tilde{\pi}_\varphi(S)^*$ for all $R, S \in \mathcal{M}_\sigma$, $x \in X$, $\varphi \in K_n$ and $n \in \mathbb{N}$. Let $a = \sum_i T_i^* \otimes \xi_i$ and $b = \sum_j S_j^* \otimes \eta_j$, where $T_i, S_j \in \mathcal{M}_\sigma$ and $\xi_i, \eta_j \in \mathbb{C}^n$. Then

$$\begin{aligned} \langle \pi_\varphi(Rx)a \mid b \rangle &= \left[\sum_i T_i^* \otimes \xi_i, \sum_j S_j^* \otimes \eta_j \right]_\varphi^{Rx} \\ &= \sum_{i,j} \langle \varphi(S_j Rx T_i^*) \xi_i \mid \eta_j \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \tilde{\pi}_\varphi(R)\pi_\varphi(x)a \mid b \rangle &= \left\langle \pi_\varphi(x) \sum_i T_i^* \otimes \xi_i \mid \Lambda_\varphi(R) \sum_j S_j^* \otimes \eta_j \right\rangle \\ &= \left\langle \pi_\varphi(x) \sum_i T_i^* \otimes \xi_i \mid \sum_j R^* S_j^* \otimes \eta_j \right\rangle \\ &= \left[\sum_i T_i^* \otimes \xi_i, \sum_j R^* S_j^* \otimes \eta_j \right]_\varphi^x \\ &= \sum_{i,j} \langle \varphi(S_j R x T_i^*) \xi_i \mid \eta_j \rangle. \end{aligned}$$

Similarly we obtain

$$\langle \pi_\varphi(xS^*)a \mid b \rangle = \sum_{i,j} \langle \varphi(S_j x S^* T_i^*) \xi_i \mid \eta_j \rangle = \langle \pi_\varphi(x)\tilde{\pi}_\varphi(S)^*a \mid b \rangle.$$

This holds for all $\varphi \in K_n$, $a, b \in \mathcal{M}_\sigma^* \otimes \mathbb{C}^n$ and $n \in \mathbb{N}$, so that the claim follows.

It is left to show that π is a complete order isomorphism. Let $x = [x_{\nu\mu}] \in M_m(X)$ such that $\pi^{(m)}(x) \geq 0$. We have to prove that x is positive. Let $\varphi \in K_n$. Then $\pi_\varphi^{(m)}(x) \geq 0$, hence $\langle \pi_\varphi^{(m)}(x) \xi | \xi \rangle \geq 0$ for all $\xi \in H_\varphi^m$. A small calculation shows

$$0 \leq \left\langle \left(\sum_{\mu} \pi_\varphi(x_{\nu\mu}) \xi_\mu \right) \middle| \xi \right\rangle = \sum_{\mu, \nu} \langle \pi_\varphi(x_{\nu\mu}) \xi_\mu | \xi_\nu \rangle = \sum_{\mu, \nu} \langle \varphi(x_{\nu\mu}) \eta_\mu | \eta_\nu \rangle = \langle \varphi^{(m)}(x) \eta | \eta \rangle,$$

where we chose $\xi_\nu = 1^* \otimes \eta_\nu \in \mathcal{M}_\sigma^* \otimes \mathbb{C}^n$ and set $\eta = (\eta_\nu) \in (\mathbb{C}^n)^m$ with arbitrary $\eta_\nu \in \mathbb{C}^n$. Thus $\varphi^{(m)}(x)$ is positive, which implies $x \geq 0$, since $\varphi \in K_n$ was arbitrary. Since π is obviously completely positive by construction, we see that π is a complete order isomorphism onto its image. \square

Proposition 2.11. *Let $(X, e) = (V, K)^*$, where (V, K) is a norm complete matrix base norm space. For the Hilbert space H and the maps $\tilde{\pi}: \mathcal{M}_\sigma \rightarrow \mathcal{B}(H)$ and $\pi: X \rightarrow \mathcal{B}(H)$ constructed in Proposition 2.10 the following holds: π is a w^* - w^* -homeomorphism and*

$$\tilde{\pi}(\mathcal{M}_\sigma) = \{ z \in \mathcal{B}(H) \mid z\pi(X) \subset \pi(X) \}. \quad (2.5)$$

In particular, $\tilde{\pi}(\mathcal{M}_\sigma)$ is w^* -closed in $\mathcal{B}(H)$.

Proof. In order to show, that π is a w^* - w^* -homeomorphism, we will show first, that π_φ is continuous with respect to the w^* -topology on X and the weak operator topology on $\mathcal{B}(H_{\pi_\varphi})$ for all $\varphi \in K_n$ and $n \in \mathbb{N}$. For this notice that given $S, T \in \mathcal{M}_\sigma$ and $x \in X$ we obtain the equation

$$\begin{aligned} \langle \psi | SxT^* \rangle &= \psi(SxT^*) = (SxT^*)(\psi) \\ &= (xT^*)(S^\delta \psi) = (Tx^*)^*(S^\delta \psi) \\ &= \overline{(Tx^*)((S^\delta \psi)^*)^*)} \\ &= \overline{x^*(T^\delta (S^\delta \psi)^*)^*)} \\ &= x((T^\delta (S^\delta \psi)^*)^*) = x(S^\delta \psi T^{\delta*}) \\ &= \langle S^\delta \psi T^{\delta*} | x \rangle \end{aligned} \quad (2.6)$$

for any $\psi \in V$. Observe that we have two different dualities. The first one between V and $V^* = X$ and the second one between V^* and V^{**} , where it is well-known, that the second duality can be interpreted as an extension of the first one. Now, let $n \in \mathbb{N}$, $\varphi \in K_n$ and $a, b \in \mathcal{M}_\sigma^* \otimes \mathbb{C}^n$. For $x \in X$ we define

$$f(x) = \langle \pi_\varphi(x) a | b \rangle_\varphi = [a, b]_\varphi^x = \sum_{i,j} \langle \varphi(S_j x T_i^*) \xi_i | \eta_j \rangle, \quad (2.7)$$

where $a = \sum_i T_i^* \otimes \xi_i$ and $b = \sum_j S_j^* \otimes \eta_j$. We will show that $f \in V$ holds, which will imply, that π_φ is continuous with respect to the w^* -topology on X and the weak operator topology on $\mathcal{B}(H_\varphi)$. Notice that $K_n \subset M_n(V)$ so that $\varphi = [\varphi_{\nu\mu}]$, where $\varphi_{\nu\mu} \in V$. Applying (2.6) to (2.7) (with $\psi = \varphi_{\nu\mu}$), we obtain immediately that

$$f(x) = \sum_{i,j} \langle \varphi(S_j x T_i^*) \xi_i | \eta_j \rangle = \sum_{i,j} \eta_j^* [\varphi_{\nu\mu}(S_j x T_i^*)] \xi_i = \sum_{i,j} \eta_j^* [\langle S_j^\delta \varphi_{\nu\mu} T_i^{\delta*} | x \rangle] \xi_i.$$

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It follows that $f \in V$, where $f(x) = \langle \pi_\varphi(x)a|b \rangle_\varphi$ for $a, b \in \mathcal{M}_\sigma^* \otimes \mathbb{C}^n$. But since by construction $\mathcal{M}_\sigma^* \otimes \mathbb{C}^n$ is dense in H_φ , we can prove that $g \in V$, where $g(x) = \langle \pi_\varphi(x)a|v \rangle_\varphi$ for arbitrary $a, b \in H_\varphi$. Fix $a, b \in H_\varphi$ and let (a_n) and (b_n) be sequences that converge in norm against a and b respectively. Then

$$|\langle \pi_\varphi(x)a|b \rangle_\varphi - \langle \pi_\varphi(x)a_n|b_n \rangle_\varphi| \leq \|x\|(\|a_n\|\|b - b_n\| + \|b\|\|a - a_n\|).$$

This shows that $f_n = \langle \pi_\varphi(\cdot)a_n|b_n \rangle_\varphi$ is norm convergent against g and thus $g \in V$, because V is norm complete. It is obvious that $g \in V$ implies that π_φ is continuous when X is given the w^* -topology and $\mathcal{B}(H_\varphi)$ is given the weak operator topology. Then the direct product $\pi = \oplus \pi_\varphi: X \rightarrow \mathcal{B}(H)$ is continuous with respect to the same topologies, cf. [52, page 42]. Recall from Proposition 2.10 that π is a complete order isomorphism onto its image, so π also is a complete isometry onto its image. Therefore the restriction $\pi: \text{Ball}(X) \rightarrow \mathcal{B}(H)$ is in particular a continuous, injective map from the w^* -compact $\text{Ball}(X)$ to $\mathcal{B}(H)$ with the weak operator topology that is a weaker Hausdorff topology than the w^* -topology (the latter coincides with the σ -weak topology on $\mathcal{B}(H)$). It follows that π is a w^* - w^* -homeomorphism between $\text{Ball}(X)$ and $\pi(\text{Ball}(X))$. Therefore, using that $\pi(\text{Ball}(X)) = \pi(X) \cap \text{Ball}(\mathcal{B}(H))$, it follows from applying the Krein-Smulian theorem that $\pi(X)$ is w^* -closed and that $\pi: X \rightarrow \pi(X)$ is a w^* - w^* -homeomorphism onto its image. It is left to verify $\tilde{\pi}(\mathcal{M}_\sigma) = \{z \in \mathcal{B}(H) \mid z\pi(X) \subset \pi(X)\}$. Let $y \in \tilde{\pi}(\mathcal{M}_\sigma)$. Then $y = \tilde{\pi}(T)$ for some $T \in \mathcal{M}_\sigma$ and obviously $y\pi(x) = \tilde{\pi}(T)\pi(x) = \pi(Tx) \subset \pi(X)$ for all $x \in X$. For the other direction let $y \in \mathcal{B}(H)$ such that $y\pi(X) \subset \pi(X)$. Then we can define a linear mapping $T: X \rightarrow X$ by $Tx = \pi^{-1}(y\pi(x))$. We only have to verify, that T is in \mathcal{M}_σ . A short calculation shows that $xT^* = \pi^{-1}(\pi(x)y^*)$ and therefore $TxT^* = \pi^{-1}(y\pi(x)y^*)$. It follows that $T \in \mathcal{M}_\sigma$. So, we have proved that equation (2.5) holds, and since $\pi(X)$ is w^* -closed, it is obvious that $\tilde{\pi}(\mathcal{M}_\sigma)$ is w^* -closed, too. \square

Projective faces

Our starting point in this chapter was the duality between a given m -base norm space (V, K) and its dual $(V, K)^* = A_b(K)$. The question is under what conditions on K will $A_b(K)$ be a W^* -algebra? After constructing the multiplier algebra of an operator system and concretely representing both spaces as bounded operators on the same Hilbert space in Proposition 2.10 and 2.11, we still need a condition on K that ensures that the embedding of Proposition 2.6 is surjective, so that we can identify $A_b(K)$ with its multiplier algebra, which will turn $A_b(K)$ into a W^* -algebra. The work of Alfsen and Shultz, cf. [6], contains a projection axiom that ensures essentially that the constructed algebra contains enough projections. We will need such an axiom for the multiplier algebra, too. This is the topic of the current section.

So let us first define what we will call projections in the multiplier algebra. As usual (X, e) is an operator system that is the dual of a complete m -base norm space (V, K) and \mathcal{M}_σ is its multiplier algebra.

Definition 2.12. An element $p \in \mathcal{M}_\sigma$ of the multiplier algebra is a (multiplier) projection, if $p^2 = p$ and pe is self-adjoint in X .

Notice from Remark 2.8 that if (X, e) is a W^* -algebra, then the multiplier projections correspond with the projections in X . Moreover, it is known that there is a relation

between projections in X and certain faces of the normal state space of X . This is the origin of the projection axiom that will be introduced soon.

Remark 2.13. To get somewhat closer to [6, 9], where P -projections (called compressions in [6]) are defined without mentioning an algebra, we could easily avoid to talk about the multiplier algebra in the preceding definition; we could equivalently define a multiplier projection to be a w^* -continuous map $p \in L(X)$ such that $p^2 = p$ and pe is self-adjoint and such that our Jordan product $\llbracket \text{diag}(p, \mathbf{1}, \dots, \mathbf{1})x \text{diag}(p^*, \mathbf{1}, \dots, \mathbf{1}) \rrbracket$ is positive for all $x \in M_n(X)_+$ and $n \in \mathbb{N}$.

A multiplier projection is a mapping from X to X . Let $p \in \mathcal{M}_\sigma$ be a projection and consider another mapping $P: X \rightarrow X$ defined by $Px = pxp^*$ for $x \in X$. Then P is weakly continuous, because p is weakly continuous as element of \mathcal{M}_σ , positive and idempotent. We will next have a short look at mappings with these properties. The following is taken from [9]:

Let $\langle X|Y \rangle$ be a dual pair of real ordered vector spaces. We consider in the following only linear mappings from X to X or from Y to Y that are $\sigma(X, Y)$ -continuous or $\sigma(Y, X)$ -continuous, respectively. We call such mappings also weakly continuous. These linear mappings have dual mappings, that are given by $\langle f(x)|y \rangle = \langle x|f^\delta(y) \rangle$ for $x \in X$ and $y \in Y$, and these dual mappings are continuous as well. A (weakly continuous) mapping $P: X \rightarrow X$ is called a positive projection, if $P^2 = P$ and if P is positive, i.e., $P(X_+) \subset X_+$. We present some simple mainly algebraic observations about positive projections, that we will need later on for the discussion of multiplier projections.

We define the annihilator of a set $D \subset X$ as

$$D^\perp = \{ y \in Y \mid \langle x|y \rangle = 0 \text{ for all } x \in D \}.$$

For $C \subset X_+$ we denote $C^\diamond = (C^\perp \cap Y_+)^\perp$. For a projection $P: X \rightarrow X$ the dual mapping $P^\delta: Y \rightarrow Y$ is also an projection. We have the formulas

$$(\text{kern } P)^\perp = \text{im } P^\delta \text{ and } (\text{im } P)^\perp = \text{kern } P^\delta.$$

If X is directed or positively generated, i.e., $X = X_+ - X_+$, then $\text{im } P = \text{im}^+ P - \text{im}^+ P$ and so $(\text{im } P)^\perp = (\text{im}^+ P)^\perp$. This implies immediately

$$(\text{kern}^+ P^\delta)^\perp = (\text{kern } P^\delta \cap Y_+)^\perp = ((\text{im}^+ P)^\perp \cap Y_+)^\perp = (\text{im}^+ P)^\diamond.$$

Remark 2.14. Let $Q: X \rightarrow X$ be some positive projection. Then we have the formula

$$(\text{kern}^+ Q)^\diamond \cap X_+ = \text{kern}^+ Q.$$

Proof. If $x \in \text{kern}^+ Q$ then of course $\langle x|y \rangle = 0$ for all $y \in (\text{kern}^+ Q)^\perp$, which implies directly $x \in (\text{kern}^+ Q)^\diamond$. So what we must show is that $(\text{kern}^+ Q)^\diamond$ is contained in $\text{kern } Q$. To this end, let $x \in (\text{kern}^+ Q)^\diamond$, i.e., $\langle x|z \rangle = 0$ for all $z \in (\text{kern}^+ Q)^\perp \cap Y_+$. Let $y \in Y$. Since Y is directed, there are $y_1, y_2 \in Y_+$ such that $y = y_1 - y_2$. Of course $Q^\delta y_i \in (\text{kern}^+ Q)^\perp$ and $Q^\delta y_i \geq 0$ because Q^δ is positive. So we have

$$\langle Qx|y \rangle = \langle x|Q^\delta y_1 \rangle - \langle x|Q^\delta y_2 \rangle = 0,$$

which means $Qx = 0$. □

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Remark 2.15. Let $P: X \rightarrow X$ be a positive projection that admits a positive projection $Q: X \rightarrow X$ such that $\text{im}^+ P = \text{kern}^+ Q$. Then we have

$$(\text{kern}^+ P^\delta)^\perp \cap X_+ = (\text{kern} P^\delta)^\perp \cap X_+$$

Proof. We have

$$(\text{kern}^+ P^\delta)^\perp = (\text{kern} P^\delta \cap X_+)^\perp = (\text{im}^+ P)^\circ,$$

by definition and noting that $\text{im} P$ is directed when X is. On the other hand

$$(\text{kern} P^\delta)^\perp = (\text{im} P)^{\perp\perp} = \text{im} P,$$

where the last equality holds, because $\text{im} P$ is σ -closed and convex. Together this implies

$$\begin{aligned} (\text{kern} P^\delta)^\perp \cap X_+ &= \text{im}^+ P = \text{kern}^+ Q \\ &= (\text{kern}^+ Q)^\circ \cap X_+ \\ &= (\text{im}^+ P)^\circ \cap X_+ = (\text{kern}^+ P^\delta)^\perp \cap X_+. \quad \square \end{aligned}$$

After these simple observations, we will discuss projective faces, projections of the multiplier algebra and a condition, which ensures that the multiplier algebra will be big enough, i.e., will be all of the operator system.

Let (V, K) be a complete matrix base norm space. Let its dual be the operator system (X, e) . We construct its multiplier algebra \mathcal{M}_σ .

Definition 2.16. A face F of K_1 is called *norm-exposed*, if there is $a \in X_+$ such that $F = \{\varphi \in K_1 \mid \langle \varphi | a \rangle = 0\}$. A face F of K_1 is called *projective*, if there is a multiplier projection $p \in \mathcal{M}_\sigma$ such that $F = \{\varphi \in K_1 \mid \langle \varphi | pe \rangle = 0\}$.

At this point, notice that to any *multiplier* projection $p \in \mathcal{M}_\sigma$ we have a *positive* projection $Px = pxp^*$ as discussed at the beginning of this section. Notice also, that $Pe = pep^* = p((pe)^*) = pe$, because pe is self-adjoint. This also shows directly, that pe is positive.

Lemma 2.17. Let $p \in \mathcal{M}_\sigma$ be a multiplier projection and $Px = pxp^*$. Then the positive projection P admits a positive projection $Q: X \rightarrow X$ such that $\text{im}^+ P = \text{kern}^+ Q$.

Proof. Let $q = \mathbb{1} - p \in \mathcal{M}_\sigma$ and $Qx = qxq^*$. Then Q is a positive projection, since q is a multiplier projection. We claim $\text{im}^+ P = \text{kern}^+ Q$, so let $x \in \text{im}^+ P$, i.e., x is positive and $Px = x$. This leads to

$$Qx = Q(Px) = Q(pxp^*) = (\mathbb{1} - p)(pxp^*)(\mathbb{1} - p)^*.$$

Thus a short calculation shows that $Qx = 0$.

Let $x \in \text{kern}^+ Q$. Since x is positive and q is a multiplier, we have

$$0 \leq \begin{pmatrix} q & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} q^* & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} qxq^* & qx \\ xq^* & x \end{pmatrix}.$$

Since $0 = Qx = qxq^*$, this implies $0 = qx = (\mathbb{1} - p)x$, i.e., $px = x$. So $Px = pxp^* = xp^* = (px)^* = x$, which means $x \in \text{im} P$, what was to be shown. \square

Proposition 2.18. *Let \mathcal{P} denote the set of the multiplier projections of \mathcal{M}_σ . If every norm-exposed face of K_1 is projective, then the w^* -closure of $\text{conv}(\mathcal{P}e)$, abbreviated by $\text{conv}(\mathcal{P}e)^-$, equals the interval $[0, e] = \{x \in X \mid 0 \leq x \leq e\}$. (Cf. [10, Prop. 1.7])*

Proof. Let $p \in \mathcal{P}$. Since \mathcal{M}_σ is an algebra containing the unit, it is obvious that $q = 1 - p \in \mathcal{M}_\sigma$ and q is also a multiplier projection, because $q^2 = q$ and qe is self-adjoint in X , because pe is self-adjoint. But then qe is even positive and thus we get $\mathcal{P}e \subset [0, e]$. Also $[0, e]$ is a w^* -compact and convex subset of X , so $\text{conv}(\mathcal{P}e)^- \subset [0, e]$.

Notice that since the mapping $x \mapsto e - 2x$ is an affine homeomorphism from $[0, e]$ to $[-e, e]$, the claim is equivalent to $\text{conv}(Se)^- = [-e, e]$, where $Se = \{e - 2pe \mid p \in \mathcal{P}\}$.

Let $x \in [-e, e]$. It suffices to show that $x \in \text{conv}(Se)^-$. Assume for contradiction that x is not contained in the w^* -closure of $\text{conv}(Se)$. Then there is a self-adjoint $\psi \in X'_h = V_h$ such that $\psi(\text{conv}(Se)^-) \leq 1$ and $\psi(x) > 1$. Therefore, $1 < \psi(x) \leq \|\psi\|$ and $\sup\{|\psi(s)| \mid s \in Se\} \leq 1$. We are going to find $s \in Se$ such that $\|\psi\| = \psi(s)$, which gives a contradiction. Since (V_h, K_1) is a base-norm space, there are $\varrho, \varphi \in V_+$ such that $\psi = \varrho - \varphi$ and $\|\psi\| = \|\varrho\| + \|\varphi\|$. Since $[-e, e]$ is w^* -compact, there is $y \in [-e, e]$ such that $\psi(y) = \|\psi\|$. Define $a = \frac{1}{2}(e + y)$ and $b = e - a$. Notice that $0 \leq a, b \leq e$ and $y = a - b$. We find

$$\|\psi\| = \psi(y) = \varrho(a) + \varphi(b) - \varrho(b) - \varphi(a) \leq \|\varrho\| + \|\varphi\| - \varrho(b) - \varphi(a), \quad (2.8)$$

which implies $\varrho(b), \varphi(a) = 0$, $\varrho(a) = \varrho(e)$ and $\varphi(b) = \varphi(e)$. Let $F = \{f \in K_1 \mid f(a) = 0\}$. Then F is a norm-exposed face of K_1 . So, by assumption there is a multiplier projection $p \in \mathcal{M}_\sigma$ such that $F = \{f \in K_1 \mid f(pe) = 0\}$. Let $q = 1 - p \in \mathcal{M}_\sigma$ and define the positive projections $Px = pxp^*$ and $Qx = qxq^*$ for all $x \in X$. We claim that $a \in (\text{kern}^+ P^\delta)^\perp$. If $g \in \text{kern}^+ P^\delta$, then $g(e)^{-1}g \in K_1$ and $\langle g|Pe \rangle = \langle g|pe \rangle = 0$. Consequently, $g \in F$, which gives $g(a) = 0$, showing the claim. Now, recall from Lemma 2.17 that P and Q are positive projections with the property $\text{im}^+ P = \text{kern}^+ Q$. So it follows from Remark 2.15 that $(\text{kern}^+ P^\delta)^\perp \cap X_+ = (\text{kern} P^\delta)^\perp \cap X_+$. Therefore, we obtain $a \in (\text{kern} P^\delta)^\perp$. Let $\phi \in V$. Since P^δ , being a positive projection, is in particular idempotent, there are $h \in \text{kern} P^\delta$ and $f \in \text{im} P^\delta$ such that $\phi = h + f$. Thus

$$\phi(Pa - a) = \langle P^\delta h|a \rangle - h(a) + \langle P^\delta f|a \rangle - f(a) = 0,$$

which implies $a = Pa \leq Pe = pe$. We have $\varphi(pe) = 0$, because $\varphi(a) = 0$. This implies $\varphi(qe) = \varphi(e - pe) = \varphi(e)$. Moreover, it follows that $\varrho(e) = \varrho(a) \leq \varrho(pe) \leq \varrho(e)$, which shows $\varrho(pe) = \varrho(e)$. We also find $\psi(pe) = \varrho(pe)$ and

$$\psi(qe) = \psi(e) - \psi(pe) = \varrho(e) - \varphi(e) - \varrho(pe) = -\varphi(qe).$$

Hence we see from equation (2.8) that

$$\|\psi\| = \varrho(a) + \varphi(b) = \varrho(e) + \varphi(e) = \varrho(pe) + \varphi(qe) = \psi(pe) - \psi(qe) = \psi(e - 2qe),$$

which yields the desired contradiction $1 < \|\psi\| = \psi(e - 2qe) \leq 1$, and the proof is complete. \square

State spaces of W^* -algebras

After the thorough preparations in the preceding sections we are now able to state and proof our first main result. We characterize abstractly the normal m -convex state spaces of W^* -algebras. (Compare with [34, Theorem 2.10] or [6, Theorem 10.25].)

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Theorem 2.19. *Let (V, K) be a matrix base norm space. Then K is the normal state space of a W^* -algebra if and only if*

- (i) V is complete in the matrix base norm,
- (ii) K is norm-closed, and
- (iii) every norm-exposed face of K_1 is projective.

Proof. If (X, e) is a W^* -algebra and (V, K) is its (matrix) predual such that the normal m -convex state space K is the m -base of V , then V is complete and K is closed in the m -base norm, cf. Proposition 1.48. Moreover, $\mathcal{M}(X) = \mathcal{M}_\sigma(X)$ can be identified with X , see Remark 2.8. Therefore the multiplier projections correspond with the projections of X . It is well-known that every norm-exposed face of the normal state space K_1 of the W^* -algebra X is projective, cf. [5].

Conversely, let (V, K) be an m -base norm space fulfilling the conditions (i) to (iii). Then $(V, K)^* = (X, e)$ is an operator system, which is isomorphic to $A_b(K)$, see Proposition 1.42. We construct its multiplier algebra \mathcal{M}_σ and claim first that the complete isometry $\Omega: \mathcal{M}_\sigma \rightarrow X$, given by $\Omega(T) = Te$, see Proposition 2.6, is surjective. There exist a Hilbert space H and representations $\pi: X \rightarrow \mathcal{B}(H)$ and $\tilde{\pi}: \mathcal{M}_\sigma \rightarrow \mathcal{B}(H)$, where π is a w^* - w^* -homeomorphism into $\mathcal{B}(H)$, see Proposition 2.11. By construction of the mappings, we have $\pi(\Omega(T)) = \pi(Te) = \tilde{\pi}(T)$ for $T \in \mathcal{M}_\sigma$. This shows, that $\Omega(\mathcal{M}_\sigma) = \pi^{-1}(\tilde{\pi}(\mathcal{M}_\sigma))$. So $\Omega(\mathcal{M}_\sigma)$ is a w^* -closed subset of X , because $\tilde{\pi}(\mathcal{M}_\sigma)$ is w^* -closed in $\mathcal{B}(H)$. Since we postulate that every norm-exposed face of K_1 is projective, we see from Proposition 2.18, that

$$[0, e] \subset \text{conv}(\Omega(\mathcal{P}))^- \subset \Omega(\mathcal{M}_\sigma).$$

Since X is an operator system, we have $X = X_h + iX_h$, $X_h = X_+ - X_+$ and $x \leq \|x\|e$ for $x \in X_h$. So, we see immediately that Ω is surjective. Now we can carry over the multiplication of \mathcal{M}_σ to X by setting $st = S(Te)$, where $s, t \in X$ and S, T are the unique elements of \mathcal{M}_σ such that $Se = s$ and $Te = t$. It is obvious that $st = St$. We have to verify next, that X is a $*$ -algebra under this multiplication. Let $s, t \in X$ and $S, T \in \mathcal{M}_\sigma$ such that $Se = s$ and $Te = t$. Then $eT^* = (Te)^* = t^*$, and hence we obtain $st^* = St^* = S(eT^*) = (Se)T^* = (T(Se)^*)^*$, which shows $(st^*)^* = T(Se)^* = Ts^* = ts^*$. Since this holds for arbitrarily chosen $s, t \in X$, we get $(st)^* = t^*s^*$, so X is a $*$ -algebra. Then $M_n(X)$ is a $*$ -algebra under matrix multiplication for all $n \in \mathbb{N}$. It follows from equation (2.2) in the proof of Proposition 2.6 that the order unit norm on $M_n(X)$ is a C^* -norm, i.e., satisfies $\|t\|^2 \leq \|tt^*\|$ for all $t \in M_n(X)$ and $n \in \mathbb{N}$. Hence X is a C^* -algebra under the above product with the same matrix ordering structure. (It is obvious that tt^* is in $M_n(X)_+$ for all $t \in M_n(X)$. Conversely, given $x \in M_n(X)_+$ with $\|x\| \leq 1$ we see that $\|e_n - x\| \leq 1$, so that the spectrum of x is positive, i.e., x is positive in the C^* -sense.) Obviously, (V, K) is the complete predual of (X, e) and the claim follows from Proposition 1.48. \square

Remark 2.20. So far we have worked only in the multiplier algebra of an operator system itself, and defined projections in the multiplier algebra, cf. Definition 2.12. However we can also consider matrices over the multiplier algebra. Let (X, e) be an operator system and \mathcal{M} its multiplier algebra. For each $n \in \mathbb{N}$ we define $p = [p_{ij}] \in M_n(\mathcal{M})$ to be a multiplier projections if $[p_{ij}(e)] = pe_n$ is self-adjoint in $M_n(X)$ and $p^2 = p$, where

we let the multiplication on $M_n(\mathcal{M})$ be the canonical matrix multiplication. Obviously, $\Omega^{(n)}: M_n(\mathcal{M}) \rightarrow M_n(X)$ is given by $\Omega^{(n)}T = [T_{ij}e] = Te_n$.

With the preceding remark we can use Theorem 2.19 to obtain a first characterization of the matrix convex state spaces of C^* -algebras. Let $K = (K_n)_{n \in \mathbb{N}}$ be a compact matrix convex set. We embed K as matrix base in the dual (V, K) of the operator system $A(K)$ of all continuous matrix affine mappings on K , cf. Corollary 1.45. Let the dual of (V, K) be the operator system (X, e) , which can be identified with the space $A_b(K)$ of all bounded matrix affine mappings on K , cf. Proposition 1.42. We construct the matrix multiplier algebra \mathcal{M}_σ of X .

Theorem 2.21. *Let $K = (K_n)_{n \in \mathbb{N}}$ be a compact matrix convex set. Then K is matrix affinely homeomorph to the m -convex state space of a unital C^* -algebra if and only if the following two conditions hold:*

- (i) *Every norm-exposed face of K_1 projective.*
- (ii) *For $a \in M_2(A(K))_h$ there are $x, y \in M_2(A(K))_+$ and a multiplier projection $P \in M_2(\mathcal{M}_\sigma)$ such that $a = x - y$, $x \leq Pe_2$ and $y \leq e_2 - Pe_2$.*

Proof. Since the dual $(V, K) = A(K)^*$ is naturally a complete m -base norm space and the m -base K (identified with a subset of V) is norm-closed, by Theorem 2.19 the first condition is equivalent to $A_b(K)$ being a W^* -algebra.

If K is m -affinely homeomorph to the m -convex state space of a unital C^* -algebra \mathcal{A} , then it follows from Proposition 1.26 that there is a complete order isomorphism between \mathcal{A} and $A(K)$. Then $M_2(A(K))$ is a C^* -algebra, and noting that the multiplier projections are just the projections, see Remark 2.8, condition (ii) is fulfilled, since any $a \in M_2(A(K))_h$ has a unique decomposition into positive and negative parts $a = a^+ - a^-$ such that $a^+, a^- \in M_2(A(K))_+$ and $a^+a^- = 0$.

Conversely, assume the compact m -convex set K fulfills condition (i) and (ii). Then $A_b(K)$ is a W^* -algebra, as noted already. We claim that the self-adjoint subspace $A(K) \subset A_b(K)$ is closed under the multiplication of $A_b(K)$. Since $A_b(K)$ is a C^* -algebra, $M_2(A_b(K))$ is a C^* -algebra, too. So, to any self-adjoint $a \in M_2(A(K))$, there is the unique decomposition into positive and negative parts $a = a^+ - a^-$ such that $a^+, a^- \in M_2(A_b(K))_+$ and $a^+a^- = a^-a^+ = 0$. Let $a = x - y$ be the decomposition that exists by condition (ii). Since $p = Pe_2$ is a projection in $M_2(A_b(K))$, we have $x = xp = px$ and also $y = yp' = p'y$, where $p' = e_2 - p$. So, $xy = xpp'y = 0$ which implies $x = a^+$ and $y = a^-$. Especially, we have $a^+ \in M_2(A(K))_h$, so it follows from Lemma A.7 that $M_2(A(K))_h$ is closed under squares. Since

$$\begin{pmatrix} * & yx \\ * & * \end{pmatrix} = \begin{pmatrix} y & x \\ x^* & 0 \end{pmatrix}^2 \in M_2(A(K))_h,$$

for $x \in A(K)$ and $y \in A(K)_h$ shows that $yx \in A(K)$ (and analogously $xy \in A(K)$), it follows that $A(K)$ is closed under the multiplication inherited from $A_b(K)$. Then $A(K)$ is a C^* -algebra and K is m -affinely homeomorph with the m -convex state space $CS(A(K))$, cf. Proposition 1.26, which shows the claim. \square

Conclusions

Theorem 2.21 looks quite similar to [6, Thm. 11.59] or [7, Cor. 8.6]. In addition to defining the projections in a different way, cf. Remark 2.13, where we can make use of matrix

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orderings, we lifted the second axiom of Alfsen's and Shultz' theorem to the second matrix level. However this second axiom, that is, 'Every $a \in A(K)$ admits a decomposition $a = b - c$ with $b, c \in A(K)^+$ and $b \perp c$.'¹, does not seem to be too enlightening what the structure on the state space of C^* -algebras is concerned. Lifting it to the second matrix level only makes it worse.

In my opinion what we have seen so far is just a W^* -result. Theorem 2.19 characterizes the m -convex normal state spaces of W^* -algebras in a simple and satisfying way. By using matrix orderings it is possible to avoid the complications and additional requirements of [6, Theorem 10.25]. Notice that all our requirements, except for the matrix ordering, are contained in the word 'spectral' in Iochum's and Shultz' theorem. The interested reader may have noticed that curiously [34, Theorem 2.10] was found some years after [7, Corollary 8.6]. Normally, one would expect to find the W^* -result first, since W^* -algebras have a richer structure than C^* -algebras. However, for the concept of orientation, as defined in [7], extreme points, or pure states, are required. But the normal state spaces of W^* -algebras do not contain extreme points in general.

Notice that we did not mention extreme points in Theorem 2.21. Actually, we need a different approach to find out more about the structure of the state spaces of C^* -algebras. We need to consider the pure matrix states. This is what the next chapter is all about.

¹Where K is a compact convex set and $b \perp c$ is defined via the orthogonality of P -projections.

3. Matrix Convex Simplexes

Everybody knows from his early days in analysis that a commutative unital C^* -algebra can be represented as the space $C(X)$ of continuous complex valued functions on some compact Hausdorff space X (or in the non-unital case as the space $C_0(X)$ of continuous functions vanishing at infinity on some locally compact Hausdorff space). Of course, if there is a homeomorphism between compact Hausdorff spaces X and Y then there is a unital $*$ -isomorphism between the C^* -algebras $C(X)$ and $C(Y)$. In the converse direction, a unital $*$ -isomorphism between $C(X)$ and $C(Y)$ implies that X and Y are homeomorphic. It was this observation that has inspired mathematicians in C^* -theory ever since then, because it means that all algebraic properties of the C^* -algebra $C(X)$ are topologically stored in X . However, the general situation for C^* -algebras is as elusive as the meaning of the term ‘non-commutative topology’. The reason for this is that a commutative C^* -algebra is completely determined by its order structure, whereas in the general case we need the *matrix* order structure to determine a C^* -algebra, as we have seen already in the previous sections.

It is well-known that the state space of $C(X)$ is affinely homeomorphic to the set of probability measures on the compact Hausdorff space X and that X is homeomorphic to the Dirac measures, which are the extreme points in the set of probability measures on X . This shows that the state space of $C(X)$ is a so-called *Bauer simplex*, i.e., a Choquet simplex such that the set of its extreme points is closed. Conversely, every Bauer simplex is affinely homeomorphic to the probability measures on its set of extreme points, which is a compact Hausdorff space, cf. [4, Cor. II 4.2]. Hence the state spaces of commutative C^* -algebras are exactly the Bauer simplexes.

In this chapter we will define *matrix convex (Bauer) simplexes* in such a way that the m -convex hull of a Bauer simplex is a (trivial) matrix convex simplex. Then we will prove that the matrix convex state spaces of C^* -algebras are exactly the matrix convex simplexes, including the commutative situation as an easy special case. So we will find another way to characterize which compact matrix convex sets are the state spaces of C^* -algebras. This can be seen as an improved version of Theorem 2.21 obtained by considering the (matrix) affine maps only on the extreme points of the state space.

We start with the following abstract definitions, that we will motivate soon.

Definition 3.1 (Equivariant Matrix Sets). Let W be a complex vector space and let $X = (X_n)_n$ be a sequence of subsets such that $X_n \subset M_n(W)$ for all $n \in \mathbb{N}$. Then X is called a matrix subset of W , or simply a matrix set. If moreover $u^* X_n u \subset X_m$ for all isometries $u \in M_{n,m}^1$ and $n, m \in \mathbb{N}$, where $n \geq m$, then X is called an *equivariant* matrix subset of W , or simply an equivariant matrix set. In particular, m -convex sets are equivariant matrix sets. We will consider also the case, where W is a locally convex space. Then we endow $M_n(W)$ with the product topology for all $n \in \mathbb{N}$. A matrix set X of W is called *compact*, if $X_n \subset M_n(W)$ is compact for all $n \in \mathbb{N}$.

¹That is, for all $u \in M_{n,m}$ such that $u^*u = \mathbb{1}_m$.

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Definition 3.2 (Matricial Relation). Let X be an equivariant matrix subset of a vector space W . Let $n, m \in \mathbb{N}$, $x \in X_n$ and $y \in X_m$. Then x is *matrix related* to y , in symbols $x \curvearrowright y$, if there exist $l \geq n, m$, isometries $u \in M_{l,n}$, $v \in M_{l,m}$, and $z \in X_l$ such that $u^*zu = x$ and $v^*zv = y$. We will also write $x \preceq z$ or $z \succeq x$, if $u^*zu = x$ for some isometry $u \in M_{l,n}$. Moreover, we use the negations $x \not\curvearrowright y$ and $x \not\preceq y$, if $x \curvearrowright y$ and $x \succeq y$ do not hold, respectively. Obviously, the matricial relation on X is reflexive and symmetric. In case the matrix relation should also be transitive, and hence is an equivalence relation, we will also say x is matrix equivalent to y , if $x \curvearrowright y$.

Definition 3.3 (Matrix Orthogonal). Let X be an equivariant matrix subset of some vector space. Then for arbitrary $n, m \in \mathbb{N}$ two elements $x \in X_n$ and $y \in X_m$ are called *matrix orthogonal*, in symbols $x \perp y$, if x and y are not matrix related or there exists an element $z \in X_{n+m}$ such that $z = \begin{pmatrix} x & z_{12} \\ z_{21} & y \end{pmatrix}$ (which means in particular that x and y are m-related). For a subset Y of X the *m-orthogonal complement* $Y^\perp = (Y_n^\perp)$ is defined by

$$Y_n^\perp = \{x \in X_n \mid x \perp y \ \forall y \in Y_m, m \in \mathbb{N}\}.$$

The next propositions indicates where the above definitions come from.

Proposition 3.4. *Let \mathcal{A} be a C^* -algebra. Then $X = \text{str}(CQ(\mathcal{A}))$ is equivariant and the m-relation on X coincides with the equivalence of pure states, and hence is transitive.*

Proof. We will first show that X is invariant under isometries. Let $x \in X_n$ and let $u \in M_{n,m}$ be an isometry. Given a minimal Stinespring representation $x = \mathcal{V}^*\pi\mathcal{V}$, we know that π must be irreducible, since x is pure. Then $u^*\mathcal{V}^*\pi\mathcal{V}u = y$ is a minimal Stinespring representation of y , because π is irreducible. It follows, that y is pure and approximately unital ($u^*u = \mathbb{1}_m$), so that $y \in X_m$.

Next we will show that the m-relation is equivalent to the equivalence of pure states. Let $x_i \in X_{n_i}$ for some $n_i \in \mathbb{N}$, $i = 1, 2$. Define $x_1 \simeq x_2$, if π_{x_1} and π_{x_2} are unitarily equivalent, in symbols $\pi_{x_1} \simeq \pi_{x_2}$, where π_{x_1} and π_{x_2} are representations of a minimal Stinespring representation of x_1 and x_2 , respectively. We have to show that $x_1 \curvearrowright x_2$ if and only if $x_1 \simeq x_2$. Let $x_1 \curvearrowright x_2$. By definition, there are $l \geq n, m$ and $z \in X_l$ and isometries $u_i \in M_{l,n_i}$, such that $x_i = u_i^*zu_i$ for $i = 1, 2$. But then, replacing x_1, x_2 and z with minimal Stinespring representations, it is obvious that $\pi_{x_1} \simeq \pi_z$ and $\pi_z \simeq \pi_{x_2}$, because π_z is irreducible (and hence $u_i^*\mathcal{V}^*\pi_z\mathcal{V}u_i$ are minimal Stinespring representation of x_i for $i = 1, 2$, where $z = \mathcal{V}^*\pi_z\mathcal{V}$ is the minimal Stinespring representation for z). Since \simeq is an equivalence relation, we obtain $\pi_{x_1} \simeq \pi_{x_2}$.

For the converse, we assume that $x_1 \simeq x_2$. Let $x_1 = \mathcal{V}_1^*\pi\mathcal{V}_1$ be a minimal Stinespring representation of x_1 . Since $x_1 \simeq x_2$, we find a minimal Stinespring representation of x_2 , such that $x_2 = \mathcal{V}_2^*\pi\mathcal{V}_2$. Let $L = \text{lin}\{\mathcal{V}_i(C^{n_i}) \mid i = 1, 2\}$ and $l = \dim(L)$. We can identify L with \mathbb{C}^l . So let $\mathcal{W}: \mathbb{C}^l \rightarrow L \subset H_\pi$ be an isometry. Then $z = \mathcal{W}^*\pi\mathcal{W} \in X_l$. Notice that $\mathcal{W}\mathcal{W}^* = p_L$, the projection onto L and that $\mathcal{V}_i(C^{n_i}) \subset L$. Thus we can define isometries $u_i = \mathcal{W}^*|_L \mathcal{V}_i$ for $i = 1, 2$. Notice, that $\mathcal{W}u_i = \mathcal{V}_i$, because $\mathcal{W}\mathcal{W}^* = p_L$ and $\mathcal{V}_i(C^{n_i}) \subset L$. Thus $u_i^*\mathcal{W}^* = \mathcal{V}_i^*$. Then we see immediately that

$$u_i^*zu_i = u_i^*\mathcal{W}^*\pi\mathcal{W}u_i = \mathcal{V}_i^*\pi\mathcal{V}_i = x_i,$$

for $i = 1, 2$, which shows the claim. \square

Corollary 3.5. *Let \mathcal{M} be an atomic W^* -algebra. Then $\text{str}(CS^\sigma(\mathcal{M}))$ is equivariant and transitive.*

Proof. We can assume that $\mathcal{M} = \oplus \mathcal{B}(H_\kappa)$. Let $\mathcal{A} = \oplus \mathcal{C}(H_\kappa)$. Then $\mathcal{A}^* = \oplus_1 \mathcal{T}(H_\kappa)$ and $\mathcal{A}^{**} = \mathcal{M}$. From Corollary 1.45 it follows that \mathcal{A}^* is an m-base norm space with m-base $K = CS(\mathcal{A})$. So, $\mathcal{A}^{**} =_{cp} A_b(K)$ by Proposition 1.42. Then it follows from Proposition 1.48 that there is an m-affine isomorphism between K and $CS^\sigma(\mathcal{M})$, so that the claim follows immediately from Proposition 3.4. \square

Definition 3.6. Let V be a vector space. Let X be an equivariant matrix subset of V . Then we will call X *transitive*, if the m-relation on X is transitive. Furthermore, for x in some X_l , we define the matrix set $[x] = ([x]_n)_{n \in \mathbb{N}}$, where

$$[x]_n = \{ y \in X_n \mid y \curvearrowright x \}$$

for all $n \in \mathbb{N}$. If X is transitive, and so the m-relation on X is an equivalence relation, then $[x]$ is called the equivalence class of x .

Proposition 3.7. *Let \mathcal{A} be a C^* -algebra and $X = \text{str}(CS(\mathcal{A}))$. Then two pure states $x, y \in X_1$ of \mathcal{A} are orthogonal if and only if x and y are m-orthogonal.*

Proof. Let $\pi_a: \mathcal{A} \rightarrow \mathcal{B}(H_a)$, where $\pi_a = \oplus \pi$ for a maximal family of pairwise non-equivalent irreducible representations $\pi: \mathcal{A} \rightarrow H_\pi$ and $H_a = \oplus H_\pi$, be the reduced atomic representation of \mathcal{A} . Then to any pure state $x \in X_1$ there is a unique π_x of the family and a unit vector $\xi_x \in H_{\pi_x} \subset H_a$ that is unique up to a factor of modulus 1 such that $x(a) = \langle \pi_a(a)\xi_x, \xi_x \rangle$. Two pure states $x, y \in X_1$ are orthogonal if and only if the corresponding unit vectors ξ_x and ξ_y are orthogonal. This is always the case if H_{π_x} and H_{π_y} are distinct, i.e., if π_x and π_y are not unitarily equivalent. In this case x and y are not unitarily equivalent and hence they are not m-related by Proposition 3.4, so they are m-orthogonal by definition. If on the other hand ξ_x and ξ_y are in $H_{\pi_x} = H_\pi$ (for short) and they are orthogonal, then $\mathcal{V}\varepsilon_1 = \xi_x$ and $\mathcal{V}\varepsilon_2 = \xi_y$, where $\{\varepsilon_1, \varepsilon_2\}$ is the standard basis of \mathbb{C}^2 , defines an isometry $\mathcal{V}: \mathbb{C}^2 \rightarrow H_\pi$. Since π is irreducible, it is obvious that $z = \mathcal{V}^* \pi \mathcal{V}$ is a pure matrix state and therefore in X_2 . Moreover, $z_{11} = \varepsilon_1^* z \varepsilon_1 = x$ and $z_{22} = y$, so x and y are m-orthogonal. Conversely, if x and y are m-related and m-orthogonal, there is $z \in X_2$ such that $z_{11} = x$ and $z_{22} = y$ by definition. Since z is pure and the minimal Stinespring representation is essentially unique, there is a π in the family (of the reduced atomic representation) such that $z = \mathcal{V}^* \pi \mathcal{V}$ for an isometry $\mathcal{V}: \mathbb{C}^2 \rightarrow H_\pi$. Then we see immediately that $\xi_x = \mathcal{V}\varepsilon_1$ and $\xi_y = \mathcal{V}\varepsilon_2$ are orthogonal, so that x and y are orthogonal pure states, and the proof is complete. \square

Apart from the fact that the set of structural elements of the state space of C^* -algebras is equivariant and the m-relation is equivalent to the unitary equivalence of representations (and hence an equivalence relation itself in this case), equivariant matrix sets have some further interesting properties, as the following results will show.

Proposition 3.8. *Let $X = (X_n)_{n \in \mathbb{N}}$ be an equivariant matrix subset of a $*$ -vector space V such that $X_n \subset M_n(V)_h$ for all $n \in \mathbb{N}$ and such that X_1 consists entirely of extreme points and X_2 consists entirely of m-irreducible points. Let $n \in \mathbb{N}$ and $y \in X_n$. If $u^* y u = v^* y v$ for isometries $u, v \in M_{n,m}$, then there is $\lambda \in \mathbb{C}$ such that $u = \lambda v$.*

Proof. We first assume $n = 2$ and $m = 1$. So let $y = [y_{ij}] \in X_2$, $x \in X_1$ and let ξ and η be unit vectors in \mathbb{C}^2 , such that $x = \eta^* y \eta = \xi^* y \xi$. Suppose for contradiction that η and ξ would be linearly independent. Observe that $e^{ir} \otimes \mathbb{1}_2$, where $r \in \mathbb{R}$, is a unitary matrix

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in M_2 such that $y = (e^{-ir} \otimes 1_2)y(e^{ir} \otimes 1_2)$. Therefore we may assume without loss of generality that $x = y_{11}$, $\eta = (1, 0)^{\text{tr}}$ and $\xi = (\xi_1, \xi_2)^{\text{tr}}$ such that $\xi_2 \in \mathbb{R} \setminus \{0\}$. Then we obtain $x = |\xi_1|^2 x + 2 \operatorname{Re}(\bar{\xi}_1 \xi_2 y_{12}) + |\xi_2|^2 y_{22}$. With the unit vectors

$$\zeta_{\pm} = \frac{\eta \pm \xi}{\|\eta \pm \xi\|} = \frac{1}{\sqrt{2(1 \pm \operatorname{Re} \xi_1)}} (1 \pm \xi_1, \pm \xi_2)^{\text{tr}}$$

we let

$$\begin{aligned} z_{\pm} &= \zeta_{\pm}^* y \zeta_{\pm} = \frac{|1 \pm \xi_1|^2 x \pm 2 \operatorname{Re}((1 \pm \bar{\xi}_1) \xi_2 y_{12}) + |\xi_2|^2 y_{22}}{2(1 \pm \operatorname{Re} \xi_1)} \\ &= \frac{(1 \pm 2 \operatorname{Re} \xi_1)x + x \pm 2 \operatorname{Re}(\xi_2 y_{12})}{2(1 \pm \operatorname{Re} \xi_1)} \\ &= x \pm \frac{\operatorname{Re}(\xi_2 y_{12})}{1 \pm \operatorname{Re} \xi_1} \in K_1. \end{aligned}$$

Now we see that

$$x = \frac{1 + \operatorname{Re} \xi_1}{2} z_+ + \frac{1 - \operatorname{Re} \xi_1}{2} z_- ,$$

is a convex combination of z_+ , $z_- \in X_1$. Since x is an extreme point, it follows that $x = z_+$ or $x = z_-$. Thus $\operatorname{Re}(\xi_2 y_{12}) = 0$. Since $\xi_2 \neq 0$ this implies $\operatorname{Re} y_{12} = 0$. Performing the preceding calculation with $i\xi_2$ in place of ξ_2 yields $-\operatorname{Im}(\xi_2 y_{12}) = \operatorname{Re}(i\xi_2 y_{12}) = 0$ and thus $\operatorname{Im} y_{12} = 0$, so that $y_{12} = 0$. It follows $y = x \oplus y_{22}$, because y is assumed to be self-adjoint. This is a contradiction to the assumption that $y \in X_2$ is m-irreducible. So, we have proved that η and ξ are linearly dependent. It is left to reduce the general case to this special case. To do so, let $n \in \mathbb{N}$ arbitrary. Given $y \in X_n$, suppose for contradiction that there are linearly independent unit vectors $\xi_1, \xi_2 \in \mathbb{C}^n$ such that $\xi_1^* y \xi_1 = \xi_2^* y \xi_2$. Then letting $L = \operatorname{lin}\{\xi_1, \xi_2\}$ we have $\dim L = 2$. Hence there is a isometry $\gamma: \mathbb{C}^2 \rightarrow \mathbb{C}^n$ such that $\gamma^* \gamma = 1_2$ and $\gamma \gamma^* = p_L$, where p_L denotes the orthogonal projection onto the 2-dimensional subspace L . It follows that

$$\xi_1^* \gamma \gamma^* y \gamma \gamma^* \xi_1 = \xi_1^* y \xi_1 = \xi_2^* y \xi_2 = \xi_2^* \gamma \gamma^* y \gamma \gamma^* \xi_2,$$

which is a contradiction, because $\gamma^* y \gamma \in X_2$ and $\gamma^* \xi_1$ and $\gamma^* \xi_2$ are linearly independent. Thus we have shown that there is $\lambda \in \mathbb{C}$ such that $\xi_1 = \lambda \xi_2$ and $|\lambda| = 1$, because ξ_1 and ξ_2 are unit vectors. Now, let $n, m \in \mathbb{N}$. Given $y \in X_n$ and isometries $u, v \in M_{n,m}$ such that $u^* y u = v^* y v$. Letting u_i be the i -th column vector of u for $i = 1, \dots, m$, i.e., $u_i = (u_{1i}, \dots, u_{ni})$, and v_i the i -th column vector of v , we have u_i and v_i are unit vectors, such that $u_i^* y u_i = v_i^* y v_i$ for $i = 1, \dots, m$. By the above there are $\lambda_i \in \mathbb{C}$ such that $u_i = \lambda_i v_i$ for $i = 1, \dots, m$. Since $x = [x_{\mu\nu}] = u^* y u$ is in X_m , the matrices $[x_{\mu\nu}]_{\mu, \nu=i}^{i+1}$ are in X_2 for $i = 1, \dots, m-1$. Hence $x_{i,i+1} \neq 0$ for $i = 1, \dots, m-1$, because X_2 consists entirely of m-irreducible elements by assumption. Evaluating the $(i, i+1)$ entries, we get

$$x_{i,i+1} = u_i^* y u_{i+1} = v_i^* y v_{i+1} = \bar{\lambda}_i \lambda_{i+1} u_i^* y u_{i+1}.$$

This implies $\bar{\lambda}_i \lambda_{i+1} = 1$ and since $|\lambda_i| = 1$ we get $\lambda_i = \lambda_{i+1}$ for all $i = 1, \dots, m-1$. So we have shown that there is $\lambda \in \mathbb{C}$ such that $u = \lambda v$ and the proof is complete. \square

Motivated by Proposition 3.8 we define what we will call the uniqueness property.

Definition 3.9 (Uniqueness Property). Let $X = (X_n)_{n \in \mathbb{N}}$ be an equivariant matrix subset of a vector space W . We will say that X fulfills the *uniqueness property*, if whenever $u^*xu = v^*xv$ for $x \in X_n$, isometries $u, v \in M_{n,m}$ and $n, m \in \mathbb{N}$, there is $\lambda \in \mathbb{C}$ such that $u = \lambda v$.

Remark 3.10. If K is a compact and m -convex subset of a locally convex vector space V , then we can embed K as m -base into the dual of $A(K)$, i.e., we have a matrix affine homeomorphism between K and $CS(A(K))$. Thus, if $\text{str}(K)$ is equivariant, Proposition 3.8 applies to the equivariant matrix subset $X = \text{str}(K)$ of $V = A(K)^*$, since structural elements are in particular m -irreducible. In addition it is obvious that the matrix sets of the pure m -states of C^* -algebras and of the normal pure m -states of atomic W^* -algebras fulfill the uniqueness property.

Remark 3.11. If $X = (X_n)_n$ is an equivariant matrix set fulfilling the uniqueness property, then an isometry $u \in M_{m,n}$ such that $x = u^*yu$, where $x \in X_n$ and $y \in X_m$, is uniquely determined up to a complex factor of modulus one. Since in the calculations that we will perform all isometries will be accompanied by their adjoint matrices, a factor of modulus one will not matter. We indicate this situation with the notation $x = u_{xy}^*yu_{xy}$, as if there would be a unique isometry u_{xy} that transforms y into x .

Equivariant mappings

Equivariant matrix sets have an additional structure, namely the equivariance, and in mathematics it is usual to consider not all, but only those functions, that are compatible with the additional structure on the set. We haven't defined yet what these functions are in the case of equivariant matrix sets. So here is the definition of equivariant mappings.

Definition 3.12. Let V and W be vector spaces. Let X be an equivariant matrix subset of V and let $f = (f_n)_n$ be a sequence of maps $f_n: X_n \rightarrow M_n(W)$ for all $n \in \mathbb{N}$. If $f_n(u^*xu) = u^*f_m(x)u$ for all $x \in X_n$, isometries $u \in M_{n,m}$ and all $n, m \in \mathbb{N}$, where $n \geq m$, then f is called an *equivariant map* from X to W . The vector space of all equivariant maps from X to W with pointwise operations will be denoted by $\mathcal{F}^E(X, W)$. In case $W = \mathbb{C}$ we let $\mathcal{F}^E(X) = \mathcal{F}^E(X, \mathbb{C})$. We call an equivariant map $f \in \mathcal{F}^E(X)$ *bounded*, if f_1 is bounded. We let $\mathcal{F}_b^E(X)$ denote the vector space of all bounded equivariant maps from X to \mathbb{C} . If in addition V is a locally convex topological vector space, we let $\mathcal{C}^E(X)$ be the set of $f \in \mathcal{F}^E(X)$ such that $f_n: X_n \rightarrow M_n$ is continuous for all $n \in \mathbb{N}$, where we endow $M_n(V)$ with the product topology. Notice that it suffices by equivariance to require that f_1 is continuous. Moreover, we will consider also the situation where the closure X_n^- is compact for all $n \in \mathbb{N}$. Then we write $\mathcal{C}_u^E(X)$ for the set of all $f \in \mathcal{F}^E(X)$ such that f_n is uniformly continuous on X_n for all $n \in \mathbb{N}$. Notice that $\mathcal{C}_u^E(X) \subset \mathcal{F}_b^E(X)$ and that we can identify $\mathcal{C}_u^E(X)$ with $\mathcal{C}^E(X^-)$, where X^- is the equivariant matrix set $(X_n^-)_n$.

Let V be a vector space. Given an equivariant subset $X = (X_n)_n$ of V we have defined in particular the vector spaces $\mathcal{F}^E(X, M_l)$ for all $l \in \mathbb{N}$. Using the $*$ -operation and the order on $M_n(M_l)$, we conclude that $\mathcal{F}^E(X, M_l)$ is an ordered vector space. This means that for $f = (f_n)_n \in \mathcal{F}^E(X, M_l)$ we define f^* by $f_n^*(x) = f_n(x)^*$ and we set $f \geq 0$ if $f_n(x) \geq 0$ for all $x \in X_n$ and $n \in \mathbb{N}$. We define a matrix ordering by setting $M_l(\mathcal{F}^E(X))_+ = \mathcal{F}^E(X, M_l)_+$. We let $\mathbb{1}_n(x) = \mathbb{1}_n$ for all $x \in X_n$ and $n \in \mathbb{N}$ and $\mathbb{1} = (\mathbb{1}_n) \in \mathcal{F}^E(X)$.

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Lemma 3.13. *Let $l \in \mathbb{N}$ and $f \in M_l(\mathcal{F}^E(X)) = \mathcal{F}^E(X, M_l)$. If $f_l(y) \geq 0$ for all $y \in X_l$, then $f_n(x) \geq 0$ for all $x \in X_n$ and $n \in \mathbb{N}$ with $n \geq l$.*

Proof. Let $n \geq l$. We have to show that $f_n(x) \geq 0$ for all $x \in X_n$. Notice that we identify $M_n(M_l) = M_{nl}$. So we have to show that $\xi^* f_n(x) \xi$ must be positive for all $\xi = (\xi_i) \in \mathbb{C}^{nl}$. We define $\eta_\nu = (\xi_{\nu+\mu l})_{\mu=0}^{n-1} \in \mathbb{C}^n$ for $\nu = 1, \dots, l$. The linear hull $\text{lin}\{\eta_1, \dots, \eta_l\}$ is contained in a subspace $L \subset \mathbb{C}^n$ of dimension l . Let $v: \mathbb{C}^r \rightarrow L$ be an isometry. Since $\eta_\nu \in L$, there are $\varrho_\nu \in \mathbb{C}^r$ such that $v\varrho_\nu = \eta_\nu$ for $\nu = 1, \dots, l$. Let $\varrho_\nu = (\zeta_{\nu+\mu l})_{\mu=0}^{l-1}$ for $\nu = 1, \dots, l$. Then with $\zeta = (\zeta_i)_{i=1}^{l^2}$ we obtain

$$\xi^* f_n(x) \xi = ([v_{ij} \otimes \mathbb{1}_l] \zeta)^* f_n(x) ([v_{ij} \otimes \mathbb{1}_l] \zeta) = \zeta^* (v^* f_n(x) v) \zeta = \zeta^* f_l(v^* x v) \zeta \geq 0.$$

Notice that the matrix product $f_n(x)v$, where $f_n(x) \in M_n(M_l)$ and $v \in M_{n,l}$, cf. equation (1.1), is equal to $f_n(x)[v_{ij} \otimes \mathbb{1}_l]$, where $f_n(x) \in M_{nl}$ and $[v_{ij} \otimes \mathbb{1}_l] \in M_{nl, l^2}$. \square

Lemma 3.14. *Let $X = (X_n)_{n \in \mathbb{N}}$ be an equivariant matrix set in a vector space V . Let $f_i: X_i \rightarrow M_i$, where $i = 1, 2$, be an equivariant pair of maps, i.e., $f_i(u^* y u) = u^* f_j(y) u$ for all $y \in X_j$ and isometries $u \in M_{j,i}$, where $1 \leq i \leq j \leq 2$. Then the map h_x defined by $h_x(0) = 0$ and $h_x(\xi) = \|\xi\|^2 f_1(\xi_1^* x \xi_1)$, where $\xi_1 = \xi / \|\xi\|$ and $\xi \in \mathbb{C}^n$, $\xi \neq 0$, is a quadratic form on \mathbb{C}^n for all $x \in X_n$ and $n \in \mathbb{N}$. Moreover, if f_1 is bounded then there is $r > 0$ such that $\|h_x\| \leq r$ for all $x \in X_n$ and $n \in \mathbb{N}$.*

Proof. Let $x \in X_n$. Obviously, h_x is a well-defined map, and if f_1 is bounded by $r > 0$ then $\|h_x\| = \sup \{ |f_1(\xi^* x \xi)| \mid \xi \in \mathbb{C}^n, \|\xi\| = 1 \} \leq r$. Hence we have to prove only that h_x is a quadratic form. Let ξ and η be vectors of \mathbb{C}^n . They are contained in a two dimensional subspace $L \subset \mathbb{C}^n$. Let $\{e_1, e_2\} \subset L$ be an orthonormal basis and define an isometry $u: \mathbb{C}^2 \rightarrow \mathbb{C}^n$ by $\mu e_1 + \nu e_2 \mapsto \mu e_1 + \nu e_2$, where $\{e_1, e_2\}$ denotes the standard basis of \mathbb{C}^2 . Then for arbitrary $\zeta = \mu e_1 + \nu e_2 \in L$ such that $\zeta \neq 0$ we obtain

$$h_x(\zeta) = \|\zeta\|^2 f_1(\zeta_1^* x \zeta_1) = \|\zeta\|^2 f_1(v_0^* u^* x u v_0) = v^* f_2(u^* x u) v = u^*(\zeta)^* f_2(u^* x u) u^*(\zeta),$$

where $v = u^*(\zeta)$, $v_0 = v / \|\zeta\|$ and $\zeta_1 = \zeta / \|\zeta\|$. Since $h_x(0) = 0$ by definition, the preceding equation holds for all $\zeta \in \mathbb{C}^n$. Therefore the calculation

$$\begin{aligned} h_x(\xi + \eta) + h_x(\xi - \eta) &= u^*(\xi + \eta)^* f_2(u^* x u) u^*(\xi + \eta) + u^*(\xi - \eta)^* f_2(u^* x u) u^*(\xi - \eta) \\ &= 2u^*(\xi)^* f_2(u^* x u) u^*(\xi) + 2u^*(\eta)^* f_2(u^* x u) u^*(\eta) \\ &= 2(h_x(\xi) + h_x(\eta)), \end{aligned}$$

shows that h_x is a quadratic form on \mathbb{C}^n for all $x \in X_n$ and $n \in \mathbb{N}$. \square

Proposition 3.15. *Let $X = (X_n)_{n \in \mathbb{N}}$ be an equivariant matrix subset in some vector space V . Let $Y = (Y_n)_{n \in \mathbb{N}}$ be the equivariant matrix set defined by $Y_1 = X_1$, $Y_2 = X_2$ and $Y_n = \emptyset$ for $n > 2$. Then there is a 2-bipositive order isomorphism between $\mathcal{F}^E(X)$ and $\mathcal{F}^E(Y)$. Moreover, if Z is an equivariant matrix subset of a vector space W such that there is an equivariant isomorphism $\phi = (\phi_i)_{i=1}^n$, where $\phi_i: Z_i \rightarrow X_i$ for $i = 1, \dots, n$, and $n \geq 2$, then there is an n -positive order isomorphism between $\mathcal{F}^E(X)$ and $\mathcal{F}^E(Z)$.*

Proof. We consider the map $\psi: \mathcal{F}^E(X) \rightarrow \mathcal{F}^E(Y)$ given by $\psi((f_l)_l) = (f_1, f_2)$. Let $n > 2$ and $x \in X_n$. Given $(f_1, f_2) \in \mathcal{F}^E(Y)$ there is a matrix $f_n(x) \in M_n$ such that

$$\langle f_n(x) \xi | \xi \rangle = \|\xi\|^2 f_1(\xi_1^* x \xi_1) \quad (3.1)$$

for all $\xi \in \mathbb{C}^n$, where . For the sequence $f = (f_l)_l$ we obtain

$$\langle u^* f_n(x) u \xi | \xi \rangle = \langle f_n(x) u \xi | u \xi \rangle = \|u \xi\|^2 f_1(\xi_1^* u^* x u \xi_1) = \langle f_m(u^* x u) \xi | \xi \rangle,$$

for all isometries $u \in M_{n,m}$, and $x \in X_n$, $\xi \in \mathbb{C}^m$ and $n, m \in \mathbb{N}$, so that $f \in \mathcal{F}^E(X)$. Obviously, $\psi(f) = (f_1, f_2)$, so ψ is surjective. From equation (3.1) we see directly that ψ is injective and positive and that the inverse of ψ is also positive. Thus ψ is an order isomorphism. Let $f = [f^{ij}] \in M_2(\mathcal{F}^E(X))_+ = \mathcal{F}^E(X, M_2)_+$, so that $f_n(x) = [f_n^{ij}(x)] \geq 0$ for all $x \in X_n$ and $n \in \mathbb{N}$. Then in particular $f_1(x), f_2(x') \geq 0$ for all $x \in X_1$ and $x' \in X_2$. Therefore the amplification $\psi^{(2)}$ is positive. Conversely, if $f_1(x), f_2(x') \geq 0$ for all $x \in X_1$ and $x' \in X_2$, i.e., if $(f_1, f_2) \in \mathcal{F}^E(Y, M_2)_+ = M_2(\mathcal{F}^E(Y))_+$, we conclude from Lemma 3.13 that $f_n(x) \geq 0$ for all $x \in X_n$ and $n \in \mathbb{N}$. Thus ψ^{-1} is also 2-positive.

If Z is another matrix set such that there is an equivariant isomorphism $\phi = (\phi_i)_{i=1}^n$, where $\phi_i: Z_i \rightarrow X_i$ for $i = 1, \dots, n$, then we define an isomorphism $\psi: \mathcal{F}_b^E(X) \rightarrow \mathcal{F}_b^E(Z)$ by $\psi(f)_i(z) = f_i(\phi_i(z))$ for all $z \in Z_i$ and $i = 1, 2$. Notice from the preceding paragraph that an equivariant map $(g_l)_{l \in \mathbb{N}}$ is determined by the pair g_1 and g_2 . So in particular we have $\psi(f)_i(z) = f_i(\phi_i(z))$ for all $z \in Z_i$ and all $i = 1, \dots, n$, because

$$\langle \psi(f)_j(z) \xi | \xi \rangle = \xi^* \psi(f)_j(z) \xi = \psi(f)_1(\xi^* z \xi) = f_1(\phi_1(\xi^* z \xi)) = \xi^* f_j(\phi_j(z)) \xi,$$

for all unit vectors $\xi \in \mathbb{C}^j$ and $2 \leq j \leq n$. Since we can identify X_i with Z_i for $i = 1, \dots, n$, it is clear from the argumentation in the preceding paragraph (which was just for the special case $n = 2$), using Lemma 3.13 again, that ψ is an n -positive order isomorphism. \square

Proposition 3.16. *Let V be a vector space and let $X = (X_n)_{n \in \mathbb{N}}$ be an equivariant subset of V . Then under the above pointwise structures $(\mathcal{F}_b^E(X), \mathbb{1})$ is an operator system. Furthermore, the (matrix) order unit norm on $\mathcal{F}_b^E(X)$ coincides with the pointwise supremum norm, so that $\mathcal{F}_b^E(X)$ is a complete operator system.*

Proof. It is obvious that the cone $\mathcal{F}_b^E(X)_+$ is proper. We have to show that the the matrix orderings are archimedean, that $\mathbb{1} \in \mathcal{F}_b^E(X)$ is an order unit and that $\mathcal{F}_b^E(X)$ is complete under the order unit norm.

Let $f = [f^{ij}] \in M_l(\mathcal{F}_b^E(X))_h$ and suppose there is $g \in M_l(\mathcal{F}_b^E(X))_h = \mathcal{F}_b^E(X, M_l)_h$, such that $rf \leq g$ for all $r \geq 0$. We have to show that $f \leq 0$ holds. But this is clear, since $rf_n(x) \leq g_n(x)$ for all $x \in X_n$, $n \in \mathbb{N}$ and $r > 0$ implies by the archimedean property of $M_l(M_n) \approx M_n(M_l)$ that $f_n(x) \leq 0$ for all $x \in X_n$ and $n \in \mathbb{N}$.

In order to see that $\mathbb{1} \in \mathcal{F}_b^E(X)$ is an order unit, let $f \in \mathcal{F}_b^E(X)$ be a self-adjoint element. Notice, that

$$\|f_n(x)\| = \sup \{ |\langle f_n(x) \xi | \xi \rangle| \mid \|\xi\| = 1 \},$$

because $f_n(x)$ is self-adjoint in M_n . We have $1 = \|\xi\| = \langle \xi | \xi \rangle = \xi^* \xi$. This implies by the property of f that

$$\langle f_n(x) \xi | \xi \rangle = \xi^* f_n(x) \xi = f_1(\xi^* x \xi).$$

Thus

$$\|f_n(x)\| = \sup \{ |f_1(\xi^* x \xi)| \mid \|\xi\| = 1 \} \leq \|f_1\| \quad (3.2)$$

for all $x \in X_n$ and hence $\|f_n\| \leq \|f_1\|$, where $\|f_m\| = \sup \{ \|f_m(x)\| \mid x \in X_m \}$ for all $m \in \mathbb{N}$. This implies immediately

$$-\|f_1\| \mathbb{1}_n \leq -\|f_n\| \mathbb{1}_n \leq f_n(x) \leq \|f_n\| \mathbb{1}_n \leq \|f_1\| \mathbb{1}_n \quad (3.3)$$

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for all $x \in X_n$ and $n \in \mathbb{N}$. This shows that $\mathbb{1} = (\mathbb{1}_n)$ is an order unit of $\mathcal{F}_b^E(X)$.

So far we have an order unit and Archimedian cones. Thus we can define the matrix order unit norm on $\mathcal{F}_b^E(X)$ by

$$\|f\|_e = \inf \left\{ r \in \mathbb{R} \mid \begin{pmatrix} r\mathbb{1} & f \\ f^* & r\mathbb{1} \end{pmatrix} \geq 0 \right\},$$

for all $f \in \mathcal{F}_b^E(X)$. Notice that the positivity of the matrix above means pointwise positive, i.e., $\begin{pmatrix} r\mathbb{1}_n(x) & f_n(x) \\ f_n(x)^* & r\mathbb{1}_n(x) \end{pmatrix} \geq 0$ for all $x \in X_n$ and $n \in \mathbb{N}$. We have to show now, that $\mathcal{F}_b^E(X)$ is complete in this norm. To this end we will show that $\|f\|_e = \|f\|$ for all $f \in \mathcal{F}_b^E(X)$, where $\|f\| = \sup \{ \|f_n(x)\| \mid x \in X_n, n \in \mathbb{N} \}$. Notice, that $\|f\| < \infty$, since

$$\|f_n(x)\| \leq \|\operatorname{Re} f_n\| + \|\operatorname{Im} f_n\| \leq \|\operatorname{Re} f_1\| + \|\operatorname{Im} f_1\|,$$

for all $x \in X_n$ and $n \in \mathbb{N}$.

Obviously, we have

$$\begin{pmatrix} \|f\|\mathbb{1}_n & f_n(x) \\ f_n(x)^* & \|f\|\mathbb{1}_n \end{pmatrix} \geq 0,$$

because $\|f_n(x)\| \leq \|f\|$ and hence $\|f\|_e \leq \|f\|$. On the other hand, if we suppose that $\|f\|_e < \|f\|$, then there is $r > 0$, such that $\|f\|_e < r < \|f\|$ and $\begin{pmatrix} r\mathbb{1} & f \\ f^* & r\mathbb{1} \end{pmatrix} \geq 0$. But this implies $\|f_n(x)\| \leq r$ for all $x \in X_n$ and $n \in \mathbb{N}$, which leads to a contradiction. Hence we have shown that $\|f\|_e = \|f\|$ and we will simply write $\|f\|$ for the matrix order unit norm. It is now obvious, that $\mathcal{F}_b^E(X)$ is complete in the order unit norm. \square

Non-commutative product of functions

So far we have defined what an equivariant matrix set is and we have seen that the space $\mathcal{F}_b^E(X)$ of bounded equivariant maps on an equivariant set X is an operator system under pointwise structures. If the m -relation (Definition 3.2) is transitive and the matrix set fulfills the uniqueness property (Definition 3.9), then much more will be true. Indeed, in this case the operator system $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra. In order to prove this claim, we need to define a product on $\mathcal{F}_b^E(X)$ that is compatible with the matrix order structure of $\mathcal{F}_b^E(X)$. But before thinking of the matrix order structure of $\mathcal{F}_b^E(X)$, we need a non-commutative product of two functions $f, g \in \mathcal{F}_b^E(X)$. We cannot simply set $(fg)_n(x) = f_n(x)g_n(x)$. While the ordering on $\mathcal{F}_b^E(X)$ was pointwise defined, a pointwise product does not make sense – one reason is that $(fg)_1(x)$ should not be $(gf)_1(x)$ in general, unless $\mathcal{F}_b^E(X)$ is a commutative W^* -algebra, i.e., the bounded functions on some set; another reason is that the function $fg = ((fg)_n)$ defined by a pointwise product would not be equivariant in general.

Instead of defining $(fg)_1(x) = f_1(x)g_1(x)$, the simple idea is to multiply the larger matrices $f_n(y)$ and $g_n(y)$ for all $n \in \mathbb{N}$ and $y \in X_n$ such that $y \succ x$, and then to cut down these products with the isometry u_{xy} that transforms y into x , for the notation recall Remark 3.11. Of course, we need to show now that our idea makes sense mathematically.

Remark 3.17. Notice that given an equivariant map $f = (f_n)_n \in \mathcal{F}_b^E(X, M_l)$ we will often omit indices and simply write $f(x)$, where $x \in X_n$ for some $n \in \mathbb{N}$, as abbreviation for $f_n(x)$. This will cause no confusion, because you can read f as mapping defined on the disjoint union of the sets X_n , $n \in \mathbb{N}$.

Remark 3.18. Suppose that $X = (X_l)_{l \in \mathbb{N}}$ is an equivariant and transitive matrix set fulfilling the uniqueness property. Let $f, g \in \mathcal{F}_b^E(X)$, $n \in \mathbb{N}$ and $x \in X_n$. Then the map $y \mapsto u_{xy}^* f(y) g(y) u_{xy}$ defined on the preordered set $\mathcal{S} = \{y \in \cup_l X_l \mid y \succ x\}$ with the preorder \preceq is a net in M_n . By Remark 3.11 the map is well-defined, so we need to verify only that (\mathcal{S}, \preceq) is directed. For $y_1, y_2 \in \mathcal{S}$ we have $y_1 \succ x$ and $y_2 \succ x$ and we conclude that $y_1 \sim y_2$ by transitivity of the m-relation. Hence by definition of the m-relation there is $l \in \mathbb{N}$ and $z \in X_l$ such that $z \succ y_1$ and $z \succ y_2$. Obviously then $z \in \mathcal{S}$.

Proposition 3.19. *Let V be a vector space. Suppose that X is an equivariant and transitive matrix subset of V such that X fulfills the uniqueness property. Then the limit*

$$(fg)_n(x) = \lim_{y \succ x} u_{xy}^* f(y) g(y) u_{xy}, \quad (3.4)$$

where $f, g \in \mathcal{F}_b^E(X, M_l)$ and $l \in \mathbb{N}$, exists for all $x \in X_n$ and $n \in \mathbb{N}$. Moreover, the function $fg = ((fg)_n)_n$ is an element of $\mathcal{F}_b^E(X, M_l)$.

Proof. Let $l \in \mathbb{N}$. We have to show first that the limit in equation (3.4) exists. Let $f \in \mathcal{F}_b^E(X, M_l)$. We start by showing that the net $(u_{xy}^* f(y) f(y)^* u_{xy})_{y \succ x}$, cf. Remark 3.18, is monotone increasing. Let $y_2 \succ y_1 \succ x$. There is an isometry $u_{y_1 y_2}$ (unique up to a factor of modulus 1) such that $y_1 = u_{y_1 y_2}^* y_2 u_{y_1 y_2}$. We find that

$$u_{x y_1}^* f(y_1) f(y_1)^* u_{x y_1} = u_{x y_2}^* f(y_2) u_{y_1 y_2} u_{y_1 y_2}^* f(y_2)^* u_{x y_2}.$$

Since $u_{y_1 y_2} u_{y_1 y_2}^*$ is a projection, we get

$$h(y_2) - h(y_1) = u_{x y_2}^* f(y_2) (1 - u_{y_1 y_2} u_{y_1 y_2}^*) f(y_2)^* u_{x y_2} \geq 0,$$

where $h(y_i) = u_{x y_i}^* f(y_i) f(y_i)^* u_{x y_i}$ for $i = 1, 2$. So, we have a monotone increasing net of positive matrices, which is bounded above by $\|f\|^2$ and hence is convergent. We have still to show that the map defined by equation (3.4) is equivariant. For $y \succ x$ we find immediately

$$\begin{aligned} (ff^*)_n(x) &= \lim_{z \succ x} u_{xz}^* f(z) f(z)^* u_{xz} \\ &= \lim_{z \succ y} u_{xz}^* f(z) f(z)^* u_{xz} \\ &= u_{xy}^* \lim_{z \succ y} u_{yz}^* f(z) f(z)^* u_{yz} u_{xy} \\ &= u_{xy}^* f f^*(y) u_{xy}. \end{aligned}$$

This shows that ff^* is equivariant and thus $ff^* \in \mathcal{F}_b^E(X, M_l)$. Now a simple calculation gives

$$u_{xy}^* f(y) g(y)^* u_{xy} = u_{xy}^* \frac{1}{4} \sum_{\nu=0}^3 i^\nu (f + i^\nu g)(y) (f + i^\nu g)^*(y) u_{xy},$$

which implies immediately that the limit

$$\begin{aligned} fg^*(x) &= \lim_{y \succ x} u_{xy}^* f(y) g(y)^* u_{xy} \\ &= \lim_{y \succ x} u_{xy}^* \frac{1}{4} \sum_{\nu=0}^3 i^\nu (f + i^\nu g)(y) (f + i^\nu g)^*(y) u_{xy} \\ &= \frac{1}{4} \sum_{\nu=0}^3 i^\nu (f + i^\nu g)(f + i^\nu g)^*(x) \end{aligned} \quad (3.5)$$

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exists. Moreover, since $(f + i^\nu g)(f + i^\nu g)^* \in \mathcal{F}_b^E(X, M_l)$, it is obvious that $fg^* \in \mathcal{F}_b^E(X, M_l)$ and the proof is complete. \square

Remark 3.20. If $x \in X_n$ and $y \in X_m$ such that $y \succ x$, then it is obvious that

$$u_{xy}^* f_m(y) g_m(y) u_{xy} = u_{xz}^* f_m(z) g_m(z) u_{xz}$$

for all $z \in X_m$ that are unitarily equivalent to y . Consequently, if there is $x \in X_n$ such that there is $z \in X_m$ with $z \succ x$ and $y \not\succ x$ for all $y \in X_l$ and all $l > m$, then $\lim_{y \succ x} u_{xy}^* f(y) g(y) u_{xy} = u_{xz}^* f_m(z) g_m(z) u_{xz}$. In particular, if $n = m$, that is, if $y \not\succ x$ for all $y \in X_l$ and all $l > n$, then $\lim_{y \succ x} u_{xy}^* f(y) g(y) u_{xy} = f_n(x) g_n(x)$. In the special situation, where $X = (X_n)_n$ is a matrix set such that $X_n = \emptyset$ for all $n \geq 2$, for instance if X are the pure m-states of a commutative C^* -algebra, then the limit in equation (3.4) reduces to the pointwise product of functions.

Does the limit in equation (3.4) define a C^* -product, and, if so, is the matrix order structure determined by the product, i.e., the squares, the same as the pointwise ordering? The answer is yes. The hard parts of the proof will be first to see that the product is associative and second that the matrix orderings do indeed coincide. To show associativity, it will be good to have the following technical lemma at hand.

Lemma 3.21. *Let X be a matrix set such that X is equivariant, transitive and fulfills the uniqueness property. If $l, n \in \mathbb{N}$, $h \in \mathcal{F}_b^E(X, M_l)$, $x \in X_n$ and $\xi \in \mathbb{C}^{n \times l}$, then for any $\varepsilon > 0$ there is $y \succ x$ such that for all $z \succ y$ we have*

$$\|h(z)u_{yz}u_{xy}\xi - u_{yz}h(y)u_{xy}\xi\|^2 < \varepsilon,$$

where u_{xy} and u_{yz} are isometries such that $x = u_{xy}^* y u_{xy}$ and $y = u_{yz}^* z u_{yz}$.

Proof. Given $\varepsilon > 0$ there is $y \succ x$ such that $|c - \|h(y)u_{xy}\xi\|^2| < \varepsilon/2$, where we have set $c = \lim_{y \succ x} \|h(y)u_{xy}\xi\|^2 = \langle (h^*h)(x)\xi | \xi \rangle$. Then for any $z \succ y$ we have $c - \|h(z)u_{xz}\xi\|^2 < \varepsilon/2$, because the net is monotone increasing to its limit c , i.e.,

$$0 \leq \langle u_{xy}^* h(y)^* h(y) u_{xy} \xi | \xi \rangle \leq \langle u_{xz}^* h(z)^* h(z) u_{xz} \xi | \xi \rangle \leq c.$$

Hence we see that

$$\begin{aligned} \|h(z)u_{yz}u_{xy}\xi - u_{yz}h(y)u_{xy}\xi\|^2 &= \|h(z)u_{yz}u_{xy}\xi\|^2 + \|u_{yz}h(y)u_{xy}\xi\|^2 \\ &\quad - 2 \operatorname{Re} \langle u_{yz}h(y)u_{xy}\xi | h(z)u_{yz}u_{xy}\xi \rangle \\ &\leq \|h(z)u_{yz}u_{xy}\xi\|^2 + \|h(y)u_{xy}\xi\|^2 \\ &\quad - 2 \operatorname{Re} \langle h(y)u_{xy}\xi | h(y)u_{xy}\xi \rangle \\ &= \|h(z)u_{yz}u_{xy}\xi\|^2 - \|h(y)u_{xy}\xi\|^2 \\ &\leq \varepsilon \end{aligned}$$

for all $z \succ y$. \square

Proposition 3.22. *Let V be a vector space. Suppose that X is a matrix subset of V such that X is an equivariant, transitive and fulfills the uniqueness property. Then $\mathcal{F}_b^E(X, M_l)$ is a C^* -algebra for all $l \in \mathbb{N}$ under the product*

$$(fg)_n(x) = \lim_{y \succ x} u_{xy}^* f(y) g(y) u_{xy}. \quad (3.6)$$

The order structure coming from the multiplication coincides with the pointwise order structure of $\mathcal{F}_b^E(X, M_l)$.

Proof. By Proposition 3.19 the limit in equation (3.6) exists and the product $fg = ((fg)_n)_n$ is an element of $\mathcal{F}_b^E(X, M_l)$. Moreover, given f, g and $h \in \mathcal{F}_b^E(X, M_l)$ it is obvious from equation (3.5) that the product is distributive and that $(fg)^* = g^*f^*$. We claim that the product is associative. The claim will be proved if we can show that the limit $\lim_{y \succ x} u_{xy}^* f(y)g(y)h(y)u_{xy}$ exists and

$$(fg)h(x) \stackrel{!}{=} \lim_{y \succ x} u_{xy}^* f(y)g(y)h(y)u_{xy} \stackrel{!}{=} f(gh)(x)$$

holds for all $x \in X_n$ and $n \in \mathbb{N}$. In order to show the left hand side of the previous equation, let $\varepsilon > 0$ and $c > \|f\|, \|g\|, \|h\|$ and fix a unit vector $\xi \in \mathbb{C}^{n \times l}$. From Lemma 3.21 there is $y' \succ x$ such that for all $z \succ y'$ we have

$$\|u_{y'z}h(y')u_{xy'}\xi - h(z)u_{y'z}u_{xy'}\xi\| \leq \frac{\varepsilon}{3c^2}. \quad (3.7)$$

Now by definition of the product there is $y_1 \succ x$ such that

$$\|(fg)h(x) - u_{xy}^* f(y)g(y)h(y)u_{xy}\| \leq \frac{\varepsilon}{3} \quad (3.8)$$

for all $y \succ y_1$. Fix a y such that $y \succ y', y_1$. Then again by definition of the multiplication there is $z_1 \succ y$ such that $\|f(y) - u_{yz}^* f(z)g(z)u_{yz}\| \leq \frac{\varepsilon}{3c}$ for all $z \succ z_1$. This implies

$$\|u_{xy}^* (f(y) - u_{yz}^* f(z)g(z)u_{yz})h(y)u_{xy}\| \leq \frac{\varepsilon}{3c} \|h(y)\| \leq \frac{\varepsilon}{3}. \quad (3.9)$$

for all $z \succ z_1$. So, adding the inequalities (3.8) and (3.9) gives

$$\|(fg)h(x) - u_{xy}^* u_{yz}^* f(z)g(z)u_{yz}h(y)u_{xy}\| \leq \frac{2\varepsilon}{3} \quad (3.10)$$

for all $z \succ z_1$. From inequality (3.7) we see that

$$\|u_{xy}^* u_{yz}^* f(z)g(z)(u_{yz}h(y)u_{xy} - h(z)u_{yz}u_{xy})\xi\| \leq \|f(z)\| \|g(z)\| \frac{\varepsilon}{3c^2} \leq \frac{\varepsilon}{3} \quad (3.11)$$

for all $z \succ y$. Adding the inequalities (3.10) and (3.11) we see that for $\varepsilon > 0$ we have found $y \succ x$, such that $\|(fg)h(x)\xi - u_{xz}^* f(z)g(z)h(z)u_{xz}\xi\| \leq \varepsilon$ for all $z \succ y$. Since $\xi \in \mathbb{C}^{n \times l}$ and $\varepsilon > 0$ were arbitrarily chosen, the claim follows. Similarly we can prove that $f(gh)(x) = \lim_{y \succ x} u_{xy}^* f(y)g(y)h(y)u_{xy}$, which shows that the product is associative.

Moreover, since the order unit norm coincides with the pointwise supremum norm by Proposition 3.16, the inequality

$$\|u_{xy}^* f_l(y)g_l(y)u_{xy}\| \leq \|f_l(y)\| \|g_l(y)\| \leq \|f\| \|g\|,$$

holding for all $y \in X_l$ and $l \in \mathbb{N}$, implies

$$\|(fg)_n(x)\| = \|\lim_{y \succ x} u_{xy}^* f(y)g(y)u_{xy}\| = \lim_{y \succ x} \|u_{xy}^* f(y)g(y)u_{xy}\| \leq \|f\| \|g\|,$$

for all $x \in X_n$ and $n \in \mathbb{N}$. So, we have shown that $\|fg\| \leq \|f\| \|g\|$, and it is obvious that $\|f\| = \|f^*\|$. To get the C^* -norm equality, we need only to verify that $\|ff^*\| \geq \|f\|^2$. But since

$$\begin{aligned} \|(ff^*)_n(x)\| &= \|\sup_{y \succ x} u_{xy}^* f(y)f(y)^* u_{xy}\| \\ &= \sup_{y \succ x} \|u_{xy}^* f(y)f(y)^* u_{xy}\| \\ &\geq \|f_n(x)f_n(x)^*\| = \|f_n(x)\|^2, \end{aligned}$$

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holds for all $x \in X_n$ and $n \in \mathbb{N}$, it follows directly that $\|ff^*\| \geq \|f\|^2$. At this point we have shown that $\mathcal{F}_b^E(X, M_l)$ with the order unit is a unital C^* -algebra under the above product. It is also obvious by definition of the product that ff^* is a (pointwise) positive element. However the cone of positive elements generated by the multiplication, i.e., the squares, could be smaller as the given pointwise positive cone.

So, we still have to show that a pointwise positive element $f \in \mathcal{F}_b^E(X, M_l)_+$ has a square root. As in [50, Lemma 3.2.10] let $(q_n)_{n \in \mathbb{N}}$ be the inductively defined sequence of polynomials such that $q_0 = 0$ and $q_n(t) = \frac{1}{2}(t + q_{n-1}(t)^2)$ for all $n \in \mathbb{N}$. Then the monotone increasing sequence (q_n) converges uniformly on the interval $[0, 1]$ to $q(t) = 1 - (1 - t)^{\frac{1}{2}}$. Notice that $0 \leq h^n \leq 1$ for any $h \in \mathcal{F}_b^E(X, M_l)$ such that $0 \leq h \leq 1$. Hence with the polynomials $p_n = q_n - q_{n-1}$ we can repeat the proof of [50, Proposition 3.2.11] for $f \in \mathcal{F}_b^E(X, M_l)$ such that $0 \leq f \leq 1$, i.e., $0 \leq f_k(x) \leq 1_{k \times l}$ for all $x \in X_k$ and $k \in \mathbb{N}$. This means for $g = 1 - f$ we define the maps

$$p_n(g)(x) = \lim_{y \succ x} u_{xy}^* p_n(g(y)) u_{xy} \in \mathcal{F}_b^E(X, M_l)_+$$

and see that $\sum p_n(g)$ converges in the supremum norm to $h \in \mathcal{F}_b^E(X, M_l)$ with $0 \leq h \leq 1$ (pointwise order). Exactly the same calculation as in the proof of [50, Proposition 3.2.11] shows that $(1 - h)^2 = f$. Therefore the pointwise positive f is also positive with respect to the multiplication. We have shown that the ordering defined by the multiplication coincides with the given pointwise ordering on $\mathcal{F}_b^E(X, M_l)$, and the proof is complete. \square

Corollary 3.23. *Let X be an equivariant matrix subset of some vector space V , such that X fulfills the assumptions of Proposition 3.22. Then the operator system $\mathcal{F}_b^E(X)$ is a C^* -algebra under the given (pointwise) matrix order structure.*

Proof. Based on the natural identification of $M_n(M_l)$ and $M_l(M_n)$ for all $l, n \in \mathbb{N}$, which is a unital $*$ -isomorphism, there is a unital $*$ -isomorphism between $M_l(\mathcal{F}_b^E(X))$ and $\mathcal{F}_b^E(X, M_l)$ for all $l \in \mathbb{N}$. Thereby we have given $\mathcal{F}_b^E(X, M_l)$ the product constructed in Proposition 3.22, and $M_l(\mathcal{F}_b^E(X))$ carries the product of the tensor product of the C^* -algebras $M_l \otimes \mathcal{F}_b^E(X)$, i.e., the canonical matrix multiplication. Now the assertion becomes obvious, because the pointwise ordering of the operator system $\mathcal{F}_b^E(X)$ on the l -th matrix level is by definition the pointwise ordering of $\mathcal{F}_b^E(X, M_l)$, that is, $M_l(\mathcal{F}_b^E(X))_+ = \mathcal{F}_b^E(X, M_l)_+$. From Proposition 3.22 the pointwise order cone $\mathcal{F}_b^E(X, M_l)_+$ coincides with the ordering of the C^* -algebra $\mathcal{F}_b^E(X, M_l)$ for all $l \in \mathbb{N}$. This means that the pointwise cone $M_l(\mathcal{F}_b^E(X))_+$ coincides with the positive cone of the C^* -algebra $M_l(\mathcal{F}_b^E(X))$ for all $l \in \mathbb{N}$, and the proof is complete. \square

Proposition 3.24. *Let X be an equivariant and transitive matrix subset of some vector space V , such that X fulfills the uniqueness property. Then $\mathcal{F}_b^E(X)$ is a W^* -algebra, and for each $x \in X_1$ there is a uniquely determined minimal projection $p \in \mathcal{F}_b^E(X)$ such that $p_1(x) = 1$.*

Proof. By the preceding corollary $\mathcal{F}_b^E(X)$ is a C^* -algebra. We will verify that $\mathcal{F}_b^E(X)$ is monotone complete with a separating family of normal states, so that $\mathcal{F}_b^E(X)$ is a W^* -algebra, see for instance [49, Thm. 3.9.3]. Let $(f^\nu)_\nu$ be a bounded monotone increasing net in $\mathcal{F}_b^E(X)_h$. Then for all $n \in \mathbb{N}$ and $x \in X_n$ the limit $f_n(x) = \lim_\nu f_n^\nu(x)$ exists. It is obvious that $f = (f_l)_l \in \mathcal{F}_b^E(X)$. Moreover, $f_n(x)$ is the lowest upper bound of the bounded monotone increasing net $(f_n^\nu(x))_\nu$ in $(M_n)_h$. Thus in particular for all $x \in X_1$ the state

\hat{x} defined by $\hat{x}(g) = g_1(x)$ for $g \in \mathcal{F}_b^E(X)$ is normal. Furthermore the set $\{\hat{x} \mid x \in X_1\}$ is separating for $\mathcal{F}_b^E(X)$. We conclude that $\mathcal{F}_b^E(X)$ is a W^* -algebra.

We are going to show next that $\mathcal{F}_b^E(X)$ contains minimal projections. Let $x_1 \in X_1$. For any $x \in X_n$ with $x \succ x_1$, so that x is in the equivalence class $[x_1]_n$ of x_1 (see Definition 3.6), there is a unitary $u \in M_n$ such that $uxu^* = \begin{pmatrix} x_1 & * \\ * & * \end{pmatrix}$. Notice that for another unitary $v \in M_n$ such that $v xv^* = \begin{pmatrix} x_1 & * \\ * & * \end{pmatrix}$ we find $\mathbb{1}_{n,1}^* v xv^* \mathbb{1}_{n,1} = x_1 = \mathbb{1}_{n,1}^* u x u^* \mathbb{1}_{n,1}$. Since X fulfills the uniqueness property, there is a real number r such that $v^* \mathbb{1}_{n,1} = e^{ir} u^* \mathbb{1}_{n,1}$. It follows that $u^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u = v^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v$. Hence we can define a map $p(x) = u^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u$ for all $x \succ x_1$. The map p is equivariant on its domain. Indeed, given $z \succ y \succ x_1$, where $z \in X_n$ and $y \in X_l$. There is an isometry $u_{yz} \in M_{n,l}$ that is determined up to a complex factor with absolute value 1 such that $y = u_{yz}^* z u_{yz}$ and there is a unitary $v \in M_l$ such that $vyv^* = \begin{pmatrix} x_1 & * \\ * & * \end{pmatrix}$. Since u_{yz} is an isometry there is a unitary $u \in M_n$ such that $u_{yz} = u \mathbb{1}_{n,l}$. Hence, noting that $\mathbb{1}_{n,l} v^* = \begin{pmatrix} v^* & 0 \\ 0 & \mathbb{1}_{n-l} \end{pmatrix} \mathbb{1}_{n,l}$, we find

$$\begin{aligned} \begin{pmatrix} x_1 & * \\ * & * \end{pmatrix} &= vyv^* = v u_{yz}^* z u_{yz} v^* \\ &= v \mathbb{1}_{n,l}^* u^* z u \mathbb{1}_{n,l} v^* \\ &= \mathbb{1}_{n,l}^* \begin{pmatrix} v & 0 \\ 0 & \mathbb{1}_{n-l} \end{pmatrix} u^* z u \begin{pmatrix} v^* & 0 \\ 0 & \mathbb{1}_{n-l} \end{pmatrix} \mathbb{1}_{n,l}. \end{aligned}$$

We conclude by the definition of $p(z)$ that

$$p(z) = u \begin{pmatrix} v^* & 0 \\ 0 & \mathbb{1}_{n-l} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & \mathbb{1}_{n-l} \end{pmatrix} u^*.$$

This implies immediately

$$u_{yz}^* p(z) u_{yz} = \mathbb{1}_{n,l}^* u^* p(z) u \mathbb{1}_{n,l} = v^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v = p(y),$$

which shows that p is equivariant on its domain. Now we can extend p on all of $[x_1]$. Given $x \in [x_1]_n$ such that $x \not\succ x_1$, there is $l \in \mathbb{N}$ and $y \in X_l$ such that $y \succ x$ and $y \succ x_1$. We define $p(x) = u_{xy}^* p(y) u_{xy}$. We have to show that p is well defined on $[x_1]$. Given another element $y' \in X_{l'}$ such that $y' \succ x$ and $y' \succ x_1$, we see that $y' \wedge x_1 \wedge y$. So there is $m \in \mathbb{N}$ and $z \in X_m$ such that $z \succ y, y'$. We have shown already that $p(y) = u_{yz}^* p(z) u_{yz}$ and $p(y') = u_{y'z}^* p(z) u_{y'z}$. Therefore

$$u_{xy'}^* p(y') u_{xy'} = u_{xy'}^* u_{y'z}^* p(z) u_{y'z} u_{xy'} = u_{xz}^* p(z) u_{xz} = u_{xy}^* u_{yz}^* p(z) u_{yz} u_{xy} = u_{xy}^* p(y) u_{xy},$$

which shows that p is well defined on $[x_1]$. We set $p(y) = 0$ for all $y \in X \setminus [x_1]$. Obviously, p is a bounded map. Moreover, let $x \in X_l$ and $y \in X_n$ such that $y \succ x$. If x is not equivalent to x_1 then y , which is equivalent to x , is also not equivalent to x_1 . Hence $p(x) = 0 = p(y)$. If $x \succ x_1$ then we have already proved that $p(x) = u_{xy}^* p(y) u_{xy}$. So we assume that $x \not\succ x_1$ and $x \in [x_1]$. Then there is $m \in \mathbb{N}$ and $z \in X_m$ such that $z \succ y, x, x_1$. Hence by definition of p , we have $p(y) = u_{yz}^* p(z) u_{yz}$. Then

$$u_{xy}^* p(y) u_{xy} = u_{xy}^* u_{yz}^* p(z) u_{yz} u_{xy} = u_{xz}^* p(z) u_{xz} = p(x),$$

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which shows that p is equivariant and so $p \in \mathcal{F}_b^E(X)$. Obviously, p is a positive map and especially self-adjoint. Let $x \in X_n$. If $x \not\succ x_1$ then $p(y) = 0$ for all $y \succ x$ and hence

$$pp(x) = \lim_{y \succ x} u_{xy}^* p(y) p(y) u_{xy} = \sup_{y \succ x} u_{xy}^* p(y) p(y) u_{xy} = 0.$$

If $x \prec x_1$ there is $y \succ x, x_1$ and to calculate the above supremum we can restrict to elements y satisfying $y \succ x_1$. Hence we get

$$\begin{aligned} pp(x) &= \sup_{y \succ x} u_{xy}^* p(y) p(y) u_{xy} \\ &= \sup_{y \succ x, x_1} u_{xy}^* v^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v v^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v u_{xy} \\ &= \sup_{y \succ x, x_1} u_{xy}^* v^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v u_{xy} \\ &= \sup_{y \succ x, x_1} u_{xy}^* p(y) u_{xy} \\ &= p(x), \end{aligned}$$

where v is a unitary such that $vyv^* = \begin{pmatrix} x_1 & * \\ * & * \end{pmatrix}$ and hence by definition $p(y) = v^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v$. This shows that p is a projection in $\mathcal{F}_b^E(X)$.

We will prove that p is a minimal projection. To this end let $x \in X_1$ such that $p(x) = 1$. Then $x \in [x_1]_1$ by definition of p . So, there is $n \in \mathbb{N}$ and $y \in [x_1]_n$ such that $y \succ x, x_1$. This means we can find a unitary $u = [u_{ij}] \in M_n$ such that $uyu^* = \begin{pmatrix} x_1 & * \\ * & * \end{pmatrix}$. Hence by definition $p(y) = u^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u$, and we obtain

$$1 = p(x) = u_{xy}^* p(y) u_{xy} = \xi^* u^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u \xi = \sum_j \xi_j^* u_{1j}^* \sum_i u_{1i} \xi_i,$$

where $u_{xy} = \xi = (\xi_1, \dots, \xi_n)^{\text{tr}}$. Consequently $\sum_i u_{1i} \xi_i = e^{ir}$ for some real number r . Since u is unitary and $\|\xi\| = 1$, it follows that $u\xi = (e^{i\varphi}, 0, \dots, 0)^{\text{tr}}$. This leads to

$$x = u_{xy}^* y u_{xy} = \xi^* u^* \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix} u \xi = (e^{-1\varphi} \ 0) \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi} \\ 0 \end{pmatrix} = x_1.$$

So, we have shown that $p(x) = 1$ if and only if $x = x_1$.

Let $q \in \mathcal{F}_b^E(X)$ be a projector such that $q \leq p$. Notice that for any $y = [y_{ij}] \in X_n$ such that $y_{11} = x_1$ it follows from $0 \leq q(y) \leq p(y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the equivariance of q that $q(y) = \begin{pmatrix} q(x_1) & 0 \\ 0 & 0 \end{pmatrix}$. Hence we obtain

$$\begin{aligned} q(x_1) &= qq(x_1) = \sup_{y \succ x_1} u_{x_1 y}^* q(y)^2 u_{x_1 y} \\ &= \sup_{y \succ x_1} u_{x_1 y}^* u^* \begin{pmatrix} q(x_1) & 0 \\ 0 & 0 \end{pmatrix}^2 u u_{x_1 y} \\ &= \sup_{y \succ x_1} (1 \ 0) \begin{pmatrix} q(x_1)^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = q(x_1)^2, \end{aligned}$$

where u is a unitary, such that $y = u^* \begin{pmatrix} x_1 & * \\ * & * \end{pmatrix} u$. This implies $q(x_1) \in \{0, 1\}$. In case $q(x_1) = 1$ we find $q \begin{pmatrix} x_1 & * \\ * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then it follows from the equivariance of q that $q(y) = p(y)$ for all $y \succ x_1$ and hence $q = p$. In case $q(x_1) = 0$ we find $q = 0$. So, we have shown that p is a minimal projection. \square

Proposition 3.25. *Let X be a matrix set such that X is equivariant, transitive and fulfills the uniqueness property, so $\mathcal{F}_b^E(X)$ is a W^* -algebra, cf. Proposition 3.24. Let the center of $\mathcal{F}_b^E(X)$ be $Z(\mathcal{F}_b^E(X))$. If $f \in Z(\mathcal{F}_b^E(X))$ then given $x_1 \in X_1$ there is $\lambda \in \mathbb{C}$ such that $f|_{[x_1]} = \lambda \mathbb{1}|_{[x_1]}$.*

Proof. Let $f \in Z(\mathcal{F}_b^E(X))$ and $x = [x_{ij}] \in X_n$. For $i \in \{1, \dots, n\}$ we construct the minimal projection p_{ii} on x_{ii} such that $p_{ii}(x_{ii}) = 1$. We let $\varepsilon_{ij} \in M_n$ denote the matrix with entry 1 on the i -th row and j -th column and 0 elsewhere. By assumption we have especially that $p_{ii}f(x) = fp_{ii}(x)$ for $i \in \{1, \dots, n\}$. We calculate

$$\begin{aligned} p_{ii}f(x) &= \lim_{y \succ x} u_{xy}^* p_{ii}(y) f(y) u_{xy} \\ &= \lim_{y \succ x} u_{xy}^* u^* p \begin{pmatrix} x & * \\ * & * \end{pmatrix} f \begin{pmatrix} x & * \\ * & * \end{pmatrix} u u_{xy} \\ &= \lim_{y \succ x} (\mathbb{1} \ 0) \begin{pmatrix} \varepsilon_{ii} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(x) & * \\ * & * \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ 0 \end{pmatrix} \\ &= \varepsilon_{ii} f(x), \end{aligned}$$

where u is a unitary such that $y = u^* \begin{pmatrix} x & * \\ * & * \end{pmatrix} u$. Similarly we find $fp_{ii}(x) = f(x)\varepsilon_{ii}$. Hence $\varepsilon_{ii}f(x) = f(x)\varepsilon_{ii}$ for all $i \in \{1, \dots, n\}$, from which we see that $f(x)_{ij} = 0$ for $i \neq j$. This holds for arbitrary $x \in X_n$ and $n \in \mathbb{N}$. Suppose there would be $x \in X_2$ such that $f(x_{11}) \neq f(x_{22})$. Then we can find a unitary $u \in M_2$ such that $u^*f(x)u = u^* \begin{pmatrix} f(x_{11}) & 0 \\ 0 & f(x_{22}) \end{pmatrix} u$ is not a diagonal matrix. Hence $f(u^*xu)$ would not be diagonal, which is not possible. Thus we obtain $f(x_{11}) = f(x_{22})$. This shows that $f = \lambda \mathbb{1}$ on each equivalence class $[x_1]$. \square

Lemma 3.26. *Let X be a matrix set such that X is equivariant, transitive and fulfills the uniqueness property, so $\mathcal{F}_b^E(X)$ is a W^* -algebra, cf. Proposition 3.24. Let $m \in \mathbb{N}$ and $z \in X_m$. The function $c^{[z]} = (c_n^{[z]})_n \in \mathcal{F}_b^E(X)$ defined by*

$$c_n^{[z]}(x) = \begin{cases} \mathbb{1}_n & \text{if } x \in [z]_n, \\ 0 & \text{if } x \in X_n \setminus [z]_n. \end{cases}$$

for all $x \in X_n$ and $n \in \mathbb{N}$ is a minimal central projection. Hence $Z(\mathcal{F}_b^E(X))$ is a atomic commutative W^ -algebra and $c^{[z]}\mathcal{F}_b^E(X)$ is a factor.*

Proof. Obviously, $c^{[z]} \in \mathcal{F}_b^E(X)$. Let $n \in \mathbb{N}$, $f \in \mathcal{F}_b^E(X)$ and $x \in X_n$. If $x \sim z$ we obtain

$$(c^{[z]}f)_n(x) = \lim_{y \succ x} u_{xy}^* c^{[z]}(y) f(y) u_{xy} = \lim_{y \succ x} u_{xy}^* f(y) u_{xy} = f_n(x) = (fc^{[z]})_n(x),$$

since $y \sim z$ for all $y \succ x$ (which especially means $y \sim x$). If $x \not\sim z$ then $y \not\sim z$ for all $y \succ x$ and hence $(c^{[z]}f)_n(x) = 0 = (fc^{[z]})_n(x)$. This shows that $c^{[z]} \in Z(\mathcal{F}_b^E(X))$. It is obvious that $c^{[z]}$ is bounded, positive and idempotent. Thus $c^{[z]}$ is a projection in the center of $\mathcal{F}_b^E(X)$. Given a non-zero projection $q \in Z(\mathcal{F}_b^E(X))$ such that $0 \leq q \leq c^{[z]}$. Obviously, q vanishes on $X \setminus [z]$. By Proposition 3.25 there is $\lambda \in \mathbb{C}$ such that $q = \lambda \mathbb{1}$ on $[z]$. It follows $\lambda \in (0, 1]$ and since q is a projection, we must have $\lambda = 1$. Hence $q = c^{[z]}$, which shows that $c^{[z]}$ is a minimal projection. Since $c^{[z]}$ is a minimal projection, $c^{[z]}\mathcal{F}_b^E(X)$ is a factor. \square

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Corollary 3.27. *Let X be a matrix set such that X is equivariant, transitive and fulfills the uniqueness property, so $\mathcal{F}_b^E(X)$ is a W^* -algebra, cf. Proposition 3.24. Then for each equivalence class $[x]$ of X there is a Hilbert space $H_{[x]}$ such that $c^{[x]}\mathcal{F}_b^E(X)$ can be identified with $\mathcal{B}(H_{[x]})$. So $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra.*

Proof. It is obvious that for distinct equivalence classes $[x] \neq [y]$, where $x, y \in \cup_l X_l$, we have $c^{[x]}c^{[y]} = c^{[y]}c^{[x]} = 0$ by definition of the minimal central projections $c^{[x]}$ and $c^{[y]}$, cf. Lemma 3.26. Since in addition $X = \cup[x]$ it follows $\sum c^{[x]} = 1$, where the union and the sum run over all equivalence classes. Consequently, $\mathcal{F}_b^E(X) = \oplus c^{[x]}\mathcal{F}_b^E(X)$. Moreover, each factor $c^{[x]}\mathcal{F}_b^E(X)$ contains minimal projections, cf. Proposition 3.24, so there is a Hilbert space $H_{[x]}$ such that $c^{[x]}\mathcal{F}_b^E(X) = \mathcal{B}(H_{[x]})$. It is now obvious that $\mathcal{F}_b^E(X) = \oplus \mathcal{B}(H_{[x]})$ is an atomic W^* -algebra. \square

Finite matrix convex simplexes

In order to have a simple example of m-convex state spaces, the current section is devoted to the study of the m-convex state spaces of finite dimensional C^* -algebras. In addition, the results in the special case of finite dimensions will help us to prove our main results later on.

Definition 3.28. Let W be some vector space. For $v \in M_n(W)$ we define the sets $[v]_l = \{w \in M_l(W) \mid v \succ w\}$ for $l \leq n$, and we set $[v]_m = \emptyset$ for $m > n$. We call the equivariant matrix set $[v] = ([v]_l)_{l \in \mathbb{N}}$ the *compressions* of v .

Proposition 3.29. *Let $m \in \mathbb{N}$ and let $\mathcal{A} = \oplus_{i=1}^m M_{n_i}$ be a finite dimensional C^* -algebra. Let $K = CS(\mathcal{A})$ and $X = \text{str}(K)$. Let $x_i: \mathcal{A} \rightarrow M_{n_i}$ be the irreducible representations onto the i -th summand, so that $x_i \in X_{n_i}$ for $1 \leq i \leq m$. Then X is the disjoint union $\cup_{i=1}^m [x_i]$. Moreover, the restriction map from $A(K) \rightarrow \mathcal{F}_b^E(X)$ is surjective, so that we have a complete order isomorphism between $A(K)$ and $\mathcal{F}_b^E(X)$.*

Proof. Notice that $A(K) = A_b(K)$, because \mathcal{A} is finite dimensional. It follows from Lemma 1.22 that $\text{mext}(K) = \cup_{i=1}^m \mathcal{U}(x_i)$. Observing that the irreducible representations x_i are pairwise not unitarily equivalent, we conclude $X = \cup_{i=1}^m [x_i]$, where $[x_i] \cap [x_j]$ is empty for $i \neq j$. By the Krein-Milman theorem any $\varphi \in K_1$ can be written as convex combination of extreme points, i.e., elements of X_1 , so it follows from Remark 1.24 that the restriction map $A(K) \rightarrow \mathcal{F}_b^E(X)$ is injective. (Alternatively we could apply the matrix version of the Krein-Milman theorem, cf. [30].) So, to show that the restriction is a complete order isomorphism it is sufficient to prove that it is a surjection. Given $f = (f_n)_n \in \mathcal{F}_b^E(X)$ we define maps $g_n: K_n \rightarrow M_n$ by $g_n(\varphi) = \varphi(\oplus_{i=1}^m f_{n_i}(x_i))$, for all $\varphi \in K_n$ and $n \in \mathbb{N}$. Obviously $g = (g_n)_n$ is well-defined, and we claim that the restriction of g to X is f and that $g \in A(K)$. If $y \in X_n$ then there is $1 \leq j \leq m$ such that $y \in [x_j]$. Hence there is an isometry $u \in M_{n_j, n}$ such that $y = u^* x_j u$. Then we obtain

$$g_n(y) = y \left(\bigoplus_{i=1}^m f_{n_i}(x_i) \right) = u^* x_j \left(\bigoplus_{i=1}^m f_{n_i}(x_i) \right) u = u^* f_{n_j}(x_j) u = f_n(u^* x_j u) = f_n(y),$$

showing that $g|_X = f$. Moreover, for any matrix convex combination $\varphi = \sum_j \alpha_j^* \varphi_j \alpha_j$ we have

$$g_n \left(\sum_j \alpha_j^* \varphi_j \alpha_j \right) = \varphi \left(\bigoplus_{i=1}^m f_{n_i}(x_i) \right) = \sum_j \alpha_j^* \varphi_j \left(\bigoplus_{i=1}^m f_{n_i}(x_i) \right) \alpha_j = \sum_j \alpha_j^* g_{l_j}(\varphi_j) \alpha_j,$$

where $\varphi \in K_n$, $\varphi_j \in K_{l_j}$, $\alpha_j \in M_{l_j, n}$ such that $\sum_j \alpha_j^* \alpha_j = \mathbb{1}_n$ and $n, l_j \in \mathbb{N}$. Therefore $g \in A(K)$ and the proof is complete. \square

Remark 3.30. We know already that the operator system $A_b(K)$ ($A(K)$) of all bounded (continuous) matrix affine maps on the (compact) matrix convex set K is unittally order isomorphic to the order unit space of all bounded (continuous) affine functions on the (compact) convex set K_1 , cf. Remark 1.24. Notice from the preceding proposition that a similar result for equivariant mappings, namely that $\mathcal{F}_b^E(X)$ would be unittally order isomorphic to all bounded maps on X_1 , cannot hold. Otherwise we could conclude from the preceding proposition that *every* continuous function on the pure states of M_l could be extended to a continuous affine function on the whole state space of M_l , which is impossible.

We would like to give an abstract description of the state spaces of finite dimensional C^* -algebras. Recall that a finite simplex is the convex hull of finitely many affinely independent points y_1, \dots, y_n . The extreme points of such a simplex are y_1, \dots, y_n and every point of the simplex has a unique representation by a convex combination of these extreme points. We define a matrix version of such simplexes, where we have to modify the uniqueness condition a bit.

Definition 3.31. Let $K = (K_n)_n$ be an m -convex subset of some vector space V . Then K is called a *finite m -convex simplex* if there are $n \in \mathbb{N}$, $n_\nu \in \mathbb{N}$ and $x_\nu \in K_{n_\nu}$ for $1 \leq \nu \leq n$ such that

- (i) $K = \text{mco}(x_1, \dots, x_n)$ and
- (ii) whenever $\sum_{\nu=1}^n \sum_{i=1}^{l_\nu} \alpha_{\nu,i}^* x_\nu \alpha_{\nu,i} = \sum_{\nu=1}^n \sum_{j=1}^{m_\nu} \beta_{\nu,j}^* x_\nu \beta_{\nu,j}$ are two equal m -convex combinations, then $\sum_{i=1}^{l_\nu} \alpha_{\nu,i}^* \cdot \alpha_{\nu,i} = \sum_{j=1}^{m_\nu} \beta_{\nu,j}^* \cdot \beta_{\nu,j}$ for all $1 \leq \nu \leq n$ read as completely positive maps from M_{n_ν} to M_{n_ν} .

We claim that the finite m -convex simplexes are exactly the m -convex state spaces of finite dimensional C^* -algebras. In order to prove this we need the following preparing results that will be useful also later on to obtain results in the infinite dimensional case.

Proposition 3.32. *Let X be an equivariant and transitive matrix subset of some vector space V such that X fulfills the uniqueness property. Let $S = CS^\sigma(\mathcal{F}_b^E(X))$ be the normal m -convex state space of the atomic W^* -algebra $\mathcal{F}_b^E(X)$. Then the map $\Delta = (\Delta_n)_n$, where $\Delta_n: X_n \rightarrow S_n$ is defined by $\Delta_n(x)(f) = \hat{x}(f) = f_n(x)$ for all $f = (f_l)_l \in \mathcal{F}_b^E(X)$, $x \in X_n$ and $n \in \mathbb{N}$, is an equivariant isomorphism onto its image $\hat{X} = (\hat{X}_n)_n = (\Delta_n(X_n))_n$.*

Proof. First let $x, y \in X_1$, such that $\hat{x} = \hat{y}$. Then $f_1(x) = f_1(y)$ for all $\mathcal{F}_b^E(X)$. Recall that to any $z \in X_1$ there is a minimal projection $p^z \in \mathcal{F}_b^E(X)$, such that $p_1^z(z') = 1$ if and only if $z = z'$, where $z' \in X_1$. (Such minimal projections of $\mathcal{F}_b^E(X)$ were explicitly constructed in the proof of Proposition 3.24.) Thus $p_1^y(x) = p_1^y(y) = 1$ implies $x = y$. This proves that the map Δ_1 is injective. Now let $n \in \mathbb{N}$, such that $n \geq 2$, and assume $\hat{x} = \hat{y}$ for $x, y \in X_n$. Given a unit vector $\xi \in \mathbb{C}^n$, we see immediately that $f_1(\xi^* x \xi) = \xi^* f_n(x) \xi = \xi^* f_n(y) \xi = f_1(\xi^* y \xi)$ for all $f \in \mathcal{F}_b^E(X)$. Using especially the minimal projection on the element $\xi^* y \xi \in X_1$, we obtain $\xi^* x \xi = \xi^* y \xi$ for all unit vectors $\xi \in \mathbb{C}^n$. Notice that $\langle \xi, \eta \rangle_z = \eta^* z \xi$ is a sesquilinear (i.e., linear in the first and anti-linear in the second variable) mapping from $\mathbb{C}^n \times \mathbb{C}^n$ to V for all $z \in M_n(V)$. Thus the polarization identity

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$4 \langle \eta, \xi \rangle_z = \sum_{k=0}^3 i^k \langle \xi + i^k \eta, \xi + i^k \eta \rangle_z$ holds for all $\xi, \eta \in \mathbb{C}^n$ and $z \in M_n(V)$. Setting $c_k = \|\xi + i^k \eta\|$ and applying the polarization identity to x and y gives

$$\begin{aligned} \eta^* x \xi &= \langle \xi, \eta \rangle_x = \frac{1}{4} \sum_{k=0}^3 i^k c_k^2 \left\langle \frac{\xi + i^k \eta}{c_k}, \frac{\xi + i^k \eta}{c_k} \right\rangle_x \\ &= \frac{1}{4} \sum_{k=0}^3 i^k c_k^2 \left\langle \frac{\xi + i^k \eta}{c_k}, \frac{\xi + i^k \eta}{c_k} \right\rangle_y = \langle \xi, \eta \rangle_y = \eta^* y \xi, \end{aligned}$$

for all $\xi, \eta \in \mathbb{C}^n$. Thus $x = y$, which shows that Δ_n is injective for $n \geq 2$. So far we have shown that Δ is injective. We have still to verify that $\widehat{X}_n \subset S_n$ for all $n \in \mathbb{N}$ and that Δ is an equivariant map. Given $x \in X_n$ it is obvious that $\hat{x}(\mathbb{1}) = \mathbb{1}_n(x) = \mathbb{1}_n$, so that \hat{x} is unital. Moreover, $\hat{x}^{(l)}([f^{ij}]) = [f_l^{ij}(x)]$ is positive whenever $[f^{ij}] \in M_l(\mathcal{F}_b^E(X))_+ = \mathcal{F}_b^E(X, M_n)_+$, which shows that \hat{x} is completely positive. To see that \hat{x} is normal, recall that we have shown already in the proof of Proposition 3.24 that \hat{z} is normal for all $z \in X_1$. So for $n > 1$ let $(f^\nu)_\nu$ be a net in $\mathcal{F}_b^E(X)$ that converges to $f \in \mathcal{F}_b^E(X)$ in the w^* -topology. We have to verify that the net $(\hat{x}(f^\nu))_\nu$ converges to $\hat{x}(f) \in M_n$. But this is clear, because for all unit vectors $\xi \in \mathbb{C}^n$ the net $\langle \hat{x}(f^\nu) \xi | \xi \rangle = \xi^* \hat{x}(f^\nu) \xi$ converges to $\xi^* \hat{x}(f) \xi$ by the normality of the state $\xi^* \hat{x} \xi \in \widehat{X}_1$. So altogether it follows that $\widehat{X}_n \subset S_n$. Moreover, it is obvious that $\widehat{v^* x v}(f) = f_n(v^* x v) = v^* f_l(x) v = v^* \hat{x}(f) v$ and $\Delta_n^{-1}(v^* \hat{x} v) = \Delta_l^{-1}(\widehat{v^* x v}) = v^* x v = v^* \Delta_l^{-1}(\hat{x}) v$, which shows that Δ is an equivariant isomorphism onto its image. \square

Remark 3.33. The preceding result shows that the given vector space V that contains X as matrix subset does not matter. We can identify X with \widehat{X} , which is a matrix subset of $\mathcal{F}_b^E(X)^*$, and $\mathcal{F}_b^E(X)$ depends only on X . This is similar to the situation where you consider a compact convex set C in a locally convex vector space. Then you build the order unit space $A(C)$ of real valued continuous functions on C , which depends only on C , and you embed C into the dual $A(C)^*$ canonically, that is, you identify $C = S(A(C))$ affinely and homeomorphically, where the state space $S(A(C))$ carries the w^* -topology.

Lemma 3.34. *Let W be a vector space and let $x \in M_n(W)$. Notice that $[x]$ is trivially transitive because all its elements are m -related. Now, if $[x]$ fulfills the uniqueness property, then $\mathcal{F}_b^E([x])$ is an atomic W^* -algebra by Corollary 3.27 and we have $\mathcal{F}_b^E([x]) = M_n$. Moreover, $[\hat{x}] = \text{str}(CS^\sigma(\mathcal{F}_b^E([x])))$.*

Proof. We show that the map $\psi: \mathcal{F}_b^E([x]) \rightarrow M_n$ defined by $\psi(f) = f_n(x)$ for all $f = (f_l) \in \mathcal{F}_b^E([x])$ is a $*$ -isomorphism. First, ψ is obviously a linear map, and $\psi(f^*) = f_n^*(x) = f_n(x)^* = \psi(f)^*$. If $\psi(f) = f_n(x) = 0$, then $f_l(y) = 0$ for all $y \preceq x$, which shows that ψ is injective. For $\gamma \in M_n$ we define maps $f_l(y) = u_{yx}^* \gamma u_{yx}$ for all $y \preceq x$. $\psi(fg) = (fg)_n(x) = \lim_{y \succ x} u_{xy}^* f_m(y) g_m(y) u_{xy} = f_n(x) g_n(x) = \psi(f) \psi(g)$.

Since $\psi: \mathcal{F}_b^E([x]) \rightarrow M_n$ is a $*$ -isomorphism, the sequence of the amplifications $(\psi_*^{(l)})$ of (the restriction of) the dual map ψ_* of ψ is an m -affine isomorphism between $CS(M_n)$ and $CS^\sigma(\mathcal{F}_b^E([x]))$. Let $\gamma = [\gamma_{ij}] \in M_n(M_n^*)$. Then

$$\psi_*^{(n)}(\gamma)(f) = [\psi_*(\gamma_{ij})(f)] = [\gamma_{ij}(\psi(f))] = [\gamma_{ij}](\psi(f)) = \gamma(f_n(x)) = \gamma(\hat{x}(f)),$$

for all $f \in \mathcal{F}_b^E([x])$. Thus for $\gamma = \text{id}: M_n \rightarrow M_n$ we obtain $\psi_*^{(n)}(\text{id}) = \hat{x}$. Since the identity is a matrix extreme point in $CS(M_n)$ and all structural elements are compressions

of the identity, it follows that \hat{x} is a matrix extreme point of $CS^\sigma(\mathcal{F}_b^E([x]))$ and $[\hat{x}] = \text{str}(CS^\sigma(\mathcal{F}_b^E([x])))$. \square

Theorem 3.35. *The matrix convex state spaces of finite dimensional C^* -algebras are exactly the finite matrix convex simplexes.*

Proof. If $\mathcal{A} = \oplus_{\nu=1}^n M_{n_\nu}$ and $K = CS(\mathcal{A})$ we know already from the proof of Proposition 3.29 that $\text{mext}(K) = \cup_{\nu=1}^n \mathcal{U}(x_\nu)$, where x_ν is the irreducible representation from \mathcal{A} onto the summand M_{n_ν} for $\nu = 1, \dots, n$. Then by the matrix version of the Krein-Milman theorem in finite dimensions we have $K = \text{mco}(\text{mext}(K))$, so that $K = \text{mco}(x_1, \dots, x_n)$, cf. [30] (note that in this special case we could also use the representation results for completely positive maps on matrices contained in [16]). Given two equal m-convex combinations $\sum_{\nu=1}^n \sum_{i=1}^{l_\nu} \alpha_{\nu,i}^* x_\nu \alpha_{\nu,i} = \sum_{\nu=1}^n \sum_{j=1}^{m_\nu} \beta_{\nu,j}^* x_\nu \beta_{\nu,j}$ we evaluate them on elements $\oplus_{\mu=1}^n a_\mu \in \mathcal{A}$ such that $a_\mu = 0$ for all $\mu \neq \nu$, where $\nu \in \{1, \dots, n\}$ is fixed, and obtain directly that $\sum_{i=1}^{l_\nu} \alpha_{\nu,i}^* \cdot \alpha_{\nu,i} = \sum_{j=1}^{m_\nu} \beta_{\nu,j}^* \cdot \beta_{\nu,j}$, so K is a finite m-convex simplex.

In the converse direction, let $K = \{x_1, \dots, x_n\}$ be a finite m-convex simplex. If $y \in [x_i] \cap [x_j]$ then there are isometries α and β such that $y = \alpha^* x_i \alpha = \beta^* x_j \beta$. It follows directly from applying condition (ii) of Definition 3.31 that $i = j$. Thus $[x_i] \cap [x_j] = \emptyset$ if $i \neq j$, and applying condition (ii) again we have $\alpha^* \cdot \alpha = \beta^* \cdot \beta$ as completely positive maps on M_{n_i} . Then there must be $\lambda \in \mathbb{C}$ such that $\alpha = \lambda \beta$, cf. [16]. Consequently, the equivariant and transitive matrix set $X = \cup_{i=1}^n [x_i]$ fulfills the uniqueness property. So, $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra. Moreover, since $[x_i] \cap [x_j] = \emptyset$ for $i \neq j$, it is obvious that $\mathcal{F}_b^E(X) = \oplus_i \mathcal{F}_b^E([x_i])$. We have $\mathcal{F}_b^E([x_i]) = M_{n_i}$ for all $1 \leq i \leq n$ by Lemma 3.34, so $\mathcal{F}_b^E(X) = \oplus_i M_{n_i}$. Now it can be seen easily from condition (ii) of Definition 3.31 that we can extend all bounded equivariant maps on X to bounded m-affine maps on K . Therefore the restriction map from $A_b(K)$ to $\mathcal{F}_b^E(X)$ is surjective and hence a complete order isomorphism. We conclude that $A_b(K) =_{cp} \mathcal{F}_b^E(X) =_{cp} \oplus_i M_{n_i}$, which implies $K = CS^\sigma(\mathcal{F}_b^E(X)) = CS(\oplus_i M_{n_i})$ completing the proof. \square

Proposition 3.36. *Let K be a finite m-convex simplex, so that $K = \text{mco}(x_1, \dots, x_n)$, where $x_i \in K_{n_i}$ for $1 \leq i \leq n$ fulfill condition (ii) of Definition 3.31. Then $\text{str}(K)$ is equal to the disjoint union $\cup_{i=1}^n [x_i]$, and $\text{mext}(K)$ is equal to the matrix set $\cup_{i=1}^n \mathcal{U}(x_i)$.*

Proof. From the proof of Theorem 3.35 we know that the disjoint union $X = \cup_i [x_i]$ fulfills the uniqueness property and $\mathcal{F}_b^E(X) = \oplus \mathcal{F}_b^E([x_i]) = \oplus M_{n_i}$. Moreover, we identified $K = CS^\sigma(\mathcal{F}_b^E(X))$. Let $C^i = CS^\sigma(\mathcal{F}_b^E([x_i])) = CS(M_{n_i})$ for all $1 \leq i \leq n$. Notice that the predual of M_{n_i} can be identified with the m-base norm space (M_{n_i}, C^i) , so that C^i is an m-convex split face of K by Proposition 1.52. Using the identification $X = \hat{X}$ of Proposition 3.32 it follows from Lemma 3.34 that $[x_i] = \text{str}(C^i)$, so that $\cup_i [x_i] \subset \text{str}(K)$ by Proposition 1.55. Obviously, $\text{str}(K) \subset \cup_i [x_i]$, because $K = \text{mco}(x_1, \dots, x_n)$. Thus $\text{str}(K) = \cup_i [x_i]$, which yields immediately $\text{mext}(K) = \cup_i \mathcal{U}(x_i)$ by the definition of matrix extreme points. \square

Remark 3.37. While the preceding results stress the similarity between convex and m-convex sets, there is an obvious difference: The convex hull of a single point consists trivially of this single point, while the m-convex hull of a single point can be rather large. For instance, the m-convex hull of the identity mapping from M_n to M_n , where $n \in \mathbb{N}$, is $CS(M_n)$, cf. [16]. If V is some vector space and $x \in M_n(V)$ is m-irreducible, then the matrix extreme points of $\text{mco}(x)$ are $\mathcal{U}(x)$, cf. [30]. However, notice that $\text{mco}(x)$ need

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not be a finite m -convex simplex, so the requirement (ii) of Definition 3.31 cannot be omitted even in the simplest case. Consider for example the irreducible operator system $\mathcal{L} = \text{lin}\{1_2, \alpha, \alpha^*\} \subset M_2$, where $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Suppose the identity on \mathcal{L} would be m -reducible, that is, suppose $\text{id}_{\mathcal{L}} = u^* \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} u$ for a unitary $u \in M_2$ and states $x_1, x_2: \mathcal{L} \rightarrow \mathbb{C}$. Then it is a consequence of Arveson's boundary theorem, cf. [62, Prop. 1.5], that the identity on M_2 would be m -reducible, which is obviously wrong. Therefore $\text{id}_{\mathcal{L}}$ is m -irreducible, so $\mathcal{U}(\text{id}_{\mathcal{L}})$ are the matrix extreme points of $\text{mco}(\text{id}_{\mathcal{L}}) = CS(\mathcal{L})$, but $CS(\mathcal{L})$ cannot be a finite m -convex simplex.

Let V be a vector space. Recall that the elements of a subset $Y \subset V$ are called extreme points if no element of Y can be written as non-trivial convex combination of elements of Y , so $Y = \text{ex}(\text{conv}(Y))$. We end the current algebraic section with describing a matrix set X such that $X = \text{str}(\text{mco}(X))$.

Remark 3.38. Let X be an equivariant matrix set that fulfills the uniqueness property. Then X consists entirely of m -irreducible elements.

Proof. Notice first that for $x = [x_{ij}] \in X_n$ the uniqueness property ensures that the diagonal elements $x_{ii} \in X_1$, where $1 \leq i \leq n$, are pairwise distinct. Assume without loss of generality that there would be $x \in X_2$ such that $x_{11} = x_{22}$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_{11} = x_{22} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

contradicting the uniqueness property. Now assume there would be $x \in X_2$ such that x is m -reducible. Without loss of generality $x = \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix}$, where $y, z \in X_1$, such that $y \neq z$. It is obvious that

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\pi}{2}}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} e^{i\frac{\pi}{2}}/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

contradicting the uniqueness property. Thus all $x \in X_2$ must be m -irreducible. If there would be an m -reducible $x = \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \in X_{n+m}$, where $y \in X_n$ and $z \in X_m$, then by equivariance $\begin{pmatrix} y_{11} & 0 \\ 0 & z_{11} \end{pmatrix} \in X_2$ is obviously an m -reducible element, such that $y_{11} \neq z_{11}$. But this is impossible, hence the claim follows. \square

Definition 3.39. Let X be an equivariant matrix set. Then X is called *m -convex independent* if whenever $x = \sum_{i=1}^l \alpha_i^* x_i \alpha_i$ is a proper² m -convex combination, where $x \in X_n$, $x_i \in X_{n_i}$, $\alpha_i \in M_{n_i, n}$ and $n, n_i \in \mathbb{N}$, it follows that $n_i \geq n$ and there are isometries $v_i \in M_{n_i, n}$, such that x is unitarily equivalent to $v_i^* x_i v_i$ for all $1 \leq i \leq l$.

Proposition 3.40. Let X be an equivariant and m -convex independent matrix set. Then $X = \text{str}(\text{mco}(X))$.

Proof. Notice first that X consists entirely of m -irreducible elements, since if there would be $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in X_{n+m}$, where $x \in X_n$ and $y \in X_m$ for some $n, m \in \mathbb{N}$, then

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} x \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} y \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

²That is, $\alpha_i \neq 0$ for all $1 \leq i \leq l$.

is an m -convex combination such that $n + m > n$, m , which contradicts the m -convex independence of X . Let $C = \text{mco}(X)$. Given $y \in \text{str}(C_n) \subset C_n$ there is a proper m -convex combination $y = \sum_{i=1}^l \alpha_i^* x_i \alpha_i$, where $x_i \in X_{n_i}$ for $n_i \in \mathbb{N}$. If $n_i < n$ we could replace $\alpha_i^* x_i \alpha_i$ with $(\alpha_i^* 0) \begin{pmatrix} x_i & 0 \\ 0 & c_i \end{pmatrix} \begin{pmatrix} \alpha_i \\ 0 \end{pmatrix}$ for some $c_i \in C_{n-n_i}$. Since y is structural it would follow that x is unitarily equivalent to $\begin{pmatrix} x_i & 0 \\ 0 & c_i \end{pmatrix}$ and hence is m -irreducible, which is impossible. So, $n_i \geq n$ for all $i = 1, \dots, l$. If $n_i > n$ we can rewrite $\alpha_i^* x_i \alpha_i$ as $|\alpha_i| v_i^* x_i v_i |\alpha_i|$, where $v_i^* v_i = 1_n$, so that $v_i^* x_i v_i \in X_n$, because X is equivariant. Hence y must be unitarily equivalent to an element of X_n and again by equivariance of X we see that $y \in X_n$. This shows that $\text{str}(C_n) \subset X_n$.

For the converse direction let $x \in X_n$ and assume that $x = \sum_{i=1}^2 \alpha_i^* y_i \alpha_i$ is an m -convex combination, where $y_1, y_2 \in C_n$ and $\alpha_2 \in M_n$ is invertible. Again for $i = 1, 2$ there are proper m -convex combinations $y_i = \sum_j \beta_{ij}^* \tilde{x}_{ij} \beta_{ij}$, where $\tilde{x}_{ij} \in X_{n_{ij}}$. Using the polar decomposition $\beta_{ij} = v_{ij} |\beta_{ij}|$, where $|\beta_{ij}| = (\beta_{ij}^* \beta_{ij})^{1/2} \in M_n$, we obtain

$$x = \sum_{i,j} \alpha_i^* \beta_{ij}^* \tilde{x}_{ij} \beta_{ij} \alpha_i = \sum_{i,j} \alpha_i^* |\beta_{ij}| x_{ij} |\beta_{ij}| \alpha_i,$$

where $x_{ij} = v_{ij}^* \tilde{x}_{ij} v_{ij} \in X_n$. Omitting those indices i and j from the sum for which $|\beta_{ij}| \alpha_i = 0$, noting that at least $|\beta_{2j}| \alpha_2 \neq 0$ for all j , there are by assumption unitary $u_{ij} \in M_n$ such that $x_{ij} = u_{ij}^* x u_{ij}$. Hence $x = \sum_{i,j} \alpha_i^* |\beta_{ij}| u_{ij}^* x u_{ij} |\beta_{ij}| \alpha_i$. Since x is m -irreducible it follows from [30] that $u_{ij} |\beta_{ij}| \alpha_i = \lambda_{ij} \mathbb{1}_n$ for all i, j such that $|\beta_{ij}| \alpha_i \neq 0$. Because the latter is the case for all j if $i = 2$, we can write

$$y_2 = \sum_j \beta_{2j}^* \tilde{x}_{2j} \beta_{2j} = \sum_j |\beta_{2j}| x_{2j} |\beta_{2j}| = \sum_j |\beta_{2j}| u_{2j}^* x u_{2j} |\beta_{2j}| = \left(\sum_j |\lambda_{2j}|^2 \right) (\alpha_2^{-1})^* x \alpha_2^{-1}.$$

Moreover from $\sum_j |\lambda_{2j}|^2 \mathbb{1}_n = \sum_j |\beta_{2j}| u_{2j}^* u_{2j} |\beta_{2j}| = \alpha_2^* \sum_j \beta_{2j}^* \beta_{2j} \alpha_2 = \alpha_2^* \alpha_2$ we see that $\alpha_2 / a_2 \in M_n$ is unitary, where $a_2^2 = \sum_j |\lambda_{2j}|^2$. Thus we have shown that x is unitarily equivalent to y_2 . Since x is m -irreducible, this implies $x \in \text{str}(C_n)$. Indeed, given a proper m -convex combination $x = \sum_{i=1}^2 \alpha_i^* y_i \alpha_i$, where $y_1, y_2 \in C_n$ and $\alpha_1, \alpha_2 \in M_n$, we can assume that $\alpha_1, \alpha_2 \geq 0$ by applying the polar decomposition. Notice that the matrices α_1 and α_2 commute, because $\sum_{i=1}^2 \alpha_i^2 = \mathbb{1}_n$. Therefore there is a unitary $u \in M_n$ such that $u^* \alpha_1 u$ and $u^* \alpha_2 u$ are diagonal matrices. Suppose that α_1 and α_2 would be both not invertible. Then obviously now x would be unitarily equivalent to a block matrix and hence m -reducible. So, we can assume that α_2 is invertible. Then we obtain $x = \alpha_1 y_1 \alpha_1 + a_2^2 x$ by the previous part of the proof, where $a_2^2 \mathbb{1}_n = \alpha_2^* \alpha_2$. If α_1 is not invertible, then we can assume $\alpha_1 = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$, where d is a diagonal matrix. We obtain $x = \begin{pmatrix} \tilde{y} & 0 \\ 0 & 0 \end{pmatrix} + a_2^2 x$. Since x is m -irreducible, it follows $a_2^2 = 1$ and hence $\alpha_2^* \alpha_2 = \mathbb{1}_n$ and $\alpha_1 = 0$. But we started with a proper m -convex combination. So α_1 can be assumed to be invertible, too. Then by the previous part of the proof x is unitarily equivalent to y_1 , and we conclude $x \in \text{str}(C_n)$. \square

Pure states and non-commutative sets

In the previous sections we learned about the importance of the conditions ‘equivariant, transitive and fulfills the uniqueness property’ on a matrix set X . We have proved that these conditions ensure that $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra. A natural question now is

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whether the set X can be identified with the normal pure matrix states of $\mathcal{F}_b^E(X)$. In the current section we will indeed abstractly characterize the set of the normal pure matrix states of atomic W^* -algebras as those matrix sets that are equivariant, transitive, fulfill the uniqueness property and have one additional property that will be introduced later. We conclude that the normal pure matrix states of atomic W^* -algebras can be seen as non-commutative sets, since any set S can be identified (via point-evaluation) with the normal pure states of the *commutative* W^* -algebra of the bounded functions on S .

Recall that atomic W^* -algebras are just direct sums of type I factors. So in concrete terms we only consider the normal (pure) matrix states of $\mathcal{B}(H)$, where H is some Hilbert space. We start with stating the known identification between normal states and trace class operators, which will help us to prove the abstract characterization results later on.

Let H be a Hilbert space. We let $\mathcal{T}(H) = \{r \in \mathcal{B}(H) \mid \text{trace}(|r|) < \infty\}$ denote the trace class operators of H . For $\xi_i, \xi_j \in H$ we define the $\xi_i \odot \xi_j$ by $(\xi_i \odot \xi_j)\eta = \langle \eta | \xi_i \rangle \xi_j$. Recall that we can identify $M_n(\mathcal{B}(H))$ with $\mathcal{B}(H^n)$ for all $n \in \mathbb{N}$ so that $M_n(\mathcal{B}(H))_+ = \mathcal{B}(H^n)_+$. We define a matrix ordering on $\mathcal{T}(H)$ by setting

$$M_n(\mathcal{T}(H))_+ = \{r \in M_n(\mathcal{T}(H)) \mid r^{\text{tr}} \in M_n(\mathcal{B}(H))_+\}.$$

Recall that there is an isometric order isomorphism between $\mathcal{T}(H)$ and the predual $\mathcal{B}(H)_*$ given by

$$\Xi: \mathcal{T}(H) \rightarrow \mathcal{B}(H)_*; \quad r \mapsto \Xi(r) = \text{trace}(r \cdot) = \text{trace}(\cdot r). \quad (3.12)$$

To show that Ξ is an complete (isometric) order isomorphism, we need the following lemma.

Lemma 3.41. *Let H be a Hilbert space and let $n \in \mathbb{N}$. For all $d = [d_{ij}] \in M_n(\mathcal{T}(H))$ such that $d \in M_n(\mathcal{B}(H))_+$ the matrix $[\text{trace}(d_{ij})]$ is positive in M_n .*

Proof. Let $n \in \mathbb{N}$ and $\{\varepsilon_l \mid l \in L\}$ be an orthonormal basis of H . Given $\lambda = (\lambda_1, \dots, \lambda_n)^{\text{tr}}$, where $\lambda_i \in \mathbb{C}$, we define $\xi_i = \lambda_i \varepsilon_l$ for some fixed $l \in L$ and $i = 1, \dots, n$. By assumption we obtain

$$0 \leq \langle d\xi | \xi \rangle = \sum_{i,j} \langle d_{ij} \lambda_j \varepsilon_l | \lambda_i \varepsilon_l \rangle = \sum_{i,j} \bar{\lambda}_i \langle d_{ij} \varepsilon_l | \varepsilon_l \rangle \lambda_j = \langle [\langle d_{ij} \varepsilon_l | \varepsilon_l \rangle] \lambda, \lambda \rangle,$$

and consequently

$$0 \leq \sum_{l \in L} \langle [\langle d_{ij} \varepsilon_l | \varepsilon_l \rangle] \lambda | \lambda \rangle = \left\langle \left[\sum_{l \in L} \langle d_{ij} \varepsilon_l | \varepsilon_l \rangle \right] \lambda \middle| \lambda \right\rangle = \langle [\text{trace}(d_{ij})] \lambda | \lambda \rangle.$$

This shows that the matrix $[\text{trace}(d_{ij})]$ is positive. \square

Proposition 3.42. *Let H be a Hilbert space. The order isomorphism $\Xi: \mathcal{T}(H) \rightarrow \mathcal{B}(H)_*$ between the trace class operators and the predual of $\mathcal{B}(H)$ is a complete order isomorphism.*

Proof. Let $\{\varepsilon_l \mid l \in L\}$ be an orthonormal basis of H . We show first that Ξ^{-1} is completely positive. Given $n \in \mathbb{N}$ and $r = [r_{\mu\nu}] \in M_n(\mathcal{T}(H))$ such that $\Xi^{(n)}(r) \geq_{cp} 0$ we have to show that $r \in M_n(\mathcal{T}(H))_+$, i.e., that the transpose r^{tr} is a positive operator matrix. By assumption

$$0 \leq_{cp} \Xi^{(n)}([r_{\mu\nu}]) = [\Xi(r_{\mu\nu})] = [\text{trace}(r_{\mu\nu} \cdot)]$$

so that the $n^2 \times n^2$ matrix $[\text{trace}(r_{\mu\nu}a_{ij})]$ is positive for any $a = [a_{ij}] \in M_n(\mathcal{B}(H))_+$. Let $\xi = (\xi_1, \dots, \xi_n)^{\text{tr}} \in H^n$. We claim that the matrix $[\xi_j \odot \xi_i]$ is positive in $\mathcal{B}(H^n)$. Indeed for $\eta = (\eta_1, \dots, \eta_n)^{\text{tr}} \in H^n$ we find

$$\begin{aligned} \langle [\xi_j \odot \xi_i] \eta | \eta \rangle &= \sum_{ij} \langle (\xi_j \odot \xi_i) \eta_j | \eta_i \rangle \\ &= \sum_{ij} \langle \eta_j | \xi_j \rangle \langle \xi_i | \eta_i \rangle \\ &= \sum_i \langle \eta_i | \xi_i \rangle \sum_i \langle \xi_i | \eta_i \rangle = |c|^2 \geq 0, \end{aligned}$$

where $c = \sum_i \langle \xi_i | \eta_i \rangle$. Hence setting $a_{ij} = \xi_j \odot \xi_i$ the matrix $[\text{trace}(r_{\mu\nu}(\xi_j \odot \xi_i))]$ is positive by assumption. We evaluate the entries of this matrix as

$$\begin{aligned} \text{trace}(r_{\mu\nu}(\xi_j \odot \xi_i)) &= \sum_{l \in L} \langle r_{\mu\nu}(\xi_j \odot \xi_i) \varepsilon_l | \varepsilon_l \rangle \\ &= \sum_{l \in L} \langle \varepsilon_l | \xi_j \rangle \langle r_{\mu\nu} \xi_i | \varepsilon_l \rangle \\ &= \left\langle r_{\mu\nu} \xi_i \left| \sum_{l \in L} \langle \xi_j | \varepsilon_l \rangle \varepsilon_l \right. \right\rangle = \langle r_{\mu\nu} \xi_i | \xi_j \rangle. \end{aligned}$$

Thus $[\langle r_{\mu\nu} \xi_i | \xi_j \rangle]$ is positive. Let $\alpha = \underbrace{(1, 0, \dots, 0, 1, 0, \dots, 0, \dots, 1)}_{(n-1)\text{-times}}^{\substack{n\text{-times} \\ n\text{-times}}} \in \mathbb{C}^{n^2}$. Then

$$0 \leq \langle [\langle r_{\mu\nu} \xi_i | \xi_j \rangle] \alpha | \alpha \rangle = \sum_{\mu, \nu=1}^n \langle r_{\nu\mu} \xi_\nu | \xi_\mu \rangle = \langle r^{\text{tr}} \xi | \xi \rangle,$$

which implies that $r^{\text{tr}} \in M_n(\mathcal{B}(H))_+$ so that $r \in M_n(\mathcal{T}(H))_+$.

Now we will prove that Ξ is completely positive. This means given $n \in \mathbb{N}$ and $r \in M_n(\mathcal{T}(H))_+$, i.e., $r^{\text{tr}} \in M_n(\mathcal{B}(H))_+$, we have to show that $\Xi^{(n)}(r)$ is completely positive. So given $a = [a_{ij}] \in M_n(\mathcal{B}(H))_+$ we must prove that $[\text{trace}(r_{\mu\nu}a_{ij})] \geq 0$. Since a is positive there are n elements $b_k = (b_{k1}, \dots, b_{kn}) \in M_{1,n}(\mathcal{B}(H))$ such that $a = \sum_{k=1}^n b_k^* b_k$. Hence we obtain $a_{ij} = \sum_{k=1}^n b_{ki}^* b_{kj}$ for all $i, j \in \{1, \dots, n\}$. Now we calculate

$$\begin{aligned} [\text{trace}(r_{\mu\nu}a_{ij})] &= \left[\text{trace} \left(r_{\mu\nu} \sum_{k=1}^n b_{ki}^* b_{kj} \right) \right] \\ &= \sum_{k=1}^n [\text{trace}(r_{\mu\nu} b_{ki}^* b_{kj})] \\ &= \sum_{k=1}^n [\text{trace}(b_{kj} r_{\mu\nu} b_{ki}^*)]. \end{aligned} \tag{3.13}$$

We consider the matrix $[b_{kj} r_{\mu\nu} b_{ki}^*] \in M_{n^2}(\mathcal{B}(H))$. For its transpose we find that

$$[b_{ki} r_{\nu\mu} b_{kj}^*] = \text{diag}(b_{k1}, \dots, b_{kn}) [r_{\nu\mu}] \text{diag}(b_{k1}, \dots, b_{kn})^* \geq 0,$$

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since $[r_{\nu\mu}] = r^{\text{tr}}$ is positive. By Lemma 3.41 we obtain $[\text{trace}(b_{ki}r_{\nu\mu}b_{kj}^*)] \geq 0$. Since the transposition is positive on M_{n^2} , it follows that $[\text{trace}(b_{kj}r_{\mu\nu}b_{ki}^*)] \geq 0$. This holds for all $k \in \{1, \dots, n\}$ so that we obtain from equation (3.13) that

$$[\text{trace}(r_{\mu\nu}a_{ij})] = \sum_{k=1}^n [\text{trace}(b_{kj}r_{\mu\nu}b_{ki}^*)] \geq 0,$$

which proves that Ξ is completely positive. \square

We can now identify the normal pure matrix states of $\mathcal{B}(H)$ with certain trace class operators.

Proposition 3.43. *Let H be a Hilbert space. Let $r \in M_n(\mathcal{T}(H))$ for some $n \in \mathbb{N}$. Then $\Xi^{(n)}(r): \mathcal{B}(H) \rightarrow M_n$ is a normal pure matrix state if and only if there is an orthonormal system $\{\xi_1, \dots, \xi_n\} \subset H$ such that $r = [\xi_i \odot \xi_j]$.*

Proof. If $\{\xi_1, \dots, \xi_n\} \subset H$ is an orthonormal system, we claim that $\varphi: \mathcal{B}(H) \rightarrow M_n$ given by $\varphi(a) = [\Xi(\xi_i \odot \xi_j)(a)] = [\text{trace}(a(\xi_i \odot \xi_j))]$, is a normal pure matrix state. There is an orthonormal basis of H extending $\{\xi_1, \dots, \xi_n\}$. We simply write this base as $\{\xi_l \mid l \in L\}$, where L is an index set containing $\{1, \dots, n\}$. Then we obtain

$$\varphi_{ij}(a) = \text{trace}(a(\xi_i \odot \xi_j)) = \sum_{l \in L} \langle a(\xi_i \odot \xi_j)\xi_l | \xi_l \rangle = \langle a\xi_j | \xi_i \rangle \quad (3.14)$$

for all $a \in \mathcal{B}(H)$. Defining an isometry $\mathcal{V}: \mathbb{C}^n \rightarrow H$ by $\mathcal{V}\varepsilon_i = \xi_i$, where $(\varepsilon_i)_{i=1}^n$ denotes the standard basis of \mathbb{C}^n , we see from equation (3.14) that $\varphi(a) = \mathcal{V}^* \text{id}(a) \mathcal{V}$, where id means the identity on $\mathcal{B}(H)$. This shows immediately that φ is a normal pure matrix state of $\mathcal{B}(H)$, because id is clearly an irreducible representation. For the converse direction we need only to notice that any normal pure matrix state $\psi: \mathcal{B}(H) \rightarrow M_n$ can be written in the form $\psi = \mathcal{W} \text{id} \mathcal{W}$, where $\mathcal{W}: \mathbb{C}^n \rightarrow H$ is an isometry. Setting $\xi_i = \mathcal{W}\varepsilon_i$ for $i = 1, \dots, n$ gives an orthonormal system, and calculating equation (3.14) backward shows $\psi = [\xi_i \odot \xi_j]$. \square

After these concrete results we come back to the abstract situation, where we have a matrix subset X of some given vector space V such that X is equivariant, the m-relation is transitive and X fulfills the uniqueness property. As mentioned in the introduction to the current section these properties of X do not suffice to recover X , or more precisely \widehat{X} , cf. Proposition 3.32, as normal pure matrix states of $\mathcal{F}_b^E(X)$. Up to now all we can show is that \widehat{X} is contained in the normal pure matrix state space of $\mathcal{F}_b^E(X)$.

Proposition 3.44. *Let X be an equivariant and transitive matrix subset of some vector space V such that X fulfills the uniqueness property. Then the matrix set of the normal pure matrix states of the W^* -algebra $\mathcal{F}_b^E(X)$ contains \widehat{X} .*

Proof. We identify X with \widehat{X} in $K = CS^\sigma(\mathcal{F}_b^E(X))$ via the map $x \mapsto \Delta(x) = \hat{x}$, where $\hat{x}(f) = f_n(x)$ for $f = (f_n) \in \mathcal{F}_b^E(X)$, cf. Proposition 3.32. Given $x \in X_1$ let $\hat{x} = \lambda\varphi + (1 - \lambda)\psi$ be a proper convex combination, where $\varphi, \psi: \mathcal{F}_b^E(X) \rightarrow \mathbb{C}$ are normal states. We identify the atomic W^* -algebra $\mathcal{F}_b^E(X)$ with $\oplus_{\kappa \in \mathfrak{K}} \mathcal{B}(H_\kappa)$, where the sum runs over the set \mathfrak{K} of equivalence classes in X with respect to the m-relation. Notice first that $\hat{x}(c^{[x]}) = c^{[x]}(x) = 1$, where $c^{[x]}$ is the minimal projection of the class $[x]$.

It follows that $\varphi(c^{[x]}) = \psi(c^{[x]}) = 1$ and so \hat{x} , φ and ψ vanish on $\mathbb{1} - c^{[x]}$. Thus we can read these states as states of $c^{[x]}\mathcal{F}_b^E(X) = \mathcal{B}(H_{[x]})$. From Proposition 3.24 there is the minimal projection $p \in \mathcal{F}_b^E(X)$ such that $p_1(x) = 1$. So by definition we have $\hat{x}(p) = p_1(x) = 1$, and consequently $\varphi(p) = \psi(p) = 1$. Reading p as minimal projection in $\mathcal{B}(H_{[x]})$, there is a unit vector $\xi \in \mathcal{B}(H_{[x]})$, such that $p(H_{[x]}) = \mathbb{C}\xi$. Let $p' = \mathbb{1} - p$. Then $T = pTp + p'Tp + pTp' + p'Tp'$ uniquely for $T \in \mathcal{B}(H_{[x]})$. Since \hat{x} , φ and ψ vanish on p' , applying the Cauchy-Schwarz inequality we obtain $\hat{x}(T) = \hat{x}(pTp)$, $\varphi(T) = \varphi(pTp)$ and $\psi(T) = \psi(pTp)$ for $T \in \mathcal{B}(H_{[x]})$, where we read \hat{x} , φ and ψ as mappings on $\mathcal{B}(H_{[x]})$. Since $p\mathcal{B}(H_{[x]})p = \mathbb{C}$ and $\hat{x}(p) = \varphi(p) = \psi(p)$, it follows by linearity that $\hat{x} = \varphi = \psi$. Hence we have proved that $\hat{X}_1 \subset \text{ex}(K_1)$.

Let $x \in X_n$. Let $(\hat{x})_{ij}$ denote the entry on the i -th row and j -th column of the matrix $\hat{x} \in \hat{X}_n$. Let $\{\varepsilon_i \mid i = 1, \dots, n\}$ be the standard basis of \mathbb{C}^n . For the diagonal entries of \hat{x} we obtain

$$\hat{x}_{ii}(f) = f_1(x_{ii}) = f_1(\varepsilon_i^* x \varepsilon_i) = \varepsilon_i^* f_n(x) \varepsilon_i = \varepsilon_i^* \hat{x}(f) \varepsilon_i = (\varepsilon_i^* \hat{x} \varepsilon_i)(f) = (\hat{x})_{ii}(f),$$

which shows $(\hat{x})_{ii} = \hat{x}_{ii}$, where \hat{x}_{ii} is the normal pure state of $c^{[x]}\mathcal{F}_b^E([x]) = \mathcal{B}(H_{[x]})$ corresponding to the minimal projection $p^{x_{ii}} = p_{ii}$. Hence there are unit vectors $\xi_i \in H_{[x]}$ such that $\hat{x}_{ii}(a) = \text{trace}(a(\xi_i \odot \xi_i)) = \langle a\xi_i | \xi_i \rangle$ for all $a \in \mathcal{B}(H_{[x]})$. Notice from the definition of the minimal projections p_{ii} that $p_{ii}(x)$ is the $n \times n$ matrix with entry 1 on the i -th row and i -th column and zero elsewhere ($p_{ii}(x) = E_{ii}$). Hence $p_{ii}(x_{jj}) = \delta_{ij}$, so that $\delta_{ij} = \hat{x}_{ii}(p_{jj}) = \langle p_{jj}\xi_i | \xi_i \rangle$. This implies $p_{jj}\xi_i = 0$ for $i \neq j$ and $p_{jj}\xi_j = \xi_j$. Thus from $\langle \xi_i | \xi_j \rangle = \langle p_{ii}\xi_i | \xi_j \rangle$ we see that the set $\{\xi_1, \dots, \xi_n\}$ is orthonormal, so that we can identify $L = \text{lin}\{\xi_1, \dots, \xi_n\}$ with \mathbb{C}^n . Let $p \in \mathcal{B}(H_{[x]})$ be the orthogonal projector onto the closed subspace L . Then

$$\varepsilon_i^* p(x) \varepsilon_i = \hat{x}_{ii}(p) = \text{trace}(p(\xi_i \odot \xi_i)) = \langle p\xi_i | \xi_i \rangle = 1$$

for $1 \leq i \leq n$. Since $0 \leq \hat{x}(p) \leq \mathbb{1}_n$, it follows that $\hat{x}(p) = \mathbb{1}_n$. Thus $\hat{x}(p') = 0$, where $p' = \mathbb{1} - p$. Let $y \in X_l$ for some $l \in \mathbb{N}$, such that $y \wedge x$ and $y \perp x$. We claim that $\hat{y}(p) = 0$. By definition of \perp there is $z \in X_{l+n}$ such that $z = \begin{pmatrix} x^* \\ y \end{pmatrix}$. Since the diagonal entries of \hat{z} are pure states, we know already from the previous part of the proof that $(\hat{z})_{ii} = \hat{z}_{ii}$, and that there is a orthonormal system $(\zeta_i)_{i=1}^{l+n}$ such that $\hat{z}_{ii}(a) = \text{trace}(a(\zeta_i \odot \zeta_i)) = \langle a\zeta_i | \zeta_i \rangle$ for all $a \in \mathcal{B}(H_{[x]})$. Since $\hat{z}_{ii} = \hat{x}_{ii}$ for $1 \leq i \leq n$, there are $\lambda_i \in \mathbb{C}$ such that $\zeta_i = \lambda_i \xi_i$. Therefore we have $\zeta_j \perp L = p(H_{[x]})$ for all $n < j \leq l+n$. Since $\hat{z}_{n+i, n+i} = \hat{y}_{ii}$ for $1 \leq i \leq l$, we obtain $\hat{y}_{ii}(p) = \text{trace}(p(\zeta_{n+i} \odot \zeta_{n+i})) = \langle p\zeta_{n+i} | \zeta_{n+i} \rangle = 0$. Consequently $\hat{y}(p) = 0$, and the claim is shown.

We want to show that \hat{x} is a pure map, which is equivalent to \hat{x} being a structural element, see Proposition 1.19. So, let $\phi: \mathcal{F}_b^E(X) \rightarrow M_n$ be completely positive, such that $\phi \leq_{cp} \hat{x}$. Then obviously $\phi(p') = 0$. Given $a \in \mathcal{B}(H_{[x]})$ such that $0 \leq a \leq \mathbb{1}$, it is obvious that $0 \leq p'ap' \leq p'$ so that we obtain $0 \leq \phi(p'ap') \leq \phi(p') = 0$. Moreover, the matrix $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$ is positive, and it follows that $\begin{pmatrix} pap & pap' \\ p'ap & p'ap' \end{pmatrix}$ is also positive. Using that ϕ is completely positive we conclude that $\begin{pmatrix} \phi(pap) & \phi(pap') \\ \phi(p'ap) & 0 \end{pmatrix}$ is a positive matrix. Therefore we must have $\phi(pap') = \phi(p'ap) = 0$. Since this holds for all positive operators $0 \leq a \leq \mathbb{1}$, it is clear that $\phi(p'ap')$, $\phi(pap')$ and $\phi(p'ap)$ vanish for all $a \in \mathcal{B}(H_{[x]})$. Of course, the same argumentation applies also to \hat{x} . Hence ϕ and \hat{x} can be interpreted as maps on $p\mathcal{B}(H_{[x]})p = pc^{[x]}\mathcal{F}_b^E(X)p$. We claim that the map $\psi: pfp \mapsto f|_{[x]}$ is a *-isomorphism

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from $p\mathcal{F}_b^E(X)p$ onto $\mathcal{F}_b^E([x])$. Notice that

$$\begin{aligned}(pfp)_n(x) &= \lim_{y \succ x} u_{xy}^* p(y) f(y) p(y) u_{xy} \\ &= \lim_{y \succ x} \mathbb{1}_{l_y n}^* \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_n(x) & * \\ * & * \end{pmatrix} \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{pmatrix} \mathbb{1}_{l_y n} = f_n(x),\end{aligned}$$

since we can restrict to those elements $y \succ x$ such that $y = \begin{pmatrix} x & * \\ * & * \end{pmatrix}$. Thus if $pfp = pgp$ then in particular $f_n(x) = g_n(x)$, so that $f|_{[x]} = g|_{[x]}$ and we conclude that the map ψ is well-defined. We have to show that ψ is injective, surjective and multiplicative. Let $f, g \in \mathcal{F}_b^E(X)$ such that $f|_{[x]} = g|_{[x]}$, i.e., $f_n(x) = g_n(x)$. We claim that $pfp = pgp$. Notice that $p(\mathbb{1} - c^{[x]}) = 0$ by definition of p , so that we can restrict our attention to $[x]$. For $y \in X_1$ such that $y \preccurlyeq x$ we have obviously $f_1(y) = g_1(y)$. If $y \not\preccurlyeq x$ and $y \wedge x$ there is $l \in \mathbb{N}$ and $z \in X_l$ such that $z \succ y, x$. Then

$$(pfp)_l(z) = \lim_{z' \succ z} u_{zz'}^* p(z') f(z') p(z') u_{zz'}.$$

Since $z' \succ z \succ x$ we can assume that z' is unitarily equivalent to a matrix with left upper corner equal to x . We have proved already that p vanishes on elements orthogonal to x . Therefore it follows $(pfp)_l(z) = \begin{pmatrix} f_n(x) & 0 \\ 0 & 0 \end{pmatrix}$, and since we can apply the same argumentation to pgp we obtain $(pgp)_l(z) = \begin{pmatrix} g_n(x) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f_n(x) & 0 \\ 0 & 0 \end{pmatrix} = (pfp)_l(z)$. It follows that $(pfp)_1(y) = (pgp)_1(y)$ for all $y \in X_1$ such that $y \wedge x$, and consequently $pfp = pgp$. Thus ψ is injective. The fact that $(pfp)_l(z) = \begin{pmatrix} f_n(x) & 0 \\ 0 & 0 \end{pmatrix}$ for all $z = \begin{pmatrix} x & * \\ * & * \end{pmatrix} \in X_l$ and $l > n$ indicates also that ψ is surjective. Indeed, let $g \in \mathcal{F}_b^E([x])$. We claim that there is $f \in \mathcal{F}_b^E(X)$ such that $pfp|_{[x]} = g$. Assume we have $l > n$ and $y, z \in X_l$ such that $\mathbb{1}_{l,n}^* y \mathbb{1}_{l,n} = x = \mathbb{1}_{l,n}^* z \mathbb{1}_{l,n}$ and $y = u^* z u$ for a unitary $u \in M_l$. Then obviously $\mathbb{1}_{l,n}^* u^* z u \mathbb{1}_{l,n} = \mathbb{1}_{l,n}^* z \mathbb{1}_{l,n}$, so it follows from the uniqueness property that there is $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\lambda u \mathbb{1}_{l,n} = \mathbb{1}_{l,n}$. Since u is unitary, we see that $u = \begin{pmatrix} \lambda \mathbb{1}_n & 0 \\ 0 & u_{22} \end{pmatrix}$. Therefore in particular $u^* \begin{pmatrix} g_n(x) & 0 \\ 0 & 0 \end{pmatrix} u = \begin{pmatrix} g_n(x) & 0 \\ 0 & 0 \end{pmatrix}$, so we can define $f_l(y) = \begin{pmatrix} g_n(x) & 0 \\ 0 & 0 \end{pmatrix}$ for all $y = \begin{pmatrix} x & * \\ * & * \end{pmatrix} \in X_l$ and all $l > n$. Of course, for $y \in [x]$ we set $f(y) = g(y)$. Now, given $y \in [x] \setminus [x]$ there is by definition of the m-relation $l \in \mathbb{N}$ and $z \in X_l$ such that $z \succ y, x$. Since $z \succ x$, it is obvious that z is unitarily equivalent to a matrix with upper left corner equal to x , so that f is defined on z by the preceding considerations. Therefore we can define $f(y) = u_{yz}^* f_l(z) u_{yz}$. It is clear from the definition that $f \in \mathcal{F}_b^E([x])$ and $pfp|_{[x]} = g$. (Notice that we can extend f easily to an element of $\mathcal{F}_b^E(X)$ by setting $f(y) = 0$ for all $y \in X \setminus [x]$.) Thus ψ is surjective. It is still left to show that ψ is multiplicative. Since $\psi(pfppgp) = \psi(pfppgp) = fpg|_{[x]}$ and $\psi(pfp)\psi(pgp) = f|_{[x]}g|_{[x]}$ we need to verify that $(fpg)_n(x) = f_n(x)g_n(x)$ to prove that ψ is multiplicative. By definition we have $(fpg)_n(x) = \lim_{y \succ x} u_{yx}^* f(y) p(y) g(y) u_{yx}$. Assuming that y is unitarily equivalent to an element with upper left corner equal to x we obtain immediately $(fpg)_n(x) = f_n(x)g_n(x)$. So, we have shown that $p\mathcal{F}_b^E(X)p$ is unitally *-isomorph to $\mathcal{F}_b^E([x])$ via the mapping $pfp \mapsto f|_{[x]}$. Notice that restricting \hat{x} to $\mathcal{F}_b^E([x])$ is the same as reading x as element of $\mathcal{F}_b^E([x])^*$ by pointwise evaluation, because $\hat{x}(pfp) = f_n(x) = \hat{x}(f|_{[x]})$, where we denoted both maps simply by \hat{x} . Furthermore, \hat{x} is a pure map in $CS(\mathcal{F}_b^E([x]))$ by Lemma 3.34, and we still have $\phi \leq_{cp} \hat{x}$ read as maps on $\mathcal{F}_b^E([x])$. Thus there is $0 < r \leq 1$ such that $\phi(f|_{[x]}) = r\hat{x}(f|_{[x]})$. Then $\phi(pfp) = r\hat{x}(pfp)$ and so $\phi(f) = r\hat{x}(f)$ for all $f \in \mathcal{F}_b^E(X)$. This shows that \hat{x} is a pure state, and the proof is complete. \square

In order to show the converse of the preceding proposition, namely that \widehat{X} consists of all the normal pure matrix states of $\mathcal{F}_b^E(X)$, we still need an additional structure on X . We will show that we can define an (inner) metric of X , if X is an equivariant and transitive matrix set that fulfills the uniqueness property. Then the condition that X is a complete metric space will ensure that \widehat{X} contains all normal pure matrix states of $\mathcal{F}_b^E(X)$. To define the metric we will need the following simple observations.

Lemma 3.45. *Let $\xi, \eta \in H$ be unit vectors of a Hilbert space H . Then*

$$\|\xi \odot \xi - \eta \odot \eta\|_1 = 2\sqrt{1 - |\langle \xi | \eta \rangle|^2}.$$

Proof. We need to calculate the trace norm $\|\xi \odot \xi - \eta \odot \eta\|_1 = \text{trace}(|\xi \odot \xi - \eta \odot \eta|)$. To do this, we read the trace class operator $\xi \odot \xi - \eta \odot \eta$ as linear map on $L = \text{lin}\{\xi, \eta\}$. Let $\zeta = \eta - \langle \eta | \xi \rangle \xi$ and $\zeta_1 = \zeta / \|\zeta\|$. Then $\xi \odot \xi$ has the matrix representation $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the orthonormal basis $\{\xi, \zeta_1\}$. By evaluating $(\eta \odot \eta)\xi$ and $(\eta \odot \eta)\zeta_1$ we find the respective matrix representation of $\eta \odot \eta$ to be

$$\eta \odot \eta = \begin{pmatrix} |\langle \xi | \eta \rangle|^2 & \|\zeta\| \langle \eta | \xi \rangle \\ \|\zeta\| \langle \xi | \eta \rangle & \|\zeta\|^2 \end{pmatrix}.$$

A simple calculation gives $\|\zeta\| = \sqrt{1 - |\langle \xi | \eta \rangle|^2}$. Consequently,

$$\begin{aligned} (\xi \odot \xi - \eta \odot \eta)^2 &= \begin{pmatrix} 1 - |\langle \xi | \eta \rangle|^2 & -\|\zeta\| \langle \eta | \xi \rangle \\ -\|\zeta\| \langle \xi | \eta \rangle & -\|\zeta\|^2 \end{pmatrix}^2 \\ &= \begin{pmatrix} \|\zeta\|^2 & -\|\zeta\| \langle \eta | \xi \rangle \\ -\|\zeta\| \langle \xi | \eta \rangle & -\|\zeta\|^2 \end{pmatrix}^2 \\ &= \begin{pmatrix} \|\zeta\|^2 & 0 \\ 0 & \|\zeta\|^2 \end{pmatrix}, \end{aligned}$$

so that $\text{trace}(|\xi \odot \xi - \eta \odot \eta|) = 2\|\zeta\| = 2\sqrt{1 - |\langle \xi | \eta \rangle|^2}$, which was the claim. \square

Corollary 3.46. *Let φ, ψ pure normal states of $\mathcal{B}(H)$, so that $\varphi, \psi \in \mathcal{T}(H)$. Then $\|\varphi - \psi\| = 2\sqrt{1 - |\langle \xi_\varphi | \xi_\psi \rangle|^2}$, where $\xi_\varphi, \xi_\psi \in H$ are the essentially unique unit vectors such that φ and ψ are the vector states determined by ξ_φ and ξ_ψ , respectively.*

Proof. Obviously we have $\varphi = \xi_\varphi \odot \xi_\varphi$ and $\psi = \xi_\psi \odot \xi_\psi$, so that

$$\|\varphi - \psi\| = \|\xi_\varphi \odot \xi_\varphi - \xi_\psi \odot \xi_\psi\|_1 = 2\sqrt{1 - |\langle \xi_\varphi | \xi_\psi \rangle|^2}$$

by the preceding lemma. \square

Lemma 3.47. *Let X be some equivariant matrix subset of a vector space W . If $x, y \in X_1$ are two distinct m -related points, then there is $z \in X_2$ such that $z \succ x, y$.*

Proof. For two distinct points $x, y \in X_1$ that are m -related there are $n \in \mathbb{N}$, $\tilde{z} \in X_n$ and $\xi, \eta \in \mathbb{C}^n$ such that $x = \xi^* \tilde{z} \xi$, $y = \eta^* \tilde{z} \eta$ and $n \geq 2$. Let $L = \text{lin}\{\xi, \eta\} \subset \mathbb{C}^n$. Since $x \neq y$ we see that $\dim(L) = 2$. Let $v: \mathbb{C}^2 \rightarrow L$ be an isometry. Then, since X is equivariant, $z = v^* \tilde{z} v \in X_2$. Obviously, $z \succ x_1, x_2$ and the proof is complete. \square

3. Matrix Convex Simplexes

Proposition 3.48. *Let X be an equivariant and transitive matrix subset of some vector space W such that X fulfills the uniqueness property. We set $d(x, x) = 0$ for $x \in X_1$, and for distinct $x, y \in X_1$ we let $d(x, y) = 2$ if x and y are matrix non-equivalent, and otherwise we let $d(x, y) = \text{trace}(|\xi \odot \xi - \eta \odot \eta|)$, where $\xi, \eta \in \mathbb{C}^2$ are unit vectors such that $x = \xi^* z \xi$ and $y = \eta^* z \eta$ for some $z \in X_2$. Then $d: X_1 \times X_1 \rightarrow \mathbb{R}_+$ is a well-defined metric on X_1 .*

Proof. We show first that d is well-defined. If $x, y \in X_1$ are m-equivalent, there are $z \in X_2$ and unit vectors $\xi_1, \xi_2 \in \mathbb{C}^2$ such that $x = \xi_1^* z \xi_1$ and $y = \xi_2^* z \xi_2$ by Lemma 3.47. If $l \geq 2$ and $z' \in X_l$ such that $x = \zeta_1^* z' \zeta_1$ and $y = \zeta_2^* z' \zeta_2$ for unit vectors $\zeta_1, \zeta_2 \in \mathbb{C}^l$, then by transitivity of the m-relation $z \sim z'$. Thus there is some $n \in \mathbb{N}$ and $\psi \in X_n$ such that $z = u_{z\psi}^* \psi u_{z\psi}$ and $z' = u_{z'\psi}^* \psi u_{z'\psi}$. Hence we obtain

$$\begin{aligned} x &= \xi_1^* u_{z\psi}^* \psi u_{z\psi} \xi_1 = \xi_1^* u_{z'\psi}^* \psi u_{z'\psi} \zeta_1 \\ y &= \xi_2^* u_{z\psi}^* \psi u_{z\psi} \xi_2 = \xi_2^* u_{z'\psi}^* \psi u_{z'\psi} \zeta_2. \end{aligned}$$

Applying the uniqueness property to the preceding equations, we find $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $|\lambda_1| = |\lambda_2| = 1$, $u_{z\psi} \xi_1 = \lambda_1 u_{z'\psi} \zeta_1$ and $u_{z\psi} \xi_2 = \lambda_2 u_{z'\psi} \zeta_2$. Therefore

$$|\langle \xi_1 | \xi_2 \rangle| = |\langle u_{z\psi} \xi_1 | u_{z\psi} \xi_2 \rangle| = |\langle u_{z'\psi} \zeta_1 | u_{z'\psi} \zeta_2 \rangle| = |\langle \zeta_1 | \zeta_2 \rangle|,$$

which shows that $2\sqrt{1 - |\langle \xi_1 | \xi_2 \rangle|^2} = 2\sqrt{1 - |\langle \zeta_1 | \zeta_2 \rangle|^2}$. Thus it follows from Lemma 3.45 that d is well-defined. By definition of d it is obvious that $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X_1$. So to see that d is a metric on X_1 , we only need to verify the triangle inequality. Let $x, y, z \in X_1$. We want to show that $d(x, z) \leq d(x, y) + d(y, z)$. Notice that there is nothing to prove if $x \not\sim y$ or $y \not\sim z$. Hence we assume $x \sim y$ and $y \sim z$. Then it follows $x \sim z$ by transitivity. By definition of the m-relation there is $n \in \mathbb{N}$ and $\phi \in X_n$ such that $x = \xi^* \phi \xi$, $y = \eta^* \phi \eta$ and $z = \zeta^* \phi \zeta$ for unit vectors $\xi, \eta, \zeta \in \mathbb{C}^n$. For these unit vectors it is obvious that

$$\|\xi \odot \xi - \zeta \odot \zeta\|_1 \leq \|\xi \odot \xi - \eta \odot \eta\|_1 + \|\eta \odot \eta - \zeta \odot \zeta\|_1,$$

so that by definition of d we obtain directly $d(x, z) \leq d(x, y) + d(y, z)$. Thus (X_1, d) is a metric space, and the proof is complete. \square

Definition 3.49 (Inner Metric). Let X be an equivariant and transitive matrix subset of some vector space W such that X fulfills the uniqueness property. Then the metric d on X_1 defined by $d(x, x) = 0$ and

$$d(x, y) = \begin{cases} \text{trace}(|\xi \odot \xi - \eta \odot \eta|) & \text{if } x \sim y \text{ and } x \neq y \\ 2 & \text{if } x \not\sim y, \end{cases}$$

where $\xi, \eta \in \mathbb{C}^2$ are some unit vectors such that $x = \xi^* z \xi$ and $y = \eta^* z \eta$ for some $z \in X_2$ (that exists in the case $x \sim y$), is called the *inner metric* of X .

Lemma 3.50. *Let \mathcal{M} be an atomic W^* -algebra. Let $X = \text{str}(S)$, where $S = CS^\sigma(\mathcal{M})$ and let d be the inner metric of X . Then $d(x, y) = \|x - y\|$ for all normal pure states $x, y \in X_1$.*

Proof. Given $x, y \in X_1$ we know that $x \sim y$ if and only if x and y are equivalent as states. In case x and y are not equivalent, we know that $\|x - y\| = 2$, and by definition of the inner metric we also have $d(x, y) = 2$ in this case. So, assume that x and y are equivalent. Then we can represent x and y using the same normal irreducible representation, i.e., there is a normal and irreducible representation $\pi: \mathcal{M} \rightarrow H_\pi$ such that $x = \xi_x^* \pi \xi_x$ and $y = \eta_y^* \pi \eta_y$ for unit vectors $\xi_x, \eta_y \in H_\pi$. Since x and y are m-equivalent there is also $z \in X_2$, such that $x = \xi^* z \xi$ and $y = \eta^* z \eta$ for unit vectors $\xi, \eta \in \mathbb{C}^2$. It follows that z is equivalent to x and y , so that the minimal Stinespring representation of z can be written as $z = \mathcal{V}^* \pi \mathcal{V}$, where $\mathcal{V}: \mathbb{C}^2 \rightarrow H_\pi$ is an isometry. Then, since the vector in the GNS representation is unique up to a factor, from $x = \xi^* \mathcal{V}^* \pi \mathcal{V} \xi$ and $y = \eta^* \mathcal{V}^* \pi \mathcal{V} \eta$ we obtain $\mathcal{V} \xi = \lambda \xi_x$ and $\mathcal{V} \eta = \mu \eta_y$, where $\lambda, \mu \in \mathbb{C}^2$ with $|\lambda| = |\mu| = 1$. Thus $\langle \xi | \eta \rangle = \langle \mathcal{V} \xi | \mathcal{V} \eta \rangle = \lambda \bar{\mu} \langle \xi_x | \eta_y \rangle$, such that $|\langle \xi | \eta \rangle| = |\langle \xi_x | \eta_y \rangle|$. Hence reading x and y as normal pure states on $\mathcal{B}(H_\pi)$ and applying Corollary 3.46 yields

$$d(x, y) = 2\sqrt{1 - |\langle \xi | \eta \rangle|^2} = 2\sqrt{1 - |\langle \xi_x | \eta_y \rangle|^2} = \|x - y\|,$$

which shows the claim. \square

The next two lemmas will help us to prove the announced abstract characterization of the normal pure matrix states of atomic W^* -algebras.

Lemma 3.51. *Let X be a matrix set such that X is equivariant, transitive and fulfills the uniqueness property, so that $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra. If $p \in \mathcal{F}_b^E(X)$ is a projection then $\sup \{p_1(x) \mid x \in X_1\} = 1$.*

Proof. Suppose that $0 \leq s = \sup \{p_1(x) \mid x \in X_1\} < 1$. Then it follows easily by equivariance that $p_n(y) \leq s \mathbb{1}_n$ for all $y \in X_n$ and $n \geq 2$. Thus

$$p_1(x) = (pp)_1(x) = \lim_{y \succ x} u_{xy}^* p(y) p(y) u_{xy} \leq s^2 < s$$

for all $x \in X_1$ gives immediately a contradiction, and the claim is shown. \square

Lemma 3.52. *Let X be a matrix subset of some vector space V such that X is equivariant, transitive and fulfills the uniqueness property, so that $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra, cf. Corollary 3.27. For $n, m \in \mathbb{N}$ let $x = [x_{ij}] \in X_n$ and $y = [y_{rs}] \in X_m$ such that $y \succ x_{ii}$ for $i = 1, \dots, n$. Then $m \geq n$ and $y \succ x$.*

Proof. Since especially $x \sim x_{11} \sim y$ and the m-relation is assumed to be transitive, it follows that $x \sim y$, i.e., $[x] = [y]$. By Proposition 3.44 we have $X = \hat{X} \subset \text{str}(CS^\sigma(\mathcal{F}_b^E(X)))$. We identify $c^{[y]} \mathcal{F}_b^E(X) = \mathcal{B}(H_{[y]})$. So by Proposition 3.43 there are orthonormal systems $\{\xi_1, \dots, \xi_n\}$ and $\{\eta_1, \dots, \eta_m\}$ in $H_{[y]}$ such that $\hat{x} = [\Xi(\xi_i \odot \xi_j)]$ and $\hat{y} = [\Xi(\eta_i \odot \eta_j)]$. Define the isometries $\mathcal{V}: \mathbb{C}^n \rightarrow H_{[y]}$ and $\mathcal{W}: \mathbb{C}^m \rightarrow H_{[y]}$ by $\mathcal{V} \varepsilon_i^n = \xi_i$ and $\mathcal{W} \varepsilon_j^m = \eta_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, where $(\varepsilon_i^n)_i$ and $(\varepsilon_j^m)_j$ are the standard basis of \mathbb{C}^n and \mathbb{C}^m , respectively. Then we obtain $\hat{x}(a) = [\langle a \xi_j | \xi_i \rangle] = \mathcal{V}^* \text{id}(a) \mathcal{V}$ and $\hat{y}(a) = [\langle a \eta_j | \eta_i \rangle] = \mathcal{W}^* \text{id}(a) \mathcal{W}$ where id denotes the identity of $\mathcal{B}(H_{[y]})$. From $y \succ x_{ii}$ we get that $\xi_i \in \mathcal{W}(\mathbb{C}^m)$ for $i = 1, \dots, n$ and consequently $\mathcal{V}(\mathbb{C}^n) \subset \mathcal{W}(\mathbb{C}^m) \subset H_{[y]}$. Hence the linear map $u = \mathcal{W}^* \mathcal{V}: \mathbb{C}^n \rightarrow \mathbb{C}^m$ is well-defined. Then $u^* u = \mathcal{V}^* \mathcal{W} \mathcal{W}^* \mathcal{V} = \mathcal{V}^* \mathcal{V} = \mathbb{1}_n$, since $\mathcal{W} \mathcal{W}^* = \text{id}_{H_{[y]}}$. So $u \in M_{m,n}$ is an isometry and $u^* \hat{y} u = u^* \mathcal{W}^* \text{id} \mathcal{W} u = \mathcal{V}^* \text{id} \mathcal{V} = \hat{x}$ shows that $y \succ x$, completing the proof. \square

3. Matrix Convex Simplexes

We can now state and prove another main result of the thesis.

Theorem 3.53 (Non-commutative Sets). *Let X be a matrix subset of some vector space W . Then X is equivariantly isomorph to the normal pure matrix states of an atomic W^* -algebra if and only if*

- (i) X equivariant, transitive and fulfills the uniqueness property, and
- (ii) (X_1, d) is a complete metric space, where d is the inner metric of X .

Proof. Let \mathcal{M} be an atomic W^* -algebra. Then $\text{str}(CS^\sigma(\mathcal{M}))$ is equivariant and transitive by Corollary 3.5, and fulfills the uniqueness property by Proposition 3.8 and Remark 3.10. Moreover, $d(\varphi, \psi) = \|\varphi - \psi\|$ for all pure states $\varphi, \psi \in \text{str}_1(CS^\sigma(\mathcal{M}))$. Since the set of the pure states is norm-closed, it follows from Lemma 3.50 that $\text{str}_1(CS^\sigma(\mathcal{M}))$ is complete in its inner metric.

In the converse direction, let X be an equivariant and transitive matrix subset of some vector space W such that X fulfills the uniqueness property. Then $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra. Letting $S = CS^\sigma(\mathcal{F}_b^E(X))$ we know from Proposition 3.44 that $X = \hat{X} \subset \text{str}(S)$. Given a pure state $\psi \in \text{str}(S_1)$, there is a corresponding minimal projection $p \in \mathcal{F}_b^E(X)$. More precisely, identifying $\mathcal{F}_b^E(X) = \oplus_\kappa \mathcal{B}(H_\kappa)$, where the sum runs over all equivalence classes of normal pure states of $\mathcal{F}_b^E(X)$, there is a unit vector $\xi \in H_{[\psi]}$ such that $\psi(a) = \langle a\xi | \xi \rangle$ for all $a \in \mathcal{B}(H_{[\psi]})$, and the minimal projector p corresponding to ψ is given by $p = \xi \odot \xi$. Now by Lemma 3.51 there exists a sequence $(x_n)_n$ in X_1 such that $p(x_n) \rightarrow 1$. Then \hat{x}_n is a pure normal state of $\mathcal{F}_b^E(X)$ so there is a unit vector $\xi_n \in \oplus_\kappa H_\kappa$ such that $\hat{x}_n(a) = \langle a\xi_n | \xi_n \rangle$ for all $n \in \mathbb{N}$ and $a \in \oplus_\kappa \mathcal{B}(H_\kappa)$. Then we obtain

$$p(x_n) = \hat{x}_n(p) = \langle p\xi_n | \xi_n \rangle = \langle (\xi \odot \xi)\xi_n | \xi_n \rangle = |\langle \xi_n | \xi \rangle|^2 \rightarrow 1.$$

It follows from Corollary 3.46 that $\|\hat{x}_n - \psi\| \rightarrow 0$. Therefore $(x_n)_n$ is a Cauchy sequence with respect to the inner metric d of X by Lemma 3.50. Now, assuming that (X_1, d) is a complete metric space, there is $x \in X_1$ such that $d(x_n, x) \rightarrow 0$. Thus we obtain immediately that $\|\hat{x}_n - \hat{x}\| \rightarrow 0$, which yields $\psi = \hat{x}$. So far we have shown that $\text{str}(S_1) \subset \hat{X}_1$, and together with Proposition 3.44 we have $\text{str}(S_1) = \hat{X}_1$. Let $n \geq 2$ and $\psi \in \text{str}(S_n)$. Then by Proposition 3.43 there is an orthonormal system $\{\zeta_1, \dots, \zeta_n\}$ such that $\psi = [\zeta_i \odot \zeta_j]$. Now $\psi_{ii} \in \text{str}(S_1)$ for $1 \leq i \leq n$, so that there are $x_{ii} \in X_1$ such that $\hat{x}_{ii} = \psi_{ii}$. Notice that the x_{ii} are pairwise m-equivalent, because the pure states ψ_{ii} are pairwise unitarily equivalent. Thus there is some $l \in \mathbb{N}$ and $y \in X_l$ such that $y \succ x_{ii}$ for all $1 \leq i \leq n$. Then $\hat{y} \succ \hat{x}_{ii} = \psi_{ii}$, again by the identification $X = \hat{X}$. It follows now from Lemma 3.52 applied in $\text{str}(S)$ that $\hat{y} \succ \psi$, so there is an isometry $v \in M_{l,n}$ such that $\psi = v^* \hat{y} v \in \hat{X}_n$, and the proof is complete. \square

Normal state space of atomic W^* -algebras

It will turn out that characterizing the normal m-convex state space of atomic W^* -algebras is the first step toward characterizing state spaces of C^* -algebras. We prove next some properties of the normal state space of atomic W^* -algebras, before we give an abstract characterization.

Proposition 3.54. *Let $K = CS^\sigma(\mathcal{M})$ for an atomic W^* -algebra \mathcal{M} , and $X = \text{str}(K)$. If $x_\nu \in X_{n_\nu}$ for $\nu = 1, \dots, l$ are finitely many pairwise matrix non-equivalent points, then $C = \text{mco}(x_1, \dots, x_l)$ is m -affinely isomorphic to the m -convex state space of the C^* -algebra $\oplus_{\nu=1}^l M_{n_\nu}$. Consequently, C is a finite m -convex simplex. Moreover, C_1 is a projective face of K_1 .*

Proof. An atomic W^* -algebra is a sum of type I factors, hence we can assume $\mathcal{M} = \oplus_{j \in J} \mathcal{B}(H_j)$ for some index set J . Then given pairwise matrix non-equivalent $x_\nu \in X_{n_\nu}$, there are indices $j_\nu \in J$, $j_\nu \neq j_\mu$ for $\nu \neq \mu$ and $\nu, \mu \in \{1, \dots, l\}$, and isometries $\mathcal{V}_\nu: \mathbb{C}^{n_\nu} \rightarrow H_{j_\nu}$, such that $x_\nu(T) = \mathcal{V}_\nu^* \pi_{j_\nu}(T) \mathcal{V}_\nu$, where $\pi_{j_\nu}: \oplus_j \mathcal{B}(H_j) \rightarrow \mathcal{B}(H_{j_\nu})$ is the irreducible representation that maps $t = \oplus_j t_j \in \oplus_j \mathcal{B}(H_j)$ to $\pi_{j_\nu}(t) = t_{j_\nu} \in \mathcal{B}(H_{j_\nu})$. Since the indices j_ν are pairwise distinct, the map $x = \oplus_{\nu=1}^l x_\nu: \oplus_j \mathcal{B}(H_j) \rightarrow \oplus_{\nu=1}^l M_{n_\nu}$ is surjective. Hence the adjoint mapping $x_*: (\oplus_{\nu=1}^l M_{n_\nu})_* \rightarrow \mathcal{M}_*$ is injective. We have to verify that $x_*^{(n)}(CS_n(\oplus_{\nu=1}^l M_{n_\nu})) = \text{mco}_n(x_1, \dots, x_l)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $\psi \in CS_n(\oplus_{\nu=1}^l M_{n_\nu})$, i.e., $\psi: \oplus_{\nu=1}^l M_{n_\nu} \rightarrow M_n$ is completely positive and unital. Then $\psi = (\psi_\nu)_{\nu=1}^l$, where $\psi_\nu: M_{n_\nu} \rightarrow M_n$ is completely positive and $\sum_{\nu=1}^l \psi_\nu(\mathbb{1}_{n_\nu}) = \mathbb{1}_n$. We obtain

$$x_*^{(n)}(\psi)(T) = \psi(x(T)) = (\psi_\nu)_{\nu=1}^l(\oplus_{\nu=1}^l x_\nu(T)) = \sum_{\nu=1}^l \psi_\nu(x_\nu(T)),$$

for all $T \in \oplus_j \mathcal{B}(H_j)$. Since $\psi_\nu(\gamma) = \sum_{i=1}^{m_\nu} \alpha_{\nu,i}^* \gamma \alpha_{\nu,i}$ for all $\gamma \in M_{n_\nu}$ and $\nu = 1, \dots, l$ (cf. [16]), we find

$$x_*^{(n)}(\psi) = \sum_{\nu=1}^l \sum_{i=1}^{m_\nu} \alpha_{\nu,i}^* x_\nu \alpha_{\nu,i} \in \text{mco}(x_1, \dots, x_l).$$

This proves that $x_*^{(n)}(CS_n(\oplus_{\nu=1}^l M_{n_\nu})) \subset \text{mco}_n(x_1, \dots, x_l)$. For the converse direction, we note that $x_\nu = x_*^{(n_\nu)}(0 \oplus \dots \oplus 0 \oplus \text{id}_{M_{n_\nu}} \oplus 0 \oplus \dots \oplus 0)$, such that $x_\nu \in x_*^{(n_\nu)}(\oplus_{\nu=1}^l M_{n_\nu})$ for $\nu = 1, \dots, l$. Since the image of the m -affine map $(x_*^{(n)})$ is m -convex, we have shown that $\text{mco}(x_1, \dots, x_l)$ is m -affinely isomorphic to the m -convex state space of $\oplus_{\nu=1}^l M_{n_\nu}$. Recall that the finite m -convex simplexes are exactly the m -convex state spaces of finite dimensional C^* -algebras.

To show that C_1 is a projective face of K_1 , we define an orthogonal projection $p = \oplus p_j \in \oplus \mathcal{B}(H_j)$, where $p_{j_\nu} = \mathcal{V}_\nu \mathcal{V}_\nu^*$, and $p_j = 0$ for $j \neq j_\nu$, $\nu = 1, \dots, l$. We claim that $C_1 = x_*(CS_1(\oplus M_{n_\nu}))$ consist exactly of those $\varphi \in K_1$, such that $\varphi(p) = 1$. Let $\psi \in CS_1(\oplus M_{n_\nu})$, then obviously $x_*(\psi)(t) = \psi(x(t)) \geq 0$, whenever $t \in \oplus \mathcal{B}(H_j)$ is positive. Furthermore, let $\text{id} = \oplus \text{id}_j$ be the unit of $\oplus \mathcal{B}(H_j)$, then $\psi(x(\text{id})) = \psi(\oplus \mathbb{1}_{n_\nu}) = 1$, so that $x_*(\psi) \in K_1$. Evaluating

$$x(p) = \oplus x_\nu(p) = \oplus \mathcal{V}_\nu^* p_{j_\nu} \mathcal{V}_\nu = \oplus \mathcal{V}_\nu^* \mathcal{V}_\nu \mathcal{V}_\nu^* \mathcal{V}_\nu = \oplus \mathbb{1}_{n_\nu}$$

shows that $\psi(x(p)) = 1$. In the converse direction, let $\varphi \in K_1$, such that $\varphi(p) = 1$. Then obviously $\varphi(\text{id} - p) = 0$, and since φ is a state, by application of the Cauchy-Schwarz inequality ($|\varphi(ba)|^2 \leq \varphi(a)\varphi(bab^*)$ for all $a \geq 0$ and b arbitrary) we obtain $\varphi(t) = \varphi(ftp)$ for all $t \in \oplus \mathcal{B}(H_j)$ (recall that $t = ftp + (\text{id} - p)tp + pt(\text{id} - p) + (\text{id} - p)t(\text{id} - p)$ uniquely). Now there is a natural embedding $\vartheta: \oplus M_{n_\nu} \hookrightarrow p \oplus \mathcal{B}(H_j)p$ given by $\vartheta(\oplus \gamma_\nu) = \oplus s_{j_\nu}$, where $s_{j_\nu} = \mathcal{V}_\nu \gamma_\nu \mathcal{V}_\nu^*$ and $s_j = 0$ for $j \neq j_\nu$, $\nu = 1, \dots, l$. We let $\psi = \varphi \circ \vartheta$, then ψ is positive and $\psi(\oplus \mathbb{1}_{n_\nu}) = \varphi(p) = 1$. Moreover,

$$x_*(\psi)(t) = \psi(x(t)) = \varphi(\vartheta(x(t))) = \varphi(\vartheta(\oplus \mathcal{V}_\nu^* \pi_{j_\nu}(t) \mathcal{V}_\nu)) = \varphi(ftp) = \varphi(t),$$

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for all $t \in \oplus \mathcal{B}(H_j)$, which shows that $\varphi \in x_*(CS_1(\oplus M_{n_\nu}))$. \square

Definition 3.55. Let V be an operator space and let $Y = (Y_n)_n$ be a matrix subset of the unit ball of V (we will consider only this special situations). Then $\sum_{i=1}^{\infty} \alpha_i^* y_i \alpha_i$, where $y_i \in Y_{n_i}$ and $\alpha_i \in M_{n_i, n}$ for all $i \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} \alpha_i^* \alpha_i = \mathbb{1}_n$, converges in norm, and we write $\sigma\text{-mco}(Y)$ for the set of all such σ -matrix convex combinations. The σ -matrix convex hull of Y is the matrix set $\sigma\text{-mco}(Y) = (\sigma\text{-mco}_n(Y))_{n \in \mathbb{N}}$.

Proposition 3.56. Let $K = CS^\sigma(\mathcal{M})$ for an atomic W^* -algebra \mathcal{M} , and $X = \text{str}(K)$. Then $K = \sigma\text{-mco}(X)$.

Proof. It is well-known that the normal state space of the bounded operators on some Hilbert space is the σ -convex hull of the vector states. However, we have to show that $K = \sigma\text{-mco}(X)$. Since K_n is norm-closed for all $n \in \mathbb{N}$, it suffices to show that K_n is contained in $\sigma\text{-mco}_n(X)$. So, let $\psi \in K_n$. We identify $\mathcal{M} = \oplus \mathcal{B}(H_\varrho)$ and consider the C^* -algebra $\mathcal{A} = \oplus \mathcal{C}(H_\varrho)$. Then obviously

$$\mathcal{A}^{**} = (\oplus_\infty \mathcal{C}(H_\varrho))^{**} = (\oplus_1 \mathcal{T}(H_\varrho))^* = \oplus_\infty \mathcal{B}(H_\varrho) = \mathcal{M}$$

Thus the m -convex state space of \mathcal{A} is m -affine isomorphic to K . Notice that \mathcal{A} is a scattered C^* -algebra by [35, Thm. 2.2]. Thus given the minimal Stinespring representation $\psi = \mathcal{W}^* \tau \mathcal{W}$, where $\mathcal{W}: \mathbb{C}^n \rightarrow \mathcal{H}$ is a isometry and $\tau: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a (non-degenerated) representation of \mathcal{A} on the bounded operators on some Hilbert space \mathcal{H} , τ is unitarily equivalent to a subrepresentation of a sum of countably many irreducible representations of \mathcal{A} . Without loss of generality we can assume that $\tau(a) = U^* \pi(a) U$ for all $a \in \mathcal{A}$, where $U: \mathcal{H} \rightarrow H \subset \oplus H_{\varrho_l}$ is a unitary operator and H is an invariant subspace for $\pi = \oplus \pi_{\varrho_l}$, where π_{ϱ_l} is the irreducible representation of \mathcal{A} that is the restriction of the normal representation of $\mathcal{M} = \oplus \mathcal{B}(H_\varrho)$ onto the summand $\mathcal{B}(H_{\varrho_l})$. We obtain $\psi = \mathcal{W}^* \tau \mathcal{W} = \mathcal{W}^* U^* (\oplus \pi_{\varrho_l}) U \mathcal{W}$. Setting $\mathcal{V} = U \mathcal{W}$, we write $\mathcal{V} = (\mathcal{V}_{\varrho_l})_{l=1}^\infty$, where $\mathcal{V}_{\varrho_l}: \mathbb{C}^n \rightarrow H_{\varrho_l}$. Then

$$\psi = \mathcal{V}^* (\oplus \pi_{\varrho_l}) \mathcal{V} = \sum_{l=1}^{\infty} \mathcal{V}_{\varrho_l}^* \pi_{\varrho_l} \mathcal{V}_{\varrho_l}.$$

If $d_l = \dim(\mathcal{V}_{\varrho_l}(\mathbb{C}^n))$ should be less than n , i.e., if \mathcal{V}_{ϱ_l} should not be injective, we can factor out the kernel and replace \mathcal{V}_{ϱ_l} with an isometry $\mathcal{V}'_{\varrho_l}: \mathbb{C}^{d_l} \rightarrow H_{\varrho_l}$. Now $\phi'_l = \mathcal{V}_{\varrho_l}^* \pi_{\varrho_l} \mathcal{V}'_{\varrho_l}$ can be read as completely positive map from \mathcal{M} to M_{d_l} and the positive matrix $\alpha_l^2 = \phi'_l(\mathbb{1}) = \mathcal{V}_{\varrho_l}^* \mathcal{V}'_{\varrho_l} \in M_{d_l}$ is invertible. Then there is a unital and completely positive map $\phi_l: \mathcal{M} \rightarrow M_{d_l}$ such that $\phi'_l = \alpha_l \phi_l \alpha_l$, cf. [18], and $\phi_l = \alpha_l^{-1} \mathcal{V}_{\varrho_l}^* \pi_{\varrho_l} \mathcal{V}'_{\varrho_l} \alpha_l^{-1}$ is pure, since π_{ϱ_l} is irreducible. Thus $\phi_l \in X_{d_l}$, and we obtain $\psi = \sum_{l=1}^{\infty} \alpha_l \phi_l \alpha_l$, so that $\psi \in \sigma\text{-mco}_n(X)$ and the proof is complete. \square

We recall some definitions of real convexity theory. Let E be a real base norm space with base C . Then two convex subsets $B, D \subset C$ are *affinely independent*, if every point $x \in \text{conv}(B \cup D)$ is written as unique convex combination, i.e., whenever $\lambda b + (1 - \lambda)d = \nu b' + (1 - \nu)d'$ for $\lambda, \nu \in [0, 1]$, $b, b' \in B$ and $d, d' \in D$, then $\lambda = \nu$, $b = b'$ and $d = d'$.

Definition 3.57. Let E be a (real) vector space, and let $C \subset K$ be two convex subsets of E . An *affine retraction* of K onto C is an affine surjection $\psi: K \rightarrow C$, such that $\psi(c) = c$ for all $c \in C$.

We define a finite non-commutative simplex property for matrix convex sets that contain structural elements.

Definition 3.58. Let K be an m -convex set such that $X = \text{str}(K)$ is a non-empty, equivariant and transitive matrix set. Then K has the *finite m -simplex property*, if whenever $x_\nu \in X_{n_\nu}$ for $\nu = 1, \dots, l$ are finitely many pairwise matrix non-equivalent points, all of the following holds:

- (i) $C = \text{mco}(x_1, \dots, x_l)$ is m -affinely isomorphic to a finite m -convex simplex, and C_1 is a face of K_1 .
- (ii) There is a non-empty convex subset $C'_1 \subset K_1$ such that C_1 and C'_1 are affinely independent, and there is an affine retraction $\psi: K_1 \rightarrow \text{conv}(C_1 \cup C'_1)$.

We are now in a position to characterize abstractly the m -convex normal state space of atomic W^* -algebras, compare with [6, Theorem 10.2].

Theorem 3.59. *Let K be the m -base of a matrix base norm space. Then K is m -affinely isomorphic to the (m -convex) normal state space of an atomic W^* -algebra if and only if all of the following holds:*

- (i) *The σ -matrix convex hull of $\text{str}(K)$ equals K ,*
- (ii) *$\text{str}(K)$ is equivariant and transitive, and*
- (iii) *K has the finite m -simplex property.*

If in addition all elements of $\text{str}(K)$ are m -equivalent, then K is m -affinely homeomorphic to the normal state space of $\mathcal{B}(H)$ for some Hilbert space H .

Proof. If $(V, K)^* = \mathcal{M}$ is an atomic W^* -algebra, so that K is its normal state space, then $K = \sigma\text{-mco}(\text{str}(K))$ and $\text{str}(K)$ is equivariant and transitive by Proposition 3.56 and by Corollary 3.5, respectively. If $x_\nu \in \text{str}(K_{n_\nu})$ for $\nu = 1, \dots, \kappa$ are finitely many pairwise matrix non-equivalent points, then it follows from Proposition 3.54 that $C = \text{mco}(x_1, \dots, x_\kappa)$ is a finite m -convex simplex and that C_1 is a projective face of K_1 . Let $p \in \mathcal{M}$ be the projection corresponding to C_1 and let $C'_1 \subset K_1$ be the projective face corresponding to $p' = e - p$, where e is the unit of \mathcal{M} , so that C'_1 is the quasicomplementary face of C_1 . Then by definition C_1 and C'_1 are affinely independent, and there is a unique affine retraction ψ from K_1 onto $\text{conv}(C_1 \cup C'_1)$ given by $\langle a | \psi(x) \rangle = \langle pap + p'ap' | x \rangle$ for $a \in \mathcal{M}$, cf. [9, Thm. 11.5]. Thus K has the finite m -simplex property.

For the converse direction let (V, K) be an m -base norm space such that K fulfills the requirements (i) to (iii) of the theorem. Let $X = \text{str}(K)$. Since $K_1 = \sigma\text{-conv}(X_1)$ is a σ -convex base of the real base norm space V_h , the base norm is complete on V_h , cf. [40, Thm. 5.1]. Thus (V, K) is a complete m -base norm space. Moreover, X is a non-empty matrix set and the restriction map from $A_b(K)$ to $\mathcal{F}_b^E(X)$, i.e., restricting bounded m -affine maps on K to X , is injective. We have to prove that the finite simplex property of K implies that this restriction map is also surjective. Once this is proved, the restriction map is a complete order isomorphism between $A_b(K)$ and $\mathcal{F}_b^E(X)$. Since X is equivariant and transitive by condition (ii) and since structural elements fulfills the uniqueness property, cf. Remark 3.10, we know from Corollary 3.27 that $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra. By Proposition 1.42 we have $(V, K)^* = A_b(K)$, hence K is the normal state space of $\mathcal{F}_b^E(X)$ by Proposition 1.48, and the proof would be complete.

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In order to verify the surjectivity of the restriction map, let $f \in \mathcal{F}_b^E(X)$. The first step will be to extend f to a map in $A_b(\text{mco}(X))$. Let $C = \text{mco}(X)$. For $v \in C_n$ and any m -convex combination $v = \sum_{i=1}^m \alpha_i^* y_i \alpha_i$ by elements $y_i \in X_{n_i}$, we define $\tilde{f}_n(v) = \sum_{i=1}^m \alpha_i^* f_{n_i}(y_i) \alpha_i$. We have to verify that \tilde{f} is well-defined. Let $v = \sum_{j=1}^{m'} \beta_j^* y'_j \beta_j$ be another m -convex combination, where $y'_j \in X_{n'_j}$. Since finitely many equivalent elements have a common majorant, considering the set of equivalence classes

$$\{[y_i] \mid 1 \leq i \leq m\} \cup \{[y'_j] \mid 1 \leq j \leq m'\},$$

we find $l \leq m + m'$ pairwise matrix non-equivalent x_1, \dots, x_l in X such that

$$\{y_1, \dots, y_m, y'_1, \dots, y'_{m'}\} \subset \bigcup_{i=1}^l [x_i] \subset \text{mco}(x_1, \dots, x_l) = D$$

Notice that $\text{str}(D) = \cup_{i=1}^l [x_i]$, since X is equivariant. By the finite simplex property D is a finite m -convex simplex, so that $A_b(D)$ is completely order isomorphic to $\mathcal{F}_b^E(\text{str}(D))$ by Theorem 3.35 and Proposition 3.29. Thus $f|_{\text{str}(D)}$ has a unique extension to a map in $A_b(D)$, i.e., there is $g \in A_b(D)$, such that $g|_{\text{str}(D)} = f|_{\text{str}(D)}$. Therefore

$$\sum_{i=1}^m \alpha_i^* f_{n_i}(y_i) \alpha_i = g_n(v) = \sum_{j=1}^{m'} \beta_j^* f_{n'_j}(y'_j) \beta_j,$$

which shows that \tilde{f} is well-defined. Then by definition of \tilde{f} it is easy to see that $\tilde{f} \in A_b(C)$, and that $\tilde{f}|_X = f$. Thus we have shown so far that $\mathcal{F}_b^E(X) =_{cp} A_b(C)$ (recalling that $C = \text{mco}(X)$). Let $W = \text{lin } C_1$. Then C is an m -convex subset of W , and C_1 is a base of the (generating) cone $W_+ = \mathbb{R}_+ C_1$, because K_1 is a base of V_+ . Hence given $f = (f_n) \in A_b(C)$ we can apply Lemma 1.29 to find a unique linear extension $g: W \rightarrow \mathbb{C}$, such that $f_n = g^{(n)}|_{C_n}$ for all $n \in \mathbb{N}$. It is essential now to prove that g is bounded with respect to the matrix base norm on V . Although (W, C) is an m -base norm space, the norm defined by $\text{amco}(C)$ may not be equivalent to the norm on V defined by the larger unit ball $\text{amco}(K)$. So let $w \in W_h \cap \text{conv}(K_1 \cup -K_1)$. Then $w \in \text{lin}_{\mathbb{R}} C_1$ and $C_1 = \text{conv}(X_1)$. Thus there is a finite subset $F \subset X_1$ such that $w \in \text{lin}_{\mathbb{R}} F$. Since F is finite, it is contained in some finite simplex $D = \text{mco}(x_1, \dots, x_l)$, i.e., $F \subset D_1$, where x_1, \dots, x_l are pairwise matrix non-equivalent elements of X . By (the second part of) the finite simplex property, there is a convex subset $D'_1 \subset K_1$, such that D_1 and D'_1 are affinely independent, and there is an affine retraction $\psi: K_1 \rightarrow \text{conv}(D_1 \cup D'_1)$. By [9, Prop.3.3] there is a positive projection $P: V_h \rightarrow E = \text{lin}_{\mathbb{R}} \text{conv}(D_1 \cup D'_1)$ such that $P(K_1) \subset K_1$ and $\psi = P|_{K_1}$. Furthermore, since D_1 and D'_1 are affinely independent, E splits into a direct ordered sum of $E_1 = \text{lin}_{\mathbb{R}} D_1$ and $E'_1 = \text{lin}_{\mathbb{R}} D'_1$. This means there are positive projections $\pi: E \rightarrow E_1$ and $\pi': E \rightarrow E'_1$. Notice that $E_1 \subset W$, because $D \subset C$. Now, $w \in \text{lin}_{\mathbb{R}} F \subset \text{lin}_{\mathbb{R}} D = E_1$, and $w \in \text{conv}(K_1 \cup -K_1)$, i.e., there are $x, y \in K_1$ and $\lambda \in [0, 1]$, such that $w = \lambda x - (1 - \lambda)y$. Hence we obtain

$$\begin{aligned} w &= Pw = \lambda Px - (1 - \lambda)Py \\ &= \lambda \pi Px - (1 - \lambda) \pi Py + \pi' Px - (1 - \lambda) \pi' Py \\ &= \lambda \pi Px - (1 - \lambda) \pi Py. \end{aligned} \tag{3.15}$$

From $Px, Py \in K_1 \cap E$ it follows that $a = \pi Px(e)$, $b = \pi Py(e) \in [0, 1]$. Hence $a^{-1}\pi Px$, $b^{-1}\pi Py \in K_1 \cap E_1$, where we suppose $a, b \neq 0$. Since by assumption D_1 is a face of K_1 , we know that $K_1 \cap E_1 \subset D_1$. Hence equation (3.15) yields

$$\begin{aligned} |g(w)| &= |\lambda af(a^{-1}\pi Px) - (1 - \lambda)bf(b^{-1}\pi Py)| \\ &\leq (\lambda a + (1 - \lambda)b)\|f\| \leq \|f\|, \end{aligned}$$

for all $w \in W_h$ with $\|w\|_V \leq 1$. So we proved that $g: W \rightarrow \mathbb{C}$ is a bounded linear map with respect to the base norm of V . Then there is a unique bounded linear map $\tilde{g}: W^\circ \rightarrow \mathbb{C}$ (recall that V is complete) such that $g = \tilde{g}|_W$. Since $K_1 = \sigma\text{-conv}(X_1) \subset \text{conv}(X_1)^\circ$, it follows that $V = \text{lin } K_1 \subset \text{lin conv}(X_1)^\circ = W^\circ \subset V$. Thus we have extended g on V and the restriction $\tilde{g}|_{K_1}$ is a bounded affine map that extends f . This shows that the restriction map $A_b(K) \rightarrow \mathcal{F}_b^E(X)$ is surjective. Moreover, since by construction $\tilde{g}^{(n)}(\sum_{i=1}^\infty \alpha_i^* x_i \alpha_i) = \sum_{i=1}^\infty \alpha_i^* f_{n_i}(x_i) \alpha_i$, it is not hard to prove that $A_b(K) =_{cp} \mathcal{F}_b^E(X)$. As stated already this shows that K is m -affine isomorph to the normal state space of the atomic W^* -algebra $\mathcal{F}_b^E(X)$.

For the last statement of the theorem, recall from (the proof of) Corollary 3.5 that the m -relation coincides with the equivalence of pure states. So, obviously all elements of $\text{str}(K)$ (i.e., the pure m -states, cf. Corollary 1.20) are m -equivalent if and only if the pure m -states are unitarily equivalent, i.e., if and only if there is only one unitary equivalence class of irreducible representations for the atomic W^* -algebra, in which case it must be a single $\mathcal{B}(H)$. \square

Projections and certain sets of pure states

In order to characterize m -state spaces of C^* -algebras abstractly as certain compact matrix convex sets, we should pose conditions, or axioms, only on the structures that we start with. That means we can pose conditions on the m -relation or the inner metric of the structural elements. (Later we will also consider a uniformity on the structural elements.) However, starting with X we should not talk about say the projections of $\mathcal{F}_b^E(X)$, even though we have proved that $\mathcal{F}_b^E(X)$ is a W^* -algebra. So we show next that, given an equivariant and transitive matrix set X fulfilling the uniqueness property, we can identify projections of the atomic W^* -algebra $\mathcal{F}_b^E(X)$ abstractly with certain matrix subsets of X that we define now.

Definition 3.60. Let X be an equivariant and transitive matrix subset of some vector space V , such that X fulfills the uniqueness property. Then an equivariant subset Y of X is *equivariantly directed*, if Y satisfies the following conditions:

- (i) Y_1 is a closed set with respect to the inner metric d on X_1 , and
- (ii) for any finite subset $\{y_1, \dots, y_m\}$ of pairwise matrix equivalent elements of Y_1 there is $n \in \mathbb{N}$ and $y \in Y_n$ such that $y \succ y_i$ for $i = 1, \dots, m$.

Proposition 3.61. Let X be a matrix subset of some vector space V such that X is equivariant, transitive and fulfills the uniqueness property. If a subset $Y \subset X$ is equivariantly directed then there is a projection $p \in \mathcal{F}_b^E(X)$ such that $Y_n = \{x \in X_n \mid p_n(x) = 0\}$ for all $n \in \mathbb{N}$. Conversely, if in addition (X_1, d) is complete, where d is the inner metric of X , then the matrix set $Y = (Y_n)_n$, where $Y_n = \{x \in X_n \mid p_n(x) = 0\}$, is non-empty for all projections $p \in \mathcal{F}_b^E(X)$ with $p \neq 1$, and Y is equivariantly directed. Moreover,

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in this case the correspondence between equivariantly directed sets and projections is a bijection, that maps p' to Y^\perp (where p corresponds with Y).

Proof. We identify $\mathcal{F}_b^E(X) = \oplus_{\varrho \in \mathfrak{K}} \mathcal{B}(H_\varrho)$ for $\mathfrak{K} = \{[x] \mid x \in X_1\}$, cf. Corollary 3.27. Let $Y \subset X$ be equivariantly directed. For all $\varrho \in \mathfrak{K}$ we define

$$\mathcal{H}_\varrho = \text{lin} \left\{ \xi \in H_\varrho \mid \|\xi\| = 1 \text{ and } \xi^* \pi_\varrho \xi \in \widehat{Y}_1 \right\}. \quad (3.16)$$

Let \mathcal{H}_ϱ^- be the norm closure of \mathcal{H}_ϱ . Let $p_\varrho \in \mathcal{B}(H_\varrho)$ be the unique projector such that $\text{kern}(p_\varrho) = \mathcal{H}_\varrho^-$ and define $p = \oplus p_\varrho$. Recall the identification $X = \widehat{X} \subset \text{str}(CS^\sigma(\mathcal{F}_b^E(X)))$, cf. Proposition 3.44. Given $y = [y_{ij}] \in Y_n$ there is an isometry $\mathcal{V}: \mathbb{C}^n \rightarrow H_\kappa$ such that $\hat{y} = \mathcal{V}^* \pi_\kappa \mathcal{V}$, where $\kappa = [y] = [y_{11}] \in \mathfrak{K}$ and $\pi_\kappa: \mathcal{F}_b^E(X) \rightarrow \mathcal{B}(H_\kappa)$ is the normal irreducible representation onto the summand $\mathcal{B}(H_\kappa)$. Then $\varepsilon_i^* \mathcal{V}^* \pi_\kappa \mathcal{V} \varepsilon_i = \hat{y}_{ii} \in \widehat{Y}_1$ implies $\xi_i = \mathcal{V} \varepsilon_i \in \mathcal{H}_\kappa \subset \mathcal{H}_\kappa^-$ for all $i = 1, \dots, n$, where $\{\varepsilon_i \mid 1 \leq i \leq n\}$ is the standard basis of \mathbb{C}^n . Hence, reading p as element $(p_l)_{l \in \mathbb{N}}$ of $\mathcal{F}_b^E(X)$, we obtain

$$p_n(y) = \hat{y}(p) = \mathcal{V}^* \pi_\kappa(p) \mathcal{V} = [\langle p_\kappa \xi_j | \xi_i \rangle] = 0,$$

which proves that $Y_n \subset \{x \in X_n \mid p_n(x) = 0\}$ for all $n \in \mathbb{N}$. On the other hand, let $x \in X_n$ such that $p_n(x) = 0$. Again, there is an index $\kappa \in \mathfrak{K}$ and an isometry $\mathcal{V}: \mathbb{C}^n \rightarrow H_\kappa$ such that $\hat{x} = \mathcal{V}^* \pi_\kappa \mathcal{V}$. We let $\xi_i = \mathcal{V} \varepsilon_i$ for $i = 1, \dots, n$ and obtain

$$0 = p_n(x) = \hat{x}(p) = \mathcal{V}^* \pi_\kappa(p) \mathcal{V} = [\langle p_\kappa \xi_j | \xi_i \rangle],$$

which implies $p_\kappa \xi_i = 0$. Thus $\xi_i \in \text{kern}(p_\kappa) = \mathcal{H}_\kappa^-$ for all $i = 1, \dots, n$. Consequently, for each $i \in \{1, \dots, n\}$, there are sequences of unit vectors $(\xi_{i,\nu})_{\nu \in \mathbb{N}}$ in \mathcal{H}_κ , such that $\xi_{i,\nu} \rightarrow \xi_i$. Let $\hat{y}_i' = \xi_{i,\nu}^* \pi_\kappa \xi_{i,\nu} \in \widehat{Y}_1$ for $i = 1, \dots, n$. From $\xi_{i,\nu} \rightarrow \xi_i$ it follows that $\hat{y}_i' \rightarrow \hat{x}_{ii}$ in the norm of $\mathcal{F}_b^E(X)^*$. Since X and \widehat{X} are equivariantly isomorph, it is obvious from the definition of the inner metric that the inner metric of X equals the inner metric of \widehat{X} . Therefore using Lemma 3.50 we have $d(y_i', x_{ii}) = d(\hat{y}_i', \hat{x}_{ii}) = \|\hat{y}_i' - \hat{x}_{ii}\| \rightarrow 0$. Since Y_1 is closed with respect to the inner metric, we get $x_{ii} \in Y_1$ for $i = 1, \dots, n$. Then, using that Y is equivariantly directed, there is some $m \in \mathbb{N}$ and $y \in Y_m$ such that $y \succcurlyeq x_{ii}$ for $i = 1, \dots, n$. Hence by Lemma 3.52 we see that $y \succcurlyeq x$, which shows directly that $x \in Y_n$. Thus we have shown so far that $Y_n = \{x \in X_n \mid p_n(x) = 0\}$ for all $n \in \mathbb{N}$, which proves the first claim.

Now, for the converse direction, suppose additionally that X_1 is complete in the inner metric, so that by (the proof of) Theorem 3.53 we have $\widehat{X} = CS^\sigma(\mathcal{F}_b^E(X))$. Then given a projection $p = (p_l) \in \mathcal{F}_b^E(X)$ the matrix set $Y = (Y_n)_{n \in \mathbb{N}}$, where $Y_n = \{x \in X_n \mid p_n(x) = 0\}$ obviously equivariant. Moreover, if $p \neq \mathbb{1}$ then Y_1 is non-empty. Indeed, $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra, so there is a minimal projection (i.e., an atom) $q \in \mathcal{F}_b^E(X)$ such that $0 \neq q \leq \mathbb{1} - p$. There is a unique pure state, i.e., an element $x \in X_1$, such that $q_1(x_1) = 1$, which shows $x \in Y_1$. We claim that Y_1 is closed in the inner metric d . Let $z \in X_1$ and let $(z_n)_{n \in \mathbb{N}}$ be a sequence in Y_1 such that $d(z_n, z) \rightarrow 0$. Then from Lemma 3.50 we have $\|\hat{z}_n - \hat{z}\| = d(\hat{z}_n, \hat{z}) = d(z_n, z) \rightarrow 0$, so especially $0 = p_1(z_n) = \hat{z}_n(p) \rightarrow \hat{z}(p) = p_1(z)$, which yields $p_1(z) = 0$. Thus $z \in Y_1$, which shows that Y_1 is closed in the inner metric. To show that Y is directed, let $y_1, \dots, y_m \in Y_1$ be pairwise equivalent. Then with $\kappa = [y_1] = \dots = [y_m] \in \mathfrak{K}$ there are unit vectors $\xi_i \in H_\kappa$ such that $\hat{y}_i(T) = \xi_i^* \pi_\kappa(T) \xi_i$ for $T \in \oplus \mathcal{B}(H_\varrho)$ and $i = 1, \dots, m$. From

$$0 = p_1(y_i) = \hat{y}_i(p) = \xi_i^* \pi_\kappa(p) \xi_i = \langle p_\kappa \xi_i | \xi_i \rangle$$

we see that the linear hull $\mathcal{H} = \text{lin}\{\xi_1, \dots, \xi_m\}$ is contained in $\text{kern}(p_\kappa)$. Let $\{\eta_1, \dots, \eta_n\}$ be some orthonormal basis of \mathcal{H} . We can define an isometry $\mathcal{V}: \mathbb{C}^n \rightarrow \mathcal{H}$ by $\mathcal{V}\varepsilon_i = \eta_i$ for $i = 1, \dots, n$, where $\{\varepsilon_i \mid i = 1, \dots, n\}$ denotes the standard basis of \mathbb{C}^n . Then $\psi = \mathcal{V}^* \pi_\kappa \mathcal{V}$ is a normal pure matrix state of \mathcal{M} , so that by assumption there is $y \in X_n$ such that $\psi = \hat{y}$. Moreover

$$p_n(y) = \hat{y}(p) = \mathcal{V}^* \pi_\kappa(p) \mathcal{V} = [\langle p_\kappa \eta_j | \eta_i \rangle] = 0,$$

so that $y \in Y_n$. Obviously, $\xi_i \in \mathcal{H} = \mathcal{V}(\mathbb{C}^n)$, so there is $\alpha_i \in \mathbb{C}^n = M_{n,1}$, such that $\xi_i = \mathcal{V}\alpha_i$ and $\alpha_i^* \alpha_i = 1$ for $i = 1, \dots, n$. We obtain

$$\alpha_i^* \hat{y}(T) \alpha_i = \alpha_i^* \hat{y}(T) \alpha_i = \alpha_i^* \mathcal{V}^* \pi_\kappa(T) \mathcal{V} \alpha_i = \xi_i^* \pi_\kappa(T) \xi_i = \hat{y}_i(T),$$

for all $T \in \oplus \mathcal{B}(H_\varrho)$. This shows that $y \succ y_i$ for $i = 1, \dots, n$, so that Y is equivariantly directed.

Finally, if $x \in Y^\perp$ and $y \in Y$, then by definition of \perp there is $z \in X$ such that $z = \begin{pmatrix} x & * \\ * & y \end{pmatrix}$ or we have $x \not\prec y$. In case $[x] \cap Y^\perp = \emptyset$, we see that $\mathcal{H}_{[x]}$ in equation (3.16) is empty, so p is the identity projection on $[x]$. Then $p'(x) = 0$. Therefore, we can assume that given $x \in Y_l^\perp$ there is $y \in Y_n$ such that $x \smile y$, so that there is $z = \begin{pmatrix} x & * \\ * & y \end{pmatrix} \in X_{n+l}$. Then there is a finite orthonormal system (ξ_i) in $H_{[z]}$ such that $\hat{z} = [\xi_i \odot \xi_j]$ by Proposition 3.43. This implies $\hat{x} = [\xi_i \odot \xi_j]_{i,j=1}^l$ and $\hat{y} = [\xi_i \odot \xi_j]_{i,j=l+1}^{n+l}$. Now it follows from $p(y) = 0$ and the orthogonality of (ξ_i) that $p'(x) = 0$. Conversely, let $x \in X$ such that $p'(x) = 0$. Given $y \in Y$ we can assume $x \smile y$, otherwise there is nothing to prove. Then $[x] = [y]$ and there are orthonormal systems (ξ_i) and (η_i) in $H_{[x]}$ such that $\hat{x} = [\xi_i \odot \xi_j]$ and $\hat{y} = [\eta_i \odot \eta_j]$. Then it follows from $p(x) = 1$ and $p(y) = 0$ that $\xi_i \perp \eta_j$ for all i and j . Hence, there is $z \in X$ such that $z = \begin{pmatrix} x & * \\ * & y \end{pmatrix}$ (where \hat{z} corresponds to the properly ordered orthonormal system that is the union of (ξ_i) and (η_j)). This shows $Y^\perp = \{x \in X \mid p'(x) = 0\}$, and the proof is complete. \square

Facial 3-balls

For the proof of our abstract characterization of matrix convex state spaces of C^* -algebras we need the main concept of [6, 5, 7], namely the so-called *global orientation*. Since we cannot repeat the theory of Alfsen and Shultz, the reader is advised to read at least [6, p. 403ff] for the exact definitions of *facial 3-balls*, *parametrizations*, *global orientation* and related concepts. (The reader has to do so in particular to understand the proofs of Theorem 3.83 and Theorem 3.87, where we use those concepts without defining them here.) To help the reader we adopt the notation of [6]. In case [6] is not at hand, the required definitions also appeared in [7]. The purpose of the next propositions is to link parametrizations and orientations with our matrix theory. We begin with the following observation.

Remark 3.62. By [5, Thm. 4.4] the state space of M_2 is affine isomorph to the closed unit ball of \mathbb{R}^3 , which we denote by \mathbf{B}^3 . This affine isomorphism is given by the order isomorphism (3.12), namely to each $\rho \in CS_1(M_2)$ there corresponds a positive trace class operator $r \in M_2$ with $\text{trace}(r) = 1$. So, r is given by

$$r = \frac{1}{2} \begin{pmatrix} 1 + r_1 & r_2 + ir_3 \\ r_2 - ir_3 & 1 - r_1 \end{pmatrix},$$

where $\det(r) \geq 0$, and the affine isomorphism is given by $(r_1, r_2, r_3) \mapsto \rho$. Notice that we silently identify \mathbf{B}^3 with the state space $CS_1(M_2)$ of M_2 in the sequel.

3. Matrix Convex Simplexes

Remark 3.63. Seeing M_2 as operator system, the operator dual $(M_2)^*$ is an m-base norm space with m-base $CS(M_2)$. Especially, $CS_1(M_2)$ is a base of the positive cone of $(M_2)^*$, so by Lemma 1.28 each affine map on $CS_1(M_2)$ has a unique extension to a linear map on $(M_2)^*$.

Proposition 3.64. *Let K be the m-convex state space of an atomic W^* -algebra \mathcal{M} and let $X = \text{str}(K)$. If $y \succ x_1, x_2$ and $z \succ x_1, x_2$ for $y, z \in X_2$ and distinct $x_1, x_2 \in X_1$, then z is unitarily equivalent to y . Moreover, $\text{mco}_1(y)$ is the smallest face of K_1 containing x_1 and x_2 .*

Proof. We identify $\mathcal{M} = \oplus \mathcal{B}(H_\theta)$. Since y is a normal pure matrix state of \mathcal{M} , the minimal Stinespring representation of y is $y = \mathcal{V}^* \pi \mathcal{V}$, where $\pi: \oplus \mathcal{B}(H_\theta) \rightarrow \mathcal{B}(H_\kappa)$ is the normal and irreducible representation onto a summand and $\mathcal{V}: \mathbb{C}^2 \rightarrow H_\kappa$ is an isometry. By Corollary 3.5 the m-relation coincides with the (unitarily) equivalence of pure states. So from $y \succ x_1, x_2$ and $z \succ x_1, x_2$ it follows that the pure m-states y and z are equivalent. Hence we can write the minimal Stinespring representation for z as $z = \mathcal{W}^* \pi \mathcal{W}$ for some isometry $\mathcal{W}: \mathbb{C}^2 \rightarrow H_\kappa$ and moreover, since $x_1 \neq x_2$, we obtain $\mathcal{W}(\mathbb{C}^2) = \mathcal{V}(\mathbb{C}^2) = L$. Let $\xi_1, \xi_2 \in L$ be orthonormal. Then there are orthonormal $\eta_1, \eta_2 \in \mathbb{C}^2$ and orthonormal $\zeta_1, \zeta_2 \in \mathbb{C}^2$, such that $\xi_j = \mathcal{V}\eta_j = \mathcal{W}\zeta_j$ for $j = 1, 2$. There is a unique unitary $u: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by $u\eta_j = \zeta_j$ for $j = 1, 2$. Obviously, $y = u^*zu$. Moreover, by Proposition 3.54 $\text{mco}(y)$ is affine isomorph with the m-convex state space of M_2 , and $\text{mco}_1(y)$ is a face of K_1 . Let $\text{face}(x_1, x_2)$ denote the smallest face of K_1 containing x_1 and x_2 . It follows $\text{face}(x_1, x_2) \subset \text{mco}_1(y)$. It follows from Remark 3.62 that x_1 and x_2 can be identified with two distinct extreme points in \mathbf{B}^3 . The smallest face in \mathbf{B}^3 containing two distinct extreme points is all of \mathbf{B}^3 . It follows that $\text{face}(x_1, x_2) = \text{mco}_1(y)$ completing the proof. \square

Remark 3.65. Let $\mathcal{M} = \oplus_{j \in J} \mathcal{B}(H_j)$ be an atomic W^* -algebra with m-convex normal state space K and $X = \text{str}(K)$. Notice that if $x, y \in X_1$ are matrix non-equivalent (and hence non-equivalent by Corollary 3.5) pure states of \mathcal{M} , then $\text{face}(x, y)$, the smallest face of K_1 containing x and y , is the (one dimensional) line segment $[x, y]$. This follows from [5, Prop. 1.30] observing that the normal state space of $\mathcal{B}(H_j)$ can be identified with a split face $F_j \subset K_1$ for all $j \in J$, and if x, y are non-equivalent then there is $j \in J$ such that $x \in F_j$ and $y \notin F_j$. Consequently, if we have $x, y \in X_1$ such that $\text{face}(x, y)$ is affinely isomorph to \mathbf{B}^3 , then $\text{face}(x, y) \neq [x, y]$, which implies that x and y must be (matrix) equivalent.

Proposition 3.66. *Let S be the normal m-convex state space of an atomic W^* -algebra \mathcal{M} and let $X = \text{str}(S)$. Then for an affine isomorphism ψ from \mathbf{B}^3 to a face $F \subset S_1$ there are exactly two possibilities: Either there is $y \in X_2$ such that $\psi = y^*$, or there is $z \in X_2^{\text{tr}}$ such that $\psi = z^*$.*

Proof. To show this claim, let F be a face of S_1 , and let $\psi: \mathbf{B}^3 \rightarrow F$ be an affine isomorphism. Then choose two distinct elements $b_1, b_2 \in \text{ex}(\mathbf{B}^3)$, the set of extreme points of \mathbf{B}^3 . By assumption F is a face, hence $x_1 = \psi(b_1)$ and $x_2 = \psi(b_2)$ are two distinct elements of X_1 , which are the extreme points of S_1 . Notice that $F = \text{face}(x_1, x_2)$, since $\text{face}(b_1, b_2) = \mathbf{B}^3$. From Remark 3.65 we have $x_1 \sim x_2$, so by Lemma 3.47 there is $x \in X_2$ such that $x \succ x_1, x_2$ and Proposition 3.64 yields $\text{mco}_1(x) = F$. By abuse of notation we let x^* denote the restriction of the dual of x to the state space $CS_1(M_2)$ of M_2 . Then x^* is an affine isomorphism from $CS_1(M_2)$ onto F , and $(x^*)^{-1} \circ \psi$ is an affine isomorphism from the state space $CS_1(M_2)$ onto itself. Hence it must be the dual of a

unital order automorphism $\phi: M_2 \rightarrow M_2$. From [5, Thm.4.35] we know that there are exactly two possibilities for ϕ . Either ϕ is a $*$ -isomorphism, in which case the dual map $\phi^* = (x^*)^{-1} \circ \psi$ is an orientation preserving affine automorphism of $CS_1(M_2)$, or ϕ is a $*$ -anti-isomorphism, in which case ϕ^* is an orientation reversing affine automorphism of $CS_1(M_2)$. From [5, Thm.4.34] we obtain a concrete representation of ϕ for both cases. In case ϕ is a $*$ -isomorphism there is a unitary $u \in M_2$ such that $\phi(\gamma) = u^* \gamma u$ for all $\gamma \in M_2$, and in case ϕ is a $*$ -anti-isomorphism there is a unitary $v \in M_2$ such that $\phi(\gamma) = v^* \gamma^{\text{tr}} v$ for all $\gamma \in M_2$, where γ^{tr} is the transpose of γ . Consequently we obtain $\psi = x^* \circ \phi^* = (\phi \circ x)^* = y^*$, where

$$y = \phi \circ x = \begin{cases} u^* x u \in X_2 & \text{in case } \phi \text{ is a } * \text{-isomorphism} \\ v^* x^{\text{tr}} v \in X_2^{\text{tr}} & \text{in case } \phi \text{ is a } * \text{-anti-isomorphism,} \end{cases}$$

which shows the claim. \square

Remark 3.67. An affine isomorphism from \mathbf{B}^3 to a face of a convex set C is called a parametrization in [6, 7]. A face affine isomorph to \mathbf{B}^3 is a facial 3-ball. The set of all parametrizations of a facial 3-ball is divided into two equivalence classes (depending on the choice of base for \mathbb{R}^3) called orientation. A global orientation of C is a choice of orientation for each facial 3-ball of C . The content of Proposition 3.66 is that if S is the normal (matrix) state space of an atomic W^* -algebra, then there is a correspondence between the two classes of orientation of facial 3-balls and elements of X_2 and X_2^{tr} . In the sequel we give each facial 3-ball of S_1 the orientation corresponding to elements of X_2 . This choice is called the *canonical global orientation* of S_1 .

State spaces of C^* -algebras

We aim at characterizing those compact and m -convex sets that are the state spaces of C^* -algebras. As mentioned in the introduction to the chapter, the state spaces of unital and *commutative* C^* -algebras are exactly the Bauer simplexes. Let \mathcal{A} be a unital and commutative C^* -algebra with m -convex state space $K = CS(\mathcal{A})$ and $X = \text{str}(K)$. Notice that $X_n = \emptyset$ for $n \geq 2$. The fact that the state space K_1 of \mathcal{A} is a Bauer simplex means that the restriction map from $\mathcal{A} = A(K_1)$ to $C(X_1)$ is surjective. (Notice that restricting continuous affine maps on a compact convex set to the extreme points is always injective, which follows from the Krein-Milman Theorem.) As a matter of fact one can define a Bauer simplex as a compact convex set such that every (uniformly) continuous map on the extreme points has a *continuous* affine extension to all of the convex set. This implies that the set of extreme points is closed and hence compact, cf. [4, Theorem II 4.3].

Now assume that \mathcal{A} is a non-commutative C^* -algebra. Then \mathcal{A} has a (non-trivial) matrix ordering and we can identify $\mathcal{A} = {}_{cp} A(K)$, where $A(K_1) = A(K)$ order isomorphically, cf. Remark 1.24. We know already from the previous sections, that the matrix set of the pure states of a C^* -algebra is equivariant (and transitive). So when restricting *matrix affine* maps on K to the structural elements X they naturally stay equivariant. Moreover, they stay continuous not only on X but also on the closure of X , which is compact. Hence these restrictions are uniformly continuous on X . So, we have a restriction map from $A(K)$ to $\mathcal{C}_u^E(X)$, which by the Krein-Milman Theorem (or the matrix convex version of it) is injective. The question is: Is the restriction surjective just like in the commutative case? Or in other words, do the state spaces of C^* -algebras fulfill

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a non-commutative simplex property, and is such a property already characterizing the m-convex state spaces of C^* -algebras among compact m-convex sets? The results for finite dimensional C^* -algebras look promising, cf. Proposition 3.29 and Theorem 3.35.

We prove first that the restriction map above is surjective for arbitrary (unital) C^* -algebras. The following preparing results are formulated also for non-unital C^* -algebras.

Remark 3.68. Let \mathcal{A} be a C^* -algebra. Then the positive part of the open unit ball, which we denote by (e_λ) , is an approximative order unit, cf. Remark A.8. We will call (e_λ) the *canonical* approximative order unit of \mathcal{A} . From Proposition 1.19 we know that $\text{str}(CQ_n(\mathcal{A}))$ consists for $n > 1$ exactly of the pure maps of $CQ_n(\mathcal{A})$ that are approximately unital. For $n = 1$ the set $\text{str}(CQ_1(\mathcal{A}))$ is the set of the extreme points of the quasi states of \mathcal{A} . These are the pure quasi states that are approximately unital together with the zero map. If \mathcal{A} has a unit, we see easily that positive and approximately unital maps with norm less or equal 1 are unital maps. Hence, if \mathcal{A} has a unit, then $\text{str}(CQ(\mathcal{A})) \setminus \{0\} = \text{str}(CS(\mathcal{A}))$.

Lemma 3.69. *Let $X = \text{str}(CQ(\mathcal{A})) \setminus \{0\}$ for a C^* -algebra \mathcal{A} . Let $\pi: \mathcal{A} \rightarrow \mathcal{B}(H_\pi)$ be an irreducible and approximately unital (cf. Remark A.10) representation of \mathcal{A} , and let $f \in \mathcal{F}_b^E(X)$. Then the map $h: H_\pi \rightarrow \mathbb{C}$ defined by $h(0) = 0$ and $h(\xi) = \|\xi\|^2 f_1(\xi_1^* \pi \xi_1)$, where $\xi_1 = \xi/\|\xi\|$ and $\xi \in H_\pi$, $\xi \neq 0$, is a bounded quadratic form on H_π .*

Proof. Obviously, h is a well-defined and bounded map, since f_1 is bounded. We have to prove that h is a quadratic form. Let ξ and η be vectors of H_π . They are contained in a subspace $L \subset H_\pi$ of dimension 2. Let $\{e_1, e_2\} \subset L$ be an orthonormal basis and define a unitary operator $u: \mathbb{C}^2 \rightarrow L$ by $\mu e_1 + \nu e_2 \mapsto \mu e_1 + \nu e_2$, where $\{\varepsilon_1, \varepsilon_2\}$ denotes the standard basis of \mathbb{C}^2 . Then for arbitrary $\rho = \mu e_1 + \nu e_2 \in L$, $\rho \neq 0$

$$h(\rho) = \|\rho\|^2 f_1(\rho_1^* \pi \rho_1) = \|\rho\|^2 f_1(v_0^* u^* \pi u v_0) = v^* f_2(u^* \pi u) v = u^*(\rho)^* f_2(u^* \pi u) u^*(\rho),$$

where $v = u^*(\rho)$, $v_0 = v/\|v\|$ and $\rho_1 = \rho/\|\rho\|$. Since $h(0) = 0$ by definition, the equation holds for all $\rho \in H_\pi$. The calculation

$$\begin{aligned} h(\xi + \eta) + h(\xi - \eta) &= u^*(\xi + \eta)^* f_2(u^* \pi u) u^*(\xi + \eta) + u^*(\xi - \eta)^* f_2(u^* \pi u) u^*(\xi - \eta) \\ &= 2u^*(\xi)^* f_2(u^* \pi u) u^*(\xi) + 2u^*(\eta)^* f_2(u^* \pi u) u^*(\eta) \\ &= 2(h(\xi) + h(\eta)) \end{aligned}$$

shows that h is a quadratic form. □

Let \mathcal{A} be a C^* -algebra and let $X = \text{str}(CQ(\mathcal{A})) \setminus \{0\}$. For each pure m-state $x \in X$, we let $\pi_{[x]}: \mathcal{A} \rightarrow H_{[x]}$ be a representative of the unitary equivalence class of irreducible representations of \mathcal{A} corresponding to the m-equivalence class $[x]$, cf. Proposition 3.4. It is known, that $\bigoplus_{\varrho \in \mathfrak{K}} \mathcal{B}(H_\varrho)$, where the sum runs over $\mathfrak{K} = \{[x] \mid x \in X_n, n \in \mathbb{N}\}$, is the atomic part of the bidual \mathcal{A}^{**} of \mathcal{A} . Moreover, we have:

Proposition 3.70. *Let \mathcal{A} be a C^* -algebra and let $X = \text{str}(CQ(\mathcal{A})) \setminus \{0\}$. Then*

$$\mathcal{F}_b^E(X) =_{cp} \bigoplus_{\varrho \in \mathfrak{K}} \mathcal{B}(H_\varrho). \quad (3.17)$$

Proof. For $n \in \mathbb{N}$ we consider $x \in X_n$. The map $x: \mathcal{A} \rightarrow M_n$ is completely positive and bounded, so by Theorem A.9 there exist a Hilbert space H_π , an approximately unital representation $\pi: \mathcal{A} \rightarrow \mathcal{B}(H_\pi)$ and a bounded operator $\mathcal{V}_x: \mathbb{C}^n \rightarrow H_\pi$, such that $x = \mathcal{V}_x^* \pi \mathcal{V}_x$ is the minimal (non-unital) Stinespring representation of x . Since x is pure, π is irreducible by Theorem A.11. Since x is approximately unital, \mathcal{V}_x is an isometry. The minimal Stinespring representation is unique up to unitary isomorphisms. If for each element of the equivalence class $[x] \in \mathfrak{K}$ we choose and fix the single irreducible representation $\pi_{[x]}: \mathcal{A} \rightarrow H_{[x]}$ for the Stinespring representation, then the respective isometry \mathcal{V}_x is unique up to a complex factor of modulus 1. Therefore, we can define a map

$$\Gamma: \bigoplus_{\varrho \in \mathfrak{K}} \mathcal{B}(H_\varrho) \rightarrow \mathcal{F}_b^E(X) \text{ by } T = \bigoplus T_\varrho \mapsto \Gamma(T) = (f_n^T)_n,$$

where $f_n^T(x) = \mathcal{V}_x^* T_{[x]} \mathcal{V}_x$ for all $T = \bigoplus T_\varrho \in \bigoplus \mathcal{B}(H_\varrho)$, $x \in X_n$ and $n \in \mathbb{N}$. Notice that $\|f_n^T(x)\| \leq \|T_{[x]}\| \leq \|T\|$ for all $x \in X_n$ and $n \in \mathbb{N}$, so f^T is bounded. Moreover, if $x = u_{xy}^* y u_{xy}$ for $x \in X_n$ and $y \in X_m$, where $u_{xy} \in M_{m,n}$ is the essentially unique isometry that transforms y into x (see Remark 3.11), then $\mathcal{V}_x^* \pi_{[x]} \mathcal{V}_x = u_{xy}^* \mathcal{V}_y^* \pi_{[y]} \mathcal{V}_y u_{xy}$, which implies $\mathcal{V}_x = e^{i\varphi} \mathcal{V}_y u_{xy}$. Hence

$$f_n^T(x) = \mathcal{V}_x^* T_{[x]} \mathcal{V}_x = u_{xy}^* \mathcal{V}_y^* T_{[y]} \mathcal{V}_y u_{xy} = u_{xy}^* f_m^T(y) u_{xy},$$

which proves that f^T is equivariant, so that altogether we have $\Gamma(T) \in \mathcal{F}_b^E(X)$ for all $T \in \bigoplus \mathcal{B}(H_{[x]})$. It is easy to verify that Γ is linear, positive and injective.

Now let $f \in \mathcal{F}_b^E(X)$. By Lemma 3.69 there is a bounded quadratic form h such that $h(\xi) = f_1(\xi^* \pi_{[x]} \xi)$ for all unit vectors $\xi \in H_{[x]}$ and $x \in X$. Consequently there is a unique $T_{[x]}^f \in \mathcal{B}(H_{[x]})$ such that $\langle T_{[x]}^f \xi | \xi \rangle = f_1(\xi^* \pi_{[x]} \xi)$ for all $x \in X$. Since $\|T_{[x]}^f\| \leq \|f\|$ for all $x \in X$, we can build $T^f = \bigoplus T_{[x]}^f \in \bigoplus \mathcal{B}(H_{[x]})$. Therefore, we obtain a mapping

$$\Omega: \mathcal{F}_b^E(X) \rightarrow \bigoplus \mathcal{B}(H_{[x]}) \text{ defined by } \Omega(f) = T^f \text{ for all } f \in \mathcal{F}_b^E(X). \quad (3.18)$$

It is easy to see that Ω is linear, positive and the inverse map of Γ . So far we have shown that there is a bipositive linear isomorphism between the spaces $\mathcal{F}_b^E(X)$ and $\bigoplus \mathcal{B}(H_{[x]})$. It is left to show that the correspondence is completely bipositive. Let $f = [f_{ij}] \in M_n(\mathcal{F}_b^E(X))$ for some $n \in \mathbb{N}$. Then the matrix

$$\Omega^{(n)}(f) = [\Omega(f_{ij})] = [T^{f_{ij}}] = [\bigoplus T_{[x]}^{f_{ij}}] = \bigoplus [T_{[x]}^{f_{ij}}]$$

is positive if and only if the matrices $[T_{[x]}^{f_{ij}}]$ are positive for all $x \in X$. It follows that both maps are completely positive, so that we have a complete order isomorphism. \square

We will restrict to unital C^* -algebras now for convenience. Then we can identify \mathcal{A} with $A(CS(\mathcal{A}))$ in the usual way, so that $A(K)$ is a C^* -subalgebra of $\mathcal{F}_b^E(X)$ by the identification (3.17). More precisely we have:

Lemma 3.71. *Let \mathcal{A} be a unital C^* -algebra. Let $K = CS(\mathcal{A})$ and $X = \text{str}(K)$. Identifying \mathcal{A} with $A(K)$ by the map $a \mapsto \tilde{a}$, where $\tilde{a}_n(\psi) = \psi(a)$ for $\psi \in K_n$ and $n \in \mathbb{N}$, we obtain $\Omega(\tilde{a}|_X) = \bigoplus \pi_{[x]}(a)$ for all $a \in \mathcal{A}$, where Ω is the complete order isomorphism given by equation (3.18).*

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Proof. From the construction of the complete order isomorphism in the proof of Proposition 3.70, we have $\langle T_{[x]}^f \xi | \xi \rangle = f_1(\xi^* \pi_{[x]} \xi)$ for all $f \in \mathcal{F}_b^E(X)$, all the irreducible representations $\pi_{[x]}: \mathcal{A} \rightarrow \mathcal{B}(H_{[x]})$ and unit vectors $\xi \in H_{[x]}$. So, if especially $f = \tilde{a}|_X$ for $a \in \mathcal{A}$, we find

$$\langle T_{[x]}^{\tilde{a}} \xi | \xi \rangle = \tilde{a}_1(\xi^* \pi_{[x]} \xi) = \langle \pi_{[x]}(a) \xi | \xi \rangle$$

by definition of \tilde{a} . Since this holds for all unit vectors $\xi \in H_{[x]}$, it follows that $T_{[x]}^{\tilde{a}} = \pi_{[x]}(a)$ and hence

$$\Omega(\tilde{a}) = T^{\tilde{a}} = \oplus T_{[x]}^{\tilde{a}} = \oplus \pi_{[x]}(a),$$

which was to be shown. \square

Corollary 3.72. *Under the assumptions of the preceding lemma, the image of $A(K)$ under restriction is a C^* -subalgebra of $\mathcal{F}_b^E(X)$.*

Proof. It will be sufficient to prove that the image of $A(K)$ is closed under multiplication. Since the complete order isomorphism between the C^* -algebras $\mathcal{F}_b^E(X)$ and $\oplus \mathcal{B}(H_{[x]})$ is a $*$ -isomorphism, we find immediately

$$\Omega(\tilde{a}\tilde{b}) = \Omega(\tilde{a})\Omega(\tilde{b}) = \oplus \pi_{[x]}(a) \oplus \pi_{[x]}(b) = \oplus \pi_{[x]}(a)\pi_{[x]}(b) = \oplus \pi_{[x]}(ab) = \Omega(\tilde{ab}),$$

which shows that $\tilde{a}\tilde{b} = \tilde{ab} \in A(K)$. \square

In addition we need the following theorem of Brown:

Theorem 5(c) in [15]. *Let \mathcal{A} be a C^* -algebra, \mathcal{B} a C^* -subalgebra and $x \in \mathcal{A}$. If any two elements of $P(\mathcal{A})^- \cup \{0\}$ that agree on \mathcal{B} agree also on x , then $x \in \mathcal{B}$.*

Now we can state and prove our first goal, namely that the restriction of the m-affine maps on the m-state space of a C^* -algebra to the structural elements is a surjection.

Theorem 3.73. *Let \mathcal{A} be a unital C^* -algebra. Let $K = CS(\mathcal{A})$ and $X = \text{str}(K)$. Then the restriction map $A(K) \rightarrow \mathcal{C}_u^E(X)$ is surjective, and consequently $\mathcal{A} =_{cp} A(K)$.*

Proof. We identify $\mathcal{A} =_{cp} A(K)$ via the map $a \mapsto \tilde{a}$ defined by $\tilde{a}_n(\varphi) = \varphi(a)$ for all $\varphi \in K_n$ and $n \in \mathbb{N}$, cf. Proposition 1.26. Notice that the restriction map from $A(K)$ to $\mathcal{C}_u^E(X)$ is injective by the Krein-Milman theorem. So we have $A(K) \hookrightarrow \mathcal{C}_u^E(X) \subset \mathcal{F}_b^E(X)$, where by the preceding Corollary (the image of) $A(K)$ is a C^* -subalgebra of $\mathcal{F}_b^E(X)$ containing the unit. Basically, we are repeating the proof of [15, Thm. 6]. Recall first the identification $X \simeq \hat{X} = \text{str}(CS^\sigma(\mathcal{F}_b^E(X)))$, cf. Theorem 3.53, so that \hat{X}_1 are the normal pure states of the atomic W^* -algebra $\mathcal{F}_b^E(X)$. Obviously, \hat{X}_1 determines the order of the C^* -algebra $\mathcal{F}_b^E(X)$, thus the closure of the pure states³ $P(\mathcal{F}_b^E(X))^-$ is contained in the closure of $(\hat{X})^-$. Now let $f \in \mathcal{C}_u^E(X)$. Given $\psi_1, \psi_2 \in P(\mathcal{F}_b^E(X))^- \cup \{0\}$ such that the restrictions of ψ_1 and ψ_2 to (the image of) $A(K)$ are equal, i.e., $\psi_1|_{A(K)} = \psi_2|_{A(K)} = \psi$. We claim that $\psi_1(f) = \psi_2(f)$. There are nets (\hat{x}_λ) and (\hat{y}_μ) in \hat{X}_1 such that $x_\lambda \rightarrow \psi_1$ and $y_\mu \rightarrow \psi_2$ in the w^* -topology of $\mathcal{F}_b^E(X)^*$. Hence we find for $a \in \mathcal{A}$

$$x_\lambda(a) = \tilde{a}(x_\lambda) = \hat{x}_\lambda(\tilde{a}) \rightarrow \psi_1(\tilde{a}) = \psi_2(\tilde{a}) \leftarrow \hat{y}_\mu(\tilde{a}) = y_\mu(a).$$

This means that (x_λ) and (y_μ) converge in the w^* -topology of the dual \mathcal{A}^* and have the same limit ψ . Hence ψ is in the closure of the pure states of \mathcal{A} . Now f_1 is uniformly

³All pure states of $\mathcal{F}_b^E(X)$, not only the normal ones.

continuous on the pure states, so it has a unique continuous extension on the (compact) closure of the pure states. This implies that $\lim_{\lambda} f_1(x_{\lambda}) = f_1(\psi) = \lim_{\mu} f_1(y_{\mu})$ and hence

$$\psi_1(f) = \lim_{\lambda} \hat{x}_{\lambda}(f) = \lim_{\lambda} f_1(x_{\lambda}) = \lim_{\mu} f_1(y_{\mu}) = \lim_{\mu} \hat{y}_{\mu}(f) = \psi_2(f).$$

This proves the claim, so that we can apply the preceding theorem of Brown to conclude that f lies in (the image of) $A(K)$. \square

Notice that the content of the preceding result is contained in different form in [3, 31, 56]. However, describing \mathcal{A} as operator-valued functions on the set of irreducible representations of \mathcal{A} seems to be cumbersome, since the irreducible representations are not on a fixed Hilbert space in a natural way. Moreover, an abstract description of the set of irreducible representations of \mathcal{A} seems to be difficult at least.

We obtain an immediate consequence of the preceding result, cp. [56, Thm. 18].

Theorem 3.74. *For unital C^* -algebras \mathcal{A} and \mathcal{B} let $K = CS(\mathcal{A})$, $C = CS(\mathcal{B})$, $X = \text{str}(K)$ and $Y = \text{str}(C)$. Then \mathcal{A} and \mathcal{B} are unital $*$ -isomorphic if and only if (X_1, X_2) and (Y_1, Y_2) are equivariantly w^* -uniform equivalent, i.e., if there is a pair of maps $\phi_i: X_i \rightarrow Y_i$, such that ϕ_i is bijective, ϕ_i and ϕ_i^{-1} are uniformly continuous maps with respect to the w^* -uniformities on X_i and Y_i respectively (where $M_2(\mathcal{A}^*)$ and $M_2(\mathcal{B}^*)$ carry the product topology) and $\phi_j(u^*xu) = u^*\phi_i(x)u$ for isometries $u \in M_{ij}$, $j \leq i$ and $i, j \in \{1, 2\}$.*

Proof. If \mathcal{A} and \mathcal{B} are $*$ -isomorphic then there is a matrix affine homeomorphism $(\phi^{(n)})_n$ between the m -convex state spaces K and C , where $\phi^{(n)}$ is the n -th amplification of dual of the $*$ -isomorphism between \mathcal{A} and \mathcal{B} . The restriction of $(\phi^{(n)})_n$ to the structural elements is an equivariantly uniform equivalence between X and Y , in particular (X_1, X_2) and (Y_1, Y_2) are equivariantly uniform equivalent.

For the other direction, let $\phi_i: Y_i \rightarrow X_i$ be an equivariantly uniform equivalence, where $i = 1, 2$. Notice that $X^- = (X_n^-)_n$ is an equivariant matrix set such that the w^* -closure X_n^- of X_n in $M_n(\mathcal{A}^*)$ is compact for all $n \in \mathbb{N}$, because K_n is w^* -compact. Furthermore notice that we can identify $\mathcal{C}_u^E(X) =_{cp} \mathcal{C}^E(X^-)$ and $\mathcal{C}_u^E(Y) =_{cp} \mathcal{C}^E(Y^-)$. We extend ϕ_i uniquely to a homeomorphism $\tilde{\phi}_i: Y_i^- \rightarrow X_i^-$ for $i = 1, 2$. Observe that the pair $(\tilde{\phi}_1, \tilde{\phi}_2)$ is still equivariant. Now from Proposition 3.15 we have a 2-positive order isomorphism $\psi: \mathcal{F}^E(X^-) \rightarrow \mathcal{F}^E(Y^-)$ given by $\psi(f)_i(y) = f_i(\tilde{\phi}_i(y))$ for all $y \in Y_i^-$ and $i = 1, 2$. Since $f = (f_l)_l \in \mathcal{F}_b^E(X^-)$ lies in $\mathcal{C}^E(X^-)$ if and only if f_1 is continuous and since $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are homeomorphism, it follows that the restriction of ψ to the continuous maps defines an 2-positive order isomorphism $\psi: \mathcal{C}^E(X^-) \rightarrow \mathcal{C}^E(Y^-)$ that is obviously unital by definition. Consequently there is a unital and 2-positive order isomorphism between the C^* -algebras $\mathcal{A} =_{cp} A(K) =_{cp} \mathcal{C}_u^E(X)$ and $\mathcal{B} =_{cp} A(C) =_{cp} \mathcal{C}_u^E(Y)$, which must be a unital $*$ -isomorphism. \square

Returning to our final goal of characterizing the m -convex state spaces of C^* -algebras, it would be tempting to define a non-commutative analogue of a Bauer simplex, based on Theorem 3.73, as a compact m -convex set K such that that $X = \text{str}(K)$ is equivariant and transitive and such that the restriction map from $A(K)$ to $\mathcal{C}_u^E(X)$ is surjective. Although with the help of Proposition 3.24 we find that $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra, the surjectivity of the restriction map implies only that the m -convex state space of the operator system $\mathcal{C}_u^E(X) \subset \mathcal{F}_b^E(X)$ is m -affine homeomorph to K . First, it is not clear

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why the normal state space $CS^\sigma(\mathcal{F}_b^E(X))$ should be contained in K , not to mention that it must in fact be an m -convex split face of K . Second, what we need to show is actually that $\mathcal{C}_u^E(X)$ is a C^* -algebra under the product it inherits from $\mathcal{F}_b^E(X)$. For this it is essential, as we will see, that the uniformity of X (X^- is compact) fits to the algebraic structure⁴ of X .

To get a grip on the uniformity of X we need further properties of X . These will be given in the following definitions, whereby we profit from the detailed study of the pure normal m -states of atomic W^* -algebras, especially from the identification between projections in $\mathcal{F}_b^E(X)$ and equivariantly directed subsets of X , see Definition 3.60.

Definition 3.75 (Splitting Subsets). Let X be a matrix subset of a vector space V such that X is equivariant, transitive and fulfills the uniqueness property. Let $f \in \mathcal{F}_b^E(X)_h$. Then a subset Y of X is called *splitting for f* , if all of the following holds:

- (i) Y is equivariantly directed,
- (ii) $f_1(y) \geq 0$ and $f_1(z) \leq 0$ for all $y \in Y_1$ and $z \in Y_1^\perp$, and
- (iii) $f_2(x) = \begin{pmatrix} f(x_{11}) & 0 \\ 0 & f(x_{22}) \end{pmatrix}$ for all $x = [x_{ij}] \in X_2$ with $x_{11} \in Y_1$ and $x_{22} \in Y_1^\perp$.

Definition 3.76 (Jordan Property). Let V be a locally convex vector space and endow $M_n(V)$ with the product topology for all $n \in \mathbb{N}$. Let $X = (X_n)_n$ be a matrix subset of V such that X is equivariant, transitive and fulfills the uniqueness property and such that the closure of X_n is compact in $M_n(V)$ for all $n \in \mathbb{N}$. Define the set \mathfrak{E} of abelian points of X by $\mathfrak{E} = \{x \in X_1 \mid \text{If } x \curvearrowright y \text{ then } x = y\}$. Then the induced uniformity on X has the *Jordan property*, if for $f \in \mathcal{C}_u^E(X)_h$ and Y a subset of X that is splitting for f the following holds: For $\varepsilon > 0$ there is a member N_ε of the uniformity of X_1 , such that

- (i) $(e, \beta^* z \beta) \in N_\varepsilon$ implies $|\max(f_1(e), 0) - |\beta_1|^2 f_1(z_{11})| \leq \varepsilon$, where $e \in \mathfrak{E}$, $\beta = (\beta_1, \beta_2)^{\text{tr}} \in M_{2,1}$ and $z = [z_{ij}] \in X_2$, such that $\|\beta\| = 1$, $z_{11} \in Y_1$ and $z_{22} \in Y_1^\perp$, and
- (ii) $(\alpha^* x \alpha, \beta^* z \beta) \in N_\varepsilon$ implies $||\alpha_1|^2 f(x_{11}) - |\beta_1|^2 f(z_{11})| \leq \varepsilon$, where $\alpha = (\alpha_1, \alpha_2)^{\text{tr}}$, $\beta = (\beta_1, \beta_2)^{\text{tr}} \in M_{2,1}$ and $x = [x_{ij}]$, $z = [z_{ij}] \in X_2$, such that $\|\alpha\| = \|\beta\| = 1$, $x_{11}, z_{11} \in Y_1$, $x_{22}, z_{22} \in Y_1^\perp$.

To show that the intricate definition of the Jordan property makes sense, we prove first that state spaces of C^* -algebras satisfy this property. We will need the following lemma to do so.

Lemma 3.77. *Let X be a matrix set such that X is equivariant, transitive and fulfills the uniqueness property and such that (X_1, d) is complete, where d is the inner metric of X . So, $\mathcal{F}_b^E(X)$ is a W^* -algebra, cf. Corollary 3.27. Let $x \in X_1$ and $\hat{x} = \xi^* \pi \xi$ be the GNS representation of the pure state \hat{x} , cf. Proposition 3.32, where $\pi: \mathcal{F}_b^E(X) \rightarrow \mathcal{B}(H_\pi)$ and $\xi \in H_\pi$ with $\|\xi\| = 1$. Let Y be an equivariantly directed subset of X and $p \in \mathcal{F}_b^E(X)$ the projection corresponding to Y such that $Y = \{z \in X \mid p(z) = \mathbb{1}\}$, cf. Proposition 3.61. If $x = \alpha^* y \alpha$, where $\alpha = (\alpha_1, \alpha_2)^{\text{tr}} \in M_{2,1}$ and $y = [y_{ij}] \in X_2$ such that $\|\alpha\| = 1$, $y_{11} \in Y_1$ and $y_{22} \in Y_1^\perp$, then $|\alpha_1| = \|\pi(p)\xi\|$ and $\hat{y}_{11} = (\pi(p)\xi / \|\pi(p)\xi\|)^* \pi(p)\xi / \|\pi(p)\xi\|$.*

⁴With ‘algebraic structure’ we only refer to properties like equivariant, transitive,...

Proof. Since $\hat{x} = \alpha^* \hat{y} \alpha$ and by the essential uniqueness of the GNS representation, there is $\mathcal{W} : \mathbb{C}^2 \rightarrow H_\pi$ such that $\hat{y} = \mathcal{W}^* \pi \mathcal{W}$. Since

$$\xi^* \pi \xi = \hat{x} = \alpha^* \hat{y} \alpha = \alpha^* \mathcal{W}^* \pi \mathcal{W} \alpha$$

it follows again from the essential uniqueness that there is $\lambda \in \mathbb{C}$ such that $\mathcal{W} \alpha = \lambda \xi$. Since \mathcal{W} is an isometry and $\|\alpha\| = \|\xi\| = 1$, we see that $|\lambda| = 1$. Letting $\eta_i = \mathcal{W} \varepsilon_i$ for $i = 1, 2$, where $\{\varepsilon_1, \varepsilon_2\}$ is the standard basis of \mathbb{C}^2 , we obtain

$$\mathcal{W} \alpha = \alpha_1 \eta_1 + \alpha_2 \eta_2 = \lambda \xi = \lambda \pi(p) \xi + \lambda \pi(p') \xi.$$

Since $\ker(p)$ and $\ker(p')$ are complementary subspaces of H_π and since $0 = p(y_{22}) = \hat{y}_{22}(p) = \eta_2^* \pi(p) \eta_2 = \|\pi(p) \eta_2\|^2$, so that $\eta_2 \in \ker(p)$ and similarly $\eta_1 \in \ker(p')$, we see that $\alpha_1 \eta_1 = \lambda \pi(p) \xi$ and $\alpha_2 \eta_2 = \lambda \pi(p') \xi$. Thus $|\alpha_1| = \|\pi(p) \xi\|$ and $|\alpha_2| = \|\pi(p') \xi\|$. Furthermore, $\eta_1 = (\lambda/\alpha_1) \pi(p) \xi$, hence

$$\hat{y}_{11} = \eta_1^* \pi \eta_1 = |\lambda|^2 \frac{(\pi(p) \xi)^*}{\|\pi(p) \xi\|} \pi \frac{\pi(p) \xi}{\|\pi(p) \xi\|},$$

and the proof is complete. \square

Proposition 3.78. *Let $K = CS(\mathcal{A})$ be the state space of a unital C^* -algebra \mathcal{A} . Then the w^* -uniformity on $X = \text{str}(K)$ fulfills the Jordan property.*

Proof. We can identify $\mathcal{A} = A(K) = \mathcal{C}_u^E(X)$, and $\mathcal{F}_b^E(X)$ is the atomic part of \mathcal{A}^{**} . Let $f \in \mathcal{C}_u^E(X)_h$ and let Y be a subset of X that is splitting for f . Let $f^+ \in \mathcal{C}_u^E(X)_+$ be the positive part of f . Given $\delta > 0$ there is a member N_δ of the w^* -uniformity of X_1 , such that $|f_1^+(v) - f_1^+(v')| \leq \delta$ for all $v, v' \in X_1$ with $(v, v') \in N_\delta$. Let $\alpha = (\alpha_1, \alpha_2)^{\text{tr}}$, $\beta = (\beta_1, \beta_2)^{\text{tr}} \in M_{2,1}$ and $z = [z_{ij}]$, $\tilde{z} = [\tilde{z}_{ij}] \in X_2$, such that $\|\alpha\| = \|\beta\| = 1$, $z_{11}, \tilde{z}_{11} \in Y_1$, $z_{22}, \tilde{z}_{22} \in Y_1^\perp$ and $(\alpha^* z \alpha, \beta^* \tilde{z} \beta) \in N_\delta$. Then we must show that $|\alpha_1|^2 f(z_{11}) - |\beta_1|^2 f(\tilde{z}_{11})| \leq \delta$. Let $x = \alpha^* z \alpha$. Since Y is equivariantly directed, there is a projection $p \in \mathcal{F}_b^E(X)$ such that $Y = \{v \in X \mid p(v) = \mathbb{1}\}$ and $Y^\perp = \{v \in X \mid p(v) = 0\}$, cf. Proposition 3.61. Since Y is splitting for f , we see that $pfp' = 0$. Indeed, let $\phi \in X_1$, so that ϕ is a pure state of \mathcal{A} and let $\phi = \xi^* \pi \xi$ be its GNS representation. We define an isometry $\mathcal{V} : \mathbb{C}^2 \rightarrow H_\pi$ by $\mathcal{V} \varepsilon_1 = \pi(p) \xi / \|\pi(p) \xi\|$ and $\mathcal{V} \varepsilon_2 = \pi(p') \xi / \|\pi(p') \xi\|$. Then $\psi = \mathcal{V}^* \pi \mathcal{V} \in X_2$ and $\phi = \gamma^* \psi \gamma$, where $\gamma = (\|\pi(p) \xi\|, \|\pi(p') \xi\|)^{\text{tr}}$. Since $\psi_{11} \in Y_1$ and $\psi_{22} \in Y_1^\perp$, the fact that Y is splitting for f implies that $f_2(\psi) = \begin{pmatrix} f_1(\psi_{11}) & 0 \\ 0 & f_1(\psi_{22}) \end{pmatrix}$. Thus, we obtain

$$\begin{aligned} pfp'(\phi) &= \langle \pi(fp') \xi | \pi(p) \xi \rangle \\ &= \gamma_1 \gamma_2 \left\langle \pi(f) \frac{\pi(p') \xi}{\|\pi(p') \xi\|} \middle| \frac{\pi(p) \xi}{\|\pi(p) \xi\|} \right\rangle \\ &= \gamma_1 \gamma_2 (\varepsilon_2^* \mathcal{V}^* \pi(f) \mathcal{V} \varepsilon_1) \\ &= \gamma_1 \gamma_2 f_2(\psi)_{21} = 0. \end{aligned}$$

Since the preceding argumentations holds for arbitrary $\phi \in X_1$, it follows that $pfp' = p'fp = 0$. Moreover,

$$pfp(\phi) = \langle \pi(f) \pi(p) \xi | \pi(p) \xi \rangle = \gamma_1^2 f(\psi_{11}) \geq 0,$$

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which shows $pf p \geq 0$, and similarly we obtain $p' f p' \leq 0$. Hence

$$f = pf p + pf p' + p' f p + p' f p' = pf p - (-p' f p'),$$

from which we conclude that $pf p = f^+$ and $-p' f p' = f^-$. Using the GNS representation $x = \zeta^* \sigma \zeta$ an application of Lemma 3.77 yields

$$\begin{aligned} f_1^+(x) &= (pf p)_1(x) = \langle \sigma(pf p) \zeta | \zeta \rangle \\ &= \langle \sigma(f) \sigma(p) \zeta | \sigma(p) \zeta \rangle \\ &= \|\sigma(p) \zeta\|^2 \left\langle \sigma(f) \frac{\sigma(p) \zeta}{\|\sigma(p) \zeta\|} \middle| \frac{\sigma(p) \zeta}{\|\sigma(p) \zeta\|} \right\rangle = |\alpha_1|^2 f_1(z_{11}). \end{aligned}$$

Similarly we obtain $f_1^+(\tilde{x}) = |\beta_1|^2 f_1(\tilde{z}_{11})$ for $\tilde{x} = \beta^* z \beta$. Thus

$$\delta \geq |f_1^+(x) - f_1^+(\tilde{x})| = \left| |\alpha_1|^2 f_1(z_{11}) - |\beta_1|^2 f_1(\tilde{z}_{11}) \right|,$$

which shows the second part of the Jordan property. In order to show the other part of the Jordan property, let $(e, \beta^* z \beta) \in N_\delta$, where $e \in \mathfrak{E}$ and $\beta = (\beta_1, \beta_2)^{\text{tr}} \in M_{2,1}$ and $z = [z_{ij}] \in X_2$, such that $\|\beta\| = 1$, $z_{11} \in Y_1$ and $z_{22} \in Y_1^\perp$. We have shown already that $f_1^+(x) = |\beta_1|^2 f_1(z_{11})$ for $x = \beta^* z \beta$. Moreover, since $e \in \mathfrak{E}$ it is obvious that $\max(f_1(e), 0) = f_1^+(e)$. Thus

$$\delta \geq |f_1^+(e) - f_1^+(x)| = \left| \max(f_1(e), 0) - |\beta_1|^2 f_1(z_{11}) \right|,$$

which shows the first part of the Jordan property and the proof is complete. \square

The purpose of the Jordan property is to have a condition on the uniformity on X that ensures that the self-adjoint and uniformly continuous equivariant maps on X are a Jordan subalgebra of the W^* -algebra of bounded equivariant maps on X . This is the content of the next proposition, for which we need the following lemma.

Lemma 3.79. *Let X be a matrix set such that X is equivariant, transitive and fulfills the uniqueness property and such that (X_1, d) is complete, where d is the inner metric of X . If Y is an equivariantly directed subset of X , then for each $x \in X_1 \setminus \mathfrak{E}$ there is $z = [z_{ij}] \in X_2$ such that $z \succ x$ and $z_{11} \in Y_1$ and $z_{22} \in Y_1^\perp$.*

Proof. Recalling the identification $X = \widehat{X} = \text{str}(CS^\sigma(\mathcal{F}_b^E(X)))$ let $\hat{x} = \xi^* \pi \xi$ be the GNS representation of the pure normal state \hat{x} , where $\pi: \mathcal{F}_b^E(X) \rightarrow \mathcal{B}(H_\pi)$ is a normal and irreducible representation and $\xi \in H_\pi$ is a unit vector. Since Y is equivariantly directed, there is $p \in \mathcal{F}_b^E(X)$ such that $Y = \{v \in X \mid p(v) = \mathbb{1}\}$ and $Y^\perp = \{v \in X \mid p(v) = 0\}$, cf. Proposition 3.61. Since $x \notin \mathfrak{E}$, $\dim(H_\pi) > 1$, so there are unique non-zero vectors $\eta, \eta^\perp \in H_\pi$ such that $\pi(p)\eta = \eta$, $\pi(p')\eta^\perp = \eta^\perp$ and $\xi = \eta + \eta^\perp$. Define an isometry $\mathcal{V}: \mathbb{C}^2 \rightarrow H_\pi$ by $\mathcal{V}\varepsilon_1 = \eta/\|\eta\|$ and $\mathcal{V}\varepsilon_2 = \eta^\perp/\|\eta^\perp\|$, where $\{\varepsilon_1, \varepsilon_2\}$ is the standard basis of \mathbb{C}^2 . We see that $\hat{z} = \mathcal{V}^* \pi \mathcal{V}$ is a normal and pure m-state, so that $z \in X_2$. Moreover, letting $\alpha = (\|\eta\|, \|\eta^\perp\|)^{\text{tr}} \in M_{2,1}$, we obtain $\alpha^* \alpha = \|\eta\|^2 + \|\eta^\perp\|^2 = \|\xi\|^2 = 1$ and $\mathcal{V}\alpha = \|\eta\|\mathcal{V}\varepsilon_1 + \|\eta^\perp\|\mathcal{V}\varepsilon_2 = \eta + \eta^\perp = \xi$. Thus $x = \alpha^* z \alpha$, and the proof is complete. \square

Proposition 3.80. *Let X be a matrix subset of a locally convex vector space V such that X is equivariant, transitive, fulfills the uniqueness property and (X_1, d) is a complete metric space, where d is the inner metric of X (so that $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra*

and X can be identified with the pure normal m -states of $\mathcal{F}_b^E(X)$, cf. Theorem 3.53). Give $M_n(V)$ the product topology and assume that $X_n^- \subset M_n(V)$ is compact for all $n \in \mathbb{N}$. Let $f \in \mathcal{F}_b^E(X)_h$ and $f^+ \in \mathcal{F}_b^E(X)_+$ its positive part. If X satisfies the Jordan property, then $f^+ \in \mathcal{C}_u^E(X)$ whenever $f \in \mathcal{C}_u^E(X)$.

Proof. If $f = f^+ - f^-$ is the unique decomposition of f into its positive and negative parts, let $p \in \mathcal{F}_b^E(X)$ be an orthogonal projection such that $pf = pf^+ = f^+$ and $p'f = -p'f^-$. By Proposition 3.61 there exists an equivariantly directed subset $Y \subset X$ of the pure m -states such that $Y = \{x \in X \mid p(x) = 1\}$ and $Y^\perp = \{x \in X \mid p(x) = 0\}$. We claim that Y is splitting for f . Let $x \in X_2$ such that $x_{11} \in Y_1$ and $x_{22} \in Y_1^\perp$. Let $\hat{x} = \mathcal{V}^* \pi \mathcal{V}$ be the minimal Stinespring representation of \hat{x} , and define $\xi_i = \mathcal{V} \varepsilon_i \in H_\pi$ for $i = 1, 2$. Since $x_{11} \in Y_1$ and $x_{22} \in Y_1^\perp$, it follows that $\pi(p)\xi_1 = \xi_1$ and $\pi(p)\xi_2 = 0$. Hence,

$$\begin{aligned} f_2(x)_{1,2} &= \hat{x}(f)_{1,2} = \langle \pi(f)\xi_2 | \xi_1 \rangle \\ &= \langle \pi(f^+)\xi_2 | \xi_1 \rangle - \langle \pi(f^-)\xi_2 | \xi_1 \rangle \\ &= \langle \pi(pf^+) \xi_2 | \xi_1 \rangle - \langle \pi(p'f^- p') \xi_2 | \xi_1 \rangle \\ &= \langle \pi(f^+) \pi(p) \xi_2 | \pi(p) \xi_1 \rangle - \langle \pi(f^-) \pi(p') \xi_2 | \pi(p') \xi_1 \rangle = 0. \end{aligned}$$

Moreover,

$$\begin{aligned} f_1(x_{11}) &= \hat{x}(f)_{11} = \langle \pi(f)\xi_1 | \xi_1 \rangle \\ &= \langle \pi(f)\pi(p)\xi_1 | \pi(p)\xi_1 \rangle \\ &= \langle \pi(pf)p \xi_1 | \xi_1 \rangle = \langle \pi(f^+)\xi_1 | \xi_1 \rangle \geq 0, \end{aligned}$$

and similarly $f_1(x_{22}) \leq 0$. Notice that we used only $x_{11} \in Y_1$ and $x_{22} \in Y_1^\perp$ for the last two results, so that $f_1(y) \geq 0$ and $f_1(y^\perp) \leq 0$ for all $y \in Y_1$ and $y^\perp \in Y_1^\perp$. Altogether this shows that Y is splitting for f . So, by the Jordan property, given $\varepsilon > 0$ there is a member N_ε of the w^* -uniformity of X_1 fulfilling the condition of Definition 3.76. In addition, since f is uniformly continuous, we may choose N_ε such that $(x, \tilde{x}) \in N_\varepsilon$ implies $|f_1(x) - f_1(\tilde{x})| \leq \varepsilon$. We would like to prove that f^+ is uniformly continuous. Let $(x, \tilde{x}) \in N_\varepsilon$, and assume first that neither x nor \tilde{x} are in \mathfrak{E} . Then by Lemma 3.79 we have $x = \alpha^* z \alpha$ and $\tilde{x} = \beta^* \tilde{z} \beta$, where $z, \tilde{z} \in X_2$ such that $z_{11}, \tilde{z}_{11} \in Y_1$ and $z_{22}, \tilde{z}_{22} \in Y_1^\perp$, and $\alpha, \beta \in M_{2,1}$ such that $\|\alpha\| = \|\beta\| = 1$. Since $(\alpha^* z \alpha, \beta^* \tilde{z} \beta) \in N_\varepsilon$, it follows that $||\alpha_1|^2 f_1(z_{11}) - |\beta_1|^2 f_1(\tilde{z}_{11})| \leq \varepsilon$. Let $\hat{x} = \xi^* \pi \xi$ be the GNS representation of the pure normal state \hat{x} . Then by construction of α and z we obtain

$$\begin{aligned} f_1^+(x) &= (pfp)_1(x) = \hat{x}(pfp) = \langle \pi(pf p) \xi | \xi \rangle \\ &= \|\pi(p)\xi\|^2 \left\langle \pi(f) \frac{\pi(p)\xi}{\|\pi(p)\xi\|} \middle| \frac{\pi(p)\xi}{\|\pi(p)\xi\|} \right\rangle \\ &= |\alpha_1|^2 f_1(z_{11}), \end{aligned}$$

where we applied Lemma 3.77. Similarly, by construction of β and \tilde{z} we also see that $f_1^+(\tilde{x}) = |\beta_1|^2 f_1(\tilde{z}_{11})$. Hence

$$|f^+(x) - f^+(\tilde{x})| = ||\alpha_1|^2 f_1(z_{11}) - |\beta_1|^2 f_1(\tilde{z}_{11})| \leq \varepsilon.$$

This holds for all $(x, \tilde{x}) \in N_\varepsilon$, such that neither x nor \tilde{x} are in \mathfrak{E} . Notice that for $e \in \mathfrak{E}$ we have $f_1(e) = f_1^+(e)$ if $f_1(e) \geq 0$ or $f_1(e) = -f_1^-(e)$ if $f_1(e) \leq 0$, so that especially

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$f_1^+(e) = \max(f_1(e), 0)$. Thus if both $x, \tilde{x} \in \mathfrak{E}$ and $f_1(x), f_1(\tilde{x}) \geq 0$ then by choice of N_ε we see that $|f_1^+(x) - f_1^+(\tilde{x})| = |f_1(x) - f_1(\tilde{x})| \leq \varepsilon$. If both $f_1(x), f_1(\tilde{x}) \leq 0$ there is nothing to show. So assume $f_1(x) \geq 0$ and $f_1(\tilde{x}) \leq 0$. Then $|f_1^+(x) - f_1^+(\tilde{x})| = f_1(x) \leq f_1(x) - f_1(\tilde{x}) \leq \varepsilon$. Finally consider the situation $x \in \mathfrak{E}$ and $\tilde{x} \notin \mathfrak{E}$. Then $\tilde{x} = \beta^* \tilde{z} \beta$ as before, and $f_1^+(\tilde{x}) = |\beta_1|^2 f_1(\tilde{z}_{11})$. Thus we obtain from the Jordan property

$$|f_1^+(x) - f_1^+(\tilde{x})| = |\max(f_1(x), 0) - |\beta_1|^2 f_1(\tilde{z}_{11})| \leq \varepsilon.$$

So we have shown that $(x, \tilde{x}) \in N_\varepsilon$ implies $|f_1^+(x) - f_1^+(\tilde{x})| \leq \varepsilon$ for all $x, \tilde{x} \in X_1$. This shows $f^+ \in \mathcal{C}_u^E(X)$ and the proof is complete. \square

Definition 3.81 (Matrix Convex Simplex). Let V be a locally convex vector space and K a compact and matrix convex subset of V . We assume that K is embedded as m-base in $A(K)^*$, cf. Proposition 1.26. Then K is a *matrix convex simplex*, if

- (i) $\text{str}(K)$ is equivariant and transitive,
- (ii) $S = \sigma\text{-mco}(\text{str}(K))$ is an m-convex split face of K ,
- (iii) S has the finite m-simplex property,
- (iv) the induced uniformity on $\text{str}(K)$ has the Jordan property, and
- (v) the restriction map from $A(K)$ to $\mathcal{C}_u^E(\text{str}(K))$ is a surjection.

Remark 3.82. Let C be a Bauer simplex and consider its matrix convex hull $K = \text{mco}(C)$. Then $C = K_1$, $\text{ex}(C) = \text{str}(K_1)$ and $\text{str}(K_n) = \emptyset$ for all $n > 1$. Thus $\mathcal{C}_u^E(\text{str}(K))$ are nothing but all uniformly continuous maps on the extreme points $\text{ex}(C)$. Since C is a Bauer simplex, all these maps can be extended to continuous affine maps on C . So, it is obvious that a Bauer Simplex fulfills the above definition, because $A(C) = A(K_1)$ is unitaly order isomorphic to $A(K)$. So, a matrix convex simplex is a generalization of a Bauer simplex.

Theorem 3.83 (Characterization of State Spaces).

The state spaces of unital C^ -algebras are exactly the m-convex simplexes.*

Proof. Let \mathcal{A} be a unital C^* -algebra. Let $K = CS(\mathcal{A})$ and $X = \text{str}(K)$. By Proposition 3.4 the matrix set X is equivariant and transitive. Moreover $S = \sigma\text{-mco}(X)$ is the normal state space of the atomic part of \mathcal{A}^{**} , which can be identified with $\mathcal{F}_b^E(X)$ by Proposition 3.70. Hence S is an m-convex split face of K . As normal state space of an atomic W^* -algebra, S has the finite m-simplex property, cf. Theorem 3.59. By Proposition 3.78 the w^* -uniformity on X fulfills the Jordan property. Finally we have the identification $\mathcal{A} =_{cp} A(K)$, cf. Proposition 1.26, and $A(K) =_{cp} \mathcal{C}_u^E(X)$ by Theorem 3.73.

In the converse direction, let K be an m-convex simplex. Since $X = \text{str}(K)$ is equivariant, transitive and by Remark 3.10 fulfills the uniqueness property, $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra by Corollary 3.27. In order to prove that $\mathcal{C}_u^E(X) \subset \mathcal{F}_b^E(X)$ is in fact a C^* -subalgebra of $\mathcal{F}_b^E(X)$, it suffices to show that $\mathcal{C}_u^E(X)$ is closed under the multiplication of $\mathcal{F}_b^E(X)$. As first step toward this end, we will show that $\mathcal{C}_u^E(X)_h$ is a Jordan subalgebra of $\mathcal{F}_b^E(X)_h$. By assumption $S = \sigma\text{-mco}(X)$ is an m-convex split face of K and S has the finite m-simplex property. Hence we can identify $\mathcal{F}_b^E(X) = A_b(S)$, cf. the proof of Theorem 3.59, and so S is the normal m-convex state space of the atomic W^* -algebra

$\mathcal{F}_b^E(X)$. Since it follows from the proof of (1.4) that $\text{str}(S) \subset X$ (the converse is obvious), $X = \text{str}(S)$ are the pure normal m-states of $\mathcal{F}_b^E(X)$, so that we conclude from Proposition 3.80 that, whenever $f \in \mathcal{C}_u^E(X)_h \subset \mathcal{F}_b^E(X)_h$, its positive part f^+ lies in $\mathcal{C}_u^E(X)$, too. Then it follows from Lemma A.7 that $\mathcal{C}_u^E(X)_h$ is closed under squares, which implies that $\mathcal{C}_u^E(X)_h$ is a Jordan subalgebra of $\mathcal{F}_b^E(X)_h$. Since by assumption we can identify $A(K) = \mathcal{C}_u^E(X)$, the state space of the Jordan algebra $\mathcal{C}_u^E(X)_h$ is K_1 . We would like to prove next that $\mathcal{C}_u^E(X)_h$ is the self-adjoint part of a C^* -algebra. Recall that S_1 , being the normal state space of a W^* -algebra, has the 3-ball property, cf. [6, Thm. 10.2]. Now, the smallest face of K_1 containing $x, y \in X_1$, denoted as $\text{face}(x, y)$, must be contained in the (split) face S_1 . Then $\text{face}(x, y)$ is also the smallest face in S_1 containing x and y , and as such $\text{face}(x, y)$ is either a point, a line segment or affinely isomorphic to a 3-ball. Consequently, K_1 has the 3-ball property, too. So by [6, Thm. 11.58] there will be a C^* -product compatible with the Jordan product, if K_1 is orientable.

To see that K_1 is orientable, recall that we endow any facial 3-ball in K_1 with the orientation that it has as 3-ball of S_1 , where S_1 has the canonical global orientation, see Remark 3.67. Let \mathcal{X} and \mathcal{Y} be the sets of orientation preserving and orientation reversing maps in $\text{Param}(K_1)$, see [6, Def. 11.47]⁵ or [7, §. 7]. Then $\text{Param}(K_1)$ is the disjoint union of \mathcal{X} and \mathcal{Y} . We will show that both \mathcal{X} and \mathcal{Y} are closed (and therefore also open). Let $(\phi_\nu)_{\nu \in \Lambda}$ be a net in \mathcal{X} converging to $\phi \in \text{Param}(K_1)$. This means $\phi_\nu(b) \rightarrow \phi(b)$ in the topology of K_1 for all $b \in \mathbf{B}^3$. Notice again that $\phi_\nu(\mathbf{B}^3), \phi(\mathbf{B}^3) \subset S_1$. Indeed, given distinct $a, b \in \text{ex}(\mathbf{B}^3)$ we see that $\phi(a), \phi(b) \in X_1$, because $\phi(\mathbf{B}^3)$ is a face of K_1 . Moreover, $\text{face}(\phi(a), \phi(b)) \subset S_1$ and $\phi(\mathbf{B}^3) = \text{face}(\phi(a), \phi(b))$, since $\mathbf{B}^3 = \text{face}(a, b)$. Thus we can apply Proposition 3.66 to conclude that since ϕ_ν is orientation preserving, there is $y_\nu \in X_2$ such that $\phi_\nu = y_\nu^*$ for all $\nu \in \Lambda$. There exists also a y such that $\phi = y^*$, where we have either $y \in X_2$ or $y \in X_2^{\text{tr}}$. Our claim is of course that y must be in X_2 , since this implies $\phi \in \mathcal{X}$. So assume for contradiction that $y \in X_2^{\text{tr}}$. A short calculations gives $\langle y_\nu(a), \gamma \rangle = \langle a, y_\nu^*(\gamma) \rangle = \langle a, \phi_\nu(\gamma) \rangle \rightarrow \langle a, \phi(\gamma) \rangle = \langle y(a), \gamma \rangle$, for all $\gamma \in \mathbf{B}^3 = CS_1(M_2)$, cf. Remark 3.62, and $a \in A(K)$. It follows that $\langle y_\nu(a), \gamma \rangle \rightarrow \langle y(a), \gamma \rangle$ for all $\gamma \in M_2^*$. This shows that the net (y_ν) converges to y in the topology of K_2 . Then $y \in K_2$, since K_2 is compact. Thus, recalling $K = CS(A(K))$ (see Proposition 1.26), y is completely positive on $A(K)$, and consequently y is completely positive on the bidual $A_b(K)$. By assumption S is an m-convex split face of K . Therefore, $A_b(K) =_{cp} A_b(S) \oplus_\infty A_b(S')$, where S' denotes the complementary m-convex split face of S , cf. Corollary 1.54. Hence y is completely positive on $A_b(S) =_{cp} \mathcal{F}_b^E(X)$. We have assumed $y \in X_2^{\text{tr}}$, which means that there is a normal pure matrix state z of the W^* -algebra $\mathcal{F}_b^E(X)$ such that $y = z^{\text{tr}}$. But the transpose of z cannot be completely positive by Lemma A.6. This contradiction shows that $y \in X_2$. Thus \mathcal{X} is closed. Let t_2 denote the transpose map on M_2 . Since the dual of the transpose map reverses orientation [5, Lem. 4.33], the map $\phi \mapsto \phi \circ t_2^*$ exchanges \mathcal{X} and \mathcal{Y} . The map $\phi \mapsto \phi \circ t_2^*$ is its own inverse and is continuous. Thus it is a homeomorphism, and we conclude that \mathcal{Y} is also closed. By definition, \mathcal{X} and \mathcal{Y} are both saturated under the action of $SO(3)$, and thus their images in \mathcal{OB}^6 provide disjoint closed cross-sections of the bundle $\mathcal{OB} \rightarrow \mathcal{B}$. Thus this bundle is trivial. If $\phi \in \mathcal{X}$, then there is $y \in X_2$ such that $\phi = y^*$, and $[y^*]$ is the (canonical) orientation on the facial 3-ball $\phi(\mathbf{B}^3) \subset S_1$ induced by $\mathcal{F}_b^E(X)$. Thus the orientation of each facial 3-ball of K_1 gives a continuous cross-section of the bundle $\mathcal{OB} \rightarrow \mathcal{B}$, i.e., a global orientation of K_1 . Hence

⁵ $\text{Param}(K_1)$ is the set of all affine isomorphisms from \mathbf{B}^3 to faces of K_1 with the topology of pointwise convergence (cf. facial 3-balls).

⁶ $\mathcal{OB} = \text{Param}(K_1)/SO(3)$ and $\mathcal{B} = \text{Param}(K_1)/O(3)$.

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by [6, Thm. 11.58] there exists a C^* -product \star on $\mathcal{C}_u^E(X)$ such that $f \star g + g \star f = fg + gf$ for all $f, g \in \mathcal{C}_u^E(X)_h$.

Now we have a C^* -algebra $(\mathcal{C}_u^E(X), \star)$, but we need to prove that the multiplication coincides with the one inherited from $\mathcal{F}_b^E(X)$. This claim will be true, because the orientation that yields \star is the orientation of the normal state space of $\mathcal{F}_b^E(X)$, which is in correspondence with the product on $\mathcal{F}_b^E(X)$. To show the claim, let the m-convex state space of $(\mathcal{C}_u^E(X), \star)$ be C , and let $Y = \text{str}(C)$. We know that the order (on the first level) of $(\mathcal{C}_u^E(X), \star)$ coincides with the (pointwise) order of $\mathcal{C}_u^E(X)$, because \star is compatible with the Jordan product on $\mathcal{C}_u^E(X)_h$. Consequently, $C_1 = K_1$, i.e., both state spaces are the same sets. Thus we know also $Y_1 = X_1$. Given $y = [y_{ij}] \in Y_2$, we obtain from (the proof of) Proposition 3.64 applied to C as the m-convex normal state space of the atomic part of the bidual $(\mathcal{C}_u^E(X), \star)^{**}$ that $\text{face}(y_{11}, y_{22}) = \text{mco}_1(y)$ is affine isomorph to \mathbf{B}^3 . Thus by Remark 3.65 applied to $S = \sigma\text{-mco}(X)$, y_{11} and y_{22} are m-equivalent with respect to X , so that by Lemma 3.47 there is $x \in X_2$ such that $x \succ y_{11}, y_{22}$. Recall from the proof of Proposition 3.66 that we denote by x^* and y^* the restrictions of the dual maps of x and y to $CS_1(M_2)$, so that x^* and y^* are affine isomorphism from $CS_1(M_2)$ onto $\text{mco}_1(y) = \text{face}(y_{11}, y_{22}) = \text{mco}_1(x)$. Now, consider the affine automorphism $\phi^* = (x^*)^{-1} \circ y^*: CS_1(M_2) \rightarrow CS_1(M_2)$. Notice from Remark 3.63 that ϕ^* has a unique extension to a linear map on M_2^* that we still denote by ϕ^* . Then a short calculation yields

$$\langle \phi(x(g)), \gamma \rangle = \langle x(g), \phi^*(\gamma) \rangle = \langle x(g), (x^*)^{-1}(y^*(\gamma)) \rangle = \langle g, y^*(\gamma) \rangle = \langle y(g), \gamma \rangle,$$

for all $g \in \mathcal{C}_u^E(X)$ and $\gamma \in M_2^*$. Thus $y = \phi(x)$. Now the orientation of $\text{face}(y_{11}, y_{22})$ is given by the parametrization y^* as well as by x^* . Then by definition of orientation (see [6, Def. 11.45]) the determinant of $\phi^* = (x^*)^{-1} \circ y^*$ is 1, whereby we identify \mathbf{B}^3 with $CS_1(M_2)$ canonically, cf. Remark 3.62, so that x^* and y^* can be read as orthogonal transformation on \mathbb{R}^3 , (see also the proof of Proposition 3.66). Since the determinant of ϕ^* is 1, ϕ is unitarily implemented, cf. [5, Thm. 4.34], so that there is a unitary $u \in M_2$ such that $y(g) = \phi(x(g)) = u^* x(g) u$ for all $g \in \mathcal{C}_u^E(X)$. Thus $y = u^* x u \in X_2$, which shows $Y_2 \subset X_2$. Starting with $x \in X_2$, the last argumentation shows also $X_2 \subset Y_2$. Since $X_1 = Y_1$ and $X_2 = Y_2$, we conclude from Proposition 3.15 that $M_2(\mathcal{C}_u^E(X))_+ = M_2((\mathcal{C}_u^E(X), \star))_+$. In fact, we have $\mathcal{C}_u^E(X) =_{cp} \mathcal{C}^E(X^-)$ and $\mathcal{C}_u^E(Y) =_{cp} \mathcal{C}^E(Y^-)$. Moreover, the 2-bipositive order isomorphism from Proposition 3.15 between $\mathcal{F}^E(X^-)$ and $\mathcal{F}^E(Y^-)$ restricts to a 2-bipositive order isomorphism between $\mathcal{C}^E(X^-)$ and $\mathcal{C}^E(Y^-)$, because $X_1 = Y_1$ and $X_2 = Y_2$. Consequently, we obtain a 2-bipositive order isomorphism between $\mathcal{C}_u^E(X)$ and $\mathcal{C}_u^E(Y)$, and since $(\mathcal{C}_u^E(X), \star)$ is a unital C^* -algebra we can identify $\mathcal{C}_u^E(Y) =_{cp} (\mathcal{C}_u^E(X), \star)$ by Theorem 3.73. So, $M_2(\mathcal{C}_u^E(X))_+ = M_2((\mathcal{C}_u^E(X), \star))_+$. Therefore the identity mapping id on $\mathcal{C}_u^E(X)$ is 2-positive from $(\mathcal{C}_u^E(X), \star)$ to $\mathcal{C}_u^E(X) \subset \mathcal{F}_b^E(X)$, and it is obviously a Jordan homomorphism (sometimes also called C^* -homomorphism) from the C^* -algebra $(\mathcal{C}_u^E(X), \star)$ into the bounded operators, so that by [58, Theorem 3.3] there are two orthogonal central projections $p, q \in C^*(\mathcal{C}_u^E(X))^-$, such that $p + q = \mathbb{1}$, $\pi_1(g) = gp$ is a $*$ -homomorphism, $\pi_2(g) = gq$ is a $*$ -anti-homomorphism and $\text{id} = \pi_1 + \pi_2$ as linear maps. Since id is 2-positive it follows that π_2 is 2-positive. But by [17, Cor. 3.2] this implies that π_2 must also be a $*$ -homomorphism. Hence id is a $*$ -homomorphism. Thus for $f, g \in \mathcal{C}_u^E(X)$ we conclude that $fg = \text{id}(f) \text{id}(g) = \text{id}(f \star g) \in \mathcal{C}_u^E(X)$. This means $\mathcal{C}_u^E(X)$ is closed under the product it inherits from $\mathcal{F}_b^E(X)$, so that we have shown that $\mathcal{C}_u^E(X)$ is a C^* -subalgebra of $\mathcal{F}_b^E(X)$ and the proof is complete. \square

Non-commutative topological spaces

The original aim of my thesis was to characterize which compact and m -convex sets are the matrix state spaces of C^* -algebras. This aim was achieved in the preceding section in Theorem 3.83. However, looking at the axioms characterizing the matrix state space of C^* -algebras it is obvious that an essential part of the requirements is concerned only with the structural elements of compact and m -convex set, that is, with the pure matrix states of the C^* -algebra. Moreover, in Theorem 3.53 we have provided already a characterization of the matrix sets that are the normal pure matrix state spaces of atomic W^* -algebras, and so in a certain sense are non-commutative sets. We will now abstractly characterize which matrix sets are the pure matrix state spaces of C^* -algebras. Thus in a certain sense we will provide an abstract definition of non-commutative topological spaces. We start with a canonical embedding similar to Proposition 3.32.

Proposition 3.84. *Let X be an equivariant matrix subset of some locally convex vector space V such that $X_n^- \subset M_n(V)$ is a compact subset for all $n \in \mathbb{N}$, where $M_n(V)$ carries the product topology. Let $K = CS(\mathcal{C}_u^E(X))$ be the matrix convex state space of the operator system $\mathcal{C}_u^E(X)$. Then the map $\Theta = (\Theta_n)_{n \in \mathbb{N}}$, where $\Theta_n: X_n \rightarrow K_n$ is defined by $\Theta_n(x)(f) = \tilde{x}(f) = f_n(x)$ for all $f = (f_l) \in \mathcal{C}_u^E(X)$, $x \in X_n$ and $n \in \mathbb{N}$, is an equivariant uniform equivalence onto its image $\tilde{X} = (\tilde{X}_n)_{n \in \mathbb{N}} = (\Theta_n(X_n))_{n \in \mathbb{N}}$. Furthermore, we have*

Proof. Obviously, if X is equivariant, then X^- is also equivariant. Recall that uniformly continuous maps on X_n have unique continuous extensions on X_n^- , because X_n^- is compact for all $n \in \mathbb{N}$, cf. [39, 51]. Therefore we can identify $\mathcal{C}_u^E(X)$ with $\mathcal{C}^E(X^-)$. Then we define an extension of Θ on X^- , that we still call Θ , by setting $\Theta_n(y)(f) = f_n(y)$ for all $f = (f_l) \in \mathcal{C}^E(X^-)$ and $y \in X_n^-$. We will first show that Θ is injective. Assume that we have $x, y \in X_n^-$ such that $\tilde{x} = \tilde{y}$. Then $f_n(x) = f_n(y)$ for all $f \in \mathcal{C}^E(X^-)$. We conclude that $g(x_{ij}) = g(y_{ij})$ for all $i, j \in \{1, \dots, n\}$ and all $g \in V'$, where V' are the continuous linear maps from V to \mathbb{C} . This follows, since obviously $(g^{(n)}|_{X_n}) \in \mathcal{C}^E(X^-)$ for $g \in V'$. Thus we have shown that $x = [x_{ij}] = [y_{ij}] = y$, so that Θ_n is injective. Since the argument applies for all $n \in \mathbb{N}$, the map Θ is injective. From $\Theta_m(u^*xu)(f) = f_m(u^*xu) = u^*f_n(x)u = u^*\tilde{x}(f)u$ for all $f \in \mathcal{C}^E(X^-)$ we obtain immediately that Θ is an equivariant map, and since Θ is injective its inverse map is obviously equivariant, too. If $x \in X_n^-$, then there is a net (x_ν) in X_n converging to x . Obviously $\tilde{x}_\nu(f) = f_n(x_\nu) \rightarrow f_n(x) = \tilde{x}(f)$ for all $f \in \mathcal{C}^E(X^-)$, which shows that Θ_n is a continuous map from X_n^- to K_n for all $n \in \mathbb{N}$. Since X_n^- is compact, Θ_n is injective and the w^* -topology is Hausdorff, Θ_n is a homeomorphism onto its image for all $n \in \mathbb{N}$.

We claim now that $\Theta_n(X_n^-) = (\tilde{X}_n)^-$. Given $\varphi \in \Theta_n(X_n^-)$ there is $x \in X_n^-$ such that $\varphi = \Theta_n(x)$. If $(x_\nu)_\nu$ is a net in X_n such that $x_\nu \rightarrow x$, we see that $\Theta_n(x_\nu) = \tilde{x}_\nu \in \tilde{X}_n$, and obviously $\tilde{x}_\nu(f) = f_n(x_\nu) \rightarrow f_n(x) = \tilde{x}(f)$ for all $f \in \mathcal{C}^E(X)$. Therefore, $\varphi \in \tilde{X}_n^-$. Conversely, given $\varphi \in (\tilde{X}_n)^-$ there is a net (x_ν) in X_n such that $\tilde{x}_\nu \rightarrow \varphi$, that is $\tilde{x}_\nu(f) = f_n(x_\nu) \rightarrow \varphi(f)$ for all $f \in \mathcal{C}^E(X^-)$. Since X_n^- is compact there is $x \in X_n^-$ and a subnet $(x_{h(\nu)})$ of (x_ν) such that $x_{h(\nu)} \rightarrow x$. It follows immediately that $f_n(x_{h(\nu)}) \rightarrow f_n(x)$ for all $f \in \mathcal{C}^E(X^-)$ and, since $(x_{h(\nu)})$ is a subnet of (x_ν) we also have $f_n(x_{h(\nu)}) \rightarrow \varphi(f)$ for all $f \in \mathcal{C}^E(X^-)$. Hence $\varphi(f) = f_n(x) = \tilde{x}(f)$ for all $f \in \mathcal{C}^E(X^-)$, so that we have found $x \in X_n^-$ such that $\varphi = \tilde{x}$. Thus the claim $(\tilde{X}_n)^- = \Theta_n(X_n^-)$ is proved. \square

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Remark 3.85. Given an equivariant matrix set X as in Proposition 3.84, we observe that we also can embed X^{tr} into the dual of $\mathcal{C}_u^E(X) = \mathcal{C}^E(X^-)$, namely via the map $z \mapsto \tilde{z}$ defined by $\tilde{z}(f) = (f_n(z^{\text{tr}}))^{\text{tr}}$ for all $z \in X_n^{\text{tr}}$ and $n \in \mathbb{N}$. Notice further that given $n \in \mathbb{N}$, $x \in X_n^-$ and $z \in X_n^{\text{tr}}$ such that $\hat{x} = \tilde{z}$ we conclude that $x = z$. In fact, as in the proof of Proposition 3.84, it follows from $f_n(x) = \hat{x}(f) = \tilde{z}(f) = (f_n(z^{\text{tr}}))^{\text{tr}}$ for all $f \in \mathcal{C}_u^E(X^-)$ that especially $g^{(n)}(x) = (g^{(n)}(z^{\text{tr}}))^{\text{tr}} = g^{(n)}(z)$ for all $g \in V'$, which shows directly $x = z$.

To characterize the pure m-state space we will need the next essentially known lemma.

Lemma 3.86. *Let \mathcal{M} be an atomic W^* -algebra and identify $\mathcal{M} = \mathcal{F}_b^E(X)$, where $X = \text{str}(CS^\sigma(\mathcal{M}))$. If $\psi \in \mathcal{M}_*$ such that $\psi = \psi^*$ then there is a set $\{x_n \mid n \in \mathbb{N}\}$ of pairwise orthogonal elements of X_1 and a sequence of numbers $(r_n)_{n \in \mathbb{N}}$ such that $\psi(f) = \sum_n r_n f_1(x_n)$ for all $f \in \mathcal{F}_b^E(X)$.*

Proof. Since we can identify $\mathcal{M} = \oplus \mathcal{B}(H_\kappa)$, it suffices to consider the special case where $\mathcal{M} = \mathcal{B}(H)$ and $\mathcal{M}_* = \mathcal{T}(H)$. If $\psi \in \mathcal{M}_*$ and $\psi = \psi^*$, then there is $T \in \mathcal{T}(H)$ such that $\psi(f) = \text{trace}(fT)$ for all $f \in \mathcal{F}_b^E(X) = \mathcal{B}(H)$. Obviously, T is a compact and self-adjoint operator. Therefore by the spectral theorem there are sequences of orthonormal vectors $(\xi_n)_n$ in H and real numbers $(r_n)_n$ such that $T = \sum_n r_n \xi_n \odot \xi_n$. Then $\psi(f) = \sum_n r_n \langle f \xi_n | \xi_n \rangle = \sum_n r_n f_1(x_n)$, where $x_n = \langle \cdot | \xi_n \rangle \xi_n \in X_1$ is a pairwise m-orthogonal sequence of normal pure states, cf. Proposition 3.7. \square

For $x \in X_1$ we define $\mathcal{I}(x) = \{f \in \mathcal{C}_u^E(X)_+ \mid f_1(x) = 0\}$. Notice that $\mathcal{I}(x)$ is a norm-closed hereditary cone in $\mathcal{C}_u^E(X)_+$ that does not contain the unit.

Theorem 3.87 (Characterization of pure m-state spaces). *Let $X = (X_n)_n$ be a matrix subset of some locally convex vector space V and give $M_n(V)$ the product topology for all $n \in \mathbb{N}$. Then X is equivariantly and uniformly isomorphic to the pure m-states of a unital C^* -algebra if and only if all of the following axioms hold:*

- (i) $X_n^- \subset M_n(V)$ is a compact subset for all $n \in \mathbb{N}$,
- (ii) X is equivariant, transitive and fulfills the uniqueness property,
- (iii) (X_1, d) is a complete metric space, where d is the inner metric of X ,
- (iv) if $\{x_n \mid n \in \mathbb{N}\}$ is a set of pairwise orthogonal elements of X_1 and $(r_n)_{n \in \mathbb{N}}$ is a sequence of real number such that $\sum_n |r_n| < \infty$, then there is $f \in \mathcal{C}_u^E(X)_h$ such that $\sum_n r_n f_1(x_n) \neq 0$,
- (v) the uniformity on X fulfills the Jordan property,
- (vi) $\mathcal{I}(x)$ is maximal in the set of all norm-closed hereditary cones of $\mathcal{C}_u^E(X)_+$ for all $x \in X_1$ and the sets $\{x \in X_1 \mid f_1(x) = 0 \text{ for all } f \in \mathcal{I}\}$ are non-empty for all maximal norm-closed hereditary cones $\mathcal{I} \subset \mathcal{C}_u^E(X)_+$ not containing the unit, and
- (vii) $X_2^- \cap X_2^{\text{tr}} = \emptyset$.

Proof. Let \mathcal{A} be a unital C^* -algebra and $X = \text{str}(CS(\mathcal{A}))$ its pure matrix state space. Since $CS(\mathcal{A})$ is w^* -compact in the dual \mathcal{A}^* , obviously $X_n^- \subset CS_n(\mathcal{A}) \subset M_n(\mathcal{A})$ is compact for all $n \in \mathbb{N}$. We proved that X is equivariant and transitive in Proposition 3.4. It is immediate from Proposition 3.8 that X fulfills the uniqueness property. Moreover, we can identify \mathcal{A} with $\mathcal{C}_u^E(X)$ by Theorem 3.73 and the atomic part of \mathcal{A}^{**} with $\mathcal{F}_b^E(X)$

by Proposition 3.70, so that X are the normal pure m-states of $\mathcal{F}_b^E(X)$, cf Theorem 3.53, which in particular shows that axiom (iii) holds. Moreover, it is known that \mathcal{A} lies w^* -dense in the atomic part of its bidual, so $\mathcal{C}_u^E(X)$ is a w^* -dense C^* -subalgebra of $\mathcal{F}_b^E(X)$. Then Lemma 3.86 implies axiom (iv). Axiom (v) is shown by Proposition 3.78. Since $\mathcal{C}_u^E(X)$ is a unital C^* -algebra, axiom (vi) is nothing but a translation of the well-known order preserving correspondence between closed hereditary cones of $\mathcal{C}_u^E(X)_+$ and closed left ideals of $\mathcal{C}_u^E(X)$, recalling that the pure states of $\mathcal{C}_u^E(X)$ (i.e., X_1) correspond with the (regular) maximal left ideals of $\mathcal{C}_u^E(X)$. Finally, to show axiom (vii), notice that the transpose of an element of X_2 , which are the pure 2×2 matrix states of \mathcal{A} , cannot be 2-positive according to Lemma A.6. So, especially, it cannot be an element of $X_2^- \subset CS_2(\mathcal{A})$.

Conversely, let X be a matrix set such that the axioms (i) to (vii) hold. From (ii) and (iii) we know that $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra and that $\widehat{X} = \text{str}(CS^\sigma(\mathcal{F}_b^E(X)))$, cf. Corollary 3.27 and Theorem 3.53. Let $K = CS(\mathcal{C}_u^E(X))$ and let $S = \sigma\text{-mco}(\widehat{X})$ be the normal m-state space of $\mathcal{F}_b^E(X)$. Using axiom (i) and Proposition 3.84 we can identify $X = \widetilde{X} \subset K$. Notice that there is an m-affine surjection from S onto $F = \sigma\text{-mco}(\widetilde{X})$. In fact, let $\rho: \mathcal{F}_b^E(X)^* \rightarrow \mathcal{C}_u^E(X)^*$ be the surjection $\rho(\psi) = \psi|_{\mathcal{C}_u^E(X)}$ for $\psi \in \mathcal{F}_b^E(X)^*$. Then the restrictions of the amplifications $\theta_n = \rho^{(n)}|_{S_n}$ define an m-affine map $\theta = (\theta_n)_{n \in \mathbb{N}}$ from S to K . We have to show $\theta_n(S_n) = F_n$ for all $n \in \mathbb{N}$. Obviously, for $x \in X_n$ we have $\theta_n(\hat{x}) = \hat{x}|_{\mathcal{C}_u^E(X)}$ and $\hat{x}(f) = f_n(x) = \tilde{x}(f)$ for all $f \in \mathcal{C}_u^E(X)$. So $\theta_n(\widehat{X}_n) = \widetilde{X}_n$ for all $n \in \mathbb{N}$. Now for a σ -matrix convex combination $\psi = \sum_{i=1}^\infty \alpha_i^* \tilde{x}_i \alpha_i \in F_n$ such that $\tilde{x}_i \in \widetilde{X}_{n_i}$ with $n_i \leq n$ for all $i \in \mathbb{N}$ we see that $\varphi = \sum_{i=1}^\infty \alpha_i^* \tilde{x}_i \alpha_i \in S_n$ and $\theta_n(\varphi) = \psi$. Thus $\theta_n(S_n) = F_n$ for all $n \in \mathbb{N}$, which shows that θ is a surjection from S onto F . By axiom (iv) $\mathcal{C}_u^E(X)$ is w^* -dense in $\mathcal{F}_b^E(X)$. In fact, suppose for contradiction that there would be $f \in \mathcal{F}_b^E(X)_h$ such that f is not in the w^* -closure of $\mathcal{C}_u^E(X)_h$. Then there is a self-adjoint ψ in the predual $\mathcal{F}_b^E(X)_*$ such that $\psi(f) > 0$ and $\psi(\mathcal{C}_u^E(X)_h) = \{0\}$. By Lemma 3.86 there is a sequence $(x_n)_n$ of pairwise orthogonal pure states and a sequence $(r_n)_n$ of real numbers with $\sum_n r_n < \infty$ such that $\psi(g) = \sum_n r_n g_1(x_n)$ for all $g \in \mathcal{F}_b^E(X)$. Now, by axiom (iv) there is $h \in \mathcal{C}_u^E(X)_h$ such that $\psi(h) = \sum_n r_n h_1(x_n) \neq 0$, which is an obvious contradiction. Therefore, $\mathcal{C}_u^E(X)_h$ is w^* -dense in $\mathcal{F}_b^E(X)_h$, and so $\mathcal{C}_u^E(X)$ is w^* -dense in $\mathcal{F}_b^E(X)$. Then the restriction θ is an injective map and consequently θ is an m-affine isomorphism between S and F . Now axiom (v) ensures by Proposition 3.80 and Lemma A.7 that $\mathcal{C}_u^E(X)_h \subset \mathcal{F}_b^E(X)_h$ is a Jordan subalgebra. We will show next that \widetilde{X}_1 are exactly the pure states of the Jordan algebra $\mathcal{C}_u^E(X)_h$. For this let $x \in X_1$. Then axiom (vi) says that $\mathcal{I}(x)$ is a maximal norm-closed hereditary cone in $\mathcal{C}_u^E(X)_+$. Thus by [59, Thm. 7.1] in combination with [23, Thm. 2.3] the inner⁷ ideal $\mathcal{J} = \{f \in \mathcal{C}_u^E(X)_h \mid f^2 \in \mathcal{I}(x)\}$ of the Jordan algebra $\mathcal{C}_u^E(X)_h$ is maximal, so there is a pure state $\varphi: \mathcal{C}_u^E(X)_h \rightarrow \mathbb{R}$ such that the kernel of φ is \mathcal{J} . Moreover, the null space N_φ of the pure state φ is given by $N_\varphi = \mathcal{J} \circ \mathcal{C}_u^E(X)_h + \mathcal{J}$. By the Cauchy-Schwarz inequality of [46, Prop. 4.4 and Cor. 4.5] \tilde{x} vanishes on \mathcal{J} . Therefore \tilde{x} vanishes also on $\mathcal{J} \circ \mathcal{C}_u^E(X)_h + \mathcal{J}$, so that $N_\varphi \subset N_{\tilde{x}}$. Since the null spaces have codimension 1, they must coincide. Using that \tilde{x} and φ are unital it follows that $\tilde{x} = \varphi$. This shows that \widetilde{X}_1 is a subset of the pure states of $\mathcal{C}_u^E(X)_h$. Conversely, given a pure state φ of $\mathcal{C}_u^E(X)_h$ the kernel \mathcal{J} of φ is a maximal norm-closed inner ideal, cf. [59, Thm. 7.1]. Then the positive part \mathcal{J}_+ is a maximal norm-closed hereditary cone of $\mathcal{C}_u^E(X)_+$. Again by axiom (vi) there is $x \in X_1$

⁷Inner ideals are sometimes also called quadratic ideals.

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such that $f_1(x) = 0$ for all $f \in \mathcal{J}_+$. Therefore $\mathcal{I}(x) \supset \mathcal{J}_+$, and by the maximality of \mathcal{J} we obtain $\mathcal{I}(x) = \mathcal{J}_+$, which implies $\tilde{x} = \varphi$. So we have shown that \tilde{X}_1 are exactly the pure states of the Jordan algebra $\mathcal{C}_u^E(X)_h$. Now the σ -convex hull of the pure states $\sigma\text{-conv}(\tilde{X}_1) = F_1$ is a (split) face of K_1 , cf. [6, Cor. 5.63].

Thus given pure states \tilde{x}_1 and \tilde{x}_2 in K_1 the smallest face of K_1 containing \tilde{x}_1 and \tilde{x}_2 must be contained in the face F_1 , that is, we have $\text{face}(\tilde{x}_1, \tilde{x}_2) \subset F_1$. Then, in view of the m-affine isomorphism between S and F , $\text{face}(\tilde{x}_1, \tilde{x}_2)$ can be identified with $\text{face}(\hat{x}_1, \hat{x}_2)$ in S_1 . Since S_1 is the normal state space of an atomic W^* -algebra, S_1 has the 3-ball property. So either $\text{face}(\hat{x}_1, \hat{x}_2)$ is a point (if and only if $\hat{x}_1 = \hat{x}_2$), a line segment (if and only if \hat{x}_1 and \hat{x}_2 are non-equivalent pure states) or affine isomorph to a Euclidean 3-ball (if and only if \hat{x}_1 and \hat{x}_2 are distinct equivalent pure states). Hence evidently the affine isomorphism between S_1 and the face $F_1 \subset K_1$ shows directly that the state space K_1 of the Jordan algebra $\mathcal{C}_u^E(X)_h$ has the 3-ball property, too. We will prove that K_1 is globally orientable. For this we need to choose first an orientation for each facial 3-ball of K_1 . We identify a facial 3-ball of K_1 with $\text{face}(\tilde{x}_1, \tilde{x}_2)$, and give it the orientation of the affinely isomorphic $\text{face}(\hat{x}_1, \hat{x}_2) \subset S_1$. We obtain this orientation in the following way: There is $x \in X_2$ such that $x \succ x_1, x_2$. Then the orientation is given by the parametrization $\hat{x}^*(CS_1(M_2)) = \text{face}(\hat{x}_1, \hat{x}_2)$, cf. Remark 3.67.

Let \mathcal{X} and \mathcal{Y} be the sets of orientation preserving and orientation reversing maps in $\text{Param}(K_1)$. Then $\text{Param}(K_1)$ is the disjoint union of \mathcal{X} and \mathcal{Y} . We will show that both \mathcal{X} and \mathcal{Y} are closed (and therefore also open). Let $(\phi_\nu)_{\nu \in \Lambda}$ be a net in \mathcal{X} converging to $\phi \in \text{Param}(K_1)$. This means $\phi_\nu(b) \rightarrow \phi(b)$ in the topology of K_1 for all $b \in \mathbf{B}^3$. We can apply Proposition 3.66 to conclude that, since ϕ_ν is orientation preserving, there is $x_\nu \in X_2$ such that $\phi_\nu = \hat{x}_\nu^*$ for all $\nu \in \Lambda$. There exists also a x such that $\phi = \hat{x}^*$, where we have either $x \in X_2$ or $x \in X_2^{\text{tr}}$. Our claim is of course that x must be in X_2 , since this implies $\phi \in \mathcal{X}$. A short calculations gives

$$\langle \tilde{x}_\nu(f), \gamma \rangle = \langle \hat{x}_\nu(f), \gamma \rangle = \langle f, \hat{x}_\nu^*(\gamma) \rangle = \langle f, \phi_\nu(\gamma) \rangle \rightarrow \langle f, \phi(\gamma) \rangle = \langle \hat{x}(f), \gamma \rangle = \langle \tilde{x}(f), \gamma \rangle,$$

for all $\gamma \in CS_1(M_2)$ and $f \in \mathcal{C}_u^E(X)$. This shows that the net (\tilde{x}_ν) converges to \tilde{x} in the w^* -topology of the dual $\mathcal{C}_u^E(X)^*$. Thus $\tilde{x} \in (\tilde{X}_2)^-$, so we conclude by axiom (vii) and Remark 3.85 that $\tilde{x} \notin (\tilde{X}_2)^{\text{tr}}$. Consequently we must have $\tilde{x} \in \tilde{X}_2$ which shows $\phi \in \mathcal{X}$. As in the proof of Theorem 3.83, K_1 is globally orientable, so that by [6, Thm. 11.58] there exists a C^* -product \star on $\mathcal{C}_u^E(X)$ such that $f \star g + g \star f = fg + gf$ for all $f, g \in \mathcal{C}_u^E(X)_h$. Now, we have a C^* -algebra $(\mathcal{C}_u^E(X), \star)$, but we need to prove that the multiplication coincides with the one inherited from $\mathcal{F}_b^E(X)$. Notice that unlike in Theorem 3.83 we do not know that $\tilde{X}_n = \text{str}(K_n)$ for $n > 2$. However, it suffices that \hat{X} is the normal pure m-state space of the atomic W^* -algebra $\mathcal{F}_b^E(X)$. So, to show the claim, let the m-convex state space of $(\mathcal{C}_u^E(X), \star)$ be C and let $Y = \text{str}(C)$. We know that the order (on the first level) of $(\mathcal{C}_u^E(X), \star)$ coincides with the (pointwise) order of $\mathcal{C}_u^E(X)$, because \star is compatible with the Jordan product on $\mathcal{C}_u^E(X)_h$. Consequently, $C_1 = K_1$, i.e., both state spaces are the same sets. Thus we know also $Y_1 = \tilde{X}_1$. Given $y = [y_{ij}] \in Y_2$, there are $x_1, x_2 \in X_1$ such that $\tilde{x}_i = y_{ii}$ for $i = 1, 2$. We obtain from (the proof of) Proposition 3.64 applied to C as the m-convex normal state space of the atomic part of the bidual $(\mathcal{C}_u^E(X), \star)^{**}$ that $\text{mco}_1(y) = \text{face}(y_{11}, y_{22}) = \text{face}(\tilde{x}_1, \tilde{x}_2) \simeq \text{face}(\hat{x}_1, \hat{x}_2)$ is affinely isomorph to \mathbf{B}^3 . Thus by Remark 3.65 applied to $S = \sigma\text{-mco}(\hat{X})$, \hat{x}_1 and \hat{x}_2 are m-equivalent with respect to \hat{X} , so that by Lemma 3.47 there is $x \in X_2$ such that $\hat{x} \succ \hat{x}_1, \hat{x}_2$. Recall from the proof of Proposition 3.66 that we denote by \hat{x}^* and y^*

the restrictions of the dual maps of \hat{x} and y to $CS_1(M_2)$, so that \hat{x}^* and y^* are affine isomorphism from $CS_1(M_2)$ onto $\text{mco}_1(y) = \text{face}(y_{11}, y_{22}) = \text{mco}_1(\hat{x})$. Now, consider the affine automorphism $\phi^* = (\hat{x}^*)^{-1} \circ y^*: CS_1(M_2) \rightarrow CS_1(M_2)$. Notice from Remark 3.63 that ϕ^* has a unique extension to a linear map on M_2^* that we still denote by ϕ^* . Then a short calculation yields

$$\langle \phi(\tilde{x}(g)), \gamma \rangle = \langle \phi(\hat{x}(g)), \gamma \rangle = \langle \hat{x}(g), \phi^*(\gamma) \rangle = \langle \hat{x}(g), (\hat{x}^*)^{-1}(y^*(\gamma)) \rangle = \langle y(g), \gamma \rangle,$$

for all $g \in \mathcal{C}_u^E(X)$ and $\gamma \in M_2^*$. Thus $y = \phi(\tilde{x})$. Now the orientation of $\text{face}(y_{11}, y_{22})$ is given by the parametrization y^* as well as by \hat{x}^* . Then by definition of orientation (see [6, Def. 11.45]) the determinant of $\phi^* = (\hat{x}^*)^{-1} \circ y^*$ is 1, whereby we identify \mathbf{B}^3 with $CS_1(M_2)$ canonically, cf. Remark 3.62, so that \hat{x}^* and y^* can be read as orthogonal transformation on \mathbb{R}^3 , (see also the proof of Proposition 3.66). Since the determinant of ϕ^* is 1, ϕ is unitarily implemented, cf. [5, Thm. 4.34], so that there is a unitary $u \in M_2$ such that $y(g) = \phi(\tilde{x}(g)) = u^* \tilde{x}(g) u$ for all $g \in \mathcal{C}_u^E(X)$. Thus $y = u^* \tilde{x} u \in \tilde{X}_2$, which shows $Y_2 \subset \tilde{X}_2$. Starting with $x \in X_2$, the last argumentation shows also $\tilde{X}_2 \subset Y_2$. Since $\tilde{X}_1 = Y_1$ and $\tilde{X}_2 = Y_2$, we conclude that $M_2(\mathcal{C}_u^E(X))_+ = M_2((\mathcal{C}_u^E(X), \star))_+$, see the proof of Theorem 3.83. Thus the identity mapping id on $\mathcal{C}_u^E(X)$ is 2-positive from $(\mathcal{C}_u^E(X), \star)$ to $\mathcal{C}_u^E(X) \subset \mathcal{F}_b^E(X)$, and it follows exactly as in the proof of Theorem 3.83 that $\mathcal{C}_u^E(X)$ is closed under the product it inherits from $\mathcal{F}_b^E(X)$. Now, $\mathcal{C}_u^E(X)$ is a C^* -subalgebra of $\mathcal{F}_b^E(X)$ and the pointwise orderings $M_n(\mathcal{C}_u^E(X))_+$ coincide with the C^* -algebra ordering of $M_n(\mathcal{C}_u^E(X))$ for all $n \in \mathbb{N}$. In fact, by Proposition 3.22 the pointwise matrix orderings and the C^* -algebra orderings coincide for $\mathcal{F}_b^E(X)$, and in the proof of that proposition we have constructed for a given $f \in M_n(\mathcal{F}_b^E(X))_+$ a sequence of polynomials in f that converges in norm to the square root of f . Since $\mathcal{C}_u^E(X)$ is closed under multiplication and norm-closed, starting with $f \in M_n(\mathcal{C}_u^E(X))_+$ shows that the square root of f is in $M_n(\mathcal{C}_u^E(X))$, so that the matrix ordering of the C^* -algebra $\mathcal{C}_u^E(X)$ coincides with the given pointwise ordering. Thus $\tilde{C} = K$. Moreover, we know that $\tilde{X}_n = Y_n$ for $n = 1, 2$. We still have to verify that $\tilde{X}_n = Y_n$ for $n > 2$, i.e., that we can identify \tilde{X} with the pure m-states of $\mathcal{C}_u^E(X)$. Let $n > 2$ and $x \in X_n$. Then \hat{x} is a pure normal m-state of $\mathcal{F}_b^E(X)$. Hence there is a Hilbert space \mathcal{H} , an irreducible normal representation $\pi: \mathcal{F}_b^E(X) \rightarrow \mathcal{B}(\mathcal{H})$ and an isometry $\mathcal{V}: \mathbb{C}^n \rightarrow \mathcal{H}$ such that $\hat{x} = \mathcal{V}^* \pi \mathcal{V}$. Then $\pi(\mathcal{C}_u^E(X))^\perp = \mathcal{B}(\mathcal{H})$, so the restriction of π to the C^* -subalgebra $\mathcal{C}_u^E(X)$ is still an irreducible representation. Therefore $\tilde{x} = \mathcal{V}^* \pi|_{\mathcal{C}_u^E(X)} \mathcal{V}$ is pure. In the converse direction, let $y = [y_{ij}] \in Y_n$ be a pure m-state of $\mathcal{C}_u^E(X)$. Notice that $v^* y v \in Y_2 = \tilde{X}_2$ for all isometries $v \in M_{n,2}$. We conclude that y_{ij} is a normal map from $\mathcal{C}_u^E(X)$ to \mathbb{C} for all $i, j = 1, \dots, n$. Thus y is normal, so that y has a unique extension to a normal and unital map $\psi: \mathcal{F}_b^E(X) \rightarrow M_n$. Notice that ψ is n -positive, so that $\psi \in S_n$. In fact, since $\mathcal{C}_u^E(X)$ is w^* -dense in $\mathcal{F}_b^E(X)$, $M_n(\mathcal{C}_u^E(X))$ is w^* -dense in $M_n(\mathcal{F}_b^E(X))$, cf. Lemma A.5. Then by the Kaplansky density theorem the positive part of the unit ball of $M_n(\mathcal{C}_u^E(X))$ is w^* -dense in the positive part of the unit ball of $M_n(\mathcal{F}_b^E(X))$, cf. [52, (proof of) Thm. 1.9.1] (or [49]), from which it follows immediately that $\psi \in S_n$. We claim that ψ is pure. If $\psi = \sum_i \alpha_i^* \phi_i \alpha_i$ is a proper m-convex combination such that $\phi_i \in S_n$ and $\alpha_i \in M_n$, then especially $y(f) = \sum_i \alpha_i^* \phi_i(f) \alpha_i$ for all $f \in \mathcal{C}_u^E(X)$. Since y is pure, i.e., a structural element of K_n , there are unitaries $u_i \in M_n$ such that $y(f) = u_i^* \phi_i(f) u_i$ for all $f \in \mathcal{C}_u^E(X)$. Since $\mathcal{C}_u^E(X)$ is w^* -dense in $\mathcal{F}_b^E(X)$ we obtain $\psi = u_i^* \phi_i u_i$. Thus $\psi \in \text{str}(S_n) = \tilde{X}_n$, so that there is $x \in X_n$ such that $\psi = \hat{x}$. Consequently $y = \tilde{x}$. \square

Remark 3.88. For the commutative case notice that $X_n = \emptyset$ for all $n \geq 2$. Then the conditions of the theorem imply that X_1 is a compact Hausdorff space. In fact, axiom

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(vi) ensures that X_1 are exactly the extreme points in the state space of the commutative C^* -algebra $\mathcal{C}_u^E(X) = C_u(X_1)$. Moreover, we can always identify $\mathcal{C}_u^E(X) = A(K)$, where $K = CS(\mathcal{C}_u^E(X))$, and $A(K)$ and $A(K_1)$ are order isomorphic. So, in the commutative case we obtain $C_u(X) = A(K_1)$, where $X_1 = \text{ex}(K_1)$. Then it follows from [4, Thm. II.4.3] that K_1 is a Bauer Simplex, so that X_1 is closed and hence compact.

Concluding Remarks

The characterization results are open for improvements. Are there better axioms describing the matrix state spaces and the pure matrix state spaces? Can the results be proved without using the general theorem [6, Thm. 11.58] of Alfsen and Shultz, considering that we are already in the rather special situation $\mathcal{C}_u^E(X) \subset \mathcal{F}_b^E(X)$, where $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra?

The abstract description of the pure m-states of C^* -algebras as certain non-commutative topological spaces (containing the commutative situation as special case) gives mathematical content to the old fantasy that a C^* -algebra should be a non-commutative $C(X)$. (Curiously, in [25, p. 102] Effros used the notation $C_Q(X)$ for non-commutative C^* -algebras, where he called X a “virtual topological space”.)

There is a different approach to characterize the pure states abstractly as certain *Poisson spaces with transition probability*, see [43]. However, Landsman’s axioms 4 and 5 are clearly statements that should be derived from real axioms on the structures of the set he starts with (supposed to become the set of the pure states of a C^* -algebra). On the other hand, Landsman’s approach has a stronger appeal toward possible applications in quantum physics than the matrix order approach. Notice, though, that given a pure matrix state $x = [x_{ij}] \in X_n$ and an isometry $v \in M_{n,1}$, the pure state v^*xv is what physicists call a *superposition* of the orthogonal pure states x_{ii} , $1 \leq i \leq n$, which are the diagonal entries of the pure matrix state x . Moreover, notice that the conditions on a matrix set X to be equivariant, transitive and fulfilling the uniqueness property turn X_1 into a transition probability space, because $\mathcal{F}_b^E(X)$ is an atomic W^* -algebra. Then the transition probability between pure states $x, y \in X_1$ is given by $p^x(y) = p^y(x)$, where p^x and p^y are the minimal projections constructed in Proposition 3.24.

Finally, I hope that my dissertation might serve as starting point for a more systematic study of C^* -algebras. What additional, perhaps characterizing, properties do the pure matrix states have for certain special classes of C^* -algebras? As first example one could study the pure matrix state spaces of approximately finite dimensional algebras. Another question to investigate in connection with AF-algebras might be: Are there any relations between the matrix order approach and K -theory?

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This is a collection of essentially known results (mathematical ‘folklore’), where we did not find a references to the literature appropriate to our needs.

The next remark is contained in a somewhat more special version in [48, Thm. 5.1].

Remark A.1. Let V be a vector space. There is linear isomorphism between $L(V, M_n)$ and $L(M_n(V), \mathbb{C})$ given by

$$f_\phi(v) = \frac{1}{n} \alpha^* \phi^{(n)}(v) \alpha, \quad (\text{A.1})$$

where $\phi \in L(V, M_n)$ is given, $\alpha = e_1 \oplus \cdots \oplus e_n$, $v \in M_n(V)$ and $(e_i)_{i=1}^n$ denotes the standard basis of \mathbb{C}^n . The inverse mapping is

$$\phi_f(v) = n[f(v \otimes e_{ij})]. \quad (\text{A.2})$$

This isomorphism maps $CP(V, M_n)$ bijectively onto $L(M_n(V), \mathbb{C})_+$. Let V be a topological vector space. We give $M_n(V)$ the product topology. From this identification we see immediately that a net (f_λ) in $L(M_n(V), \mathbb{C})$ converges pointwise to $f \in L(M_n(V), \mathbb{C})$ if and only if (ϕ_{f_λ}) converges pointwise to $\phi_f \in L(V, M_n)$.

Remark A.2. Let V be a locally convex vector space and endow $M_n(V)$ with the product topology for all $n \in \mathbb{N}$. Suppose V is a matrix ordered vector space such that the cones $M_n(V)_+$ are closed for all $n \in \mathbb{N}$. If $v \notin M_n(V)_+$ then there is a continuous $\varphi \in CP(V, M_n)$ such that $\varphi^{(n)}(v) \not\geq 0$.

Proof. If $v \notin M_n(V)_+$, there is a continuous $f \in L(M_n(V), \mathbb{C})$ such that $f(M_n(V)_+) \geq 0$ and $f(v) < 0$. Hence ϕ_f is continuous and completely positive by Remark A.1. Moreover, it follows from equation (A.1) that $\phi_f^{(n)}(v) \not\geq 0$. \square

We need the following operator versions of well-known results in classical analysis.

Proposition A.3. *Let V be an operator space and give V^* the w^* -topology. We endow $M_n(V^*)$ with the product topology, which we call the w^* -product topology, for all $n \in \mathbb{N}$. Then $\text{Ball}(M_n(V^*))$ is compact with respect to the w^* -product topology for all $n \in \mathbb{N}$. Furthermore, the canonical image of $\text{Ball}(M_n(V))$ is dense in $\text{Ball}(M_n(V^{**}))$ with respect to the w^* -product topology for all $n \in \mathbb{N}$.*

Proof. Let (f_λ) be a universal net in $\text{Ball}(M_n(V^*))$. Let $v \in V$. Then $\|f_\lambda(v)\| \leq \|v\|$, so that the induced universal net $(f_\lambda(v))$ lies in a compact subspace of M_n and hence is convergent. Let $\lim_\lambda f_\lambda(v) = f(v)$ for all $v \in V$. We see that $f: V \rightarrow M_n$ is a linear mapping. Moreover, given $w = [w_{ij}] \in M_n(V)$, we obtain

$$f^{(n)}(w) = [f(w_{ij})] = [\lim_\lambda f_\lambda(w_{ij})] = \lim_\lambda [f_\lambda(w_{ij})] = \lim_\lambda f_\lambda^{(n)}(w).$$

It follows that $\|f^{(n)}(v)\| = \lim_\lambda \|f_\lambda^{(n)}(v)\| \leq \|v\|$ for all $v \in M_n(V)$. By [26, Proposition 2.2.2] $\|f\|_{cb} = \|f^n\| \leq 1$, so that $f \in \text{Ball}(M_n(V^*))$.

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For the second claim let C_n be the w^* -closure of the canonical image of $\text{Ball}(M_n(V))$ in $M_n(V^{**})$ for all $n \in \mathbb{N}$. Notice that $\text{Ball}(M_n(V^{**}))$ is w^* -compact for all $n \in \mathbb{N}$ by the first assertion that we have shown already. Hence the w^* -closed absolute matrix convex set $C = (C_n)_n$ is contained in $(\text{Ball}(M_n(V^{**})))_n$. Assume for contradiction that there is $\psi \in \text{Ball}(M_n(V^{**}))$ such that $\psi \notin C_n$. Then there exists $f \in M_n(V^*)$ such that $\|f^m(v)\| \leq 1$ for all $v \in \text{Ball}(M_m(V))$ and $m \in \mathbb{N}$ and $\|\psi^n(f)\| > 1$. By the first inequality $\|f\|_{cb} \leq 1$ and since $\|\psi\|_{cb} \leq 1$ it follows $\|\psi^n(f)\| \leq 1$ which is an obvious contradiction to the second inequality. Hence $C_n = \text{Ball}(M_n(V^{**}))$ for all $n \in \mathbb{N}$. \square

Proposition A.4. *Let (X, e) be an operator system that is the dual of a matrix ordered complete operator space V . Then there is a Hilbert space H and a unital complete order isomorphism $\pi: X \rightarrow \mathcal{B}(H)$ that also is a w^* - w^* -homeomorphism onto its image.*

Proof. From the proof of Theorem e is strictly positive and thus

$$K_n = \left\{ v \in M_n(V)_+ \mid e^{(n)}(v) = \mathbb{1}_n \right\}$$

for all $n \in \mathbb{N}$ defines an m-base of V . We interpret elements $v \in M_n(V)$ as maps from X to M_n by $v(x) = x^{(n)}(v)$ for $x \in X = V^*$. As in the proof of we set $M_{n_\varphi} = M_n$ for all $\varphi \in K_n$ and $n \in \mathbb{N}$. Then $\oplus M_{n_\varphi}$, where the sum runs over all $\varphi \in K_n$ and all $n \in \mathbb{N}$, is a unital C^* -algebra contained in $\mathcal{B}(H)$, where $H = \oplus \mathbb{C}^{n_\varphi}$, and we define

$$\pi: X \rightarrow \bigoplus_{\substack{\varphi \in K_n \\ n \in \mathbb{N}}} M_{n_\varphi} \subset \mathcal{B}(H) \text{ by } \pi(x) = \bigoplus_{\substack{\varphi \in K_n \\ n \in \mathbb{N}}} \varphi(x).$$

Obviously π is a unital and completely positive. Assume $\pi^{(n)}(x) \geq 0$ for $x \in M_n(X)$. Then in particular $\varphi^{(n)}(x) \geq 0$ for all $\varphi \in K_n$. Since K as m-base generates the matrix ordering of V it follows that $v^{(n)}(x) \geq 0$ for all $v \in M_n(V)_+$. Thus by Lemma A.2 we obtain $x \geq 0$. We have shown so far that π is a unital complete order isomorphism onto its image. Consequently, π is a complete isometry. Since all $\varphi \in K_n$ are obviously w^* - w^* -continuous for all $n \in \mathbb{N}$, it follows from the construction of π that π is continuous with respect to the w^* -topology on X and the weak operator topology on $\mathcal{B}(H)$, cf. [52, page 42]. Since π is injective and $\text{Ball}(X)$ is w^* -compact, the restriction $\pi: \text{Ball}(X) \rightarrow \pi(\text{Ball}(X))$ is a w^* - w^* -homeomorphism onto its image. Therefore, using that $\pi(\text{Ball}(X)) = \pi(X) \cap \text{Ball}(\mathcal{B}(H))$, it follows from applying the Krein-Smulian theorem (e.g., [50, Thm. 2.5.9]) that $\pi(X)$ is w^* -closed and that $\pi: X \rightarrow \pi(X)$ is open and thus a w^* - w^* -homeomorphism. \square

Lemma A.5. *Let \mathcal{M} be a W^* -algebra and \mathcal{M}_* its predual. \mathcal{M} carries the w^* -topology (i.e., the $\sigma(\mathcal{M}, \mathcal{M}_*)$ -topology). Then the w^* -topology of the W^* -algebra $M_n(\mathcal{M})$ coincides with the product topology on $M_n(\mathcal{M})$ for all $n \in \mathbb{N}$. That is, a net $(x^\nu)_\nu = (x_{ij}^\nu)_\nu$ in $M_n(\mathcal{M})$ converges to $x = [x_{ij}] \in M_n(\mathcal{M})$ with respect to the w^* -topology if and only if $\psi(x_{ij}^\nu) \rightarrow \psi(x_{ij})$ for all $\psi \in \mathcal{M}_*$ and $i, j = 1, \dots, n$.*

Proof. There is a Hilbert space H and a representation $\pi: \mathcal{M} \rightarrow \mathcal{B}(H)$ that is a homeomorphism onto its image with respect to the $\sigma(\mathcal{M}, \mathcal{M}_*)$ and $\sigma(\mathcal{B}(H), \mathcal{T}(H))$ topologies, where we identify the predual $\mathcal{B}(H)_*$ with the trace class operators $\mathcal{T}(H)$. Therefore \mathcal{M} is a w^* -closed C^* -subalgebra of $\mathcal{B}(H)$, and conversely all w^* -closed C^* -subalgebras of $\mathcal{B}(H)$ are W^* -algebras. Recall that on $\mathcal{B}(H)$ the w^* -topology coincides with the σ -weak

(or ultraweak) topology. We identify $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$. The σ -weak topology on $\mathcal{B}(H^n)$ is given by the functionals $a \mapsto \sum_l \langle a\xi_l | \eta_l \rangle$, where $(\xi_l)_l$ and $(\eta_l)_l$ are sequences in H^n such that $\sum_l \|\xi_l\|^2 < \infty$ and $\sum_l \|\eta_l\|^2 < \infty$. Reading $a \in \mathcal{B}(H^n)$ as matrix $[a_{ij}] \in M_n(\mathcal{B}(H))$ and letting $\xi_l = (\xi_{l,1}, \dots, \xi_{l,n})^{\text{tr}}$ and $\eta_l = (\eta_{l,1}, \dots, \eta_{l,n})^{\text{tr}}$ we obtain

$$\sum_l \langle a\xi_l | \eta_l \rangle = \sum_l \sum_{i=1}^n \left\langle \sum_{j=1}^n a_{ij} \xi_{l,j} \middle| \eta_{l,i} \right\rangle = \sum_{i,j=1}^n \sum_l \langle a_{ij} \xi_{l,j} | \eta_{l,i} \rangle, \quad (\text{A.3})$$

where $(\xi_{l,j})_l$ and $(\eta_{l,i})_l$ are sequences in H such that $\sum_l \|\xi_{l,j}\|^2 < \infty$ and $\sum_l \|\eta_{l,i}\|^2 < \infty$ for $i, j = 1, \dots, n$. Equation (A.3) shows that a net $(a^\nu)_\nu = ([a_{ij}^\nu])_\nu$ in $M_n(\mathcal{B}(H))$ converges in the product topology to $a = [a_{ij}] \in M_n(\mathcal{B}(H))$, i.e., $a_{ij}^\nu \rightarrow a_{ij}$ with respect to the $\sigma(\mathcal{B}(H), \mathcal{T}(H))$ -topology, if and only if it converges on $\mathcal{B}(H^n) = M_n(\mathcal{B}(H))$ with respect to the $\sigma(\mathcal{B}(H^n), \mathcal{T}(H^n))$ -topology. Since \mathcal{M} is a w^* -closed C^* -subalgebra of $\mathcal{B}(H)$, it follows that $M_n(\mathcal{M}) \subset M_n(\mathcal{B}(H))$ is a C^* -subalgebra that is closed with respect to the product topology, which coincides with the w^* -topology by the preceding argument. Therefore $M_n(\mathcal{M})$ is w^* -closed and hence itself a W^* -algebra in such a way that the w^* -topology of the W^* -algebra $M_n(\mathcal{M})$ coincides with the σ -weak (or w^* -) topology of $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$. \square

Lemma A.6. *Let \mathcal{A} be a unital C^* -algebra. If $y: \mathcal{A} \rightarrow M_2$ is a pure matrix state, then the transpose of y is not 2-positive.*

Proof. Let $y = \mathcal{V}^* \pi \mathcal{V}$ be the minimal Stinespring representation of y , where $\pi: \mathcal{A} \rightarrow \mathcal{B}(H)$ is a representation of \mathcal{A} on some Hilbert space H , and $\mathcal{V}: \mathbb{C}^2 \rightarrow H$ is an isometry. Since y is pure, π is irreducible. So, $\pi(\mathcal{A})$ is weakly dense in $\mathcal{B}(H)$. Consequently, $D = \pi^{(2)}(M_2(\mathcal{A}))$ is weakly dense in $M_2(\mathcal{B}(H))$. Since π is a complete isometry and a complete order isomorphism onto its image, we obtain

$$\left(\pi^{(2)}(\text{Ball}(M_2(\mathcal{A})_+)) \right)^- = (\text{Ball}(D)_+)^- \supset \text{Ball}(D^-)_+ = \text{Ball}(M_2(\mathcal{B}(H))_+) \quad (\text{A.4})$$

by applying the Kaplansky density theorem. Notice that given $\alpha \in M_2(M_2)_+$ and some isometry $\mathcal{W}: \mathbb{C}^4 \rightarrow H$, there exists $T \in M_2(\mathcal{B}(H))_+$ such that $\mathcal{W}^* T \mathcal{W} = \alpha$. Since

$$y^{(2)}(a) = [y(a_{ij})] = [\mathcal{V}^* \pi(a_{ij}) \mathcal{V}] = \begin{pmatrix} \mathcal{V}^* & 0 \\ 0 & \mathcal{V}^* \end{pmatrix} \pi^{(2)}(a) \begin{pmatrix} \mathcal{V} & 0 \\ 0 & \mathcal{V} \end{pmatrix}$$

for all $a = [a_{ij}] \in M_2(\mathcal{A})$, it follows from equation (A.4) that $y^{(2)}(\text{Ball}(M_2(\mathcal{A})_+))$ is dense in, and thus coincides with, $\text{Ball}(M_2(M_2)_+)$. Consequently there is $a \in M_2(\mathcal{A})_+$, such that

$$y^{(2)}(a) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

But then for the transpose of y we obviously obtain

$$y_{\text{tr}}^{(2)}(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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which is not a positive matrix. Hence the transpose y_{tr} cannot be 2-positive and the proof is complete. \square

Lemma A.7. *Let \mathcal{A} be a unital C^* -algebra and let $B \subset \mathcal{A}_h$ be a norm-closed subspace containing the unit of \mathcal{A} . Then the following are equivalent:*

- (i) B is closed under the map $b \mapsto b^+$.
- (ii) B is closed under the map $b \mapsto \varphi(b)$ for $\varphi \in C(\text{sp}(b))_h$.
- (iii) B is closed under the map $b \mapsto b^2$.

Proof. Let $b \in B$. Let $\text{sp}(b)$ denote the spectrum of b and define the set

$$Z_b = \{ \varphi \in C(\text{sp}(b))_h \mid \varphi(b) \in B \}.$$

The map $\varphi \mapsto \varphi(b)$ is a unital isometric $*$ -isomorphism from $C(\text{sp}(b))$ onto $C^*(b)$. So Z_b is a norm-closed self-adjoint subspace of $C(\text{sp}(b))_h$ that contains especially all linear functions $\xi \mapsto r\xi + s$, where $r, s \in \mathbb{R}$ and $\xi \in \text{sp}(b)$. Thus Z_b separates $\text{sp}(b)$.

Assume (i) and let $\varphi = \varphi^+ - \varphi^-$ be the unique decomposition of $\varphi \in C(\text{sp}(b))_h$ such that $\varphi^+, \varphi^- \geq 0$ and $\varphi^+ \varphi^- = 0$ in the C^* -algebra $C(\text{sp}(b))$. Then $\varphi(a) = \varphi^+(a) - \varphi^-(a)$ is the unique decomposition of $\varphi(a)$ into positive and negative parts in the C^* -algebra $C^*(a)$. Therefore if $\varphi \in Z_b$, then the positive and negative parts φ^+ and φ^- are in Z_b . It follows that Z_b is a sublattice of $C(\text{sp}(b))_h$, because for real-valued functions f, g the relations $|f| = f^+ + f^-$, $f \vee g = \frac{1}{2}(f + g + |f - g|)$ and $f \wedge g = \frac{1}{2}(f + g - |f - g|)$ hold. The lattice version of the Stone-Weierstrass theorem implies $Z_b = C(\text{sp}(b))_h$. So we have proved that (i) implies (ii). It is obvious that (ii) implies (iii), so we assume that (iii) holds and we will show that (i) follows. Since for $\varphi, \psi \in C(\text{sp}(b))_h$

$$(\varphi(b) + \psi(b))^2 - \varphi(b)^2 - \psi(b)^2 = \varphi(b)\psi(b) + \psi(b)\varphi(b),$$

we see that Z_b is a subalgebra of $C(\text{sp}(b))_h$. Hence by the Stone-Weierstrass theorem $Z_b = C(\text{sp}(b))_h$. Then obviously $\xi \mapsto \max(\xi, 0)$ is in Z_b , so that $b^+ \in B$, and the proof is complete. \square

About the non-unital case

Unfortunately in the literature Stinespring's theorem is proved only for unital C^* -algebras (even in new books like [26]). However, not all C^* -algebras have a unit and from discussing the m -convex state space of the compact operators $\mathcal{C}(H)$ on a Hilbert space H (to have a simple example) we experienced that it is sometimes awkward to adjoin a unit. Hence we take the occasion to give proofs of some results usually only stated for the unital case in the literature, but certainly true for the non-unital case, too.

Remark A.8. Let \mathcal{A} be a C^* -algebra and $\mathcal{A}^1 = \{ a \in \mathcal{A} \mid \|a\| < 1 \}$. Let Λ denote the positive part of the open unit ball of \mathcal{A} , i.e., $\Lambda = \mathcal{A}^1 \cap \mathcal{A}_+$. We define $e_\lambda = \lambda$ for $\lambda \in \Lambda$. Then $(e_\lambda)_{\lambda \in \Lambda}$ is an approximate order unit of \mathcal{A}_h and the C^* -norm of \mathcal{A}_h is the approximate order unit norm, i.e.,

$$\|a\| = \inf \{ r \geq 0 \mid \exists \lambda \quad -re_\lambda \leq a \leq re_\lambda \}$$

for any $a \in \mathcal{A}_h$. Moreover, (e_λ) is an approximate identity, i.e., for all $a \in \mathcal{A}$ we have $\|ae_\lambda - a\| \rightarrow 0$.

Proof. It is known from C^* -theory that the positive part $\mathcal{A}^1 \cap \mathcal{A}_+$ of the open unit ball is directed. This implies that the self-adjoint part of the open unit ball is directed. Indeed, let $a, b \in \mathcal{A}^1 \cap \mathcal{A}_h$. Then there are unique decompositions $a = a_+ - a_-$ and $b = b_+ - b_-$ such that $a_+ a_- = 0 = b_+ b_-$ and $a_+, a_-, b_+, b_- \in \mathcal{A}^1 \cap \mathcal{A}_+$. Then $|a| = a_+ + a_-$ and $|b| = b_+ + b_-$ are in $\mathcal{A}^1 \cap \mathcal{A}_+$ and there is $c \in \mathcal{A}^1 \cap \mathcal{A}_+$ such that $c \geq |a| \geq a$ and $c \geq |b| \geq b$. Thus $\mathcal{A}^1 \cap \mathcal{A}_h$ is directed. Moreover, C^* -algebras are 1-normal. To see this let $a \leq b \leq c$. Considering \mathcal{A} as subalgebra of some $\mathcal{B}(H)$, we obtain

$$\langle a\xi|\xi \rangle \leq \langle b\xi|\xi \rangle \leq \langle c\xi|\xi \rangle$$

for all $\xi \in H$. Thus $|\langle b\xi|\xi \rangle| \leq \max\{|\langle a\xi|\xi \rangle|, |\langle c\xi|\xi \rangle|\}$, which implies $\|b\| \leq \max\{\|a\|, \|c\|\}$. Hence from [47, Proposition 1] \mathcal{A}_h is an approximate order unit space with approximate order unit the positive part of the open unit ball, which by definition is the monotone increasing net (e_λ) .

The last statement, that (e_λ) is an approximate identity of \mathcal{A} , is well-known. \square

Theorem A.9 (Non-unital Stinespring). *Let \mathcal{A} be a C^* -algebra and let H be a Hilbert space. Then every completely positive and bounded mapping of $\varphi: \mathcal{A} \rightarrow \mathcal{B}(H)$ has the form $\varphi(x) = \mathcal{V}^* \pi(x) \mathcal{V}$, where π is a representation of \mathcal{A} on some Hilbert space H_π , such that $(\pi(e_\lambda))$ converges strongly to the identity of $\mathcal{B}(H_\pi)$, where (e_λ) is the canonical approximate unit of \mathcal{A} (i.e., the positive part of the open unit ball) and \mathcal{V} is a bounded operator from H to H_π . (Cf. [11, Theorem 1.1.1].)*

Proof. We consider the vector space tensor product $\mathcal{A} \otimes H$ and define a bilinear form $[\cdot, \cdot]$ on $\mathcal{A} \otimes H$ by

$$[u, v] = \sum_{i,j} \langle \varphi(y_i^* x_j) \xi_j | \eta_i \rangle,$$

where φ is the given mapping and $u = \sum_j x_j \otimes \xi_j$ and $v = \sum_i y_i \otimes \eta_i$ in $\mathcal{A} \otimes H$. Since φ is completely positive, $[\cdot, \cdot]$ is positive semi-definite. For each $x \in \mathcal{A}$ we define a linear transformation $\pi_0(x)$ on $\mathcal{A} \otimes H$ by $\sum_j x_j \otimes \xi_j \mapsto \pi_0(x) = \sum_j x x_j \otimes \xi_j$. π_0 is an algebra homomorphism for which $[u, \pi_0(x)v] = [\pi_0(x^*)u, v]$ for all $u, v \in \mathcal{A} \otimes H$. It follows that, for fixed u , $\rho(x) = [\pi_0(x)u, u]$ defines a positive linear functional on \mathcal{A} . Let $(e_\lambda)_{\lambda \in \Lambda}$ be the canonical approximate unit of \mathcal{A} . We will prove that $\lim_\lambda \rho(e_\lambda) = [u, u]$ (and hence $\rho(e_\lambda) \leq [u, u]$ for all λ , because the net of positive numbers is monotone increasing). Let $u = \sum_{j=1}^n x_j \otimes \xi_j$. Then

$$\begin{aligned} \rho(e_\lambda) - [u, u] &= [\pi_0(e_\lambda)u, u] - [u, u] \\ &= [\pi_0(e_\lambda)u - u, u] \\ &= \left[\sum_j (e_\lambda x_j - x_j) \otimes \xi_j, \sum_i x_i \otimes \xi_i \right] \\ &= \sum_{i,j=1}^n \langle \varphi(x_i^* (e_\lambda x_j - x_j)) \xi_j | \xi_i \rangle. \end{aligned}$$

Since φ is bounded and since $\|x_i^* (e_\lambda x_j - x_j)\| \leq \|x_i^*\| \|e_\lambda x_j - x_j\|$ converges against zero for the finitely many $i, j \in \{1, \dots, n\}$, we see that the sum above converges against zero. Hence $\rho(e_\lambda) \rightarrow [u, u]$. Now from Remark A.8 (e_λ) is also an approximate order unit, hence we find $\mu \in \Lambda$ such that

$$[\pi_0(x)u, \pi_0(x)u] = [\pi_0(x^*x)u, u] = \rho(x^*x) \leq \|x^*x\| \rho(e_\lambda)$$

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for all $\lambda \geq \mu$. This implies $[\pi_0(x)u, \pi_0(x)u] \leq \|x\|^2[u, u]$.

Let $N = \{u \in \mathcal{A} \otimes H \mid [u, u] = 0\}$. N is linear subspace of $\mathcal{A} \otimes H$, invariant under $\pi_0(x)$ for all $x \in \mathcal{A}$. Moreover, $[\cdot, \cdot]$ determines an inner product on the quotient space $\mathcal{A} \otimes H/N$ by $[u + N, v + N] = [u, v]$. We let H_π be the Hilbert space completion of the quotient. The preceding paragraph implies that there is a unique bounded representation $\pi: \mathcal{A} \rightarrow \mathcal{B}(H_\pi)$ that extends π_0 . Finally let $\xi \in H$. We will prove that the net $(e_\lambda \otimes \xi)$ converges to an element of H_π that we denote by $V\xi$. Letting $\lambda, \mu \in \Lambda$ such that $\lambda \geq \mu$ we find

$$\begin{aligned} \|e_\lambda \otimes \xi - e_\mu \otimes \xi\|^2 &= [(e_\lambda - e_\mu) \otimes \xi, (e_\lambda - e_\mu) \otimes \xi] \\ &= \langle \varphi((e_\lambda - e_\mu)^2) \xi | \xi \rangle \\ &\leq 2 \langle \varphi(e_\lambda - e_\mu) \xi | \xi \rangle. \end{aligned}$$

Since $\langle \varphi(e_\lambda) \xi | \xi \rangle$ converges as monotone increasing and bounded (by $\|\varphi\| \|\xi\|^2$) net, it follows that $(e_\lambda \otimes \xi)$ is a Cauchy net and hence is convergent in H_π . The convergence process is linear in ξ . Moreover,

$$\|e_\lambda \otimes \xi\|^2 = \langle \varphi(e_\lambda^2) \xi | \xi \rangle \leq \langle \varphi(e_\lambda) \xi | \xi \rangle \leq \|\varphi\| \|\xi\|^2,$$

which shows that $V\xi = \lim_\lambda (e_\lambda \otimes \xi)$ is a bounded operator. Moreover, for any $\xi, \eta \in H$ we have

$$\begin{aligned} [\pi(x)e_\lambda \otimes \xi, e_\mu \otimes \eta] &= [xe_\lambda \otimes \xi, e_\mu \otimes \eta] \\ &= \langle \varphi(e_\mu x e_\lambda) \xi | \eta \rangle \rightarrow \langle \varphi(x) \xi | \eta \rangle, \end{aligned}$$

because

$$\|e_\mu x e_\lambda - x\| \leq \|e_\mu\| \|x e_\lambda - x\| + \|e_\mu x - x\|.$$

On the other hand from another application of the triangle inequality together with the Cauchy-Schwarz inequality we get

$$|[\pi(x)e_\lambda \otimes \xi, e_\mu \otimes \eta] - [\pi(x)V\xi, V\eta]| \leq \|\pi(x)\| \|e_\lambda \otimes \xi - V\xi\| \|e_\mu \otimes \eta\| + \|\pi(x)V\xi\| \|V\eta - e_\mu \eta\|$$

Consequently

$$[\pi(x)e_\lambda \otimes \xi, e_\mu \otimes \eta] \rightarrow [\pi(x)V\xi, V\eta] = \langle V^* \pi(x) V \xi | \eta \rangle,$$

This shows $\varphi(x) = V^* \pi(x) V$ for all $x \in \mathcal{A}$. □

Remark A.10. Let $\pi: \mathcal{A} \rightarrow \mathcal{B}(H_\pi)$ be a representation of the C^* -algebra \mathcal{A} . If $(\pi(e_\lambda))$ converges strongly to the identity of $\mathcal{B}(H_\pi)$, where (e_λ) is the canonical approximate order of \mathcal{A} , then we call π an *approximately unital* representation of \mathcal{A} .

Of course, the Stinespring representation is not unique. However by passing to the so-called *minimal* Stinespring representation, we get a uniqueness result up to unitary transformation. This is completely unrelated to the algebra having a unit or not. The theorem that we are after is a correspondence between pure maps and irreducible representations.

Theorem A.11. *Let \mathcal{A} be a C^* -algebra and H a Hilbert space. If $\varphi: \mathcal{A} \rightarrow \mathcal{B}(H)$ is a completely positive and bounded mapping and $\varphi = V^*\pi V$ is the minimal Stinespring representation of φ , then φ is pure if and only if π is irreducible. (Cf. [11, Corollary 1.4.3])*

Proof. We can use the minimal Stinespring representation from Theorem A.9. Then an inspection of the proofs of [11, Lemma 1.4.1 and Theorem 1.4.2] shows that there is only a single argument where the presence of a unit required there is used. This is in Lemma 1.4.1, where for minimal Stinespring representations $\varphi_i = V_i^*\pi_i V_i$, $i = 1, 2$ such that $\varphi_1 \leq_{cp} \varphi_2$, it is shown that a contraction T exists such that $TV_2 = V_1$ and $T\pi_2(x) = \pi_1(x)T$ for all $x \in \mathcal{A}$. The contraction T that is constructed in the proof fulfills $T\pi_2(x)V_2\xi = \pi_1(x)V_1\xi$ for all $x \in \mathcal{A}$. If \mathcal{A} has a unit, it is easily seen that $TV_2 = V_1$ since π_1 and π_2 are unital. But we can easily replace the unit here: We have $T\pi_2(e_\lambda)V_2\xi = \pi_1(e_\lambda)V_1\xi$ for all λ , where (e_λ) denotes the positive part of the open unit Ball of \mathcal{A} . It follows from Theorem A.9 that $\pi_i(e_\lambda)$ converges strongly to the unit of $\mathcal{B}(H_{\pi_i})$ for $i = 1, 2$. All the other parts of the proofs of [11, Lemma 1.4.1 and Theorem 1.4.2] apply verbatim without unit (using the non-unital Stinespring representation above). \square

About the general Weierstrass conjecture

We will state here shortly the result of [32] that the general Weierstrass conjecture is not related to the matrix order structure of a C^* -algebra. Rather it seems that the conjecture is only related to the order structure, that is the Jordan structure, of the self-adjoint part of the C^* -algebra.

Let \mathcal{A} be a unital C^* -algebra. For a Hilbert space H we let $P_{\mathcal{A}}(H)$ denote the set of all completely positive maps from \mathcal{A} to $\mathcal{B}(H)$ that are unital and pure. The general Weierstrass conjecture is that a C^* -subalgebra \mathcal{B} of \mathcal{A} containing the unit and separating the pure states must be \mathcal{A} .

Now the main theorem of [32] is that if \mathcal{B} separates the pure states, then \mathcal{B} separates also $P_{\mathcal{A}}(H)$ for all Hilbert spaces H . We will give a short proof of this observation.

Proposition A.12. *Let \mathcal{A} be a unital C^* -algebra. If $\mathcal{B} \subset \mathcal{A}$ is a C^* -subalgebra containing the unit and separating the pure states of \mathcal{A} , then \mathcal{B} separates also $P_{\mathcal{A}}(H)$ for all Hilbert spaces H .*

Proof. Let $\phi, \psi \in P_{\mathcal{A}}(H)$ and assume that $\phi(b) = \psi(b)$ for all $b \in \mathcal{B}$. Obviously, $\langle \phi(\cdot)\xi, \xi \rangle$ and $\langle \psi(\cdot)\xi, \xi \rangle$ are pure states of \mathcal{A} for all unit vectors $\xi \in H$. In fact, only notice that ϕ and ψ correspond to irreducible representations via the minimal Stinespring representation, since they are pure. Now, by assumption $\langle \phi(b)\xi, \xi \rangle = \langle \psi(b)\xi, \xi \rangle$ for all $b \in \mathcal{B}$. Since \mathcal{B} separates the pure states, we obtain $\langle \phi(a)\xi, \xi \rangle = \langle \psi(a)\xi, \xi \rangle$ for all $a \in \mathcal{A}$. Since $\xi \in H$ is an arbitrary unit vector it follows immediately from the polarization identity that $\phi = \psi$, which is the claim. \square

The matrix ordering of \mathcal{A} can be described by the pure matrix states $\text{str}(CS(\mathcal{A}))$. Hence applying the last result in the special cases $H = \mathbb{C}^n$ for all $n \in \mathbb{N}$ indicates that the Weierstrass conjecture isn't related to the matrix ordering of \mathcal{A} , since separating the pure states implies already separating the pure matrix states (on all matrix levels).

A. Miscellaneous

The longer proof of the last result contained in [32] also proved at the same time that a rich C^* -subalgebra separates the pure states. However, this can also be obtained independently in the usual way. We will provide a detailed proof after recalling the definition of the term rich C^* -subalgebra.

Definition A.13. Let \mathcal{A} be a C^* -algebra. A C^* -subalgebra $\mathcal{B} \subset \mathcal{A}$ is called *rich* if the following holds:

- (i) If π is an irreducible $*$ -homomorphism of \mathcal{A} then $\pi|_{\mathcal{B}}$ is an irreducible $*$ -homomorphism of \mathcal{B} .
- (ii) If π and π' are inequivalent and irreducible $*$ -homomorphisms of \mathcal{A} then $\pi|_{\mathcal{B}}$ and $\pi'|_{\mathcal{B}}$ are inequivalent and irreducible $*$ -homomorphisms of \mathcal{B} .

Proposition A.14. Let $\mathbb{1} \in \mathcal{B} \subset \mathcal{A}$. Then \mathcal{B} separates $P(\mathcal{A})$ if and only if \mathcal{B} is a rich C^* -subalgebra of \mathcal{A} .

Proof. If \mathcal{B} separates $P(\mathcal{A})$ then \mathcal{B} is a rich C^* -subalgebra of \mathcal{A} , cf. [21, 11.1.7]. Conversely, let \mathcal{B} be a rich C^* -subalgebra of \mathcal{A} and $\psi_1, \psi_2 \in P(\mathcal{A})$ such that $\psi_1(b) = \psi_2(b)$ for all $b \in \mathcal{B}$. By Stinespring theorem there are Hilbert spaces H_i , $*$ -homomorphisms $\pi_i: \mathcal{A} \rightarrow \mathcal{B}(H_i)$ and isometries $\mathcal{V}_i: \mathbb{C} \rightarrow H_i$, such that $\psi_i(a) = \mathcal{V}_i^* \pi_i(a) \mathcal{V}_i$ for all $a \in \mathcal{A}$ and $i = 1, 2$. Since ψ_i is pure, π_i is irreducible and by assumption $\pi_i|_{\mathcal{B}}$ is irreducible. Thus $\mathcal{V}_i^* \pi_i|_{\mathcal{B}} \mathcal{V}_i$ is a minimal Stinespring representation and because

$$\mathcal{V}_1^* \pi_1(b) \mathcal{V}_1 = \psi_1(b) = \psi_2(b) = \mathcal{V}_2^* \pi_2(b) \mathcal{V}_2$$

for all $b \in \mathcal{B}$, the representations π_1 and π_2 of \mathcal{B} are unitarily equivalent by the uniqueness of the minimal Stinespring representation. By assumption this implies that π_1 and π_2 are unitarily equivalent as representations of \mathcal{A} . Together this means that there are unitary operators $U, \tilde{U}: H_1 \rightarrow H_2$ such that $\mathcal{V}_1 = U^* \mathcal{V}_2$,

$$\begin{aligned} U^* \pi_2(b) U &= \pi_1(b), \quad b \in \mathcal{B} \\ \tilde{U}^* \pi_2(a) \tilde{U} &= \pi_1(a), \quad a \in \mathcal{A}. \end{aligned}$$

We let $\mathcal{W} = U \tilde{U}^*$ and see from the equations above that $\pi_2(b) \mathcal{W} = \mathcal{W} \pi_2(b)$ for all $b \in \mathcal{B}$. Thus $\mathcal{W} \in \pi_2(\mathcal{B})' = \mathbb{C} \mathbb{1}$ such that there is $\lambda \in \mathbb{C}$ with $\mathcal{W} = U \tilde{U}^* = \lambda \mathbb{1}$. Since \mathcal{W} is unitary we have $|\lambda|^2 = 1$. We obtain

$$\begin{aligned} \varphi_1(a) &= \mathcal{V}_1^* \pi_1(a) \mathcal{V}_1 = \mathcal{V}_1^* U^* U \pi_1(a) U^* U \mathcal{V}_1 \\ &= \mathcal{V}_2^* \mathcal{W} \pi_2(a) \mathcal{W}^* \mathcal{V}_2 \\ &= |\lambda|^2 \mathcal{V}_2^* \pi_2(a) \mathcal{V}_2 = \varphi_2(a). \end{aligned}$$

This shows the claim. □

List of Symbols

$\mathcal{A}(K)$	m-affine maps on K	10
$A_b(K)$	bounded m-affine maps on K	10
$A(K)$	continuous m-affine maps on K	11
$\text{amco}(Y)$	absolutely m-convex hull of Y	14
\mathbf{B}^3	closed unit ball of \mathbb{R}^3	79
\mathcal{B}	(together with \mathcal{OB}) facial 3-balls, see [6, Def. 11.48]	91
$\text{Ball}(V)$	(closed) unit ball of V	15
$\mathcal{B}(H)$	bounded operators on Hilbert space H	4
\mathbb{C}	complex numbers	1
\mathbb{C}^n	vector space $\mathbb{C} \times \cdots \times \mathbb{C}$	9
$CS^\sigma(\mathcal{M})$	normal m-state space of dual operator system \mathcal{M}	21
$\mathcal{C}_u^E(X)$	uniformly continuous equivariant maps on X	47
$\mathcal{C}^E(X)$	continuous equivariant maps on X	47
$\text{conv}(Y)$	convex hull of Y	14
$CP(V, W)$	completely positive maps	2
$CQ(Y)$	quasi matrix states of Y	6
$CS(X)$	matrix states of X	6
\mathfrak{E}	abelian points	86
$\{\varepsilon_1, \dots, \varepsilon_n\}$	standard basis of \mathbb{C}^n	45
$\text{ex}(C)$	extreme points of convex set C	7
$\mathcal{F}_b^E(X)$	bounded equivariant maps on X	47
$\mathcal{F}^E(X)$	equivariant maps on X	47
$\text{face}(x_1, x_2)$	smallest face containing x_1 and x_2	80
$\xi \odot \eta$	one dimensional trace class operator	64
\mathfrak{K}	set of the m-equivalence classes $\{[x] \mid x \in X\}$	66

List of Symbols

$\text{lin } F$	linear hull of the set F	26
$\text{mext}(K)$	matrix extreme points of K	8
$M_n(V)$	$M_{n,n}(V)$	1
$\text{mco}(Y)$	m-convex hull of Y	7
\mathbb{N}	natural numbers	1
\mathcal{OB}	oriented facial 3-balls, see [6, Def. 11.48]	91
p'	projection $p' = \mathbb{1} - p$	67
$\text{Param}(K_1)$	see [6, Def. 11.47] or [7, §. 7]	91
\mathbb{R}, \mathbb{R}_+	real, and positive real, numbers	2
r^{tr}	transpose matrix of r	1
$\sigma\text{-mco}(X)$	σ -matrix convex hull of X	74
$\text{span } F$	norm closure of the linear hull of F	9
$\text{str}(K)$	structural elements of K	7
$\mathcal{T}(H)$	trace class operators on Hilbert space H	64
$\mathcal{U}(x)$	unitary equivalence class of x	7
u_{xy}	isometry transforming y into x	47
$W^=$	norm closure of W	77
$[x]$	matricial equivalence class of x	45
$[x]$	compressions of x	58
X^-	(weak) closure of X	47
$x \frown y$	matricial relation	44
$x \perp y$	x and y are matrix orthogonal	44
$y \succ x$	if $x = u_{xy}^* y u_{xy}$	44
Y^\perp	m-orthogonal complement of the matrix set Y	44

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