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STOCHASTIC PROCESSES ON THE BASIS OF NEW MEASURE THEORY

HEINZ KÖNIG

Abstract. The present article describes the reformulation of certain basic structures, first in measure and integration as in the previous work of the author, and on this basis then in stochastic processes. Both times the aim is to overcome certain well-known substantial difficulties.

0. Introduction

In the present article the author wants to describe the reformulation of certain basic structures, first in measure and integration in his 1997 book [12] and in subsequent papers summarized in his survey articles [14][18], and on this basis then in stochastic processes [15][16][17]. The reasons were certain substantial difficulties with the traditional theories, which we start to recall and to which we shall come back. We also refer to the treatises listed in the references below.

The traditional abstract theory of measure and integration which emerged from the achievements of Borel and Lebesgue in the first two decades of the 20th century is burdened with its total limitation to sequential procedures and its neglect of regularity. The alternative concept of Bourbaki [2] which arose in the middle of the century was able to relieve this burden but produced new ones, first of all its Procrustean bed in topology. There is also a methodical point which then reappeared in the later sequential and non-sequential abstract variants: In spite of the deliberate and innovative turn from (often unnoticed) *outer* to explicit *inner* regularity, based on the profound rôle of compactness, one went on to produce the basic entities, now intended to be of decided *inner* character, with the weapons from the *outer* arsenal - a procedure which had to be repaired at once with that unfortunate construction named the *essential* one. All this has been made clear in [14].

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Note (added 31 May 2006): In the published final version of the present article some citations from [16] are incorrect, due to late changes in the numbering system of the respective periodical. However, in the present preprint of the article these citations have been corrected and marked with $\star$.

Since the famous 1933 work of Kolmogorov [11] the mathematical theory of probability is a part of abstract measure theory, and hence exposed to its imperfections as well. The central notion ever since is that of *stochastic processes*, its first systematic treatment is due 1953 to Doob [6]. A stochastic process amounts to a probability measure (*prob measure* for short), called its *canonical measure*, on a certain σ algebra in the *path space*, composed of *time domain* and *state space*. The probabilistic context requires that a stochastic process be rooted in the finite subsets of the time domain. Combined with the above limitation to sequential procedures this forces the σ algebra in question to consist of so-called *countably determined* subsets of the path space. In the all-important case of an uncountable time domain the class of these subsets is much too narrow, a fact which is the obvious reason for well-known serious difficulties with stochastic processes. Thus it enforces the ad hoc formation of an unforeseeable multitude of measure extensions of the canonical measure, as a rule in the guise of so-called *versions* of the stochastic process under consideration. A vast crowd of them turned out to be pathological, so that one has to find out the substantial ones.

A related issue is to detect those subsets of the path space which could be named the *essential* ones for the stochastic process, that is those subsets which support the essential features of the process, and which a priori can be far from obvious. The most prominent example of such an essential subset is the set of *continuous paths* for the traditional Wiener measure, the canonical measure of one-dimensional Brownian motion. Note that this particular example comes from experimental observations without participation of mathematics! In its more than fifty years the traditional theory of stochastic processes has not been able to produce an adequate notion of an essential subset. The usual attempt due to Doob defines these subsets to be the sets of *outer* canonical measure one; equivalent for a subset is that the canonical measure has a measure extension which lives on the set. But the class of these subsets turned out to be pathological as well.

Both times it appears to be a natural idea that the collection of measure extensions of the canonical measure needs a drastic and clever reduction. One could even think of a *unique* measure extension, provided that it has a wide domain in order to expose the full breadth of the process under consideration. But the traditional theory of stochastic processes did not contribute to this idea.

However, the new structure in measure and integration quoted at the outset [12][14] was able to achieve the aim in question and to resolve the connected problems with stochastic processes [15][16][17], after that its innovative force had already been confirmed through other applications. The decisive step is a new projective limit theorem of Kolmogorov type in terms of inner premeasures and in its τ (:=nonsequential) version, which in final form is in [16] theorem $9\star$. This τ theorem inspires an immediate variant of the concept of stochastic processes: after due modification of the structure in the state space one defines the new stochastic processes in terms of *inner τ premeasures* instead of measures. Then the fundamental extension procedure, which is the heart of the new structure, provides each new stochastic process with a unique and highly distinguished prob measure on the path space, called its *maximal measure*, defined on an immense domain which in particular reaches far beyond the class of countably determined subsets. After this it is natural to define the *essential subsets* for a new process to be those subsets of the path space on which the maximal measure of the process lives. For all that the new stochastic processes remain, in view of the above projective limit theorem, rooted in the finite subsets of the time domain like the traditional ones. Thus the new concept is able to unite two aspects which seemed to be incompatible under the former ones.

After this the main question is of course how the new stochastic processes are related to the traditional ones. The answer is that the two concepts are in one-to-one correspondence whenever the state space is a *Polish topological space*, on the traditional side equipped with its Borel σ algebra and on the new side with the lattice of its compact subsets. The correspondence is kind of a restriction and is as simple and natural as it could be, so that in practice the two kinds of stochastic processes can be identified. In particular, the maximal measure for the new process is an extension of the canonical measure for the traditional process.

The next question is on the notion of *essential* subsets: it is whether in case of a Polish state space the above *well-defined* essential subsets for a new stochastic process are in reasonable connection with the *dreamt-of* essential subsets for the related traditional process. In [15][16][17] we considered the two typical examples of the Wiener and Poisson processes with state space $\mathbb{R}$. Both times it turned out that in essence the new maximal measure lives on those subsets of the path space which are the classical examples of essential subsets in the traditional intuitive sense. This appears to be a pleasant confirmation, even though some contrast remains, for example with the set of càdlàg paths in the Poisson process.

In conclusion we want to refer to some work of predecessors. In the particular frame of *compactness* the classical Kolmogorov projective limit theorem has a variant for Radon measures which first appeared 1943 in Kakutani [9]. In this frame then the foundational problems for stochastic processes have been attacked in the 1959 paper of Nelson [19] and in the 1972 and 1980 books of Tjur [21][22]. In particular [22] chapter 10 contains a number of results on the above notion of *essential* subsets. But it becomes visible that beyond *compact* Polish state spaces an adequate treatment requires the new measure-theoretic foundations laid down in [12][14]. For time-honoured evidence we invoke the discussions in [5] and in the historical note of [2] chapter IX. We also note that the work of Nelson [19] and Tjur [21][22] did not at all find due attention in the subsequent literature on stochastic processes.

To be sure, the whole enterprise requires some trace of compactness. Yet the present work makes clear that this is *not* topological compactness, but rather the different and more flexible notion of *set theoretical τ compactness*, manifested in the formation of the lattice $\mathfrak{S}$ in section 3 below. Thus the usual projective limit theorem for Radon measures on Hausdorff spaces is not nearly as good as claimed in [4] p.65, because it does not even cover the simple example 3.5 below. Rather it seems that the true adequate projective limit theorem appears first in the present 3.1 = [16] theorem 11$\star$ with its extension [16] theorem 9$\star$, of course with their roots in [15] section 4. In view of all this it is plain that in the present context of stochastic processes, detached from the traditional abstract theory of measure and integration, the need is not at all for foundation upon *topological* concepts, but rather for a kind of measure-theoretic foundation which comprises the concepts of *regularity* and τ *continuity* - in short for a conception as developed in [12][14] and sketched in section 2 below. For one more evidence, Bourbaki has still not fulfilled his promise of 1952 to develop probability in his Treatise.

1. The Traditional Stochastic Processes

We fix an infinite index set T called the *time domain*, and a measurable space $(Y, \mathfrak{B})$, that is a nonvoid set Y equipped with a σ algebra $\mathfrak{B}$ in Y, called the *state space*. One forms the T-fold product set $X := Y^T$, called the *path space*, the members of which are the *paths* $x = (x_t)_{t \in T} : T \to Y$. For $t \in T$ let $H_t : X \to Y$ be the canonical projection $x \mapsto x_t$. In $X = Y^T$ one forms the finite-based product set system

$$\mathfrak{B}^{[T]} := \{ \prod_{t \in T} B_t : B_t \in \mathfrak{B} \ \forall \, t \in T \text{ with } B_t = Y \ \forall\!\forall \, t \in T \},$$

where $\forall\!\forall$ means *for all except for finitely many*, and the generated σ algebra $\mathfrak{A} := \mathrm{A}\sigma(\mathfrak{B}^{[T]})$, which is the smallest σ algebra $\mathfrak{A}$ in X such that the $H_t : X \to Y$ for all $t \in T$ are measurable $\mathfrak{A} - \mathfrak{B}$. It is well known that for uncountable T the formation $\mathfrak{A}$ is quite narrow, because its members $A \in \mathfrak{A}$ are *countably determined* in the sense that $A = \{ x \in X : (x_t)_{t \in D} \in E \}$ for some nonvoid countable $D \subset T$ and some $E \subset Y^D$.

It is this situation where the traditional notion of stochastic processes comes into existence: A *stochastic process* with time domain T and state space $(Y, \mathfrak{B})$, for T and $(Y, \mathfrak{B})$ for short, amounts to be a prob measure $\alpha : \mathfrak{A} \to [0, \infty[$ on $\mathfrak{A}$. In view of the size of the measurable space $(X, \mathfrak{A})$ it is a nontrivial problem how to produce such stochastic processes. The standard method is via *projective limits*.

Let I consist of the nonvoid finite subsets $p, q, \cdots$ of T. For $p \in I$ one forms the product set Y^p, with $H_p : X \to Y^p$ the canonical projection $x \mapsto (x_t)_{t \in p}$, and also the canonical projections $H_{pq} : Y^q \to Y^p$ for the pairs $p \subset q$ in I. In Y^p one forms the usual product set system $\mathfrak{B}^p := \mathfrak{B} \times \cdots \times \mathfrak{B}$ and the generated σ algebra $\mathfrak{B}_p := \mathrm{A}\sigma(\mathfrak{B}^p)$. Besides the prob measures $\alpha : \mathfrak{A} \to [0, \infty[$, that is the stochastic processes for T and $(Y, \mathfrak{B})$, one considers the families $(\beta_p)_{p \in I}$ of prob measures $\beta_p : \mathfrak{B}_p \to [0, \infty[$ which are *consistent* in the sense that $\beta_p = \beta_q(H_{pq}^{-1}(\cdot)) | \mathfrak{B}_p$ for all pairs $p \subset q$ in I (which makes sense because H_{pq} is measurable $\mathfrak{B}_q - \mathfrak{B}_p$).

Each prob measure $\alpha : \mathfrak{A} \to [0, \infty[$ produces such a consistent family $(\beta_p)_{p \in I}$ via $\beta_p = \alpha(H_p^{-1}(\cdot)) | \mathfrak{B}_p$ (which as before makes sense because H_p is measurable $\mathfrak{A} - \mathfrak{B}_p$). One notes that the correspondence $\alpha \mapsto (\beta_p)_{p \in I}$ is *injective*, but it need not be *surjective*. The consistent family $(\beta_p)_{p \in I}$ is called *solvable* iff it comes from some and hence from a unique prob measure $\alpha : \mathfrak{A} \to [0, \infty[$, called the *projective limit* of the family $(\beta_p)_{p \in I}$. Thus a stochastic process for T and $(Y, \mathfrak{B})$ can also be defined as a *solvable consistent family* $(\beta_p)_{p \in I}$, called the family of *finite-dimensional distributions* of the process.

There is a famous particular situation $(Y,\mathfrak{B})$ where all consistent families $(\beta_p)_{p\in I}$ for all T are solvable: it is the situation that Y is a *Polish topological space* and $\mathfrak{B}=\mathrm{Bor}(Y)$ its Borel σ algebra. This is the *projective limit theorem* due to Kolmogorov [11] chapter III section 4. The fundamental fact behind it is that in a Polish space Y all finite (and all locally finite) measures on $\mathrm{Bor}(Y)$ are inner regular with respect to the lattice $\mathrm{Comp}(Y)$ of its compact subsets, that means are *Radon* measures. The situation will be contained in the development of section 3 as a basic special case.

1.1 EXAMPLES. Let $T=[0,\infty[$ and $Y=\mathbb{R}$ with $\mathfrak{B}=\mathrm{Bor}(\mathbb{R})$. We fix a family $(\vartheta_t)_{t\in T}$ of prob measures $\vartheta_t:\mathfrak{B}\to[0,\infty[$ with $\vartheta_0=\delta_0|\mathfrak{B}$ which under convolution fulfils $\vartheta_s\star\vartheta_t=\vartheta_{s+t}$ for all $s,t\in T$. One proves that it produces a consistent family $(\beta_p)_{p\in I}$ of prob measures $\beta_p:\mathfrak{B}_p\to[0,\infty[$, defined to be $\beta_{\{t\}}=\vartheta_t$ for $t\in T$, and via induction for $q=\{t(0),t(1),\cdots,t(n)\}$ and $p=\{t(1),\cdots,t(n)\}$ with $0\leqq t(0)<t(1)<\cdots<t(n)$ to be

$$\beta_q\big(B(0)\times B(1)\times\cdots\times B(n)\big)$$
$$=\int\limits_{B(0)}\beta_{p-t(0)}\Big((B(1)-u)\times\cdots\times(B(n)-u)\Big)d\vartheta_{t(0)}(u)$$

with $B(0),B(1),\cdots,B(n)\in\mathfrak{B}$. The Kolmogorov theorem then furnishes the stochastic process $\alpha:\mathfrak{A}\to[0,\infty[$. The most prominent examples are

the *Wiener process* α for the $\vartheta_t:\vartheta_t(B)=\frac{1}{\sqrt{2\pi t}}\int\limits_B e^{-x^2/2t}dx$, and

the *Poisson process* α for the $\vartheta_t:\vartheta_t(B)=e^{-t}\sum\limits_{l=0}^{\infty}(t^l/l!)\delta_l(B)$,

with $B\in\mathfrak{B}$ and $t>0$.

The Wiener process $\alpha:\mathfrak{A}\to[0,\infty[$ has the physical interpretation of the one-dimensional Brownian motion. The experimental observations quoted above say that "all paths are continuous". In mathematical context this statement has the intuitive sense that "the subset $\mathrm{C}(T,\mathbb{R})$ of the path space $X=\mathbb{R}^T$ is essential for the process α". This cannot mean that $\mathrm{C}(T,\mathbb{R})$ has full measure for α, for $\mathrm{C}(T,\mathbb{R})$ is not countably determined and hence not in $\mathfrak{A}$. The traditional theory of stochastic processes proves that $\mathrm{C}(T,\mathbb{R})$ has *outer* α measure one, with the implications which result from the specialities of $\mathrm{C}(T,\mathbb{R})$. However, this result is clouded by the obvious fact that the complement $X\setminus\mathrm{C}(T,\mathbb{R})$ has outer α measure one as well. Thus it is reasonable to make a problem out of the true significance of *full outer α measure.* We start to describe the condition in a few lines in terms of some suitable equivalences, in independent notations up to the end of 1.2 below.

Let $K:\Omega\to X$ be a map between nonvoid sets Ω and X. For a σ algebra $\mathfrak{P}$ in Ω one defines the *direct image* $\vec{K}\mathfrak{P}:=\{A\subset X:K^{-1}(A)\in\mathfrak{P}\}$, which is a σ algebra in X. For a measure $P:\mathfrak{P}\to[0,\infty]$ on $\mathfrak{P}$ one defines the *direct image* $\vec{K}P:\vec{K}\mathfrak{P}\to[0,\infty]$ to be $\vec{K}P(A)=P(K^{-1}(A))$ for $A\in\vec{K}\mathfrak{P}$, which is a measure on $\vec{K}\mathfrak{P}$, and a prob measure when P is one. If P *lives* on the subset $T\subset\Omega$, that is if all $N\subset\Omega\setminus T$ are in $\mathfrak{P}$ with $P(N)=0$, then one verifies that $\vec{K}P$ lives on the image set $K(T)\subset K(\Omega)\subset X$.

Next if $\mathfrak{A}$ is a σ algebra in X, then $\mathfrak{A} \subset \vec{K}\mathfrak{P}$ means that $K : \Omega \to X$ is measurable $\mathfrak{P} - \mathfrak{A}$ in the usual sense. If $\alpha : \mathfrak{A} \to [0, \infty]$ is a measure on $\mathfrak{A}$, then $\alpha = \vec{K}P|\mathfrak{A}$ means that α is the image measure of P on $\mathfrak{A}$ under K in the usual sense. In this case one also says that $K : (\Omega, \mathfrak{P}, P) \to X$ is a *version* of α. In these terms one has the equivalences [16] proposition 3$\star$ which follow.

1.2 PROPOSITION. *Let* $\alpha : \mathfrak{A} \to [0, \infty[$ *be a prob measure on the measurable space* $(X, \mathfrak{A})$*, and define its outer envelope* $\alpha^\star : \mathfrak{P}(X) \to [0, \infty[$ *to be*

$$\alpha^\star(M) = \inf\{\alpha(A) : A \in \mathfrak{A} \textit{ with } A \supset M\} \quad \textit{for } M \subset X.$$

For a subset $C \subset X$ *then the following are equivalent.*

0) $\alpha^\star(C) = 1$ (in which case C is called *thick* for α).

1) α *has a version* $K : (\Omega, \mathfrak{P}, P) \to X$ *with image* $K(\Omega) \subset C$.

1') α *has a version* $K : (\Omega, \mathfrak{P}, P) \to X$ *with image* $K(\Omega) = C$.

2) α *has a measure extension* $\rho : \mathfrak{R} \to [0, \infty[$ *with* $C \in \mathfrak{R}$ *and* $\rho(C) = 1$.

2') α *has a measure extension* $\rho : \mathfrak{R} \to [0, \infty[$ *which lives on* C.

In this case α *has a unique minimal measure extension* $\rho : \mathfrak{R} \to [0, \infty[$ *which lives on* C (minimal with respect to the inclusion $\subset$ of domains). *This is* $\rho : \rho(R) = \alpha^\star(R \cap C)$ *on* $\mathfrak{R} := \{R \subset X : R \cap C = A \cap C \textit{ for some } A \in \mathfrak{A}\}$.

After this we return to the situation of the present section. We repeat [16] theorem 4$\star$ which demonstrates without doubt that the condition of outer α measure one has *no connection* with reasonable notions of essentialness.

1.3 THEOREM. *Fix an arbitrary path* $a = (a_t)_{t \in T} \in X$ *and form*

$$C(a) := \{x \in X : x_t = a_t \textit{ for all } t \in T \textit{ except countably many ones}\}.$$

Then $C(a)$ *has* $\alpha^\star(C(a)) = 1$ *for all stochastic processes* α *for* T *and* $(Y, \mathfrak{B})$. *Thus after* 1.2 *each such* α *has versions* $K : (\Omega, \mathfrak{P}, P) \to X$ *with* $K(\Omega) \subset C(a)$ *and measure extensions* $\rho : \mathfrak{R} \to [0, \infty[$ *which live on* $C(a)$.

Note that $C(a)$ is $= X$ when T is countable, but is of obvious smallness when T is uncountable, and then X is the disjoint union of myriads of such $C(a)$.

Proof. Fix $A \in \mathfrak{A}$ with $A \supset C(a)$. We prove that $A' = \varnothing$ and hence $A = X$. Let $A' = \{x \in X : (x_t)_{t \in D} \in E\}$ for some nonvoid countable $D \subset T$ and $E \subset Y^D$, and assume that $A' \neq \varnothing$. Take $u = (u_t)_{t \in T} \in A'$, and define $x = (x_t)_{t \in T}$ to be $x_t = u_t$ for $t \in D$ and and $x_t = a_t$ for $t \in T \setminus D$. Then $x \in A' \subset (C(a))'$, whereas $x \in C(a)$ by definition. Thus we obtain a contradiction. $\square$

Under the impression of this absurd collection of thick subsets, and hence of measure extensions and of versions for *each* traditional stochastic process $\alpha : \mathfrak{A} \to [0, \infty[$ with uncountable time domain T, we proceed to our new structure in measure and integration, in the hope to find clarification and simplification. The basic step in this new structure are parallel outer and inner extension procedures for certain set functions. For historical reasons the outer versions look more familiar, but in recent years the inner versions became more and more authoritative. Thus the basis in the present context will be the inner τ version.

2. The Inner Extension Theories

Let X be a nonvoid set. We start to recall the fundamental ideas 1914 of Carathéodory [3] on the extension of set functions. On the one hand he defines for a set function $\Theta : \mathfrak{P}(X) \to [0,\infty]$ with $\Theta(\varnothing) = 0$ the set system

$$\mathfrak{C}(\Theta) := \{A \subset X : \Theta(M) = \Theta(M \cap A) + \Theta(M \cap A') \ \forall\, M \subset X\},$$

the members of which are called *measurable* Θ. It turns out that $\Theta|\mathfrak{C}(\Theta)$ is a content on an algebra in X. Beyond $\Theta(\varnothing) = 0$ we define the class $\mathfrak{C}(\Theta) \subset \mathfrak{P}(X)$ as in [12] section 4, but we shall not need the explicit definition.

On the other hand Carathéodory defines for a set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a set system $\mathfrak{S}$ in X with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ the so-called *outer measure* $\varphi^\circ : \mathfrak{P}(X) \to [0,\infty]$ to be

$$\varphi^\circ(A) = \inf\{\sum_{l=0}^{\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } \bigcup_{l=0}^{\infty} S_l \supset A\}.$$

His main theorem then reads as follows. *If $\varphi : \mathfrak{S} \to [0,\infty]$ is a content on a ring and upward σ continuous, then $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ is a measure on a σ algebra in X and an extension of φ.*

In the traditional theory this theorem is the most fundamental tool in order to produce nontrivial measures. However, it has been under quite some criticism. In the traditional frame the attacks are towards the formation $\mathfrak{C}(\cdot)$, as an unmotivated and artificial one, while as a rule no doubt falls upon the outer measure formation $\varphi \mapsto \varphi^\circ$. But the new structure to be described below will disclose that the opposite is true: There are in fact serious deficiencies around the Carathéodory theorem, but it is the particular form of his outer measure which must be blamed for them, whereas the formation $\mathfrak{C}(\cdot)$ remains the decisive methodical idea and even improves when put into the adequate context. The main deficiencies of the Carathéodory theorem can be described as follows.

1) The measure extension it produces is of an obvious *outer regular* character, like φ° itself. It is mysterious how an *inner regular* counterpart could look - while inner regular aspects become more and more important.

2) The measure extension it produces is of an obvious *sequential* character. It is mysterious how a *nonsequential* counterpart could look - while nonsequential aspects become more and more important. Both times the *sum* in the definition of φ° is a crucial obstacle.

3) The proof of the theorem suffers a complete breakdown as soon as one attempts to pass from rings $\mathfrak{S}$ to less restrictive set systems like lattices - while lattices of subsets become more and more important.

All these defects will disappear under the new structure in measure and integration, to which we proceed now. We shall be concerned with the *inner* theories, but with an obvious contrast to Bourbaki [2] from the start. Our ancestors are Kisyński [10] of 1968 and Topsøe [23] of 1970.

Let as before X be a nonvoid set. We adopt a kind of shorthand notation, in that $\bullet = \star\sigma\tau$ marks three parallel theories, where $\star$ stands for *finite*, σ for *sequential* or countable, and τ for *nonsequential* or arbitrary. As an example, let for a nonvoid set system $\mathfrak{S}$ in X denote $\mathfrak{S}_\bullet$ and $\mathfrak{S}^\bullet$ the systems of the intersections and the unions of the nonvoid $\bullet$ subsystems of $\mathfrak{S}$.

In the sequel we assume that $\mathfrak{S}$ is a lattice in X with $\varnothing \in \mathfrak{S}$ and that $\varphi : \mathfrak{S} \to [0,\infty[$ is an isotone set function with $\varphi(\varnothing) = 0$. The basic definition is as follows: We define φ to be an *inner* $\bullet$ *premeasure* iff it can be extended to a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}_\bullet$ such that

α is inner regular $\mathfrak{S}_\bullet$, and
$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (note that $\alpha|\mathfrak{S}_\bullet < \infty$).

We call these set functions α the *inner* $\bullet$ *extensions* of φ.

The subsequent *inner extension theorem* characterizes those φ which are inner $\bullet$ premeasures, and then describes all inner $\bullet$ extensions of φ. The theorem is in terms of the *inner* $\bullet$ *envelopes* $\varphi_\bullet : \mathfrak{P}(X) \to [0,\infty]$ of φ, defined to be

$$\varphi_\bullet(A) = \sup\{\inf_{M \in \mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \downarrow\subset A\},$$

where $\mathfrak{M} \downarrow\subset A$ means that $\mathfrak{M}$ is downward directed with intersection contained in A. We also need their *satellites* $\varphi_\bullet^B : \mathfrak{P}(X) \to [0,\infty]$ with $B \subset X$, defined to be

$$\varphi_\bullet^B(A) = \sup\{\inf_{M \in \mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \downarrow\subset A \text{ and } M \subset B \; \forall M \in \mathfrak{M}\}.$$

We note that $\varphi_\bullet$ is inner regular $\mathfrak{S}_\bullet$. For $A \in \mathfrak{S}$ we have $\varphi(A) \leqq \varphi_\bullet(A)$, and $\varphi(A) = \varphi_\bullet(A)$ iff φ is downward $\bullet$ continuous at A.

2.1 Inner Extension Theorem. *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$. Then the following are equivalent.*

0) *φ is an inner $\bullet$ premeasure.*
1) *φ is supermodular and downward $\bullet$ continuous, and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*
1') *$\varphi(B) = \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*
2) *φ is supermodular and downward $\bullet$ continuous at $\varnothing$, and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*
2') *$\varphi(B) = \varphi(A) + \varphi_\bullet^B(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*
3) *The set function $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extension of φ.*

In this case $\Phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an inner $\bullet$ extension of φ, and a measure *on a σ algebra when $\bullet = \sigma\tau$. All inner $\bullet$ extensions of φ are restrictions of Φ. Moreover we have the* localization principle *which reads*

for $A \subset X$: $S \cap A \in \mathfrak{C}(\varphi_\bullet)$ for all $S \in \mathfrak{S} \Longrightarrow A \in \mathfrak{C}(\varphi_\bullet)$.

Thus we have $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$. It is plain that the members of $\mathfrak{S}_\bullet$ are the most basic measurable subsets.

The prominent rôle of $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ as the *unique maximal inner* $\bullet$ *extension* of φ emphasizes the fundamental nature of Carathéodory's formation $\mathfrak{C}(\cdot)$. There is no such fact in the traditional context: If $\varphi : \mathfrak{S} \to [0,\infty]$ is an upward σ continuous content on a ring $\mathfrak{S}$ in X then $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ need not be a *maximal measure extension* of φ (for example for $\mathfrak{S} = \{\varnothing, X\}$ and $\varphi \neq 0$ one has $\varphi^\circ|\mathfrak{C}(\varphi^\circ) = \varphi$).

We also note a special case of particular importance: The nonvoid set system $\mathfrak{S}$ is called $\bullet$ *compact* iff each nonvoid $\bullet$ subsystem of $\mathfrak{S}$ with intersection $\varnothing$ has a nonvoid *finite* subsystem with intersection $\varnothing$. It is obvious that if the present $\mathfrak{S}$ is $\bullet$ compact then the above functions φ are all downward $\bullet$ continuous at $\varnothing$. Thus the equivalent condition 2) in 2.1 becomes much simpler.

The natural example is that X is a Hausdorff topological space with $\mathfrak{S} = \mathrm{Comp}(X)$. For an isotone set function $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$ then the three conditions $\bullet = \star\sigma\tau$ in 2.1 turn out to be identical, and if fulfilled produce the same $\varphi_\bullet$ and hence the same $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. In this case φ is called a *Radon premeasure* and Φ the *maximal Radon measure* which results from φ. The localization principle implies that $\mathfrak{C}(\varphi_\bullet) \supset \mathrm{Bor}(X)$.

3. The New Stochastic Processes

We fix as before an infinite set T called the *time domain.* But this time we assume the *state space* $(Y, \mathfrak{K})$ to consist of a nonvoid set Y and of a *lattice* $\mathfrak{K}$ in Y which contains the finite subsets of Y and is $\bullet$ compact. We retain the *path space* $X := Y^T$ and the projections $H_t : X \to Y$ for $t \in T$. In $X = Y^T$ we form the finite-based product set system

$$(\mathfrak{K} \cup \{Y\})^{[T]} := \{ \prod_{t \in T} S_t : S_t \in \mathfrak{K} \cup \{Y\} \ \forall\, t \in T \text{ with } S_t = Y \ \forall\forall\, t \in T \},$$

and $\mathfrak{S} := ((\mathfrak{K} \cup \{Y\})^{[T]})^\star$. Thus $\mathfrak{S}$ is a lattice in X with $\varnothing, X \in \mathfrak{S}$, and is $\bullet$ compact after [13] 2.6. This formation is the decisive step in the new enterprise.

Next we let as before I consist of the nonvoid finite subsets $p, q, \cdots$ of T. We retain for $p \in I$ the product set Y^p and the projection $H_p : X \to Y^p$, and for the pairs $p \subset q$ in I the projections $H_{pq} : Y^q \to Y^p$. In Y^p we form the usual product set system $\mathfrak{K}^p := \mathfrak{K} \times \cdots \times \mathfrak{K}$ and the generated lattice $\mathfrak{K}_p = (\mathfrak{K}^p)^\star$. Note that $H_p^{-1}(\mathfrak{K}_p) \subset \mathfrak{S}$, but as a rule $H_{pq}^{-1}(\mathfrak{K}_p) \not\subset \mathfrak{K}_q$ for $p \subset q$ in I.

We turn to the relevant set functions. These are on the one hand on $X = Y^T$ the inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(X) = 1$ (the *inner $\bullet$ prob premeasures* for short) with their maximal inner $\bullet$ extensions $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ (thus with $\Phi(X) = 1$). On the other hand we consider the families $(\varphi_p)_{p \in I}$ of inner $\bullet$ *prob* premeasures $\varphi_p : \mathfrak{K}_p \to [0, \infty[$ with their Φ_p (thus with $\Phi_p(Y^p) = 1$), which are *consistent* in the sense that $\varphi_p = (\varphi_q)_\bullet(H_{pq}^{-1}(\cdot))|\mathfrak{K}_p$ for all $p \subset q$ in I. These entities are connected via the present main result [16] theorem 11$\star$ which follows. It is a comprehensive counterpart of the classical Kolmogorov projective limit theorem invoked in section 1.

3.1 Theorem. *The family of the maps*

$$\varphi \mapsto \varphi_p := \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p \quad \textit{for } p \in I$$

defines a one-to-one correspondence between the inner $\bullet$ prob premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ and the consistent families $(\varphi_p)_{p \in I}$ of inner $\bullet$ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0, \infty[$. It fulfils

$$(\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot)) \ \textit{on } \mathfrak{P}(Y^p) \quad \textit{and} \quad \Phi_p = \overrightarrow{H}_p\Phi \quad \textit{for all } p \in I.$$

Moreover $\Phi(A) = \inf_{p \in I} \Phi_p(H_p(A))$ *for* $A \in \mathfrak{S}_\bullet$.

We want to note that this projective limit theorem appears in [16] theorem $9\star$ in an even more comprehensive version: instead of the fixed state space $(Y,\mathfrak{K})$ one admits a family of individual pairs $(Y_t,\mathfrak{K}_t)$ for the $t\in T$. But for the present context the above specialization will be adequate.

The present result appears to be much more favourable than the traditional one: This time *all* consistent families $(\varphi_p)_{p\in I}$ deserve to be called *solvable*. Also the relations between these families $(\varphi_p)_{p\in I}$ and their projective limits φ look deeper than before. But the main benefit compared with the traditional situation is that in case $\bullet=\tau$ the resultant prob measure $\Phi=\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ on X has an immense domain: In fact, even the most prominent subclass $\mathfrak{S}_\tau\subset\mathfrak{C}(\varphi_\tau)$ contains for example all $A\subset X$ of the form $A=\prod\limits_{t\in T}K_t$ with $K_t\in\mathfrak{K}\cup\{Y\}\ \forall t\in T$, and hence reaches far beyond the class of countably determined subsets. On the other side the result preserves the traditional situation in that one admits *all* inner τ prob premeasures $\varphi:\mathfrak{S}\to[0,\infty[$, and the projective limit theorem 3.1 asserts that all of them remain rooted in the finite subsets of T.

Thus we feel entitled to define a *stochastic process* with time domain T and *state space* $(Y,\mathfrak{K})$, for short for T and $(Y,\mathfrak{K})$, to be an inner τ prob premeasure $\varphi:\mathfrak{S}\to[0,\infty[$. The maximal inner τ extension $\Phi=\varphi_\tau|\mathfrak{C}(\varphi_\tau)$ of φ will be called its *maximal measure*.

We proceed to the comparison with the traditional situation in the most fundamental particular case. The result is [16] theorem $13\star$. Its proof combines the above theorems 2.1 and 3.1 with the basic properties of Polish spaces.

3.2 Theorem. *Assume that Y is a Polish space with $\mathfrak{B}=\mathrm{Bor}(Y)$ and $\mathfrak{K}=\mathrm{Comp}(Y)$. There is a one-to-one correspondence between*

the traditional stochastic processes $\alpha:\mathfrak{A}\to[0,\infty[$ for T and $(Y,\mathfrak{B})$, and
the new stochastic processes $\varphi:\mathfrak{S}\to[0,\infty[$ for T and $(Y,\mathfrak{K})$.

The correspondence rests upon $\mathfrak{S}\subset\mathfrak{A}\subset\mathfrak{C}(\varphi_\tau)$ and reads $\varphi=\alpha|\mathfrak{S}$ and $\alpha=\Phi|\mathfrak{A}$. Moreover $\varphi_\tau=(\alpha^\star|\mathfrak{S}_\tau)_\star\leqq\alpha^\star$.

Proof of the final assertion. We have $\varphi^\star\geqq\alpha^\star\geqq\Phi^\star$ and hence $\varphi^\star|\mathfrak{S}_\tau\geqq\alpha^\star|\mathfrak{S}_\tau\geqq\Phi^\star|\mathfrak{S}_\tau=\Phi|\mathfrak{S}_\tau=\varphi_\tau|\mathfrak{S}_\tau$. Now $\varphi^\star|\mathfrak{S}_\tau=\varphi_\tau|\mathfrak{S}_\tau$ because $\varphi_\tau|\mathfrak{S}_\tau$ is downward τ continuous. Therefore $\varphi_\tau|\mathfrak{S}_\tau=\alpha^\star|\mathfrak{S}_\tau$, and hence $\varphi_\tau=(\varphi_\tau|\mathfrak{S}_\tau)_\star=(\alpha^\star|\mathfrak{S}_\tau)_\star$ since φ_τ is inner regular $\mathfrak{S}_\tau$. $\square$

Thus in the present particular case the situation is as claimed in the introduction: we have a *universal extension procedure* which assigns to each traditional stochastic process $\alpha:\mathfrak{A}\to[0,\infty[$ the maximal measure $\Phi=\varphi_\tau|\mathfrak{C}(\varphi_\tau)$ of its counterpart $\varphi=\alpha|\mathfrak{S}$. These are in fact simple and natural formulae. We complete the comparison with an addendum on the new *essential* subsets $C\in\mathfrak{C}(\varphi_\tau)$ with $\Phi(C)=1$.

3.3 Addendum. *Assume that $C\in\mathfrak{C}(\varphi_\tau)$ with $\Phi(C)=1$. Then $\alpha^\star(C)=1$. Moreover the unique minimal measure extension $\rho:\mathfrak{R}\to[0,\infty[$ of α obtained in* 1.2 *fulfils $\mathfrak{R}\subset\mathfrak{C}(\varphi_\tau)$ and $\rho=\Phi|\mathfrak{R}$.*

Proof. The final assertion in 3.2 implies that $\alpha^\star(C)=1$. Next for $R\in\mathfrak{R}$ we have on the one hand $R\cap C=A\cap C$ for some $A\in\mathfrak{A}$ and hence $R\cap C\in\mathfrak{C}(\varphi_\tau)$, and on the other hand $R\cap C'\in\mathfrak{C}(\varphi_\tau)$, because $C'\in\mathfrak{C}(\varphi_\tau)$

with $\Phi(C')=0$ and Φ is complete. Thus $R\in\mathfrak{C}(\varphi_\tau)$. For $R\in\mathfrak{R}$ now $\Phi(R)=\Phi(R\cap C)\leqq\alpha^\star(R\cap C)=\rho(R)$, and hence $\Phi=\rho$ on $\mathfrak{R}$. $\square$

In the final section 4 we shall invoke the two typical examples with state space $\mathbb{R}$ defined above, in order to convince ourselves that the measure extension Φ has adequate behaviour with respect to its essential subsets.

In the remainder of the present section we continue to assume a Polish state space Y. We equip $X=Y^T$ with the *product topology* and want to describe the partial connection of the new stochastic processes with the topological species of Radon premeasures. The result is [16] corollary 14$\star$.

3.4 Theorem. *Let as before Y be a Polish space with $\mathfrak{B}=\operatorname{Bor}(Y)$ and $\mathfrak{K}=\operatorname{Comp}(Y)$, and let $X=Y^T$ be equipped with the product topology.*

0) *We have* $\mathfrak{S}_\tau\subset\operatorname{Cl}(X)$ (:= the closed subsets of X), *and*

$$\operatorname{Comp}(X)=\{S\in\mathfrak{S}_\tau:S\subset\ \textit{some } F\in\mathfrak{K}^T\}\subset\mathfrak{S}_\tau,$$

with $\mathfrak{K}^T$ the usual product set system. In particular $\operatorname{Comp}(X)=\mathfrak{S}_\tau$ *iff Y is compact.*

1) *Let $\varphi:\mathfrak{S}\to[0,\infty[$ be an inner τ prob premeasure, and assume that*

$$(\circ)\qquad\sup\{\Phi(S):S\in\operatorname{Comp}(X)\}=1.$$

Then $\phi:=\varphi_\tau|\operatorname{Comp}(X)$ is a Radon premeasure with $\phi_\tau=\varphi_\tau$. Hence $\Phi=\phi_\tau|\mathfrak{C}(\phi_\tau)$ is maximal Radon.

We shall see that the assumption $(\circ)$ is fulfilled for the two examples in section 4. But there are natural situations where this assumption is violated. We conclude with a simple example.

3.5 Example. Let the $\vartheta_t:\mathfrak{K}\to[0,\infty[$ for $t\in T$ be inner τ prob premeasures, and the $\varphi_p=\prod_{t\in p}\vartheta_t$ for $p\in I$ be their products in the sense of [13] section 1. Thus the $\varphi_p:\mathfrak{K}_p\to[0,\infty[$ are inner τ prob premeasures with

$$(\varphi_p)_\tau\Big(\prod_{t\in p}A_t\Big)=\prod_{t\in p}(\vartheta_t)_\tau(A_t)\quad\text{for } A_t\subset Y\ \forall t\in p,$$

and hence form a consistent family $(\varphi_p)_{p\in I}$. Let $\varphi:\mathfrak{S}\to[0,\infty[$ with $\Phi=\varphi_\tau|\mathfrak{C}(\varphi_\tau)$ be the resultant stochastic process for T and $(Y,\mathfrak{K})$. We claim that if T is uncountable and $\vartheta_t<1$ on $\mathfrak{K}$ for all $t\in T$ then $\Phi|\operatorname{Comp}(X)=0$, so that assumption $(\circ)$ is violated. In fact, for $S\in\operatorname{Comp}(X)$ we have $S\subset$ some $F\in\mathfrak{K}^T$, that is $F=\prod_{t\in T}K_t$ with $K_t\in\mathfrak{K}\,\forall\,t\in T$. For $p\in I$ thus

$$\Phi(S)\leqq\Phi(F)\leqq\Phi\Big(\prod_{t\in p}K_t\times Y^{T\setminus p}\Big)=\varphi_p\Big(\prod_{t\in p}K_t\Big)=\prod_{t\in p}\vartheta_t(K_t).$$

Now there exists an uncountable $M\subset T$ such that $\vartheta_t(K_t)\leqq$ some $c<1$ for all $t\in M$. It follows that $\Phi(S)\leqq c^{\operatorname{card}(p)}$ for all $p\subset M$ and hence that $\Phi(S)=0$. $\square$

4. Specializations and Examples

The present section assumes $T=[0,\infty[$, and for the initial part a Polish state space Y with $\mathfrak{B}=\operatorname{Bor}(Y)$ and $\mathfrak{K}=\operatorname{Comp}(Y)$ as before. We consider in the path space $X=Y^T$ a few subsets of particular importance

$$C=\mathrm{C}(T,Y)\subset D\subset E\subset F\subset X=Y^T,$$

defined as follows: F consists of the paths $x : T \to Y$ which possess all one-sided limits $x_t^{\pm} \in Y$ for $t \in T$, with the convention $x_0^- := x_0$. Then E consists of the paths $x \in F$ which at each $t \in T$ are either left or right continuous, and D of the paths $x \in F$ which are right continuous at all $t \in T$, the so-called *càdlàg* ones. Note that all these subsets are not countably determined, and hence are not in $\mathfrak{A}$. However, the first main result in [17] theorem 2.4 asserts that for each pair of stochastic processes $\alpha : \mathfrak{A} \to [0,\infty[$ and $\varphi : \mathfrak{S} \to [0,\infty[$ as in 3.2 one has the remarkable fact which follows. The proof combines ideas from Nelson [19] and Tjur [21][22] with the Choquet capacitability theorem.

4.1 Theorem. *Assume that Y fulfils condition*

COMP: *There exists a sequence of compact subsets $K(n) \subset Y\ \forall n \in \mathbb{N}$ such that each compact $K \subset Y$ is contained in some $K(n)$.*

For each pair of stochastic processes α and φ then $C = \mathrm{C}(T, Y)$ and E, F are members of $\mathfrak{C}(\varphi_\tau)$ and fulfil $\alpha^\star(\cdot) = \Phi(\cdot)$.

After this we specialize to $Y = \mathbb{R}$ and turn to the one-dimensional Wiener and Poisson processes as defined in section 1. We want to obtain some basic examples of essential subsets.

We start with the Wiener process α. The basic point for the sequel is the well-known relation

$$\int |H_t - H_s|^a d\alpha = (t-s)^{a/2} M(a) \quad \text{for } 0 \leqq s < t,$$

with some constant $M(a) < \infty$ for all $a > 0$. The main result [15] theorem 6.1 which follows needs but a weakened form of this relation.

4.2 Theorem (Generalized Wiener Process). *Assume that the stochastic process α fulfils*

$$\int |H_t - H_s|^a d\alpha \leqq c(t-s)^{1+b} \quad \textit{for } 0 \leqq s < t,$$

with some real constants $a, b, c > 0$. Fix $0 < \gamma \leqq 1$ with $\gamma < b/a$, and for $m \in \mathbb{N}$ define

$$E_m(\gamma) := \{x \in X : |x_0| \leqq m \text{ and } \\ |x_v - x_u| \leqq m 2^{(u \vee v)(1-\gamma)} |v-u|^\gamma\ \forall\, u, v \in T\}.$$

Then $E_m(\gamma) \in \mathrm{Comp}(X) \subset \mathfrak{S}_\tau$. For $m \to \infty$ we have $E_m(\gamma) \uparrow$ some $E(\gamma) \in (\mathfrak{S}_\tau)^\sigma \subset \mathfrak{C}(\varphi_\tau)$ with $\Phi\big(E(\gamma)\big) = 1$.

Thus the maximal measure Φ lives on $E(\gamma)$. $E(\gamma)$ is a certain class of locally Hölder continuous functions with exponent γ on T. In particular $E(\gamma) \subset \mathrm{C}(T, \mathbb{R})$, so that Φ likewise lives on $\mathrm{C}(T, \mathbb{R})$. After 3.4.1) also Φ is maximal Radon. The Wiener process α itself fulfils all this with the exponents $0 < \gamma < 1/2$.

We pass to the Poisson process α. We start to recall the main result [16] theorem 27$\star$. For $t \in \mathbb{R}$ define $[t] :=$ the largest integer $\leqq t$ and $\{t\} :=$ the smallest integer $\geqq t$.

4.3 Theorem (Poisson Process). *For $m \in \mathbb{N}$ define $Z_m \subset X$ to consist of the $x \in X$ which are integer valued with $x_0 = 0$ and increasing with*

$$x_v - x_u \leqq \{(1/2)\big(\{2^n v\} - [2^n u]\big)\} \leqq \{2^n(v-u)\}$$
$$\text{for all } 0 \leqq u < v \leqq n \text{ and } m \leqq n \in \mathbb{N}.$$

Then $Z_m \in \mathrm{Comp}(X) \subset \mathfrak{S}_\tau$, and the $x \in Z_m$ fulfil $x_t^+ - x_t^- \leqq 1$ for all $t \in T$. For $m \to \infty$ we have $Z_m \uparrow$ some $Z \in (\mathfrak{S}_\tau)^\sigma \subset \mathfrak{C}(\varphi_\tau)$ with $\Phi(Z) = 1$ (in [16] the notations were $Z_m = E_m(T)$ and $Z = E(T)$).

Thus the maximal measure Φ lives on Z. In view of 3.4.1) also Φ is maximal Radon. The properties of Z furnish at once that $Z \subset$ the above E. It follows that $\Phi(E) = 1$. Note that our treatment of the Poisson process started from the state space $Y = \mathbb{R}$, in contrast to the traditional approach which started from the discrete state space $Y = \mathbb{Z}$.

After this we turn to the above subset $D \subset E$ which is more delicate.

We recall from [17] section 5 the subset $X^\circ \subset X$ to consist of the paths $x : T \to \mathbb{R}$ which are integer valued with $x_0 = 0$ and increasing, so that $x \in F$, which fulfil $x_t^+ - x_t^- \leqq 1$ for all $t \in T$, so that $x \in E$, and which are unbounded $x_t \uparrow \infty$ for $t \uparrow \infty$. Thus $Z \cap X^\circ$ consists of the unbounded members of Z. It follows from 4.3 combined with [16] proposition 30.2)$\star$ that $Z \cap X^\circ \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(Z \cap X^\circ) = 1$. This implies that $X^\circ \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(X^\circ) = 1$.

Now we note that each path $x \in X^\circ \subset E$ has an infinite sequence of jump points $0 \leqq t(x,1) < \cdots < t(x,n) < \cdots$, each of height =1, and with $t(x,n) \uparrow \infty$ for $n \to \infty$. For $n \in \mathbb{N}$ define $L(n) := \{x \in X^\circ : x_{t(x,n)} = n-1\}$ and $R(n) := \{x \in X^\circ : x_{t(x,n)} = n\}$ to consist of those paths which are left/right continuous at the nth jump point. Thus $X^\circ = L(n) \cup R(n)$ and $L(n) \cap R(n) = \varnothing$. Then the second main result in [17] theorem 2.5 reads as follows. Its last assertion (but not the quantitative first one) has a certain precedent in Tjur [22] 10.1.2 and 10.9.4.

4.4 Theorem. *We have $\varphi_\tau(L(n)) = \varphi_\tau(R(n)) = 0$ for all $n \in \mathbb{N}$. Thus $L(n)$ and $R(n)$ are not in $\mathfrak{C}(\varphi_\tau)$.*

For $n \in \mathbb{N}$ now define $J_n : X^\circ \to [0,\infty[$ to be $J_n(x) = x_{t(x,n)}$, the value of $x \in X^\circ$ at its nth jump point. Then $\{x \in X^\circ : J_n(x) = n-1\} = L(n)$ and $\{x \in X^\circ : J_n(x) = n\} = R(n)$ show that the functions J_n are *extremely nonmeasurable* $\mathfrak{C}(\varphi_\tau)$, in spite of the immense size of $\mathfrak{C}(\varphi_\tau)$: one has to face that the values of the paths $x \in X^\circ$ at their jump points are not substantial!

Next we note that $D \cap X^\circ = \bigcap_{n\in\mathbb{N}} R(n)$ from the definition. Therefore $\varphi_\tau(D \cap X^\circ) = 0$, and combined with $X^\circ \in \mathfrak{C}(\varphi_\tau)$ and $\Phi(X^\circ) = 1$ we obtain

$$\varphi_\tau(D) = \varphi_\tau(D \cap X^\circ) + \varphi_\tau(D \cap (X^\circ)') = 0.$$

It follows that the subsets D and $D \cap X^\circ$ are either not in $\mathfrak{C}(\varphi_\tau)$, or are in $\mathfrak{C}(\varphi_\tau)$ with $\Phi(\cdot) = 0$. At present the answer is not known.

The above observations are in sharp contrast to the traditional result that $\alpha^\star(D) = 1$ and even $\alpha^\star(D \cap X^\circ) = 1$, which in the present frame results from [16] remark 29$\star$ and proposition 30.2)$\star$. The traditional treatment of the Poisson process is in terms of D or rather of $D \cap X^\circ$, that means in terms of measure extensions of the canonical measure α which live on these sets. We see that D and $D \cap X^\circ$ cannot maintain this position when the treatment is based on the present φ and its maximal measure Φ. The present author thinks that the previous emphasis on D and $D \cap X^{\circ}$ rests upon questionable former ideas and will be abandoned in the future.

At last we want to note that in both of the above examples the pathological thick subsets $C(a)$ for the $a \in X$ in theorem 1.3 are in $\mathfrak{C}(\varphi_\tau)$ with $\Phi(C(a)) = 0$.

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